SECTION 6.5 EXERCISES

SKILLS

In Exercises 1–16, find the exact value of each expression. Give the answer in radians.

1. $\arccos\left(\frac{\sqrt{2}}{2}\right)$ **2.** $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ **3.** $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ **4.** $\arcsin\left(\frac{1}{2}\right)$

5. $\cot^{-1}(-1)$ **6.** $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ **7.** $\operatorname{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$ **8.** $\operatorname{arccsc}(-1)$

9. $\csc^{-1}2$ **10.** $\sec^{-1}(-2)$ **11.** $\arctan(-\sqrt{3})$ **12.** $\operatorname{arccot}(\sqrt{3})$

13. $\sin^{-1}0$ **14.** $\tan^{-1}1$ **15.** $\sec^{-1}(-1)$ **16.** $\cot^{-1}0$

In Exercises 17–32, find the exact value of each expression. Give the answer in degrees.

17. $\cos^{-1}\left(\frac{1}{2}\right)$ **18.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **19.** $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ **20.** $\sin^{-1}0$

21. $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **22.** $\tan^{-1}(-\sqrt{3})$ **23.** $\arctan\left(\frac{\sqrt{3}}{3}\right)$ **24.** $\operatorname{arccot}1$

25. $\operatorname{arccsc}(-2)$ **26.** $\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$ **27.** $\operatorname{arcsec}(-\sqrt{2})$ **28.** $\operatorname{arccsc}(-\sqrt{2})$

29. $\sin^{-1}(-1)$ **30.** $\arctan(-1)$ **31.** $\operatorname{arccot}0$ **32.** $\operatorname{arcsec}(-1)$

In Exercises 33–42, use a calculator to evaluate each expression. Give the answer in degrees and round it to two decimal places.

33. $\cos^{-1}(0.5432)$ **34.** $\sin^{-1}(0.7821)$ **35.** $\tan^{-1}(1.895)$ **36.** $\tan^{-1}(3.2678)$

37. $\sec^{-1}(1.4973)$ **38.** $\sec^{-1}(2.7864)$ **39.** $\csc^{-1}(-3.7893)$ **40.** $\csc^{-1}(-6.1324)$

41. $\cot^{-1}(-4.2319)$ **42.** $\cot^{-1}(-0.8977)$

In Exercises 43–52, use a calculator to evaluate each expression. Give the answer in radians and round it to two decimal places.

43. $\sin^{-1}(-0.5878)$ **44.** $\sin^{-1}(0.8660)$ **45.** $\cos^{-1}(0.1423)$ **46.** $\tan^{-1}(-0.9279)$

47. $\tan^{-1}(1.3242)$ **48.** $\cot^{-1}(2.4142)$ **49.** $\cot^{-1}(-0.5774)$ **50.** $\sec^{-1}(-1.0422)$

51. $\csc^{-1}(3.2361)$ **52.** $\csc^{-1}(-2.9238)$

In Exercises 53–76, evaluate each expression exactly, if possible. If not possible, state why.

53. $\sin^{-1}\left[\sin\left(\frac{5\pi}{12}\right)\right]$ **54.** $\sin^{-1}\left[\sin\left(-\frac{5\pi}{12}\right)\right]$ **55.** $\sin[\sin^{-1}(1.03)]$ **56.** $\sin[\sin^{-1}(1.1)]$

57. $\sin^{-1}\left[\sin\left(-\frac{7\pi}{6}\right)\right]$ **58.** $\sin^{-1}\left[\sin\left(\frac{7\pi}{6}\right)\right]$ **59.** $\cos^{-1}\left[\cos\left(\frac{4\pi}{3}\right)\right]$ **60.** $\cos^{-1}\left[\cos\left(-\frac{5\pi}{3}\right)\right]$

61. $\cot[\cot^{-1}(\sqrt{3})]$ **62.** $\cot^{-1}\left[\cot\left(\frac{5\pi}{4}\right)\right]$ **63.** $\sec^{-1}\left[\sec\left(-\frac{\pi}{3}\right)\right]$ **64.** $\sec\left[\sec^{-1}\left(\frac{1}{2}\right)\right]$

65. $\csc\left[\csc^{-1}\left(\frac{1}{2}\right)\right]$ **66.** $\csc^{-1}\left[\csc\left(\frac{7\pi}{6}\right)\right]$ **67.** $\cot(\cot^{-1}0)$ **68.** $\cot^{-1}\left[\cot\left(-\frac{\pi}{4}\right)\right]$

69. $\tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]$ **70.** $\tan^{-1}\left[\tan\left(\frac{\pi}{4}\right)\right]$ **71.** $\sec(\sec^{-1}0)$ **72.** $\csc^{-1}(\csc\pi)$

73. $\cot^{-1}\left[\cot\left(\frac{8\pi}{3}\right)\right]$ **74.** $\tan^{-1}[\tan(8\pi)]$ **75.** $\csc^{-1}\left[\csc\left(\frac{15\pi}{4}\right)\right]$ **76.** $\sec^{-1}\left[\sec\left(\frac{17\pi}{2}\right)\right]$

In Exercises 77–88, evaluate each expression exactly.

77. $\cos\left[\sin^{-1}\left(\frac{3}{4}\right)\right]$

78. $\sin\left[\cos^{-1}\left(\frac{2}{3}\right)\right]$

79. $\sin\left[\tan^{-1}\left(\frac{12}{5}\right)\right]$

80. $\cos\left[\tan^{-1}\left(\frac{7}{24}\right)\right]$

81. $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)\right]$

82. $\tan\left[\cos^{-1}\left(\frac{2}{5}\right)\right]$

83. $\sec\left[\sin^{-1}\left(\frac{\sqrt{2}}{5}\right)\right]$

84. $\sec\left[\cos^{-1}\left(\frac{\sqrt{7}}{4}\right)\right]$

85. $\csc\left[\cos^{-1}\left(\frac{1}{4}\right)\right]$

86. $\csc\left[\sin^{-1}\left(\frac{1}{4}\right)\right]$

87. $\cot\left[\sin^{-1}\left(\frac{60}{61}\right)\right]$

88. $\cot\left[\sec^{-1}\left(\frac{41}{9}\right)\right]$

89. $\cos\left[\tan^{-1}\left(\frac{3}{4}\right) - \sin^{-1}\left(\frac{4}{5}\right)\right]$

90. $\cos\left[\tan^{-1}\left(\frac{12}{5}\right) + \sin^{-1}\left(\frac{3}{5}\right)\right]$

91. $\sin\left[\cos^{-1}\left(\frac{5}{13}\right) + \tan^{-1}\left(\frac{4}{3}\right)\right]$

92. $\sin\left[\cos^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{5}{12}\right)\right]$

93. $\sin\left[2\cos^{-1}\left(\frac{3}{5}\right)\right]$

94. $\cos\left[2\sin^{-1}\left(\frac{3}{5}\right)\right]$

95. $\tan\left[2\sin^{-1}\left(\frac{5}{13}\right)\right]$

96. $\tan\left[2\cos^{-1}\left(\frac{5}{13}\right)\right]$

For each of the following expressions, write an equivalent expression in terms of only the variable *u*.

97. $\cos(\sin^{-1}u)$

98. $\sin(\cos^{-1}u)$

99. $\tan(\cos^{-1}u)$

100. $\tan(\sin^{-1}u)$

APPLICATIONS

For Exercises 101 and 102, refer to the following:

Annual sales of a product are generally subject to seasonal fluctuations and are approximated by the function

$$s(t) = 4.3\cos\left(\frac{\pi}{6}t\right) + 56.2 \quad 0 \le t \le 11$$

where t represents time in months ($t = 0$ represents January) and $s(t)$ represents monthly sales of the product in thousands of dollars.

101. Business. Find the month(s) in which monthly sales are \$56,200.

102. Business. Find the month(s) in which monthly sales are \$51,900.

For Exercises 103 and 104, refer to the following:

Allergy sufferers' symptoms fluctuate with pollen levels. Pollen levels are often reported to the public on a scale of 0–12, which is meant to reflect the levels of pollen in the air. For example, a pollen level between 4.9 and 7.2 indicates that pollen levels will likely cause symptoms for many individuals allergic to the predominant pollen of the season (*Source*: http://www.pollen.com). The pollen levels at a single location were measured and averaged for each month. Over a period of 6 months, the levels fluctuated according to the model

$$p(t) = 5.5 + 1.5\sin\left(\frac{\pi}{6}t\right) \quad 0 \le t \le 6$$

where t is measured in months and $p(t)$ is the pollen level.

103. Biology/Health. In which month(s) was the monthly average pollen level 7.0?

104. Biology/Health. In which month(s) was the monthly average pollen level 6.25?

105. Alternating Current. Alternating electrical current in amperes (A) is modeled by the equation $i = I\sin(2\pi ft)$, where i is the current, I is the maximum current, t is time in seconds, and f is the frequency in hertz is the number of cycles per second. If the frequency is 5 hertz and maximum current is 115 angstrom, what time t corresponds to a current of 85 angstrom? Find the smallest positive value of t.

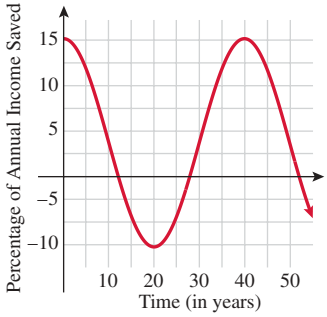
106. Alternating Current. If the frequency is 100 hertz and maximum current is 240 angstrom, what time t corresponds to a current of 100 angstrom? Find the smallest positive value of t.

107. Hours of Daylight. The number of hours of daylight in San Diego, California, can be modeled with $H(t) = 12 + 2.4\sin(0.017t - 1.377)$, where t is the day of the year (January 1, $t = 1$, etc.). For what value of t is the number of hours of daylight equal to 14.4? If May 31 is the 151st day of the year, what month and day correspond to that value of t?

108. Hours of Daylight. Repeat Exercise 107. For what value of t is the number of hours of daylight equal to 9.6? What month and day correspond to the value of t? (You may have to count backwards.)

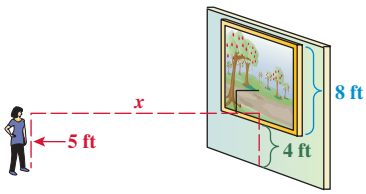
109. Money. A young couple get married and immediately start saving money. They renovate a house and are left with less and less saved money. They have children after 10 years and are in debt until their children are in college. They then save until retirement. A formula that represents the percentage of their annual income that they either save (positive) or are in debt (negative) is given by $P(t) = 12.5\cos(0.157t) + 2.5$, where $t = 0$ corresponds

to the year they were married. How many years into their marriage do they first accrue debt?



110. Money. For the couple in Exercise 109, how many years into their marriage are they back to saving 15% of their annual income?

111. Viewing Angle of Painting. A museum patron whose eye level is 5 feet above the floor is studying a painting that is 8 feet in height and mounted on the wall 4 feet above the floor. If the patron is x feet from the wall, use $\tan(\alpha + \beta)$ to express $\tan\theta$, where θ is the angle that the patron's eye sweeps from the top to the bottom of the painting.

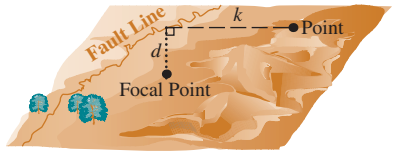


112. Viewing Angle of Painting. Using the equation for $\tan\theta$ in Exercise 111, solve for θ using the inverse tangent. Then find the measure of the angles θ for $x = 10$ and $x = 20$ (to the nearest degree).

113. Earthquake Movement. The horizontal movement of a point that is k kilometers away from an earthquake's fault line can be estimated with

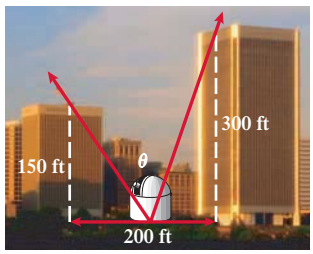
$$M = \frac{f}{2}\left[1 - \frac{2\tan^{-1}\left(\frac{k}{d}\right)}{\pi}\right]$$

where M is the movement of the point in meters, f is the total horizontal displacement occurring along the fault line, k is the distance of the point from the fault line, and d is the depth in kilometers of the focal point of the earthquake. If an earthquake produces a displacement f of 2 meters and the depth of the focal point is 4 kilometers, then what is the movement M of a point that is 2 kilometers from the fault line? of a point 10 kilometers from the fault line?



114. Earthquake Movement. Repeat Exercise 113. If an earthquake produces a displacement f of 3 meters and the depth of the focal point is 2.5 kilometers, then what is the movement M of a point that is 5 kilometers from the fault line? of a point 10 kilometers from the fault line?

115. Laser Communication. A laser communication system depends on a narrow beam, and a direct line of sight is necessary for communication links. If a transmitter/receiver for a laser system is placed between two buildings (see the figure) and the other end of the system is located on a low-earth-orbit satellite, then the link is operational only when the satellite and the ground system have a line of sight (when the buildings are not in the way). Find the angle θ that corresponds to the system being operational (i.e., find the maximum value of θ that permits the system to be operational). Express θ in terms of inverse tangent functions and the distance from the shorter building.



Corbis Digital Stock

116. Laser Communication. Repeat Exercise 115, assuming that the ground system is on top of a 20-foot tower.

CATCH THE MISTAKE

In Exercises 117–120, explain the mistake that is made.

117. Evaluate the expression exactly: $\sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right]$.

Solution:

Use the identity $\sin^{-1}(\sin x) = x$ on $0 \le x \le \pi$.

Since $\frac{3\pi}{5}$ is in the interval $[0, \pi]$, the identity can be used.

$$\sin^{-1}\left[\sin\left(\frac{3\pi}{5}\right)\right] = \frac{3\pi}{5}$$

This is incorrect. What mistake was made?

118. Evaluate the expression exactly: $\cos^{-1}\left[\cos\left(-\frac{\pi}{5}\right)\right]$.

Solution:

Use the identity $\cos^{-1}(\cos x) = x$ on $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Since $-\frac{\pi}{5}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the identity can be used.

$$\cos^{-1}\left[\cos\left(-\frac{\pi}{5}\right)\right] = -\frac{\pi}{5}$$

This is incorrect. What mistake was made?

119. Evaluate the expression exactly: $\cot^{-1}(2.5)$.

Solution:

Use the reciprocal identity. $\cot^{-1}(2.5) = \dfrac{1}{\tan^{-1}(2.5)}$

Evaluate $\tan^{-1}(2.5) = 1.19$. $\cot^{-1}(2.5) = \dfrac{1}{1.19}$

Simplify. $\cot^{-1}(2.5) = 0.8403$

This is incorrect. What mistake was made?

120. Evaluate the expression exactly: $\csc^{-1}\left(\dfrac{1}{4}\right)$.

Solution:

Use the reciprocal identity. $\csc^{-1}\left(\dfrac{1}{4}\right) = \dfrac{1}{\sin^{-1}\left(\dfrac{1}{4}\right)}$

Evaluate $\sin^{-1}\left(\frac{1}{4}\right) = 14.478$. $\csc^{-1}\left(\dfrac{1}{4}\right) = \dfrac{1}{14.478}$

Simplify. $\csc^{-1}\left(\dfrac{1}{4}\right) = 0.0691$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 121–124, determine whether each statement is true or false.

121. The inverse secant function is an even function.

122. The inverse cosecant function is an odd function.

123. $\csc^{-1}(\csc\theta) = \theta$, for all θ in the domain of cosecant.

124. $\sin^{-1}(2x)\cdot\csc^{-1}(2x) = 1$, for all x for which both functions are defined.

125. Explain why $\sec^{-1}\left(\frac{1}{2}\right)$ does not exist.

126. Explain why $\csc^{-1}\left(\frac{1}{2}\right)$ does not exist.

CHALLENGE

127. Evaluate exactly: $\sin\left[\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) + \sin^{-1}\left(-\dfrac{1}{2}\right)\right]$.

128. Determine the x-values for which

$$\sin^{-1}\left[2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)\right] = 3x$$

129. Evaluate exactly: $\sin(2\sin^{-1}1)$.

130. Let $f(x) = 2 - 4\sin\left(x - \dfrac{\pi}{2}\right)$.

a. State an accepted domain of $f(x)$ so that $f(x)$ is a one-to-one function.

b. Find $f^{-1}(x)$ and state its domain.

131. Let $f(x) = 3 + \cos\left(x - \dfrac{\pi}{4}\right)$.

a. State an accepted domain of $f(x)$ so that $f(x)$ is a one-to-one function.

b. Find $f^{-1}(x)$ and state its domain.

132. Let $f(x) = 1 - \tan\left(x + \dfrac{\pi}{3}\right)$.

a. State an accepted domain of $f(x)$ so that $f(x)$ is a one-to-one function.

b. Find $f^{-1}(x)$ and state its domain.

133. Let $f(x) = 2 + \dfrac{1}{4}\cot\left(2x - \dfrac{\pi}{6}\right)$.

a. State an accepted domain of $f(x)$ so that $f(x)$ is a one-to-one function.

b. Find $f^{-1}(x)$ and state its domain.

134. Let $f(x) = -\csc\left(\dfrac{\pi}{4}x - 1\right)$.

a. State an accepted domain of $f(x)$ so that $f(x)$ is a one-to-one function.

b. Find $f^{-1}(x)$ and state its domain.

TECHNOLOGY

135. Use a graphing calculator to plot $Y_1 = \sin(\sin^{-1}x)$ and $Y_2 = x$ for the domain $-1 \le x \le 1$. If you then increase the domain to $-3 \le x \le 3$, you get a different result. Explain the result.

136. Use a graphing calculator to plot $Y_1 = \cos(\cos^{-1}x)$ and $Y_2 = x$ for the domain $-1 \le x \le 1$.If you then increase the domain to $-3 \le x \le 3$, you get a different result. Explain the result.

137. Use a graphing calculator to plot $Y_1 = \csc^{-1}(\csc x)$ and $Y_2 = x$. Determine the domain for which the following statement is true: $\csc^{-1}(\csc x) = x$. Give the domain in terms of π.

138. Use a graphing calculator to plot $Y_1 = \sec^{-1}(\sec x)$ and $Y_2 = x$. Determine the domain for which the following statement is true: $\sec^{-1}(\sec x) = x$. Give the domain in terms of π.

139. Given $\tan x = \dfrac{40}{9}$ and $\pi < x < \dfrac{3\pi}{2}$:

a. Find $\sin(2x)$ using the double-angle identity.

b. Use the inverse of tangent to find x in quadrant III and use a calculator to find $\sin(2x)$. Round to five decimal places.

c. Are the results in (a) and (b) the same?

140. Given $\sin x = -\dfrac{1}{\sqrt{10}}$ and $\dfrac{3\pi}{2} < x < 2\pi$:

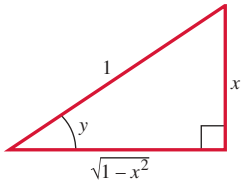
a. Find $\tan(2x)$ using the double-angle identity.

b. Use the inverse of sine to find x in quadrant IV and find $\tan(2x)$.

c. Are the results in (a) and (b) the same?

PREVIEW TO CALCULUS

In calculus, we study the derivatives of inverse trigonometric functions. In order to obtain these formulas, we use the definitions of the functions and a right triangle. Thus, if $y = \sin^{-1}x$, then $\sin y = x$; the right triangle associated with this equation is given below, where we can see that $\cos y = \sqrt{1 - x^2}$.



In Exercises 141–144, use this idea to find the indicated expression.

141. If $y = \tan^{-1}x$, find $\sec^2 y$.

142. If $y = \cos^{-1}x$, find $\sin y$.

143. If $y = \sec^{-1}x$, find $\sec y \tan y$.

144. If $y = \cot^{-1}x$, find $\csc^2 y$.

SECTION 6.6 TRIGONOMETRIC EQUATIONS

SKILLS OBJECTIVES

- Solve trigonometric equations by inspection.
- Solve trigonometric equations using algebraic techniques.
- Solve trigonometric equations using inverse functions.
- Solve trigonometric equations (involving more than one trigonometric function) using trigonometric identities.

CONCEPTUAL OBJECTIVES

- Understand that solving trigonometric equations is similar to solving algebraic equations.
- Realize that the goal in solving trigonometric equations is to find the value(s) for the independent variable that make(s) the equation a true statement.

Solving Trigonometric Equations by Inspection

The goal in solving equations in one variable is to find the values for that variable which make the equation true. For example, $9x = 72$ can be solved by inspection by asking the question, "9 times what is 72?" The answer is $x = 8$. We approach simple trigonometric equations the same way we approach algebraic equations: We inspect the equation and determine the solution.

EXAMPLE 1 Solving a Trigonometric Equation by Inspection

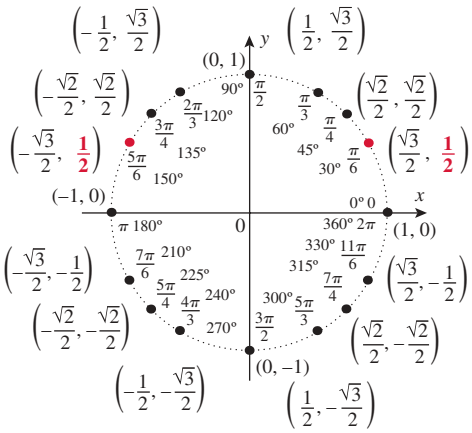
Solve each of the following equations on the interval $[0, 2\pi]$:

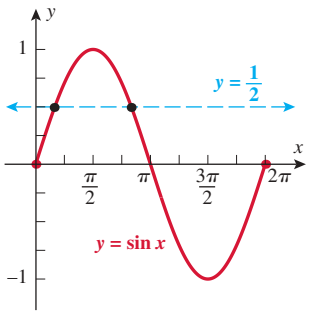
a. $\sin x = \frac{1}{2}$ **b.** $\cos(2x) = \frac{1}{2}$

Solution (a):

Ask the question, "sine of what angles is $\frac{1}{2}$?"

$$\boxed{x = \frac{\pi}{6} \quad \text{or} \quad x = \frac{5\pi}{6}}$$





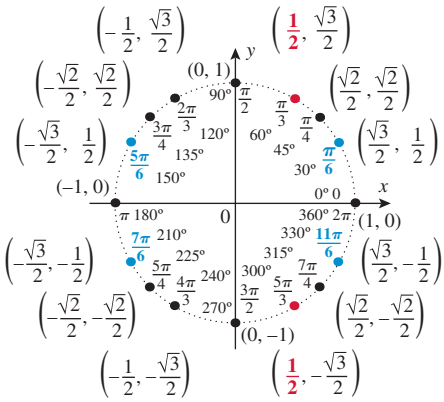
Solution (b):

Ask the question, "cosine of what angles is $\frac{1}{2}$?"

In this case, the angle is equal to $2x$.

$$2x = \frac{\pi}{3} \quad \text{or} \quad 2x = \frac{5\pi}{3}$$

Solve for x: $\boxed{x = \frac{\pi}{6} \quad \text{or} \quad x = \frac{5\pi}{6}}$.



If the solution set for x is over $[0, 2\pi)$, then the solution set for $2x$ is over $[0, 4\pi)$.

Notice that $\boxed{x = \frac{7\pi}{6} \quad \text{or} \quad x = \frac{11\pi}{6}}$ also satisfy the equation.

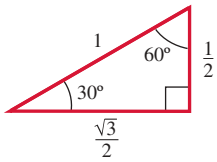
$\cos(2x) = \frac{1}{2}$.

YOUR TURN Solve each of the following equations on the interval $[0, 2\pi)$:

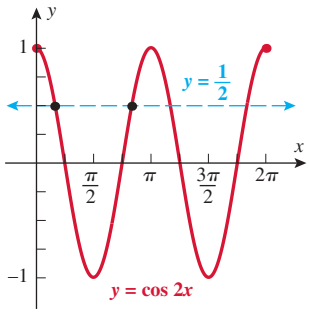
a. $\cos x = \frac{1}{2}$ **b.** $\sin(2x) = \frac{1}{2}$

Study Tip

Recall the special triangle.



$\sin 30° = \frac{1}{2}$

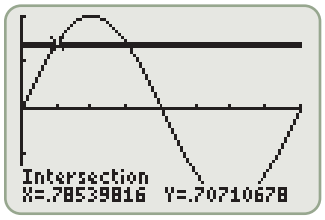


Answer: **a.** $x = \frac{\pi}{3}, \frac{5\pi}{3}$

b. $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

Technology Tip

Use the fact that a solution to the equation $\sin x = \frac{\sqrt{2}}{2}$ is the same as a point of intersection of $y = \sin x$ and $y = \frac{\sqrt{2}}{2}$ over one period, $[0, 2\pi)$ or $[0, 360°)$.

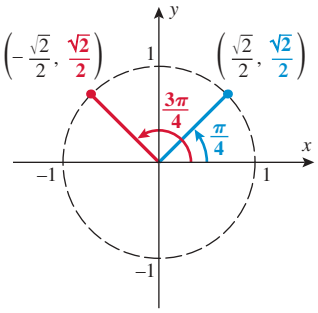


Study Tip

Find *all* solutions unless the domain is restricted.

EXAMPLE 2 Solving a Trigonometric Equation by Inspection

Solve the equation $\sin x = \frac{\sqrt{2}}{2}$.



Solution:

STEP 1 Solve over one period, $[0, 2\pi)$.

Ask the question, "sine of what angles is $\frac{\sqrt{2}}{2}$?"

The sine function is positive in quadrants I and II.

DEGREES	$x = 45°$ or $x = 135°$
RADIANS	$x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$

STEP 2 Solve over all real numbers.

Since the sine function has a period of 360° or 2π, adding integer multiples of 360° or 2π will give the other (infinitely many) solutions.

DEGREES	$x = 45° + 360°n$ or $x = 135° + 360°n$
RADIANS	$x = \frac{\pi}{4} + 2n\pi$ or $x = \frac{3\pi}{4} + 2n\pi$, where n is any integer

YOUR TURN Solve the equation $\cos x = \frac{1}{2}$.

Answer:

DEGREES	$x = 60° + 360°n$ or $x = 300° + 360°n$
RADIANS	$x = \frac{\pi}{3} + 2n\pi$ or $x = \frac{5\pi}{3} + 2n\pi$, where n is any integer

Notice that the equations in Example 2 and Your Turn have an infinite number of solutions. Unless the domain is restricted, you must find *all* solutions.

EXAMPLE 3 Solving a Trigonometric Equation by Inspection

Solve the equation $\tan(2x) = -\sqrt{3}$.

Solution:

STEP 1 Solve over one period, $[0, \pi)$.

Ask the question, "tangent of what angles is $-\sqrt{3}$?" Note that the angle in this case is $2x$.

DEGREES	$2x = 120°$
RADIANS	$2x = \frac{2\pi}{3}$

The tangent function is negative in quadrants II and IV. Since $[0, \pi)$ includes quadrants I and II, we find only the angle in quadrant II. (The solution corresponding to quadrant IV will be found when we extend the solution over all real numbers.)

STEP 2 Solve over all x.

Since the tangent function has a period of 180°, or π, adding integer multiples of 180° or π will give all of the other solutions.

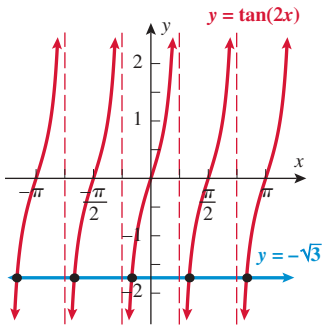
DEGREES	$2x = 120° + 180°n$
RADIANS	$2x = \frac{2\pi}{3} + n\pi$, where n is any integer

Solve for x by dividing by 2.

DEGREES	$x = 60° + 90°n$
RADIANS	$x = \frac{\pi}{3} + \frac{n}{2}\pi$, where n is any integer

Note:

- There are infinitely many solutions. If we graph $y = \tan(2x)$ and $y = -\sqrt{3}$, we see that there are infinitely many points of intersection.

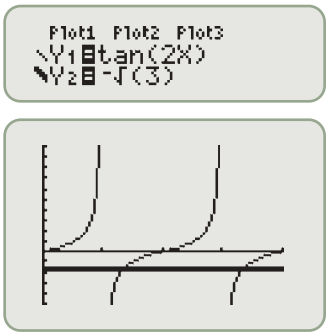


- Had we restricted the domain to $0 \le x < 2\pi$, the solutions (in radians) would be the values given to the right in the table.

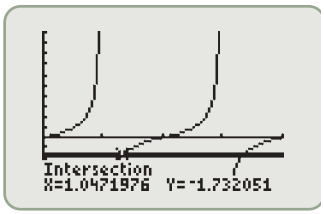
n	$x = \frac{\pi}{3} + \frac{n}{2}\pi$
0	$x = \frac{\pi}{3}$
1	$x = \frac{5\pi}{6}$
2	$x = \frac{4\pi}{3}$
3	$x = \frac{11\pi}{6}$

Notice that only $n = 0, 1, 2, 3$ yield x-values in the domain $0 \le x < 2\pi$.

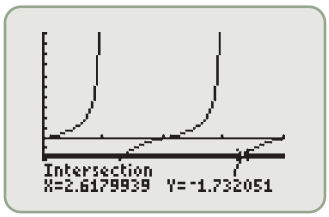
Technology Tip



To find the point of intersection, use 2nd TRACE for CALC; move the down arrow to 5: Intersect; type ENTER for the first curve, ENTER for the second curve, 0.8 for guess, and ENTER.



For the second answer, you need a number close to the answer for the guess. Type 2.5 for guess, and ENTER.



Notice that in Step 2 of Example 2, $2n\pi$ was added to get all of the solutions, whereas in Step 2 of Example 3, we added $n\pi$ to the argument of the tangent function. The reason that we added $2n\pi$ in Example 2 and $n\pi$ in Example 3 is because the sine function has period 2π, whereas the tangent function has period π.

Solving Trigonometric Equations Using Algebraic Techniques

We now will use algebraic techniques to solve trigonometric equations. Let us first start with linear and quadratic equations. For linear equations, we solve for the variable by isolating it. For quadratic equations, we often employ factoring or the quadratic formula. If we can let x represent the trigonometric function and the resulting equation is either linear or quadratic, then we use techniques learned in solving algebraic equations.

Type	Equation	Substitution	Algebraic Equation
Linear trigonometric equation	$4\sin\theta - 2 = -4$	$u = \sin\theta$	$4u - 2 = -4$
Quadratic trigonometric equation	$2\cos^2\theta + \cos\theta - 1 = 0$	$u = \cos\theta$	$2u^2 + u - 1 = 0$

It is not necessary to make the substitution, though it is convenient. Frequently, one can see how to factor a quadratic trigonometric equation without first converting it to an algebraic equation. In Example 4, we will not use a substitution. However, in Example 5, we will illustrate the use of a substitution.

EXAMPLE 4 Solving a Linear Trigonometric Equation

Solve $4\sin\theta - 2 = -4$ on $0 \le \theta < 2\pi$.

Solution:

Step 1 Solve for $\sin\theta$.

$$4\sin\theta - 2 = -4$$

Add 2.

$$4\sin\theta = -2$$

Divide by 4.

$$\sin\theta = -\frac{1}{2}$$

Step 2 Find the values of θ on $0 \le \theta < 2\pi$ that satisfy the equation $\sin\theta = -\frac{1}{2}$.

The sine function is negative in quadrants III and IV.

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2} \text{ and } \sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}. \qquad \boxed{\theta = \frac{7\pi}{6}} \text{ or } \boxed{\theta = \frac{11\pi}{6}}$$

■ **YOUR TURN** Solve $2\cos\theta + 1 = 2$ on $0 \le \theta < 2\pi$.

■ **Answer:** $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

EXAMPLE 5 Solving a Quadratic Trigonometric Equation

Solve $2\cos^2\theta + \cos\theta - 1 = 0$ on $0 \le \theta < 2\pi$.

Solution:

STEP 1 Solve for $\cos\theta$.

	$2\cos^2\theta + \cos\theta - 1 = 0$
Let $u = \cos\theta$.	$2u^2 + u - 1 = 0$
Factor the quadratic equation.	$(2u - 1)(u + 1) = 0$
Set each factor equal to 0.	$2u - 1 = 0$ or $u + 1 = 0$
Solve each for u.	$u = \frac{1}{2}$ or $u = -1$
Substitute $u = \cos\theta$.	$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

STEP 2 Find the values of θ on $0 \le \theta < 2\pi$ which satisfy the equation $\cos\theta = \frac{1}{2}$.

The cosine function is positive in quadrants I and IV.

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$. $\boxed{\theta = \frac{\pi}{3}}$ or $\boxed{\theta = \frac{5\pi}{3}}$

STEP 3 Find the values of θ on $0 \le \theta < 2\pi$ that satisfy the equation $\cos\theta = -1$.

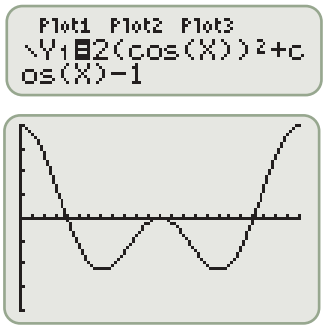
$\cos\pi = -1$. $\boxed{\theta = \pi}$

The solutions to $2\cos^2\theta + \cos\theta - 1 = 0$ on $0 \le \theta < 2\pi$ are $\theta = \frac{\pi}{3}$, $\theta = \frac{5\pi}{3}$, and $\theta = \pi$.

■ **YOUR TURN** Solve $2\sin^2\theta - \sin\theta - 1 = 0$ on $0 \le \theta < 2\pi$.

■ **Answer:** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$

Technology Tip



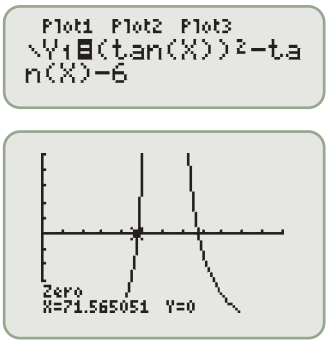
Study Tip

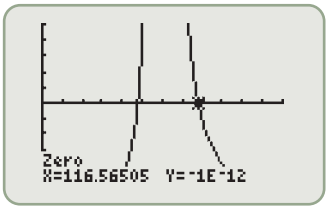
If Example 5 asked for the solution to the trigonometric equation over all real numbers, then the solutions would be $\theta = \frac{\pi}{3} \pm 2n\pi, \frac{5\pi}{3} \pm 2n\pi$, and $\pi \pm 2n\pi$.

Solving Trigonometric Equations That Require the Use of Inverse Functions

Thus far, we have been able to solve the trigonometric equations exactly. Now we turn our attention to situations that require using a calculator and inverse functions to approximate a solution to a trigonometric equation.

Technology Tip





EXAMPLE 6 Solving a Trigonometric Equation That Requires the Use of Inverse Functions

Solve $\tan^2\theta - \tan\theta = 6$ on $0° \le \theta < 180°$.

Solution:

STEP 1 Solve for $\tan\theta$.

Subtract 6. $\tan^2\theta - \tan\theta - 6 = 0$

Factor the quadratic trigonometric expression on the left. $(\tan\theta - 3)(\tan\theta + 2) = 0$

Set the factors equal to 0. $\tan\theta - 3 = 0$ or $\tan\theta + 2 = 0$

Solve for $\tan\theta$. $\tan\theta = 3$ or $\tan\theta = -2$

STEP 2 Solve $\tan\theta = 3$ on $0° \le \theta < 180°$.

The tangent function is positive on $0° \le \theta < 180°$ only in quadrant I.

Write the equivalent inverse notation to $\tan\theta = 3$. $\theta = \tan^{-1}3$

Use a calculator to evaluate (approximate) θ. $\boxed{\theta \approx 71.6°}$

STEP 3 Solve $\tan\theta = -2$ on $0° \le \theta < 180°$.

The tangent function is negative on $0° \le \theta < 180°$ only in quadrant II.

A calculator gives values of the inverse tangent in quadrants I and IV.

We will call the reference angle in quadrant IV "α."

Write the equivalent inverse notation to $\tan\alpha = -2$. $\alpha = \tan^{-1}(-2)$

Use a calculator to evaluate (approximate) α. $\alpha \approx -63.4°$

To find the value of θ in quadrant II, add 180°. $\theta = \alpha + 180°$

$\boxed{\theta \approx 116.6°}$

The solutions to $\tan^2\theta - \tan\theta = 6$ on $0° \le \theta < 180°$ are $\theta = 71.6°$ and $\theta = 116.6°$.

■ **YOUR TURN** Solve $\tan^2\theta + \tan\theta = 6$ on $0° \le \theta < 180°$.

■ **Answer:** $\theta \approx 63.4°$ or $108.4°$

Recall that in solving algebraic quadratic equations, one method (when factoring is not obvious or possible) is to use the Quadratic Formula.

$$ax^2 + bx + c = 0 \text{ has solutions } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 7 Solving a Quadratic Trigonometric Equation That Requires the Use of the Quadratic Formula and Inverse Functions

Solve $2\cos^2\theta + 5\cos\theta - 6 = 0$ on $0° \le \theta < 360°$.

Solution:

STEP 1 Solve for $\cos\theta$.

$$2\cos^2\theta + 5\cos\theta - 6 = 0$$

Let $u = \cos\theta$.

$$2u^2 + 5u - 6 = 0$$

Use the Quadratic Formula, $a = 2, b = 5, c = -6$.

$$u = \frac{-5 \pm \sqrt{5^2 - 4(2)(-6)}}{2(2)}$$

Simplify.

$$u = \frac{-5 \pm \sqrt{73}}{4}$$

Use a calculator to approximate the solution.

$$u \approx -3.3860 \quad \text{or} \quad u \approx 0.8860$$

Let $u = \cos\theta$.

$$\cos\theta \approx -3.3860 \quad \text{or} \quad \cos\theta \approx 0.8860$$

STEP 2 Solve $\cos\theta = -3.3860$ on $0° \le \theta < 360°$.

Recall that the range of the cosine function is $[-1, 1]$; therefore, the cosine function can never equal a number outside that range.

Since $-3.3860 < -1$, the equation $\cos\theta = -3.3860$ has *no solution*.

STEP 3 Solve $\cos\theta = 0.8860$ on $0° \le \theta < 360°$.

The cosine function is positive in quadrants I and IV. Since a calculator gives inverse cosine values only in quadrants I and II, we will have to use a reference angle to get the quadrant IV solution.

Write the equivalent inverse notation for $\cos\theta = 0.8860$.

$$\theta = \cos^{-1}(0.8860)$$

Use a calculator to evaluate (approximate) the solution.

$$\boxed{\theta \approx 27.6°}$$

To find the second solution (in quadrant IV), subtract the reference angle from 360°.

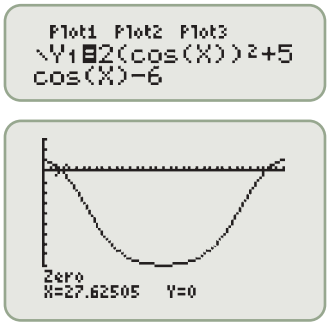
$$\theta = 360° - 27.6°$$

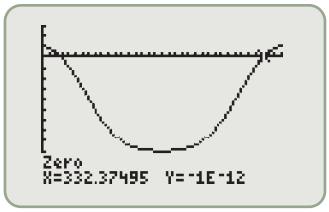
$$\boxed{\theta \approx 332.4°}$$

The solutions to $2\cos^2\theta + 5\cos\theta - 6 = 0$ on $0° \le \theta < 360°$ are $\theta \approx 27.6°$ and $\theta \approx 332.4°$.

YOUR TURN Solve $2\sin^2\theta - 5\sin\theta - 6 = 0$ on $0° \le \theta < 360°$.

Technology Tip





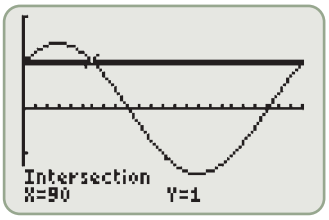
Answer: $\theta \approx 242.4°$ or $297.6°$

Using Trigonometric Identities to Solve Trigonometric Equations

We now consider trigonometric equations that involve more than one trigonometric function. Trigonometric identities are an important part of solving these types of equations.

Technology Tip

Find the points of intersection of $y = \sin x + \cos x$ and $y = 1$.



In radians, the x-coordinates will be in decimal form as opposed to multiples of π. In this screen shot degrees is used, which allows for more familiar known exact values to be illustrated.

EXAMPLE 8 Using Trigonometric Identities to Solve Trigonometric Equations

Solve $\sin x + \cos x = 1$ on $0 \le x < 2\pi$.

Solution:

Square both sides. $\sin^2 x + 2\sin x\cos x + \cos^2 x = 1$

Label the Pythagorean identity. $\underbrace{\sin^2 x + \cos^2 x}_{1} + 2\sin x\cos x = 1$

Subtract 1 from both sides. $2\sin x\cos x = 0$

Use the Zero Product Property. $\sin x = 0$ or $\cos x = 0$

Solve for x on $0 \le x < 2\pi$. $x = 0$ or $x = \pi$ or $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

Because we squared the equation, we have to check for extraneous solutions.

Check $x = 0$. $\sin 0 + \cos 0 = 0 + 1 = 1$ ✓

Check $x = \pi$. $\sin \pi + \cos \pi = 0 - 1 = -1$ ✗

Check $x = \frac{\pi}{2}$. $\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = 1 + 0 = 1$ ✓

Check $x = \frac{3\pi}{2}$. $\sin\left(\frac{3\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right) = -1 + 0 = -1$ ✗

The solutions to $\sin x + \cos x = 1$ on $0 \le x < 2\pi$ are $\boxed{x = 0}$ and $\boxed{x = \frac{\pi}{2}}$.

■ **YOUR TURN** Solve $\sin x - \cos x = 1$ on $0 \le x < 2\pi$.

■ **Answer:** $x = \frac{\pi}{2}$ or $x = \pi$

EXAMPLE 9 Using Trigonometric Identities to Solve Trigonometric Equations

Solve $\sin(2x) = \sin x$ on $0 \le x < 2\pi$.

COMMON MISTAKE

Dividing by a trigonometric function (which could be equal to zero).

CORRECT

Use the double-angle formula for sine.

$$\underbrace{\sin(2x)}_{2\sin x\cos x} = \sin x$$

Subtract $\sin x$.

$$2\sin x\cos x - \sin x = 0$$

Factor out the common $\sin x$.

$$(\sin x)(2\cos x - 1) = 0$$

Set each factor equal to 0.

$$\sin x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$

$$\sin x = 0 \quad \text{or} \quad \cos x = \frac{1}{2}$$

Solve $\sin x = 0$ for x on $0 \le x < 2\pi$.

$$\boxed{x = 0} \quad \text{or} \quad \boxed{x = \pi}$$

Solve $\cos x = \frac{1}{2}$ for x on $0 \le x < 2\pi$.

$$\boxed{x = \frac{\pi}{3}} \quad \text{or} \quad \boxed{x = \frac{5\pi}{3}}$$

INCORRECT

$$2\sin x\cos x = \sin x$$

Divide by $\sin x$. **ERROR**

$$2\cos x = 1$$

Solve for $\cos x$.

$$\cos x = \frac{1}{2}$$

Missing solutions from $\sin x = 0$.

The solutions to $\sin(2x) = \sin x$ are $x = 0, \frac{\pi}{3}, \pi$, and $\frac{5\pi}{3}$.

CAUTION

Do not divide equations by trigonometric functions, as they can sometimes equal zero.

YOUR TURN Solve $\sin(2x) = \cos x$ on $0 \le x < 2\pi$.

Answer: $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}$, or $\frac{5\pi}{6}$

EXAMPLE 10 Using Trigonometric Identities to Solve Trigonometric Equations

Solve $\sin x + \csc x = -2$.

Solution:

Use the reciprocal identity. $\sin x + \underbrace{\csc x}_{\frac{1}{\sin x}} = -2$

Add 2. $\sin x + 2 + \dfrac{1}{\sin x} = 0$

Multiply by $\sin x$. *Note*: $\sin x \neq 0$. $\sin^2 x + 2\sin x + 1 = 0$

Factor as a perfect square. $(\sin x + 1)^2 = 0$

Solve for $\sin x$. $\sin x = -1$

Solve for x on one period of the sine function, $[0, 2\pi)$. $x = \dfrac{3\pi}{2}$

Add integer multiples of 2π to obtain all solutions. $\boxed{x = \dfrac{3\pi}{2} + 2n\pi}$

Study Tip

In Example 10 the equation $\sin x + \csc x = -2$ has an implied domain restriction of $\sin x \neq 0$.

EXAMPLE 11 Using Trigonometric Identities and Inverse Functions to Solve Trigonometric Equations

Solve $3\cos^2\theta + \sin\theta = 3$ on $0° \leq \theta < 360°$.

Solution:

Use the Pythagorean identity. $3\underbrace{\cos^2\theta}_{1-\sin^2\theta} + \sin\theta = 3$

Subtract 3. $3(1 - \sin^2\theta) + \sin\theta - 3 = 0$

Eliminate the parentheses. $3 - 3\sin^2\theta + \sin\theta - 3 = 0$

Simplify. $-3\sin^2\theta + \sin\theta = 0$

Factor the common $\sin\theta$. $\sin\theta(1 - 3\sin\theta) = 0$

Set each factor equal to 0. $\sin\theta = 0$ or $1 - 3\sin\theta = 0$

Solve for $\sin\theta$. $\sin\theta = 0$ or $\sin\theta = \dfrac{1}{3}$

Solve $\sin\theta = 0$ for x on $0° \leq \theta < 360°$. $\boxed{\theta = 0°}$ or $\boxed{\theta = 180°}$

Solve $\sin\theta = \frac{1}{3}$ for x on $0° \leq \theta < 360°$.

The sine function is positive in quadrants I and II.

A calculator gives inverse values only in quadrant I.

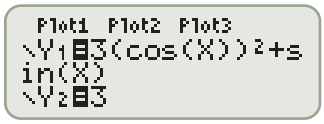
Write the equivalent inverse notation for $\sin\theta = \frac{1}{3}$. $\theta = \sin^{-1}\left(\dfrac{1}{3}\right)$

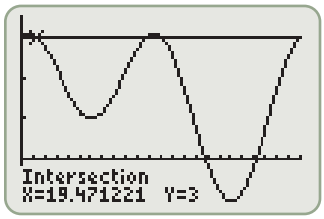
Use a calculator to approximate the quadrant I solution. $\boxed{\theta \approx 19.5°}$

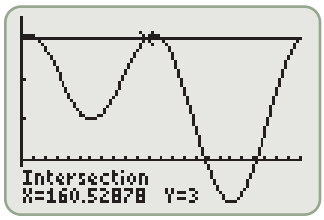
To find the quadrant II solution, subtract the reference angle from 180°. $\theta \approx 180° - 19.5°$

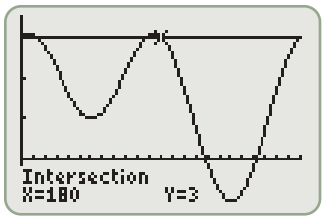
$\boxed{\theta \approx 160.5°}$

Technology Tip









Applications

EXAMPLE 12 Applications Involving Trigonometric Equations

Light bends (refracts) according to Snell's law, which states

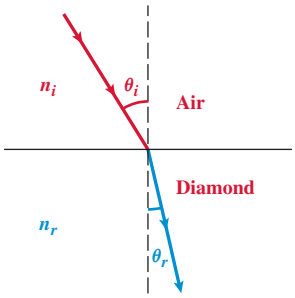
$$n_i \sin(\theta_i) = n_r \sin(\theta_r)$$

where

- n_i is the refractive index of the medium the light is leaving.
- θ_i is the incident angle between the light ray and the normal (perpendicular) to the interface between mediums.
- n_r is the refractive index of the medium the light is entering.
- θ_r is the refractive angle between the light ray and the normal (perpendicular) to the interface between mediums.

Janis Christie/Getty Images, Inc.

Assume that light is going from air into a diamond. Calculate the refractive angle θ_r if the incidence angle is $\theta_i = 32°$ and the index of refraction values for air and diamond are $n_i = 1.00$ and $n_r = 2.417$, respectively.



Solution:

Write Snell's law. $\quad n_i \sin(\theta_i) = n_r \sin(\theta_r)$

Substitute $\theta_i = 32°$, $n_i = 1.00$, and $n_r = 2.417$. $\quad \sin 32° = 2.417 \sin \theta_r$

Isolate $\sin \theta_r$ and simplify. $\quad \sin \theta_r = \dfrac{\sin 32°}{2.417} \approx 0.21925$

Solve for θ_r using the inverse sine function. $\quad \theta_r \approx \sin^{-1}(0.21925) \approx 12.665°$

Round to the nearest degree. $\quad \boxed{\theta_r \approx 13°}$

SECTION 6.6 SUMMARY

In this section, we began by solving basic trigonometric equations that contained only one trigonometric function. Some such equations can be solved exactly by inspection, and others can be solved exactly using algebraic techniques similar to those of linear and quadratic equations. Calculators and inverse functions are needed when exact values are not known. It is important to note that calculators give the inverse function in only one of the two relevant quadrants. The other quadrant solutions must be found using reference angles. Trigonometric identities are useful for solving equations that involve more than one trigonometric function. With trigonometric identities we can transform such equations into equations involving only one trigonometric function, and then we can apply algebraic techniques.

SECTION 6.6 EXERCISES

SKILLS

In Exercises 1–20, solve the given trigonometric equation exactly over the indicated interval.

1. $\cos\theta = -\frac{\sqrt{2}}{2}$, $0 \le \theta < 2\pi$

2. $\sin\theta = -\frac{\sqrt{2}}{2}$, $0 \le \theta < 2\pi$

3. $\csc\theta = -2$, $0 \le \theta < 4\pi$

4. $\sec\theta = -2$, $0 \le \theta < 4\pi$

5. $\tan\theta = 0$, all real numbers

6. $\cot\theta = 0$, all real numbers

7. $\sin(2\theta) = -\frac{1}{2}$, $0 \le \theta < 2\pi$

8. $\cos(2\theta) = \frac{\sqrt{3}}{2}$, $0 \le \theta < 2\pi$

9. $\sin\left(\frac{\theta}{2}\right) = -\frac{1}{2}$, all real numbers

10. $\cos\left(\frac{\theta}{2}\right) = -1$, all real numbers

11. $\tan(2\theta) = \sqrt{3}$, $-2\pi \le \theta < 2\pi$

12. $\tan(2\theta) = -\sqrt{3}$, all real numbers

13. $\sec\theta = -2$, $-2\pi \le \theta < 0$

14. $\csc\theta = \frac{2\sqrt{3}}{3}$, $-\pi \le \theta < \pi$

15. $\cot(4\theta) = -\frac{\sqrt{3}}{3}$, all real numbers

16. $\tan(5\theta) = 1$, all real numbers

17. $\sec(3\theta) = -1$, $-2\pi \le \theta \le 0$

18. $\sec(4\theta) = \sqrt{2}$, $0 \le \theta \le \pi$

19. $\csc(3\theta) = 1$, $-2\pi \le \theta \le 0$

20. $\csc(6\theta) = -\frac{2\sqrt{3}}{3}$, $0 \le \theta \le \pi$

In Exercises 21–40, solve the given trigonometric equation exactly on $0 \le \theta < 2\pi$.

21. $2\sin(2\theta) = \sqrt{3}$

22. $2\cos\left(\frac{\theta}{2}\right) = -\sqrt{2}$

23. $3\tan(2\theta) - \sqrt{3} = 0$

24. $4\tan\left(\frac{\theta}{2}\right) - 4 = 0$

25. $2\cos(2\theta) + 1 = 0$

26. $4\csc(2\theta) + 8 = 0$

27. $\sqrt{3}\cot\left(\frac{\theta}{2}\right) - 3 = 0$

28. $\sqrt{3}\sec(2\theta) + 2 = 0$

29. $\tan^2\theta - 1 = 0$

30. $\sin^2\theta + 2\sin\theta + 1 = 0$

31. $2\cos^2\theta - \cos\theta = 0$

32. $\tan^2\theta - \sqrt{3}\tan\theta = 0$

33. $\csc^2\theta + 3\csc\theta + 2 = 0$

34. $\cot^2\theta = 1$

35. $\sin^2\theta + 2\sin\theta - 3 = 0$

36. $2\sec^2\theta + \sec\theta - 1 = 0$

37. $\sec^2\theta - 1 = 0$

38. $\csc^2\theta - 1 = 0$

39. $\sec^2(2\theta) - \frac{4}{3} = 0$

40. $\csc^2(2\theta) - 4 = 0$

In Exercises 41–60, solve the given trigonometric equation on $0° \le \theta < 360°$ and express the answer in degrees rounded to two decimal places.

41. $\sin(2\theta) = -0.7843$

42. $\cos(2\theta) = 0.5136$

43. $\tan\left(\frac{\theta}{2}\right) = -0.2343$

44. $\sec\left(\frac{\theta}{2}\right) = 1.4275$

45. $5\cot\theta - 9 = 0$

46. $5\sec\theta + 6 = 0$

47. $4\sin\theta + \sqrt{2} = 0$

48. $3\cos\theta - \sqrt{5} = 0$

49. $4\cos^2\theta + 5\cos\theta - 6 = 0$

50. $6\sin^2\theta - 13\sin\theta - 5 = 0$

51. $6\tan^2\theta - \tan\theta - 12 = 0$

52. $6\sec^2\theta - 7\sec\theta - 20 = 0$

53. $15\sin^2(2\theta) + \sin(2\theta) - 2 = 0$

54. $12\cos^2\left(\frac{\theta}{2}\right) - 13\cos\left(\frac{\theta}{2}\right) + 3 = 0$

55. $\cos^2\theta - 6\cos\theta + 1 = 0$

56. $\sin^2\theta + 3\sin\theta - 3 = 0$

57. $2\tan^2\theta - \tan\theta - 7 = 0$

58. $3\cot^2\theta + 2\cot\theta - 4 = 0$

59. $\csc^2(3\theta) - 2 = 0$

60. $\sec^2\left(\frac{\theta}{2}\right) - 2 = 0$

In Exercises 61–88, solve the trigonometric equations exactly on the indicated interval, $0 \le x < 2\pi$.

61. $\sin x = \cos x$

62. $\sin x = -\cos x$

63. $\sec x + \cos x = -2$

64. $\sin x + \csc x = 2$

65. $\sec x - \tan x = \dfrac{\sqrt{3}}{3}$

66. $\sec x + \tan x = 1$

67. $\csc x + \cot x = \sqrt{3}$

68. $\csc x - \cot x = \dfrac{\sqrt{3}}{3}$

69. $2\sin x - \csc x = 0$

70. $2\sin x + \csc x = 3$

71. $\sin(2x) = 4\cos x$

72. $\sin(2x) = \sqrt{3}\sin x$

73. $\sqrt{2}\sin x = \tan x$

74. $\cos(2x) = \sin x$

75. $\tan(2x) = \cot x$

76. $3\cot(2x) = \cot x$

77. $\sqrt{3}\sec x = 4\sin x$

78. $\sqrt{3}\tan x = 2\sin x$

79. $\sin^2 x - \cos(2x) = -\frac{1}{4}$

80. $\sin^2 x - 2\sin x = 0$

81. $\cos^2 x + 2\sin x + 2 = 0$

82. $2\cos^2 x = \sin x + 1$

83. $2\sin^2 x + 3\cos x = 0$

84. $4\cos^2 x - 4\sin x = 5$

85. $\cos(2x) + \cos x = 0$

86. $2\cot x = \csc x$

87. $\frac{1}{4}\sec(2x) = \sin(2x)$

88. $-\frac{1}{4}\csc\left(\frac{1}{2}x\right) = \cos\left(\frac{1}{2}x\right)$

In Exercises 89–98, solve each trigonometric equation on $0° \le \theta < 360°$. Express solutions in degrees and round to two decimal places.

89. $\cos(2x) + \frac{1}{2}\sin x = 0$

90. $\sec^2 x = \tan x + 1$

91. $6\cos^2 x + \sin x = 5$

92. $\sec^2 x = 2\tan x + 4$

93. $\cot^2 x - 3\csc x - 3 = 0$

94. $\csc^2 x + \cot x = 7$

95. $2\sin^2 x + 2\cos x - 1 = 0$

96. $\sec^2 x + \tan x - 2 = 0$

97. $\dfrac{1}{16}\csc^2\left(\dfrac{x}{4}\right) - \cos^2\left(\dfrac{x}{4}\right) = 0$

98. $-\dfrac{1}{4}\sec^2\left(\dfrac{x}{8}\right) + \sin^2\left(\dfrac{x}{8}\right) = 0$

APPLICATIONS

For Exercises 99 and 100, refer to the following:

Computer sales are generally subject to seasonal fluctuations. The sales of QualComp computers during 2008–2010 is approximated by the function

$$s(t) = 0.120\sin(0.790t - 2.380) + 0.387 \quad 1 \le t \le 12$$

where t represents time in quarters ($t = 1$ represents the end of the first quarter of 2008), and $s(t)$ represents computer sales (quarterly revenue) in millions of dollars.

99. Business. Find the quarter(s) in which the quarterly sales are \$472,000.

100. Business. Find the quarter(s) in which the quarterly sales are \$507,000.

For Exercises 101 and 102, refer to the following:

Allergy sufferers' symptoms fluctuate with the concentration of pollen in the air. At one location the pollen concentration, measured in grains per cubic meter, of grasses fluctuates throughout the day according to the function:

$$p(t) = 35 - 26\cos\left(\frac{\pi}{12}t - \frac{7\pi}{6}\right), \quad 0 \le t \le 24$$

where t is measured in hours and $t = 0$ is 12:00 A.M.

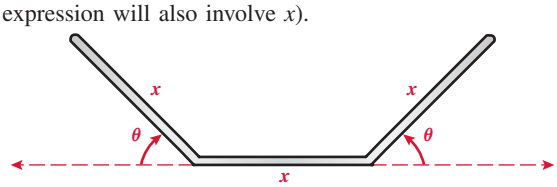
101. Biology/Health. Find the time(s) of day when the grass pollen level is 41 grains per cubic meter. Round to the nearest hour.

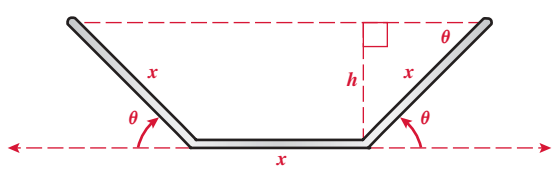
102. Biology/Health. Find the time(s) of day when the grass pollen level is 17 grains per cubic meter. Round to the nearest hour.

103. Sales. Monthly sales of soccer balls are approximated by $S = 400\sin\left(\dfrac{\pi}{6}x\right) + 2000$, where x is the number of the month (January is $x = 1$, etc.). During which month do sales reach 2400?

104. Sales. Monthly sales of soccer balls are approximated by $S = 400\sin\left(\dfrac{\pi}{6}x\right) + 2000$, where x is the number of the month (January is $x = 1$, etc.). During which two months do sales reach 1800?

105. Home Improvement. A rain gutter is constructed from a single strip of sheet metal by bending as shown below and on the right, so that the base and sides are the same length. Express the area of the cross section of the rain gutter as a function of the angle θ (note that the expression will also involve x).





106. Home Improvement. A rain gutter is constructed from a single strip of sheet metal by bending as shown above, so that the base and sides are the same length. When the area of the cross section of the rain gutter is expressed as a function of the angle θ, you can then determine the value of θ that produces the cross section with the greatest possible area. The angle is found by solving the equation $\cos^2\theta - \sin^2\theta + \cos\theta = 0$. Which angle gives the maximum area?

107. Deer Population. The number of deer on an island is given by $D = 200 + 100\sin\left(\frac{\pi}{2}x\right)$, where x is the number of years since 2000. Which is the first year after 2000 that the number of deer reaches 300?

108. Deer Population. The number of deer on an island is given by $D = 200 + 100\sin\left(\frac{\pi}{6}x\right)$, where x is the number of years since 2000. Which is the first year after 2000 that the number of deer reaches 150?

109. Optics. Assume that light is going from air into a diamond. Calculate the refractive angle θ_r if the incidence angle is $\theta_i = 75°$ and the index of refraction values for air and diamond are $n_i = 1.00$ and $n_r = 2.417$, respectively. Round to the nearest degree. (See Example 12 for Snell's law.)

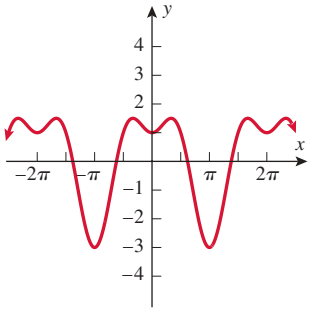
110. Optics. Assume that light is going from a diamond into air. Calculate the refractive angle θ_r if the incidence angle is $\theta_i = 15°$ and the index of refraction values for diamond and air are $n_i = 2.417$ and $n_r = 1.00$, respectively. Round to the nearest degree. (See Example 12 for Snell's law.)

111. Air in Lungs. If a person breathes in and out every 3 seconds, the volume of air in the lungs can be modeled by $A = 2\sin\left(\frac{\pi}{3}x\right)\cos\left(\frac{\pi}{3}x\right) + 3$, where A is in liters of air and x is in seconds. How many seconds into the cycle is the volume of air equal to 4 liters?

112. Air in Lungs. For the function given in Exercise 111, how many seconds into the cycle is the volume of air equal to 2 liters?

For Exercises 113 and 114, refer to the following:

The figure below shows the graph of $y = 2\cos x - \cos(2x)$ between -2π and 2π. The maximum and minimum values of the curve occur at the *turning points* and are found in the solutions of the equation $-2\sin x + 2\sin(2x) = 0$.



113. Finding Turning Points. Solve for the coordinates of the turning points of the curve between 0 and 2π.

114. Finding Turning Points. Solve for the coordinates of the turning points of the curve between -2π and 0.

115. Business. An analysis of a company's costs and revenue shows that annual costs of producing their product as well as annual revenues from the sale of a product are generally subject to seasonal fluctuations and are approximated by the function

$$C(t) = 2.3 + 0.25\sin\left(\frac{\pi}{6}t\right) \qquad 0 \le t \le 11$$

$$R(t) = 2.3 + 0.5\cos\left(\frac{\pi}{6}t\right) \qquad 0 \le t \le 11$$

where t represents time in months ($t = 0$ represents January), $C(t)$ represents the monthly costs of producing the product in millions of dollars, and $R(t)$ represents monthly revenue from sales of the product in millions of dollars. Find the month(s) in which the company breaks even. *Hint:* A company breaks even when its profit is zero.

116. Business. An analysis of a company's costs and revenue shows that the annual costs of producing its product as well as annual revenues from the sale of a product are generally subject to seasonal fluctuations and are approximated by the function

$$C(t) = 25.7 + 0.2\sin\left(\frac{\pi}{6}t\right) \qquad 0 \le t \le 11$$

$$R(t) = 25.7 + 9.6\cos\left(\frac{\pi}{6}t\right) \qquad 0 \le t \le 11$$

where t represents time in months ($t = 0$ represents January), $C(t)$ represents the monthly costs of producing the product in millions of dollars, and $R(t)$ represents monthly revenue from sales of the product in millions of dollars. Find the month(s) in which the company breaks even. *Hint:* A company breaks even when its profit is zero.

For Exercises 117 and 118, refer to the following:

By analyzing available empirical data, it has been determined that the body temperature of a species fluctuates according to the model

$$T(t) = 37.10 + 1.40\sin\left(\frac{\pi}{24}t\right)\cos\left(\frac{\pi}{24}t\right) \qquad 0 \le t \le 24$$

where T represents temperature in degrees Celsius and t represents time (in hours) measured from 12:00 A.M. (midnight).

117. Biology/Health. Find the time(s) of day the body temperature is 37.28°C. Round to the nearest hour.

118. Biology/Health. Find the time(s) of day the body temperature is 36.75°C. Round to the nearest hour.

CATCH THE MISTAKE

In Exercises 119–122, explain the mistake that is made.

119. Solve $\sqrt{2 + \sin\theta} = \sin\theta$ on $0 \le \theta \le 2\pi$.

Solution:

Square both sides.	$2 + \sin\theta = \sin^2\theta$
Gather all terms to one side.	$\sin^2\theta - \sin\theta - 2 = 0$
Factor.	$(\sin\theta - 2)(\sin\theta + 1) = 0$
Set each factor equal to zero.	$\sin\theta - 2 = 0$ or $\sin\theta + 1 = 0$
Solve for $\sin\theta$.	$\sin\theta = 2$ or $\sin\theta = -1$
Solve $\sin\theta = 2$ for θ.	no solution
Solve $\sin\theta = -1$ for θ.	$\theta = \frac{3\pi}{2}$

This is incorrect. What mistake was made?

120. Solve $\sqrt{3\sin\theta - 2} = -\sin\theta$ on $0 \le \theta \le 2\pi$.

Solution:

Square both sides.	$3\sin\theta - 2 = \sin^2\theta$
Gather all terms to one side.	$\sin^2\theta - 3\sin\theta + 2 = 0$
Factor.	$(\sin\theta - 2)(\sin\theta - 1) = 0$
Set each factor equal to zero.	$\sin\theta - 2 = 0$ or $\sin\theta - 1 = 0$
Solve for $\sin\theta$.	$\sin\theta = 2$ or $\sin\theta = 1$
Solve $\sin\theta = 2$ for θ.	no solution
Solve $\sin\theta = 1$ for θ.	$\theta = \frac{\pi}{2}$

This is incorrect. What mistake was made?

121. Solve $3\sin(2x) = 2\cos x$ on $0° \le \theta \le 180°$.

Solution:

Use the double-angle identity for the sine function.	$3\underbrace{\sin(2x)}_{2\sin x\cos x} = 2\cos x$
Simplify.	$6\sin x\cos x = 2\cos x$
Divide by $2\cos x$.	$3\sin x = 1$
Divide by 3.	$\sin x = \frac{1}{3}$
Write the equivalent inverse notation.	$x = \sin^{-1}\left(\frac{1}{3}\right)$
Use a calculator to approximate the solution.	$x \approx 19.47°$, QI solution
The QII solution is:	$x \approx 180° - 19.47° \approx 160.53°$

This is incorrect. What mistake was made?

122. Solve $\sqrt{1 + \sin x} = \cos x$ on $0 \le x \le 2\pi$.

Solution:

Square both sides.	$1 + \sin x = \cos^2 x$
Use the Pythagorean identity.	$1 + \sin x = \underbrace{\cos^2 x}_{1-\sin^2 x}$
Simplify.	$\sin^2 x + \sin x = 0$
Factor.	$\sin x(\sin x + 1) = 0$
Set each factor equal to zero.	$\sin x = 0$ or $\sin x + 1 = 0$
Solve for $\sin x$.	$\sin x = 0$ or $\sin x = -1$
Solve for x.	$x = 0, \pi, \frac{3\pi}{2}, 2\pi$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 123–126, determine whether each statement is true or false.

123. Linear trigonometric equations always have one solution on $[0, 2\pi]$.

124. Quadratic trigonometric equations always have two solutions on $[0, 2\pi]$.

125. If a trigonometric equation has all real numbers as its solution, then it is an identity.

126. If a trigonometric equation has an infinite number of solutions, then it is an identity.

CHALLENGE

127. Solve $16\sin^4\theta - 8\sin^2\theta = -1$ over $0 \le \theta \le 2\pi$.

128. Solve $\left|\cos\left(\theta + \frac{\pi}{4}\right)\right| = \frac{\sqrt{3}}{2}$ over all real numbers.

129. Solve for the smallest positive x that makes this statement true:

$$\sin\left(x + \frac{\pi}{4}\right) + \sin\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

130. Solve for the smallest positive x that makes this statement true:

$$\cos x \cos 15° + \sin x \sin 15° = 0.7$$

131. Find all real numbers x such that $\dfrac{1 - \cos\left(\frac{x}{3}\right)}{1 + \cos\left(\frac{x}{3}\right)} + 1 = 0$.

132. Find all real numbers θ such that $\sec^4\left(\frac{1}{3}\theta\right) - 1 = 0$.

133. Find all real numbers θ such that $\csc^4\left(\frac{\pi}{4}\theta - \pi\right) - 4 = 0$.

134. Find all real numbers x such that $2\tan(3x) = \sqrt{3} - \sqrt{3}\tan^2(3x)$.

TECHNOLOGY

Graphing calculators can be used to find approximate solutions to trigonometric equations. For the equation $f(x) = g(x)$, let $Y_1 = f(x)$ and $Y_2 = g(x)$. The x-values that correspond to points of intersections represent solutions.

135. With a graphing utility, solve the equation $\sin\theta = \cos(2\theta)$ on $0 \le \theta \le \pi$.

136. With a graphing utility, solve the equation $\csc\theta = \sec\theta$ on $0 \le \theta \le \frac{\pi}{2}$.

137. With a graphing utility, solve the equation $\sin\theta = \sec\theta$ on $0 \le \theta \le \pi$.

138. With a graphing utility, solve the equation $\cos\theta = \csc\theta$ on $0 \le \theta \le \pi$.

139. With a graphing utility, find all of the solutions to the equation $\sin\theta = e^\theta$ for $\theta \ge 0$.

140. With a graphing utility, find all of the solutions to the equation $\cos\theta = e^\theta$ for $\theta \ge 0$.

Find the smallest positive values of x that make the statement true. Give the answer in degrees and round to two decimal places.

141. $\sec(3x) + \csc(2x) = 5$

142. $\cot(5x) + \tan(2x) = -3$

143. $e^x - \tan x = 0$

144. $e^x + 2\sin x = 1$

145. $\ln x - \sin x = 0$

146. $\ln x - \cos x = 0$

PREVIEW TO CALCULUS

In calculus, the definite integral is used to find the area between two intersecting curves (functions). When the curves correspond to trigonometric functions, we need to solve trigonometric equations.

In Exercises 147–150, solve each trigonometric equation within the indicated interval.

147. $\cos x = 2 - \cos x,\ 0 \le x \le 2\pi$

148. $\cos(2x) = \sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{6}$

149. $2\sin x = \tan x,\ -\frac{\pi}{3} \le x \le \frac{\pi}{3}$

150. $\sin(2x) - \cos(2x) = 0,\ 0 < x \le \pi$

CHAPTER 6 INQUIRY-BASED LEARNING PROJECT

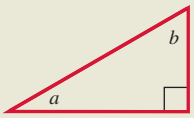
When it comes to identities, don't always let your intuition be your guide.

1. Suppose a fellow student claims that the equation $\sin(a + b) = \sin a + \sin b$ is an identity. He's wondering if you can help because he's not sure how to verify his claim, but says, "It just seems intuitively so." Can you help this student?

a. First, let's understand the student's claim. What does it *mean* to say that $\sin(a + b) = \sin a + \sin b$ "is an identity"?

b. If you decided to try and verify the student's claim, you'd start with one side of his equation and try to manipulate that side until it looks like the other side. But, that may turn out to be a lot of unnecessary work *if*, in fact, the student's claim is false. So, instead, try something else.

Consider a right triangle with angles a and b. Calculate the values in the chart below, using various values of a and b.



a	b	$a + b$	$\sin(a + b)$	$\sin a$	$\sin b$	$\sin a + \sin b$

c. What does your data tell you about the student's claim? Explain.

d. In this example, you discovered that function notation is not distributive. Now, show the student how to write the *sum identity for the sine function*; that is,

$\sin(a + b) =$ _____

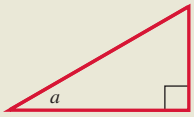
(This identity is derived in this chapter.)

2. Word has gotten out that you are really good at helping others understand trigonometric identities. Another of your fellow students asks whether $\sin(2a) = 2\sin a$ is an identity.

a. How many values of a would you need to check to determine whether the student's equation is an identity? Explain.

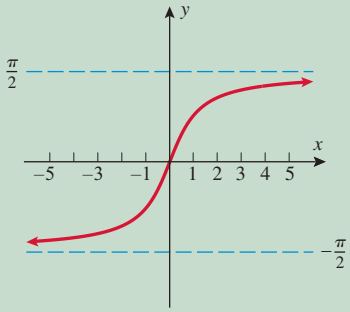
b. Show how to convince the student that his equation is *not* an identity.

c. Try to discover the *double-angle identity* $\sin(2a) =$ _____: For the right triangle below, fill out the chart (exact values) and look for a pattern.



a	$\sin(2a)$	$\sin a$	$\cos a$
30°			
45°			
60°			
90°			

MODELING OUR WORLD



In the Modeling Our World feature in Chapter 5, you modeled mean *temperatures* with sinusoidal models. Now we consider *carbon emissions*, which are greenhouse gases that have been shown to negatively affect the ozone layer. Over the last 50 years, we have increased our global carbon emissions at an alarming rate. Recall that the graph of the inverse tangent function increases rapidly and then levels off at the horizontal asymptote, $y = \frac{\pi}{2} \approx 1.57$. To achieve a similar plateau with carbon emissions, drastic environmental regulations will need to be enacted.

The carbon emissions data over the last 50 years suggest an almost linear climb (similar to the inverse tangent graph from $x = 0$ to $x = 1$). If the world started reducing carbon emissions, they might possibly reach a plateau level.

The following table summarizes average yearly temperature in degrees Fahrenheit (°F) and carbon dioxide emissions in parts per million (ppm) for Mauna Loa, Hawaii.

YEAR	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
TEMPERATURE	44.45	43.29	43.61	43.35	46.66	45.71	45.53	47.53	45.86	46.23
CO_2 EMISSIONS (PPM)	316.9	320.0	325.7	331.1	338.7	345.9	354.2	360.6	369.4	379.7

1. Plot the carbon emissions data with time on the horizontal axis and CO_2 emissions (in ppm) on the vertical axis. Let $t = 0$ correspond to 1960.

2. Find an *inverse tangent function* of the form $f(x) = A\tan^{-1}(Bx) + k$ that models the carbon emissions in Mauna Loa.

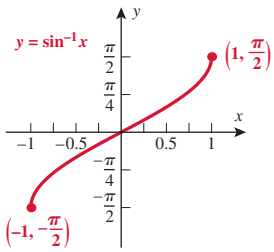
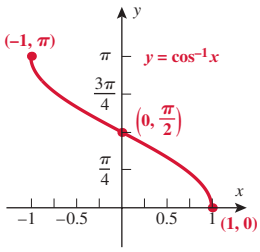
a. Use the data from 1960, 1985, and 2005.

b. Use the data from 1960, 1965, and 1995.

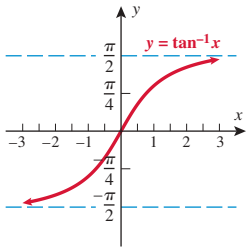
3. According to your model, what are the expected carbon emissions in 2050?

4. Describe the ways by which the world might be able to reach a plateau level of carbon emissions instead of the predicted increased rates.

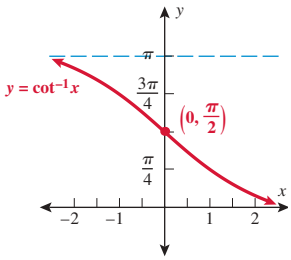
Section	Concept	Key Ideas/Formulas
6.1	**Verifying trigonometric identities**	Identities must hold for *all* values of x (not just some values of x) for which both sides of the equation are defined.
	Fundamental identities	*Reciprocal identities* $\csc\theta = \dfrac{1}{\sin\theta}$ $\quad \sec\theta = \dfrac{1}{\cos\theta}$ $\quad \cot\theta = \dfrac{1}{\tan\theta}$ *Quotient identities* $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}$ *Pythagorean identities* $\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $1 + \cot^2\theta = \csc^2\theta$ *Cofunction identities* $\sin\theta = \cos\left(\dfrac{\pi}{2} - \theta\right)$ $\quad \csc\theta = \sec\left(\dfrac{\pi}{2} - \theta\right)$ $\cos\theta = \sin\left(\dfrac{\pi}{2} - \theta\right)$ $\quad \sec\theta = \csc\left(\dfrac{\pi}{2} - \theta\right)$ $\tan\theta = \cot\left(\dfrac{\pi}{2} - \theta\right)$ $\quad \cot\theta = \tan\left(\dfrac{\pi}{2} - \theta\right)$
	Simplifying trigonometric expressions using identities	Use the reciprocal, quotient, or Pythagorean identities to simplify trigonometric expressions.
	Verifying identities	■ Convert all trigonometric expressions to sines and cosines. ■ Write all sums or differences of fractions as a single fraction.
6.2	**Sum and difference identities**	$f(A \pm B) \neq f(A) \pm f(B)$ For trigonometric functions, we have the sum and difference identities.
	Sum and difference identities for the cosine function	$\cos(A + B) = \cos A \cos B - \sin A \sin B$ $\cos(A - B) = \cos A \cos B + \sin A \sin B$
	Sum and difference identities for the sine function	$\sin(A + B) = \sin A \cos B + \cos A \sin B$ $\sin(A - B) = \sin A \cos B - \cos A \sin B$
	Sum and difference identities for the tangent function	$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
6.3	**Double-angle and half-angle identities**	
	Double-angle identities	$\sin(2A) = 2\sin A \cos A$ $\cos(2A) = \cos^2 A - \sin^2 A$ $= 1 - 2\sin^2 A = 2\cos^2 A - 1$ $\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}$

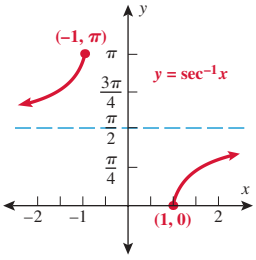
Section	Concept	Key Ideas/Formulas
	Half-angle identities	$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$ $\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$ $\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$
6.4	**Product-to-sum and sum-to-product identities**	
	Product-to-sum identities	$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$ $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ $\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
	Sum-to-product identities	$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$ $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
6.5	**Inverse trigonometric functions**	$\sin^{-1}x$ or $\arcsin x$ $\quad$ $\cos^{-1}x$ or $\arccos x$ $\tan^{-1}x$ or $\arctan x$ $\quad$ $\cot^{-1}x$ or $\operatorname{arccot} x$ $\sec^{-1}x$ or $\operatorname{arcsec} x$ $\quad$ $\csc^{-1}x$ or $\operatorname{arccsc} x$
	Inverse sine function	**Definition** $y = \sin^{-1}x$ means $x = \sin y$ $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ **Identities** $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ $\sin(\sin^{-1}x) = x$ for $-1 \le x \le 1$ 
	Inverse cosine function	**Definition** $y = \cos^{-1}x$ means $x = \cos y$ $-1 \le x \le 1$ and $0 \le y \le \pi$ **Identities** $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$ $\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$ 

Section	Concept	Key Ideas/Formulas
	Inverse tangent function	**Definition** $y = \tan^{-1}x$ means $x = \tan y$ $-\infty < x < \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$ **Identities** $\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ $\tan(\tan^{-1}x) = x$ for $-\infty < x < \infty$

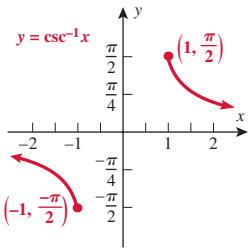


	Remaining inverse trigonometric functions	**Inverse cotangent function** **Definition** $y = \cot^{-1}x$ means $x = \cot y$ $-\infty < x < \infty$ and $0 < y < \pi$ **Identity** $\cot^{-1}x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & x < 0 \end{cases}$ **Inverse secant function** **Definition** $y = \sec^{-1}x$ means $x = \sec y$ $x \le -1$ or $x \ge 1$ and $0 \le y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \le \pi$ **Identity** $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$ for $x \le -1$ or $x \ge 1$





CHAPTER REVIEW

Section	Concept	Key Ideas/Formulas
		Inverse cosecant function **Definition** $y = \csc^{-1}x$ means $x = \csc y$ $x \le -1$ or $x \ge 1$ and $-\frac{\pi}{2} \le y < 0$ or $0 < y \le \frac{\pi}{2}$ **Identity** $\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$ for $x \le -1$ or $x \ge 1$ 
	Finding exact values for expressions involving inverse trigonometric functions	
6.6	**Trigonometric equations**	Goal: Find the values of the variable that make the equation true.
	Solving trigonometric equations by inspection	Solve: $\sin\theta = \frac{\sqrt{2}}{2}$ on $0 \le \theta \le 2\pi$. Answer: $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$. Solve: $\sin\theta = \frac{\sqrt{2}}{2}$ on all real numbers. Answer: $\theta = \begin{cases} \frac{\pi}{4} + 2n\pi \\ \frac{3\pi}{4} + 2n\pi \end{cases}$ where n is an integer.
	Solving trigonometric equations using algebraic techniques	Transform trigonometric equations into linear or quadratic algebraic equations by making a substitution such as $x = \sin\theta$. Then use algebraic methods for solving linear and quadratic equations. If an expression is squared, always check for extraneous solutions.
	Solving trigonometric equations that require the use of inverse functions	Follow the same procedures outlined by inspection or algebraic methods. Finding the solution requires the use of inverse functions and a calculator. Be careful: Calculators only give one solution (the one in the range of the inverse function).
	Using trigonometric identities to solve trigonometric equations	Use trigonometric identities to transform an equation with multiple trigonometric functions into an equation with only one trigonometric function. Then use the methods outlined above.

6.1 Verifying Trigonometric Identities

Use the cofunction identities to fill in the blanks.

1. $\sin 30° = \cos$ _____
2. $\cos A = \sin$ _____
3. $\tan 45° = \cot$ _____
4. $\csc 60° = \sec$ _____
5. $\sec 30° = \csc$ _____
6. $\cot 60° = \tan$ _____

Simplify the following trigonometric expressions:

7. $\tan x(\cot x + \tan x)$
8. $(\sec x + 1)(\sec x - 1)$
9. $\dfrac{\tan^4 x - 1}{\tan^2 x - 1}$
10. $\sec^2 x(\cot^2 x - \cos^2 x)$
11. $\cos x[\cos(-x) - \tan(-x)] - \sin x$
12. $\dfrac{\tan^2 x + 1}{2\sec^2 x}$
13. $\dfrac{\csc^3(-x) + 8}{\csc x - 2}$
14. $\dfrac{\csc^2 x - 1}{\cot x}$

Verify the trigonometric identities.

15. $(\tan x + \cot x)^2 - 2 = \tan^2 x + \cot^2 x$
16. $\csc^2 x - \cot^2 x = 1$
17. $\dfrac{1}{\sin^2 x} - \dfrac{1}{\tan^2 x} = 1$
18. $\dfrac{1}{\csc x + 1} + \dfrac{1}{\csc x - 1} = \dfrac{2\tan x}{\cos x}$
19. $\dfrac{\tan^2 x - 1}{\sec^2 x + 3\tan x + 1} = \dfrac{\tan x - 1}{\tan x + 2}$
20. $\cot x(\sec x - \cos x) = \sin x$

Determine whether each of the following equations is a conditional equation or an identity:

21. $2\tan^2 x + 1 = \dfrac{1 + \sin^2 x}{\cos^2 x}$
22. $\sin x - \cos x = 0$
23. $\cot^2 x - 1 = \tan^2 x$
24. $\cos^2 x(1 + \cot^2 x) = \cot^2 x$
25. $\left(\cot x - \dfrac{1}{\tan x}\right)^2 = 0$
26. $\csc x + \sec x = \dfrac{1}{\sin x + \cos x}$

6.2 Sum and Difference Identities

Find the exact value for each trigonometric expression.

27. $\cos\left(\dfrac{7\pi}{12}\right)$
28. $\sin\left(\dfrac{\pi}{12}\right)$
29. $\tan(-15°)$
30. $\cot 105°$

Write each expression as a single trigonometric function.

31. $\sin(4x)\cos(3x) - \cos(4x)\sin(3x)$
32. $\sin(-x)\sin(-2x) + \cos(-x)\cos(-2x)$
33. $\dfrac{\tan(5x) - \tan(4x)}{1 + \tan(5x)\tan(4x)}$
34. $\dfrac{\tan\left(\dfrac{\pi}{4}\right) + \tan\left(\dfrac{\pi}{3}\right)}{1 - \tan\left(\dfrac{\pi}{4}\right)\tan\left(\dfrac{\pi}{3}\right)}$

Find the exact value of the indicated expression using the given information and identities.

35. Find the exact value of $\tan(\alpha - \beta)$ if $\sin\alpha = -\frac{3}{5}$, $\sin\beta = -\frac{24}{25}$, the terminal side of α lies in quadrant IV, and the terminal side of β lies in quadrant III.
36. Find the exact value of $\cos(\alpha + \beta)$ if $\cos\alpha = -\frac{5}{13}$, $\sin\beta = \frac{7}{25}$, the terminal side of α lies in quadrant II, and the terminal side of β also lies in quadrant II.
37. Find the exact value of $\cos(\alpha - \beta)$ if $\cos\alpha = \frac{9}{41}$, $\cos\beta = \frac{7}{25}$, the terminal side of α lies in quadrant IV, and the terminal side of β lies in quadrant I.
38. Find the exact value of $\sin(\alpha - \beta)$ if $\sin\alpha = -\frac{5}{13}$, $\cos\beta = -\frac{4}{5}$, the terminal side of α lies in quadrant III, and the terminal side of β lies in quadrant II.

Determine whether each of the following equations is a conditional equation or an identity:

39. $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
40. $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

Graph the following functions:

41. $y = \cos\left(\frac{\pi}{2}\right)\cos x - \sin\left(\frac{\pi}{2}\right)\sin x$

42. $y = \sin\left(\frac{2\pi}{3}\right)\cos x + \cos\left(\frac{2\pi}{3}\right)\sin x$

43. $y = \dfrac{2\tan\left(\frac{x}{3}\right)}{1 - \tan^2\left(\frac{x}{3}\right)}$

44. $y = \dfrac{\tan(\pi x) - \tan x}{1 + \tan(\pi x)\tan x}$

6.3 Double-Angle and Half-Angle Identities

Use double-angle identities to answer the following questions:

45. If $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, find $\cos(2x)$.

46. If $\cos x = \frac{7}{25}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin(2x)$.

47. If $\cot x = -\frac{11}{61}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\tan(2x)$.

48. If $\tan x = -\frac{12}{5}$ and $\frac{\pi}{2} < x < \pi$, find $\cos(2x)$.

49. If $\sec x = \frac{25}{24}$ and $0 < x < \frac{\pi}{2}$, find $\sin(2x)$.

50. If $\csc x = \frac{5}{4}$ and $\frac{\pi}{2} < x < \pi$, find $\tan(2x)$.

Simplify each of the following expressions. Evaluate exactly, if possible.

51. $\cos^2 15° - \sin^2 15°$

52. $\dfrac{2\tan\left(-\frac{\pi}{12}\right)}{1 - \tan^2\left(-\frac{\pi}{12}\right)}$

53. $6\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

54. $1 - 2\sin^2\left(\frac{\pi}{8}\right)$

Verify the following identities:

55. $\sin^3 A - \cos^3 A = (\sin A - \cos A)\left[1 + \frac{1}{2}\sin(2A)\right]$

56. $2\sin A\cos^3 A - 2\sin^3 A\cos A = \cos(2A)\sin(2A)$

57. $\tan A = \dfrac{\sin(2A)}{1 + \cos(2A)}$

58. $\tan A = \dfrac{1 - \cos(2A)}{\sin(2A)}$

59. Launching a Missile. When launching a missile for a given range, the minimum velocity needed is related to the angle θ of the launch, and the velocity is determined by $V = \dfrac{2\cos(2\theta)}{1 + \cos(2\theta)}$. Show that V is equivalent to $1 - \tan^2\theta$.

60. Launching a Missile. When launching a missile for a given range, the minimum velocity needed is related to the angle θ of the launch, and the velocity is determined by $V = \dfrac{2\cos(2\theta)}{1 + \cos(2\theta)}$. Find the value of V when $\theta = \frac{\pi}{6}$.

Use half-angle identities to find the exact value of each of the following trigonometric expressions:

61. $\sin(-22.5°)$

62. $\cos 67.5°$

63. $\cot\left(\frac{3\pi}{8}\right)$

64. $\csc\left(-\frac{7\pi}{8}\right)$

65. $\sec(-165°)$

66. $\tan(-75°)$

Use half-angle identities to find each of the following values:

67. If $\sin x = -\frac{7}{25}$ and $\pi < x < \frac{3\pi}{2}$, find $\sin\left(\frac{x}{2}\right)$.

68. If $\cos x = -\frac{4}{5}$ and $\frac{\pi}{2} < x < \pi$, find $\cos\left(\frac{x}{2}\right)$.

69. If $\tan x = \frac{40}{9}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan\left(\frac{x}{2}\right)$.

70. If $\sec x = \frac{17}{15}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin\left(\frac{x}{2}\right)$.

Simplify each expression using half-angle identities. Do not evaluate.

71. $\sqrt{\dfrac{1 - \cos\left(\frac{\pi}{6}\right)}{2}}$

72. $\sqrt{\dfrac{1 - \cos\left(\frac{11\pi}{6}\right)}{1 + \cos\left(\frac{11\pi}{6}\right)}}$

Verify each of the following identities:

73. $\left[\sin\left(\frac{A}{2}\right) + \cos\left(\frac{A}{2}\right)\right]^2 = 1 + \sin A$

74. $\sec^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{A}{2}\right) = \dfrac{3 - \cos A}{1 + \cos A}$

75. $\csc^2\left(\frac{A}{2}\right) + \cot^2\left(\frac{A}{2}\right) = \dfrac{3 + \cos A}{1 - \cos A}$

76. $\tan^2\left(\frac{A}{2}\right) + 1 = \sec^2\left(\frac{A}{2}\right)$

Graph each of the following functions.
Hint: **Use trigonometric identities first.**

77. $y = \sqrt{\dfrac{1 - \cos\left(\dfrac{\pi}{12}x\right)}{2}}$

78. $y = \cos^2\left(\dfrac{x}{2}\right) - \sin^2\left(\dfrac{x}{2}\right)$

79. $y = -\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}$

80. $y = \sqrt{\dfrac{1 + \cos(3x - 1)}{2}}$

6.4 Product-to-Sum and Sum-to-Product Identities

Write each product as a sum or difference of sines and/or cosines.

81. $6\sin(5x)\cos(2x)$

82. $3\sin(4x)\sin(2x)$

Write each expression as a product of sines and/or cosines.

83. $\cos(5x) - \cos(3x)$

84. $\sin\left(\dfrac{5x}{2}\right) + \sin\left(\dfrac{3x}{2}\right)$

85. $\sin\left(\dfrac{4x}{3}\right) - \sin\left(\dfrac{2x}{3}\right)$

86. $\cos(7x) + \cos x$

Simplify each trigonometric expression.

87. $\dfrac{\cos(8x) + \cos(2x)}{\sin(8x) - \sin(2x)}$

88. $\dfrac{\sin(5x) + \sin(3x)}{\cos(5x) + \cos(3x)}$

Verify the identities.

89. $\dfrac{\sin A + \sin B}{\cos A - \cos B} = -\cot\left(\dfrac{A - B}{2}\right)$

90. $\dfrac{\sin A - \sin B}{\cos A - \cos B} = -\cot\left(\dfrac{A + B}{2}\right)$

91. $\csc\left(\dfrac{A - B}{2}\right) = \dfrac{2\sin\left(\dfrac{A + B}{2}\right)}{\cos B - \cos A}$

92. $\sec\left(\dfrac{A + B}{2}\right) = \dfrac{2\sin\left(\dfrac{A - B}{2}\right)}{\sin A - \sin B}$

6.5 Inverse Trigonometric Functions

Find the exact value of each expression. Give the answer in radians.

93. $\arctan 1$

94. $\operatorname{arccsc}(-2)$

95. $\cos^{-1} 0$

96. $\sin^{-1}(-1)$

97. $\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$

98. $\cot^{-1}(-\sqrt{3})$

Find the exact value of each expression. Give the answer in degrees.

99. $\csc^{-1}(-1)$

100. $\arctan(-1)$

101. $\operatorname{arccot}\left(\dfrac{\sqrt{3}}{3}\right)$

102. $\cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right)$

103. $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

104. $\sec^{-1} 1$

Use a calculator to evaluate each expression. Give the answer in degrees and round to two decimal places.

105. $\sin^{-1}(-0.6088)$

106. $\tan^{-1}(1.1918)$

107. $\sec^{-1}(1.0824)$

108. $\cot^{-1}(-3.7321)$

Use a calculator to evaluate each expression. Give the answer in radians and round to two decimal places.

109. $\cos^{-1}(-0.1736)$

110. $\tan^{-1}(0.1584)$

111. $\csc^{-1}(-10.0167)$

112. $\sec^{-1}(-1.1223)$

Evaluate each expression exactly, if possible. If not possible, state why.

113. $\sin^{-1}\left[\sin\left(-\dfrac{\pi}{4}\right)\right]$

114. $\cos\left[\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)\right]$

115. $\tan\left[\tan^{-1}(-\sqrt{3})\right]$

116. $\cot^{-1}\left[\cot\left(\dfrac{11\pi}{6}\right)\right]$

117. $\csc^{-1}\left[\csc\left(\dfrac{2\pi}{3}\right)\right]$

118. $\sec\left[\sec^{-1}\left(-\dfrac{2\sqrt{3}}{3}\right)\right]$

Evaluate each expression exactly.

119. $\sin\left[\cos^{-1}\left(\dfrac{11}{61}\right)\right]$

120. $\cos\left[\tan^{-1}\left(\dfrac{40}{9}\right)\right]$

121. $\tan\left[\cot^{-1}\left(\dfrac{6}{7}\right)\right]$

122. $\cot\left[\sec^{-1}\left(\dfrac{25}{7}\right)\right]$

123. $\sec\left[\sin^{-1}\left(\dfrac{1}{6}\right)\right]$

124. $\csc\left[\cot^{-1}\left(\dfrac{5}{12}\right)\right]$

6.6 Trigonometric Equations

Solve the given trigonometric equation on the indicated interval.

125. $\sin(2\theta) = -\frac{\sqrt{3}}{2}, 0 \le \theta \le 2\pi$

126. $\sec\left(\frac{\theta}{2}\right) = 2, -2\pi \le \theta \le 2\pi$

127. $\sin\left(\frac{\theta}{2}\right) = -\frac{\sqrt{2}}{2}, -2\pi \le \theta \le 2\pi$

128. $\csc(2\theta) = 2, 0 \le \theta \le 2\pi$

129. $\tan\left(\frac{1}{3}\theta\right) = -1, 0 \le \theta \le 6\pi$

130. $\cot(4\theta) = -\sqrt{3}, -\pi \le \theta \le \pi$

Solve each trigonometric equation exactly on $0 \le \theta \le 2\pi$.

131. $4\cos(2\theta) + 2 = 0$

132. $\sqrt{3}\tan\left(\frac{\theta}{2}\right) - 1 = 0$

133. $2\tan(2\theta) + 2 = 0$

134. $2\sin^2\theta + \sin\theta - 1 = 0$

135. $\tan^2\theta + \tan\theta = 0$

136. $\sec^2\theta - 3\sec\theta + 2 = 0$

Solve the given trigonometric equations on $0° \le \theta \le 360°$ and express the answer in degrees to two decimal places.

137. $\tan(2\theta) = -0.3459$

138. $6\sin\theta - 5 = 0$

139. $4\cos^2\theta + 3\cos\theta = 0$

140. $12\cos^2\theta - 7\cos\theta + 1 = 0$

141. $\csc^2\theta - 3\csc\theta - 1 = 0$

142. $2\cot^2\theta + 5\cot\theta - 4 = 0$

Solve each trigonometric equation exactly on the interval $0 \le \theta \le 2\pi$.

143. $\sec x = 2\sin x$

144. $3\tan x + \cot x = 2\sqrt{3}$

145. $\sqrt{3}\tan x - \sec x = 1$

146. $2\sin(2x) = \cot x$

147. $\sqrt{3}\tan x = 2\sin x$

148. $2\sin x = 3\cot x$

149. $\cos^2 x + \sin x + 1 = 0$

150. $2\cos^2 x - \sqrt{3}\cos x = 0$

151. $\cos(2x) + 4\cos x + 3 = 0$

152. $\sin(2x) + \sin x = 0$

153. $\tan^2\left(\frac{1}{2}x\right) - 1 = 0$

154. $\cot^2\left(\frac{1}{3}x\right) - 1 = 0$

Solve each trigonometric equation on $0° \le \theta \le 360°$. Give the answers in degrees and round to two decimal places.

155. $\csc^2 x + \cot x = 1$

156. $8\cos^2 x + 6\sin x = 9$

157. $\sin^2 x + 2 = 2\cos x$

158. $\cos(2x) = 3\sin x - 1$

159. $\cos x - 1 = \cos(2x)$

160. $12\cos^2 x + 4\sin x = 11$

Technology Exercises

Section 6.1

161. Is $\cos 73° = \sqrt{1 - \sin^2 73°}$? Use a calculator to find each of the following:

a. $\cos 73°$

b. $1 - \sin 73°$

c. $\sqrt{1 - \sin^2 73°}$

Which results are the same?

162. Is $\csc 28° = \sqrt{1 + \cot^2 28°}$? Use a calculator to find each of the following:

a. $\cot 28°$

b. $1 + \cot 28°$

c. $\sqrt{1 + \cot^2 28°}$

Which results are the same?

Section 6.2

Recall that the difference quotient for a function f is given by $\frac{f(x + h) + f(x)}{h}$.

163. Show that the difference quotient for $f(x) = \sin(3x)$ is

$$[\cos(3x)]\left[\frac{\sin(3h)}{h}\right] - [\sin(3x)]\left[\frac{1 - \cos(3h)}{h}\right].$$

Plot $Y_1 = [\cos(3x)]\left[\frac{\sin(3h)}{h}\right] - [\sin(3x)]\left[\frac{1 - \cos(3h)}{h}\right]$

for

a. $h = 1$

b. $h = 0.1$

c. $h = 0.01$

What function does the difference quotient for $f(x) = \sin(3x)$ resemble when h approaches zero?

164. Show that the difference quotient for $f(x) = \cos(3x)$ is

$$[-\sin(3x)]\left[\frac{\sin(3h)}{h}\right] - [\cos(3x)]\left[\frac{1 - \cos(3h)}{h}\right].$$

Plot $Y_1 = [-\sin(3x)]\left[\frac{\sin(3h)}{h}\right] - [\cos(3x)]\left[\frac{1 - \cos(3h)}{h}\right]$

for

a. $h = 1$

b. $h = 0.1$

c. $h = 0.01$

What function does the difference quotient for $f(x) = \cos(3x)$ resemble when h approaches zero?

Section 6.3

165. With a graphing calculator, plot $Y_1 = \tan(2x)$, $Y_2 = 2\tan x$, and $Y_3 = \dfrac{2\tan x}{1 - \tan^2 x}$ in the same viewing rectangle $[-2\pi, 2\pi]$ by $[-10, 10]$. Which graphs are the same?

166. With a graphing calculator, plot $Y_1 = \cos(2x)$, $Y_2 = 2\cos x$, and $Y_3 = 1 - 2\sin^2 x$ in the same viewing rectangle $[-2\pi, 2\pi]$ by $[-2, 2]$. Which graphs are the same?

167. With a graphing calculator, plot $Y_1 = \cos\left(\dfrac{x}{2}\right)$, $Y_2 = \dfrac{1}{2}\cos x$, and $Y_3 = -\sqrt{\dfrac{1 + \cos x}{2}}$ in the same viewing rectangle $[\pi, 2\pi]$ by $[-1, 1]$. Which graphs are the same?

168. With a graphing calculator, plot $Y_1 = \sin\left(\dfrac{x}{2}\right)$, $Y_2 = \dfrac{1}{2}\sin x$, and $Y_3 = -\sqrt{\dfrac{1 - \cos x}{2}}$ in the same viewing rectangle $[2\pi, 4\pi]$ by $[-1, 1]$. Which graphs are the same?

Section 6.4

169. With a graphing calculator, plot $Y_1 = \sin(5x)\cos(3x)$, $Y_2 = \sin(4x)$, and $Y_3 = \frac{1}{2}[\sin(8x) + \sin(2x)]$ in the same viewing rectangle $[0, 2\pi]$ by $[-1, 1]$. Which graphs are the same?

170. With a graphing calculator, plot $Y_1 = \sin(3x)\cos(5x)$, $Y_2 = \cos(4x)$, and $Y_3 = \frac{1}{2}[\sin(8x) - \sin(2x)]$ in the same viewing rectangle $[0, 2\pi]$ by $[-1, 1]$. Which graphs are the same?

Section 6.5

171. Given $\cos x = -\dfrac{1}{\sqrt{5}}$ and $\dfrac{\pi}{2} < x < \pi$:

a. Find $\cos(2x)$ using the double-angle identity.
b. Use the inverse of cosine to find x in quadrant II and to find $\cos(2x)$.
c. Are the results in (a) and (b) the same?

172. Given $\cos x = \dfrac{5}{12}$ and $\dfrac{3\pi}{2} < x < 2\pi$:

a. Find $\cos\left(\frac{1}{2}x\right)$ using the half-angle identity.
b. Use the inverse of cosine to find x in quadrant IV and to find $\cos\left(\frac{1}{2}x\right)$. Round to five decimal places.
c. Are the results in (a) and (b) the same?

Section 6.6

Find the smallest positive value of x that makes each statement true. Give the answer in radians and round to four decimal places.

173. $\ln x + \sin x = 0$

174. $\ln x + \cos x = 0$

1. For what values of x does the quotient identity $\tan x = \dfrac{\sin x}{\cos x}$ not hold?

2. Is the equation $\sqrt{\sin^2 x + \cos^2 x} = \sin x + \cos x$ a conditional equation or an identity?

3. Evaluate $\sin\left(-\dfrac{\pi}{8}\right)$ exactly.

4. Evaluate $\tan\left(\dfrac{7\pi}{12}\right)$ exactly.

5. If $\cos x = \dfrac{2}{5}$ and $\dfrac{3\pi}{2} < x < 2\pi$, find $\sin\left(\dfrac{x}{2}\right)$.

6. If $\sin x = -\dfrac{1}{5}$ and $\pi < x < \dfrac{3\pi}{2}$, find $\cos(2x)$.

7. Write $\cos(7x)\cos(3x) - \sin(3x)\sin(7x)$ as a cosine or sine of a sum or difference.

8. Write $-\dfrac{2\tan x}{1 - \tan^2 x}$ as a single tangent function.

9. Write $\sqrt{\dfrac{1 + \cos(a + b)}{2}}$ as a single cosine function if $a + b$ is an angle in quadrant II. $\left(\text{Assume } \dfrac{\pi}{2} < a + b < \pi.\right)$

10. Write $2\sin\left(\dfrac{x+3}{2}\right)\cos\left(\dfrac{x-3}{2}\right)$ as a sum of two sine functions.

11. Write $10\cos(3 - x) + 10\cos(x + 3)$ as a product of two cosine functions.

12. In the expression $\sqrt{9 - u^2}$, let $u = 3\sin x$. What is the resulting expression?

Solve the trigonometric equations exactly, if possible. Otherwise, use a calculator to approximate solution(s).

13. $2\sin\theta = -\sqrt{3}$ on all real numbers.

14. $2\cos^2\theta + \cos\theta - 1 = 0$ on $0 \le \theta \le 2\pi$

15. $\sin 2\theta = \dfrac{1}{2}\cos\theta$ over $0 \le \theta \le 360°$

16. $\sqrt{\sin x + \cos x} = -1$ over $0 \le \theta \le 2\pi$

17. Determine whether $(1 + \cot x)^2 = \csc^2 x$ is a conditional or an identity.

18. Evaluate $\csc\left(-\dfrac{\pi}{12}\right)$ exactly.

19. If $\sin x = -\dfrac{5}{13}$ and $\pi < x < \dfrac{3\pi}{2}$, find $\cos\left(\dfrac{x}{2}\right)$.

20. If $\cos x = -0.26$ and $\dfrac{\pi}{2} < x < \pi$, find $\sin(2x)$.

21. Express $y = \sqrt{\dfrac{1 + \frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{3}x\right) + \sin\left(\frac{\pi}{3}x\right)\right]}{1 - \frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{3}x\right) + \sin\left(\frac{\pi}{3}x\right)\right]}}$ as a cotangent function.

22. Calculate $\csc\left(\csc^{-1}\sqrt{2}\right)$.

23. Determine an interval on which $f(x) = a + b\csc(\pi x + c)$ is one-to-one, and determine the inverse of $f(x)$ on this interval. Assume that a, b, and c are all positive.

24. Find the range of $y = -\dfrac{\pi}{4} + \arctan(2x - 3)$.

25. Solve $\cos\left(\dfrac{\pi}{4}\theta\right) = -\dfrac{1}{2}$, for all real numbers.

26. Solve $\sqrt{\dfrac{1 - \cos(2\pi x)}{1 + \cos(2\pi x)}} = -\dfrac{1}{\sqrt{3}}$, for all real numbers.

27. Solve $\dfrac{\sqrt{3}}{\csc\left(\frac{x}{3}\right)} = \cos\left(\dfrac{x}{3}\right)$, for all real numbers.

28. Show that the difference quotient for $f(x) = \cos\left(\frac{1}{2}x\right)$ is

$$-\sin\left(\frac{1}{2}x\right)\left[\frac{\sin\left(\frac{1}{2}h\right)}{h}\right] - \cos\left(\frac{1}{2}x\right)\left[\frac{1 - \cos\left(\frac{1}{2}h\right)}{h}\right].$$

Plot $Y_1 = -\sin\left(\frac{1}{2}x\right)\left[\dfrac{\sin\left(\frac{1}{2}h\right)}{h}\right] - \cos(2x)\left[\dfrac{1 - \cos\left(\frac{1}{2}h\right)}{h}\right]$

for

a. $h = 1$

b. $h = 0.1$

c. $h = 0.01$

What function does the difference quotient for $f(x) = \cos\left(\frac{1}{2}x\right)$ resemble when h approaches zero?

29. Given $\tan x = \dfrac{3}{4}$ and $\pi < x < \dfrac{3\pi}{2}$:

a. Find $\sin\left(\frac{1}{2}x\right)$ using the half-angle identity.

b. Use the inverse of tangent to find x in quadrant III and to find $\sin\left(\frac{1}{2}x\right)$. Round to five decimal places.

c. Are the results in (a) and (b) the same?

1. Find the exact value of the following trigonometric functions:

a. $\sin\left(\frac{7\pi}{3}\right)$

b. $\tan\left(-\frac{5\pi}{3}\right)$

c. $\csc\left(\frac{11\pi}{6}\right)$

2. Find the exact value of the following trigonometric functions:

a. $\sec\left(\frac{5\pi}{6}\right)$

b. $\cos\left(-\frac{3\pi}{4}\right)$

c. $\cot\left(\frac{7\pi}{6}\right)$

3. Find the exact value of the following inverse trigonometric functions:

a. $\cos^{-1}\left(-\frac{1}{2}\right)$

b. $\csc^{-1}(-2)$

c. $\cot\left(-\sqrt{3}\right)$

4. For the relation $x^2 - y^2 = 25$, determine whether y is a function of x.

5. Determine whether the function $g(x) = \sqrt{2 - x^2}$ is odd or even.

6. For the function $y = 5(x - 4)^2$, identify all of the transformations of $y = x^2$.

7. Find the composite function, $f \circ g$, and state the domain for $f(x) = x^3 - 1$ and $g(x) = \frac{1}{x}$.

8. Find the inverse of the function $f(x) = \sqrt[3]{x} - 1$.

9. Find the vertex of the parabola associated with the quadratic function $f(x) = \frac{1}{4}x^2 + \frac{3}{5}x - \frac{6}{25}$.

10. Find a polynomial of minimum degree that has the zeros $x = -\sqrt{7}$ (multiplicity 2), $x = 0$ (multiplicity 3), $x = \sqrt{7}$ (multiplicity 2).

11. Use long division to find the quotient $Q(x)$ and the remainder $r(x)$ of $(5x^3 - 4x^2 + 3) \div (x^2 + 1)$.

12. Given the zero $x = 4i$ of the polynomial $P(x) = x^4 + 2x^3 + x^2 + 32x - 240$, determine all the other zeros and write the polynomial in terms of a product of linear factors.

13. Find the vertical and horizontal asymptotes of the function $f(x) = \dfrac{0.7x^2 - 5x + 11}{x^2 - x - 6}$.

14. If \$5400 is invested at 2.25% compounded continuously, how much is in the account after 4 years?

15. Use interval notation to express the domain of the function $f(x) = \log_4(x + 3)$.

16. Use properties of logarithms to simplify the expression $\log_\pi 1$.

17. Give an exact solution to the logarithmic equation $\log_5(x + 2) + \log_5(6 - x) = \log_5(3x)$.

18. If money is invested in a savings account earning 4% compounded continuously, how many years will it take for the money to triple?

19. Use a calculator to evaluate $\cos 62°$. Round the answer to four decimal places.

20. **Angle of Inclination (Skiing).** The angle of inclination of a mountain with triple black diamond ski trails is 63°. If a skier at the top of the mountain is at an elevation of 4200 feet, how long is the ski run from the top to the base of the mountain?

21. Convert $-105°$ to radians. Leave the answer in terms of π.

22. Find all of the exact values of θ, when $\tan\theta = 1$ and $0 \le \theta \le 2\pi$.

23. Determine whether the equation $\cos^2 x - \sin^2 x = 1$ is a conditional equation or an identity.

24. Simplify $\dfrac{2\tan\left(-\frac{\pi}{8}\right)}{1 - \tan^2\left(-\frac{\pi}{8}\right)}$ and evaluate exactly.

25. Evaluate exactly the expression $\tan\left[\sin^{-1}\left(\frac{5}{13}\right)\right]$.

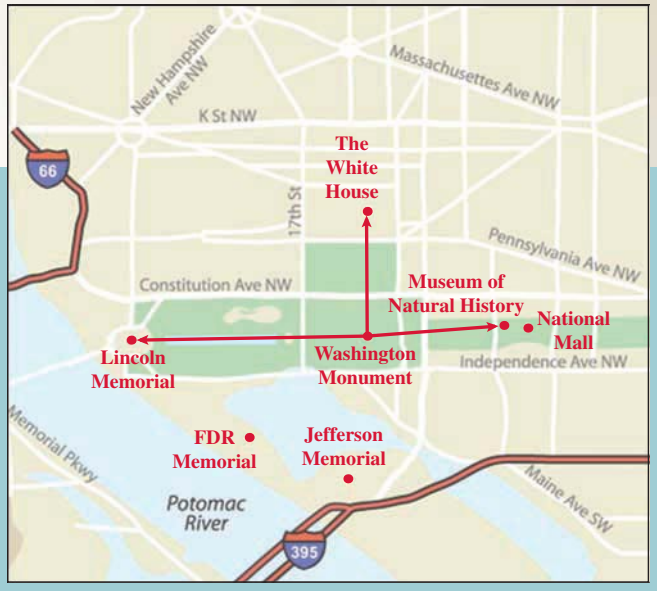
26. With a graphing calculator, plot $Y_1 = \sin x \cos(3x)$, $Y_2 = \cos(4x)$, and $Y_3 = \frac{1}{2}[\sin(4x) - \sin(2x)]$ in the same viewing rectangle $[0, 2\pi]$ by $[-1, 1]$. Which graphs are the same?

27. Find the smallest positive value of x that makes the statement true. Give the answer in radians and round to four decimal places.

$$\ln x - \sin(2x) = 0$$

7

Vectors, the Complex Plane, and Polar Coordinates



Jamirae/iStockphoto

A coordinate system is used to locate a point in a plane. In the Cartesian plane, rectangular coordinates (x, y) are used to describe the location of a point. For example, we can say that the Museum of Natural History in Washington, D.C., is at the corner of Constitution Avenue and 12th Street. But we can also describe the location of the Museum of Natural History as being $\frac{1}{4}$ mile east-northeast of the Washington Monument. Instead of using a grid of streets running east-west and north-south, it is sometimes more convenient to give a location with respect to a distance and direction from a fixed point. In the *polar coordinate system,* the location of a point is given in *polar coordinates* as (r, θ), where r is the distance and θ is the direction angle of the point from a fixed reference point (origin).

IN THIS CHAPTER vectors will be defined and combined with the Law of Sines and the Law of Cosines to find resulting velocity and force vectors. The dot product (product of two vectors) is defined and used in physical problems like calculating work. Trigonometric functions are then used to define complex numbers in polar form. Lastly, we define polar coordinates and examine polar equations and their corresponding graphs.

VECTORS, THE COMPLEX PLANE, AND POLAR COORDINATES

7.1 Vectors

- Magnitude and Direction of Vectors
- Vector Operations
- Horizontal and Vertical Components of a Vector
- Unit Vectors
- Resultant Vectors

7.2 The Dot Product

- The Dot Product
- Angle Between Two Vectors
- Work

7.3 Polar (Trigonometric) Form of Complex Numbers

- Complex Numbers in Rectangular Form
- Complex Numbers in Polar Form

7.4 Products, Quotients, Powers, and Roots of Complex Numbers

- Products of Complex Numbers
- Quotients of Complex Numbers
- Powers of Complex Numbers
- Roots of Complex Numbers

7.5 Polar Coordinates and Graphs of Polar Equations

- Polar Coordinates
- Converting Between Polar and Rectangular Coordinates
- Graphs of Polar Equations

LEARNING OBJECTIVES

- Find the direction and magnitude of a vector.
- Find the dot product of two vectors.
- Express complex numbers in polar form.
- Use De Moivre's theorem to find a complex number raised to a power.
- Convert between rectangular and polar coordinates.

SECTION 7.1 VECTORS

SKILLS OBJECTIVES

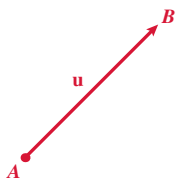
- Represent vectors geometrically and algebraically.
- Find the magnitude and direction of a vector.
- Add and subtract vectors.
- Perform scalar multiplication of a vector.
- Find unit vectors.
- Express a vector in terms of its horizontal and vertical components.

CONCEPTUAL OBJECTIVES

- Understand the difference between scalars and vectors.
- Relate the geometric and algebraic representations of vectors.

Magnitude and Direction of Vectors

What is the difference between velocity and speed? Speed (55 miles per hour) has only *magnitude*, whereas velocity (55 miles per hour west) has both *magnitude* and *direction*. We use **scalars**, which are real numbers, to denote magnitudes such as speed and mass. We use **vectors**, which have magnitude *and* direction, to denote quantities such as velocity (speed in a certain direction) and force (weight in a certain direction).



A vector quantity is geometrically denoted by a **directed line segment**, which is a line segment with an arrow representing direction. There are many ways to denote a vector. For example, the vector shown in the margin can be denoted as $\mathbf{u}$, $\vec{u}$, or $\overrightarrow{AB}$, where A is the **initial point** and B is the **terminal point**.

It is customary in books to use the bold letter to represent a vector and when handwritten (as in your class notes and homework) to use the arrow on top to denote a vector.

In this section, we will limit our discussion to vectors in a plane (two-dimensional). It is important to note that geometric representation can be extended to three dimensions and algebraic representation can be extended to any higher dimension, as you will see in the exercises.

Study Tip

The magnitude of a vector is the distance between the initial and terminal points of the vector.

Geometric Interpretation of Vectors

The *magnitude* of a vector can be denoted one of two ways: $|\mathbf{u}|$ or $\|\mathbf{u}\|$. We will use the former notation.

MAGNITUDE: $|\mathbf{u}|$

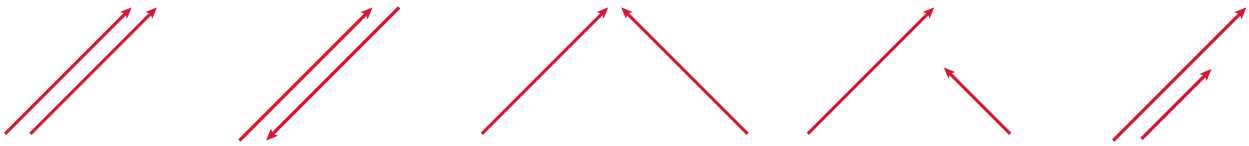
The **magnitude** of a vector $\mathbf{u}$, denoted $|\mathbf{u}|$, is the length of the directed line segment, that is the distance between the initial and terminal points of the vector.

Two vectors have the **same direction** if they are parallel and point in the same direction. Two vectors have **opposite direction** if they are parallel and point in opposite directions.

EQUAL VECTORS: $\mathbf{u} = \mathbf{v}$

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **equal** ($\mathbf{u} = \mathbf{v}$) if and only if they have the same magnitude ($|\mathbf{u}| = |\mathbf{v}|$) and the same direction.

Equal Vectors $\mathbf{u} = \mathbf{v}$	Same Magnitude but Opposite Direction $\mathbf{u} = -\mathbf{v}$	Same Magnitude $\lvert\mathbf{u}\rvert = \lvert\mathbf{v}\rvert$	Different Magnitude	Same Direction Different Magnitude



It is important to note that vectors do not have to coincide to be equal.

Study Tip

Equal vectors can be translated (shifted) so that they coincide.

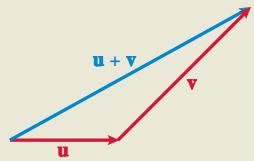
Vector Addition: u + v

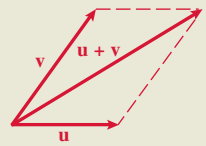
Two vectors, **u** and **v**, can be added together using either of the following approaches:

- The **tail-to-tip** (or head-to-tail) method: Sketch the initial point of one vector at the terminal point of the other vector. The **sum**, $\mathbf{u} + \mathbf{v}$, is the **resultant** vector from the tail end of **u** to the tip end of **v**.

  [or]

- The parallelogram method: Sketch the initial points of the vectors at the same point. The sum $\mathbf{u} + \mathbf{v}$ is the diagonal of the parallelogram formed by **u** and **v**.



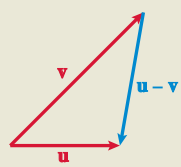


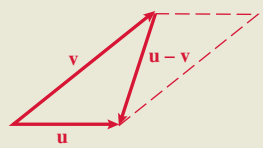
The difference, $\mathbf{u} - \mathbf{v}$, is the

- Resultant vector from the tip of **v** to the tip of **u**, when the tails of **v** and **u** coincide.

  [or]

- The other diagonal formed by the parallelogram method.



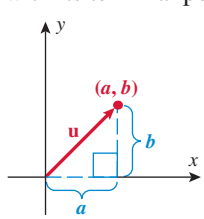


Algebraic Interpretation of Vectors

Since vectors that have the same direction and magnitude are equal, any vector can be translated to an equal vector with its initial point located at the origin in the Cartesian plane. Therefore, we will now consider vectors in a rectangular coordinate system.

A vector with its initial point at the origin is called a **position vector**, or a vector in **standard position**. A position vector **u** with its terminal point at the point (a, b) is denoted:

$$\mathbf{u} = \langle a, b \rangle$$



where the real numbers a and b are called the **components** of vector **u**.

Study Tip

$\langle a, b\rangle$ denotes a vector
(a, b) denotes a point

Notice the subtle difference between coordinate notation and vector notation. The point is denoted with parentheses, (a, b), whereas the vector is denoted with angled brackets, $\langle a, b\rangle$. The notation $\langle a, b\rangle$ denotes a vector whose initial point is $(0, 0)$ and terminal point is (a, b).

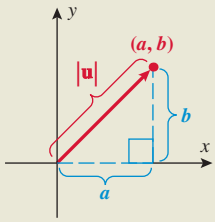
The vector with initial point $(3, 4)$ and terminal point $(8, 10)$ is equal to the vector $\langle 5, 6\rangle$, which has initial point $(0, 0)$ and terminal point $(5, 6)$.

Recall that the geometric definition of the *magnitude* of a vector is the *length* of the vector.

MAGNITUDE: $|\mathbf{u}|$

The **magnitude** (or norm) of a vector, $\mathbf{u} = \langle a, b\rangle$, is

$$|\mathbf{u}| = \sqrt{a^2 + b^2}$$



EXAMPLE 1 Finding the Magnitude of a Vector

Find the magnitude of the vector $\mathbf{u} = \langle 3, -4\rangle$.

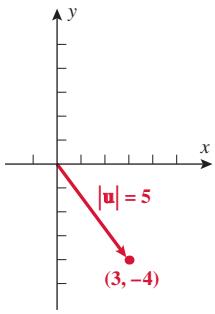
Solution:

Write the formula for magnitude of a vector. $|\mathbf{u}| = \sqrt{a^2 + b^2}$

Let $a = 3$ and $b = -4$. $|\mathbf{u}| = \sqrt{3^2 + (-4)^2}$

Simplify. $|\mathbf{u}| = \boxed{\sqrt{25} = 5}$

Note: If we graph the vector $\mathbf{u} = \langle 3, -4\rangle$, we see that the distance from the origin to the point $(3, -4)$ is five units.



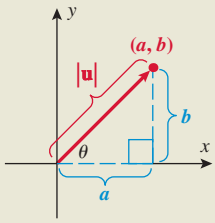
■ **YOUR TURN** Find the magnitude of the vector $\mathbf{v} = \langle -1, 5\rangle$.

■ **Answer:** $\sqrt{26}$

DIRECTION ANGLE OF A VECTOR

The positive angle between the x-axis and a position vector is called the **direction angle**, denoted θ.

$$\tan\theta = \frac{b}{a}, \text{ where } a \neq 0$$



EXAMPLE 2 Finding the Direction Angle of a Vector

Find the direction angle of the vector $\mathbf{v} = \langle -1, 5 \rangle$.

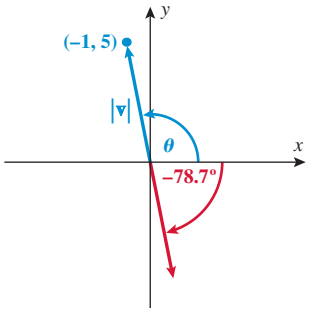
Solution:

Start with $\tan\theta = \dfrac{b}{a}$ and let $a = -1$ and $b = 5$. $\tan\theta = \dfrac{5}{-1}$

With a calculator, approximate $\tan^{-1}(-5)$. $\tan^{-1}(-5) \approx -78.7°$

The calculator gives a **quadrant IV** angle.

The point $(-1, 5)$ lies in quadrant II.



Add 180°. $\theta = -78.7° + 180° = 101.3°$

$\boxed{\theta = 101.3°}$

Technology Tip

Use the calculator to find θ.

```
tan⁻¹(-5)
    -78.69006753
```

■ **YOUR TURN** Find the direction angle of the vector $\mathbf{u} = \langle 3, -4 \rangle$.

■ **Answer:** 306.9°

Recall that two vectors are equal if they have the same magnitude and direction. Algebraically, this corresponds to their corresponding vector components (a and b) being equal.

EQUAL VECTORS: U = V

The vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ are **equal** (that is, $\mathbf{u} = \mathbf{v}$) if and only if $a = c$ and $b = d$.

Vector Operations

Vector addition is done geometrically with the tail-to-tip rule. Algebraically, vector addition is performed component by component.

VECTOR ADDITION: U + V

If $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$, then $\mathbf{u} + \mathbf{v} = \langle a + c, b + d \rangle$.

EXAMPLE 3 Adding Vectors

Let $\mathbf{u} = \langle 2, -7 \rangle$ and $\mathbf{v} = \langle -3, 4 \rangle$. Find $\mathbf{u} + \mathbf{v}$.

Solution:

Let $\mathbf{u} = \langle 2, -7 \rangle$ and $\mathbf{v} = \langle -3, 4 \rangle$ in the addition formula. $\mathbf{u} + \mathbf{v} = \langle 2 + (-3), -7 + 4 \rangle$

Simplify. $\mathbf{u} + \mathbf{v} = \boxed{\langle -1, -3 \rangle}$

■ **YOUR TURN** Let $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{v} = \langle -5, -4 \rangle$. Find $\mathbf{u} + \mathbf{v}$.

■ **Answer:** $\mathbf{u} + \mathbf{v} = \langle -4, -2 \rangle$

We now summarize vector operations. As we have seen, addition and subtraction are performed algebraically component by component. Multiplication, however, is not as straightforward. To perform **scalar multiplication** of a vector (to multiply a vector by a real number), we multiply each component by the scalar. In Section 7.2, we will study a form of multiplication for two vectors that is defined as long as the vectors have the same number of components; it gives a result known as the *dot product*, and is useful in solving common problems in physics.

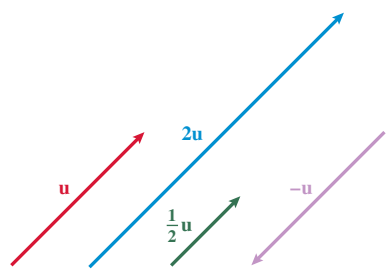
SCALAR MULTIPLICATION: $k\mathbf{u}$

If k is a scalar (real number) and $\mathbf{u} = \langle a, b\rangle$, then

$$k\mathbf{u} = k\langle a, b\rangle = \langle ka, kb\rangle$$

Scalar multiplication corresponds to

- Increasing the length of the vector: $|k| > 1$
- Decreasing the length of the vector: $|k| < 1$
- Changing the direction of the vector: $k < 0$



The following box is a summary of vector operations:

VECTOR OPERATIONS

If $\mathbf{u} = \langle a, b\rangle$, $\mathbf{v} = \langle c, d\rangle$, and k is a scalar, then

$$\mathbf{u} + \mathbf{v} = \langle a + c, b + d\rangle$$

$$\mathbf{u} - \mathbf{v} = \langle a - c, b - d\rangle$$

$$k\mathbf{u} = k\langle a, b\rangle = \langle ka, kb\rangle$$

The zero vector, $\mathbf{0} = \langle 0, 0\rangle$, is a vector in any direction with a magnitude equal to zero. We now can state the algebraic properties (associative, commutative, and distributive) of vectors.

ALGEBRAIC PROPERTIES OF VECTORS

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$(k_1k_2)\mathbf{u} = k_1(k_2\mathbf{u})$$

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$$

$$(k_1 + k_2)\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$$

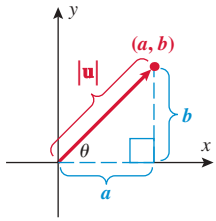
$$0\mathbf{u} = \mathbf{0} \qquad 1\mathbf{u} = \mathbf{u} \qquad -1\mathbf{u} = -\mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

Horizontal and Vertical Components of a Vector

The **horizontal component** a and **vertical component** b of a vector $\mathbf{u}$ are related to the magnitude of the vector, $|\mathbf{u}|$, through the sine and cosine of the direction angle.

$$\cos\theta = \frac{a}{|\mathbf{u}|} \qquad \sin\theta = \frac{b}{|\mathbf{u}|}$$



Horizontal and Vertical Components of a Vector

The horizontal and vertical components of vector $\mathbf{u}$, with magnitude $|\mathbf{u}|$ and direction angle θ, are given by

$$\text{horizontal component: } a = |\mathbf{u}|\cos\theta$$
$$\text{vertical component: } b = |\mathbf{u}|\sin\theta$$

The vector $\mathbf{u}$ can then be written as $\mathbf{u} = \langle a, b\rangle = \langle |\mathbf{u}|\cos\theta, |\mathbf{u}|\sin\theta\rangle$.

EXAMPLE 4 Finding the Horizontal and Vertical Components of a Vector

Find the vector that has a magnitude of 6 and a direction angle of 15°.

Solution:

Write the horizontal and vertical components of a vector $\mathbf{u}$.	$a = |\mathbf{u}|\cos\theta$ and $b = |\mathbf{u}|\sin\theta$
Let $|\mathbf{u}| = 6$ and $\theta = 15°$.	$a = 6\cos 15°$ and $b = 6\sin 15°$
Use a calculator to approximate the sine and cosine functions of 15°.	$a \approx 5.8$ and $b \approx 1.6$
Let $\mathbf{u} = \langle a, b\rangle$.	$\mathbf{u} = \boxed{\langle 5.8, 1.6\rangle}$

■ **YOUR TURN** Find the vector that has a magnitude of 3 and direction angle of 75°.

■ **Answer:** $\mathbf{u} = \langle 0.78, 2.9\rangle$

Unit Vectors

A **unit vector** is any vector with magnitude equal to 1 or $|\mathbf{u}| = 1$. It is often useful to be able to find a unit vector in the same direction of some vector $\mathbf{v}$. A unit vector can be formed from any nonzero vector as follows:

Finding a Unit Vector

If $\mathbf{v}$ is a nonzero vector, then

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|}\cdot\mathbf{v}$$

is a **unit vector** in the same direction as $\mathbf{v}$. In other words, multiplying any nonzero vector by the reciprocal of its magnitude results in a unit vector.

Study Tip

Multiplying a nonzero vector by the reciprocal of its magnitude results in a unit vector.

It is important to notice that since the magnitude is always a scalar, then the reciprocal of the magnitude is always a scalar. A scalar times a vector is a vector.

EXAMPLE 5 Finding a Unit Vector

Find a unit vector in the same direction as $\mathbf{v} = \langle -3, -4 \rangle$.

Solution:

Find the magnitude of the vector $\mathbf{v} = \langle -3, -4 \rangle$.	$\lvert\mathbf{v}\rvert = \sqrt{(-3)^2 + (-4)^2}$
Simplify.	$\lvert\mathbf{v}\rvert = 5$
Multiply $\mathbf{v}$ by the reciprocal of its magnitude.	$\frac{1}{\lvert\mathbf{v}\rvert} \cdot \mathbf{v}$
Let $\lvert\mathbf{v}\rvert = 5$ and $\mathbf{v} = \langle -3, -4 \rangle$.	$\frac{1}{5}\langle -3, -4 \rangle$
Simplify.	$\boxed{\left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle}$

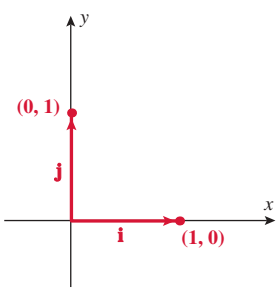
Check: The unit vector, $\left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle$, should have a magnitude of 1.

$$\sqrt{\left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{25}{25}} = 1$$

■ **YOUR TURN** Find a unit vector in the same direction as $\mathbf{v} = \langle 5, -12 \rangle$.

■ **Answer:** $\left\langle \frac{5}{13}, -\frac{12}{13} \right\rangle$

Two important unit vectors are the horizontal and vertical unit vectors **i** and **j**. The unit vector **i** has an initial point at the origin and terminal point at (1, 0). The unit vector **j** has an initial point at the origin and terminal point at (0, 1). We can use these unit vectors to represent vectors algebraically. For example, the vector $\langle 3, -4 \rangle = 3\mathbf{i} - 4\mathbf{j}$.



Resultant Vectors

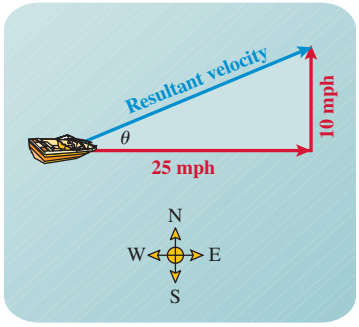
Vectors arise in many applications. **Velocity vectors** and **force vectors** are two that we will discuss. For example, suppose that you are at the beach and "think" that you are swimming straight out at a certain speed (magnitude and direction). This is your **apparent velocity** with respect to the water. After a few minutes you turn around to look at the shore, and you are farther out than you thought and appear to have drifted down the beach. This is because of the current of the water. When the **current velocity** and the apparent velocity are added together, the result is the **actual** or **resultant velocity**.

EXAMPLE 6 Resultant Velocities

A boat's speedometer reads 25 miles per hour (which is relative to the water) and sets a course due east (90° from due north). If the river is moving 10 miles per hour due north, what is the resultant (actual) velocity of the boat?

Solution:

Draw a picture.



Label the horizontal and vertical components of the resultant vector. $\langle 25, 10 \rangle$

Determine the magnitude of the resultant vector. $\sqrt{25^2 + 10^2} = 5\sqrt{29} \approx 27 \text{ mph}$

Determine the direction angle. $\tan\theta = \dfrac{10}{25}$

Solve for θ. $\theta = \tan^{-1}\left(\dfrac{2}{5}\right) \approx 22°$

The actual velocity of the boat has magnitude $\boxed{\text{27 miles per hour}}$ and the boat is headed $\boxed{\text{22° north of east or 68° east of north}}$.

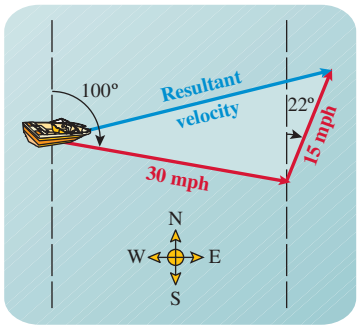
In Example 6, the three vectors formed a right triangle. In Example 7, the three vectors form an oblique triangle.

EXAMPLE 7 Resultant Velocities

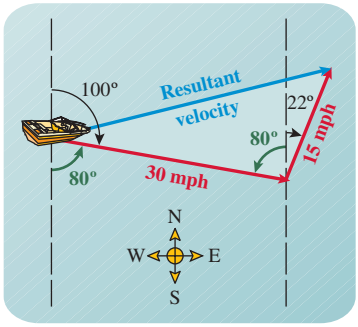
A speedboat traveling 30 miles per hour has a compass heading of 100° east of north. The current velocity has a magnitude of 15 miles per hour and its heading is 22° east of north. Find the resultant (actual) velocity of the boat.

Solution:

Draw a picture.



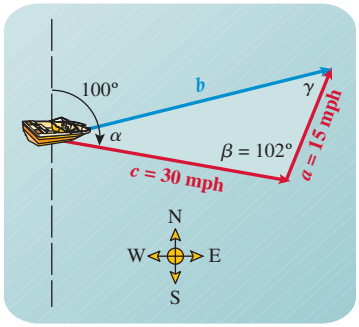
Label the **supplementary angles** to 100°.



Draw and label the oblique triangle.

The magnitude of the actual (resultant) velocity is b.

The heading of the actual (resultant) velocity is $100° - \alpha$.



Use the Law of Sines and the Law of Cosines to solve for α and b.

Find b: Apply the Law of Cosines. $b^2 = a^2 + c^2 - 2ac\cos\beta$

Let $a = 15$, $c = 30$, and $\beta = 102°$. $b^2 = 15^2 + 30^2 - 2(15)(30)\cos 102°$

Solve for b. $\boxed{b \approx 36 \text{ mph}}$

Find α: Apply the Law of Sines. $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

Isolate $\sin\alpha$. $\sin\alpha = \dfrac{a}{b}\sin\beta$

Let $a = 15$, $b = 36$, and $\beta = 102°$. $\sin\alpha = \dfrac{15}{36}\sin 102°$

Apply the inverse sine function to solve for α. $\alpha = \sin^{-1}\left(\dfrac{15}{36}\sin 102°\right)$

Approximate α with a calculator. $\alpha \approx 24°$

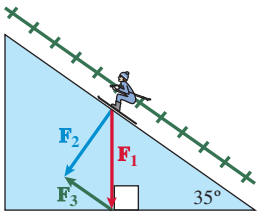
Actual heading: $100° - \alpha = 100° - 24° = \boxed{76°}$

The actual velocity vector of the boat has magnitude $\boxed{\text{36 miles per hour}}$ and the boat is headed $\boxed{\text{76° east of north}}$.

Two vectors combine to yield a resultant vector. The opposite vector to the resultant vector is called the **equilibrant**.

EXAMPLE 8 Finding an Equilibrant

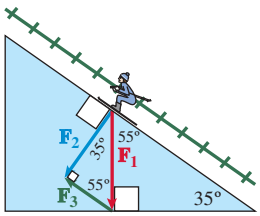
A skier is being pulled up a slope by a handle lift. Let $\mathbf{F}_1$ represent the vertical force due to gravity and $\mathbf{F}_2$ represent the force of the skier pushing against the side of the mountain, at an angle of 35° to the horizontal. If the weight of the skier is 145 pounds, that is, $|\mathbf{F}_1| = 145$, find the magnitude of the equilibrant force $\mathbf{F}_3$ required to hold the skier in place (i.e., to keep the skier from sliding down the mountain). Assume that the side of the mountain is a frictionless surface.



Solution:

The angle between vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ is 35°.

The magnitude of vector $\mathbf{F}_3$ is the force required to hold the skier in place.



Relate the magnitudes (side lengths) to the given angle using the sine ratio.

$$\sin 35° = \frac{|\mathbf{F}_3|}{|\mathbf{F}_1|}$$

Solve for $|\mathbf{F}_3|$.

$$|\mathbf{F}_3| = |\mathbf{F}_1| \sin 35°$$

Let $|\mathbf{F}_1| = 145$.

$$|\mathbf{F}_3| = 145 \sin 35°$$

$$|\mathbf{F}_3| = 83.16858$$

A force of approximately $\boxed{\text{83 pounds}}$ is required to keep the skier from sliding down the hill.

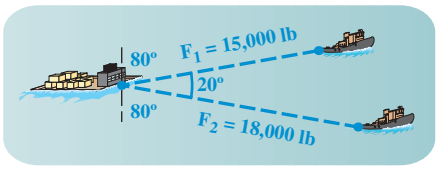
Technology Tip

Use a TI calculator to find $|\mathbf{F}_3| = 145 \sin 35°$.

```
145sin(35)
      83.16858327
```

EXAMPLE 9 Resultant Forces

A barge runs aground outside the channel. A single tugboat cannot generate enough force to pull the barge off the sandbar. A second tugboat comes to assist. The following diagram illustrates the force vectors, $\mathbf{F}_1$ and $\mathbf{F}_2$, from the tugboats. What is the resultant force vector of the two tugboats?



Technology Tip

Use the calculator to find b.

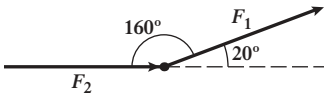
```
15000²+18000²-2*
15000*18000cos(1
60)
        1056434015
√(Ans)
       32502.83088
```

Use the calculator to find α.

```
15000/32503sin(1
60)
        .1578408808
sin⁻¹(Ans)
        9.08159538
```

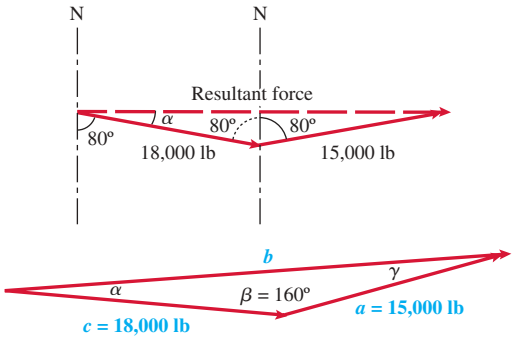
Study Tip

In Example 9, $\beta = 160°$ because the angle between the paths of the tugboats is 20°.



Solution:

Using the tail-to-tip rule, we can add these two vectors and form a triangle:



Find b: Apply the Law of Cosines. $\quad b^2 = a^2 + c^2 - 2ac\cos\beta$

Let $a = 15{,}000$, $c = 18{,}000$, and $\beta = 160°$. $\quad b^2 = 15{,}000^2 + 18{,}000^2 - 2(15{,}000)(18{,}000)\cos 160°$

Solve for b. $\quad \boxed{b = 32{,}503 \text{ lb}}$

Find α: Apply the Law of Sines. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

Isolate $\sin\alpha$. $\quad \sin\alpha = \dfrac{a}{b}\sin\beta$

Let $a = 15{,}000$, $b = 32{,}503$, and $\beta = 160°$. $\quad \sin\alpha = \dfrac{15{,}000}{32{,}503}\sin 160°$

Apply the inverse sine function to solve for α. $\quad \alpha = \sin^{-1}\left(\dfrac{15{,}000}{32{,}503}\sin 160°\right)$

Approximate α with a calculator. $\quad \boxed{\alpha \approx 9.08°}$

The resulting force is $\boxed{\text{32,503 pounds}}$ at an angle of $\boxed{\text{9° from the tug pulling with a force of 18,000 pounds}}$.

SECTION 7.1 SUMMARY

In this section, we discussed scalars (real numbers) and vectors. Scalars have only magnitude, whereas vectors have both magnitude and direction.

$$\text{Vector:} \quad \mathbf{u} = \langle a, b\rangle$$

$$\text{Magnitude:} \quad |\mathbf{u}| = \sqrt{a^2 + b^2}$$

$$\text{Direction } (\theta): \quad \tan\theta = \frac{b}{a}$$

We defined vectors both algebraically and geometrically and gave interpretations of magnitude and vector addition in both methods.

Vector addition is performed algebraically component by component.

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle.$$

The trigonometric functions are used to express the horizontal and vertical components of a vector.

$$\text{Horizontal component:} \quad a = |\mathbf{u}|\cos\theta$$

$$\text{Vertical component:} \quad b = |\mathbf{u}|\sin\theta$$

Velocity and force vectors illustrate applications of the Law of Sines and the Law of Cosines.

SECTION 7.1 EXERCISES

SKILLS

In Exercises 1–6, find the magnitude of the vector $\overrightarrow{AB}$.

1. $A = (2, 7)$ and $B = (5, 9)$ **2.** $A = (-2, 3)$ and $B = (3, -4)$ **3.** $A = (4, 1)$ and $B = (-3, 0)$

4. $A = (-1, -1)$ and $B = (2, -5)$ **5.** $A = (0, 7)$ and $B = (-24, 0)$ **6.** $A = (-2, 1)$ and $B = (4, 9)$

In Exercises 7–16, find the magnitude and direction angle of the given vector.

7. $\mathbf{u} = \langle 3, 8\rangle$ **8.** $\mathbf{u} = \langle 4, 7\rangle$ **9.** $\mathbf{u} = \langle 5, -1\rangle$ **10.** $\mathbf{u} = \langle -6, -2\rangle$ **11.** $\mathbf{u} = \langle -4, 1\rangle$

12. $\mathbf{u} = \langle -6, 3\rangle$ **13.** $\mathbf{u} = \langle -8, 0\rangle$ **14.** $\mathbf{u} = \langle 0, 7\rangle$ **15.** $\mathbf{u} = \langle \sqrt{3}, 3\rangle$ **16.** $\mathbf{u} = \langle -5, -5\rangle$

In Exercises 17–24, perform the indicated vector operation, given $\mathbf{u} = \langle -4, 3\rangle$ and $\mathbf{v} = \langle 2, -5\rangle$.

17. $\mathbf{u} + \mathbf{v}$ **18.** $\mathbf{u} - \mathbf{v}$ **19.** $3\mathbf{u}$ **20.** $-2\mathbf{u}$

21. $2\mathbf{u} + 4\mathbf{v}$ **22.** $5(\mathbf{u} + \mathbf{v})$ **23.** $6(\mathbf{u} - \mathbf{v})$ **24.** $2\mathbf{u} - 3\mathbf{v} + 4\mathbf{u}$

In Exercises 25–34, find the vector, given its magnitude and direction angle.

25. $|\mathbf{u}| = 7, \theta = 25°$ **26.** $|\mathbf{u}| = 5, \theta = 75°$ **27.** $|\mathbf{u}| = 16, \theta = 100°$ **28.** $|\mathbf{u}| = 8, \theta = 200°$ **29.** $|\mathbf{u}| = 4, \theta = 310°$

30. $|\mathbf{u}| = 8, \theta = 225°$ **31.** $|\mathbf{u}| = 9, \theta = 335°$ **32.** $|\mathbf{u}| = 3, \theta = 315°$ **33.** $|\mathbf{u}| = 2, \theta = 120°$ **34.** $|\mathbf{u}| = 6, \theta = 330°$

In Exercises 35–44, find a unit vector in the direction of the given vector.

35. $\mathbf{v} = \langle -5, -12\rangle$ **36.** $\mathbf{v} = \langle 3, 4\rangle$ **37.** $\mathbf{v} = \langle 60, 11\rangle$ **38.** $\mathbf{v} = \langle -7, 24\rangle$ **39.** $\mathbf{v} = \langle 24, -7\rangle$

40. $\mathbf{v} = \langle -10, 24\rangle$ **41.** $\mathbf{v} = \langle -9, -12\rangle$ **42.** $\mathbf{v} = \langle 40, -9\rangle$ **43.** $\mathbf{v} = \langle \sqrt{2}, 3\sqrt{2}\rangle$ **44.** $\mathbf{v} = \langle -4\sqrt{3}, -2\sqrt{3}\rangle$

In Exercises 45–50, express the vector in terms of unit vectors i and j.

45. $\langle 7, 3\rangle$ **46.** $\langle -2, 4\rangle$ **47.** $\langle 5, -3\rangle$ **48.** $\langle -6, -2\rangle$ **49.** $\langle -1, 0\rangle$ **50.** $\langle 0, 2\rangle$

In Exercises 51–56, perform the indicated vector operation.

51. $(5\mathbf{i} - 2\mathbf{j}) + (-3\mathbf{i} + 2\mathbf{j})$ **52.** $(4\mathbf{i} - 2\mathbf{j}) + (3\mathbf{i} - 5\mathbf{j})$ **53.** $(-3\mathbf{i} + 3\mathbf{j}) - (2\mathbf{i} - 2\mathbf{j})$

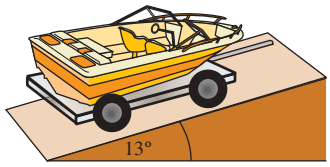
54. $(\mathbf{i} - 3\mathbf{j}) - (-2\mathbf{i} + \mathbf{j})$ **55.** $(5\mathbf{i} + 3\mathbf{j}) + (2\mathbf{i} - 3\mathbf{j})$ **56.** $(-2\mathbf{i} + \mathbf{j}) + (2\mathbf{i} - 4\mathbf{j})$

APPLICATIONS

57. Bullet Speed. A bullet is fired from ground level at a speed of 2200 feet per second at an angle of 30° from the horizontal. Find the magnitude of the horizontal and vertical components of the velocity vector.

58. Weightlifting. A 50-pound weight lies on an inclined bench that makes an angle of 40° with the horizontal. Find the component of the weight directed perpendicular to the bench and also the component of the weight parallel to the inclined bench.

59. Weight of a Boat. A force of 630 pounds is needed to pull a speedboat and its trailer up a ramp that has an incline of 13°. What is the combined weight of the boat and its trailer?

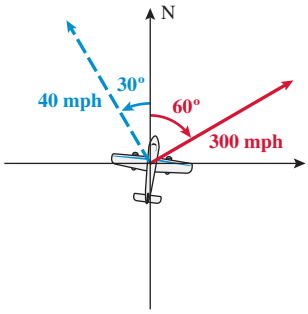


60. Weight of a Boat. A force of 500 pounds is needed to pull a speedboat and its trailer up a ramp that has an incline of 16°. What is the weight of the boat and its trailer?

61. Speed and Direction of a Ship. A ship's captain sets a course due north at 10 miles per hour. The water is moving at 6 miles per hour due west. What is the actual velocity of the ship, and in what direction is it traveling?

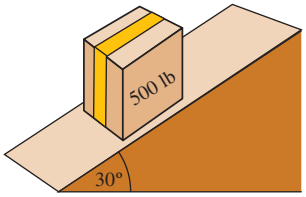
62. Speed and Direction of a Ship. A ship's captain sets a course due west at 12 miles per hour. The water is moving at 3 miles per hour due north. What is the actual velocity of the ship, and in what direction is it traveling?

63. Heading and Airspeed. A plane has a compass heading of 60° east of due north and an airspeed of 300 miles per hour. The wind is blowing at 40 miles per hour with a heading of 30° west of due north. What are the plane's actual heading and airspeed?



64. Heading and Airspeed. A plane has a compass heading of 30° east of due north and an airspeed of 400 miles per hour. The wind is blowing at 30 miles per hour with a heading of 60° west of due north. What are the plane's actual heading and airspeed?

65. Sliding Box. A box weighing 500 pounds is held in place on an inclined plane that has an angle of 30°. What force is required to hold it in place?



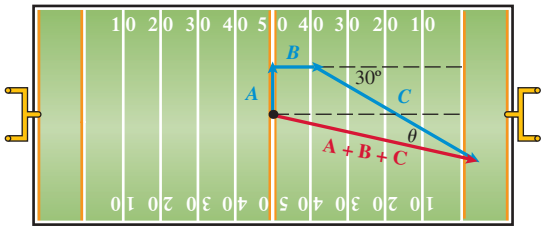
66. Sliding Box. A box weighing 500 pounds is held in place on an inclined plane that has an angle of 10°. What force is required to hold it in place?

67. Baseball. A baseball player throws a ball with an initial velocity of 80 feet per second at an angle of 40° with the horizontal. What are the vertical and horizontal components of the velocity?

68. Baseball. A baseball pitcher throws a ball with an initial velocity of 100 feet per second at an angle of 5° with the horizontal. What are the vertical and horizontal components of the velocity?

For Exercises 69 and 70, refer to the following:

In a post pattern in football, the receiver in motion runs past the quarterback parallel to the line of scrimmage (A), runs perpendicular to the line of scrimmage (B), and then cuts toward the goal post (C).



69. Football. A receiver runs the post pattern. If the magnitudes of the vectors are $|A| = 4$ yd, $|B| = 12$ yd, and $|C| = 20$ yd, find the magnitude of the resultant vector $\mathbf{A} + \mathbf{B} + \mathbf{C}$.

70. Football. A receiver runs the post pattern. If the magnitudes of the vectors are $|A| = 4$ yd, $|B| = 12$ yd, and $|C| = 20$ yd, find the direction angle θ.

71. Resultant Force. A force with a magnitude of 100 pounds and another with a magnitude of 400 pounds are acting on an object. The two forces have an angle of 60° between them. What is the direction of the resultant force with respect to the force of 400 pounds?

72. Resultant Force. A force with a magnitude of 100 pounds and another with a magnitude of 400 pounds are acting on an object. The two forces have an angle of 60° between them. What is the magnitude of the resultant force?

73. Resultant Force. A force of 1000 pounds is acting on an object at an angle of 45° from the horizontal. Another force of 500 pounds is acting at an angle of −40° from the horizontal. What is the magnitude of the resultant force?

74. Resultant Force. A force of 1000 pounds is acting on an object at an angle of 45° from the horizontal. Another force of 500 pounds is acting at an angle of −40° from the horizontal. What is the angle of the resultant force?

75. Resultant Force. Forces with magnitudes of 200 N and 180 N act on a hook. The angle between these two forces is 45°. Find the direction and magnitude of the resultant of these forces.

76. Resultant Force. Forces with magnitudes of 100 N and 50 N act on a hook. The angle between these two forces is 30°. Find the direction and magnitude of the resultant of these forces.

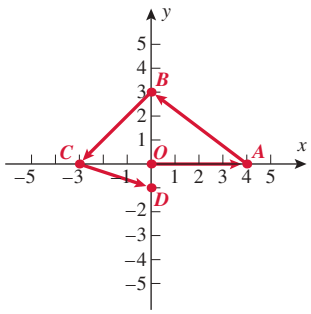
77. Exercise Equipment. A tether ball weighing 5 pounds is pulled outward from a pole by a horizontal force $\mathbf{u}$ until the rope makes a 45° angle with the pole. Determine the resulting tension (in pounds) on the rope and magnitude of $\mathbf{u}$.

78. Exercise Equipment. A tether ball weighing 8 pounds is pulled outward from a pole by a horizontal force $\mathbf{u}$ until the rope makes a 60° angle with the pole. Determine the resulting tension (in pounds) on the rope and magnitude of $\mathbf{u}$.

79. Recreation. A freshman wishes to sign up for four different clubs during orientation. Each club is positioned at a different table in the gym and the clubs of interest to him are positioned at A, B, C, and D, as pictured below. He starts at the entrance way O and walks directly toward A, followed by B and C, then to D, and then back to O.

a. Find the resultant vector of all his movement.

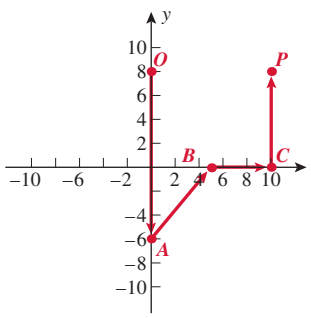
b. How far did he walk during this sign-up adventure?



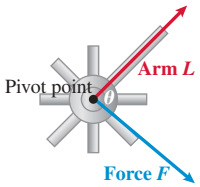
80. Recreation. A freshman wishes to sign up for three different clubs during orientation. Each club is positioned at a different table in the gym and the clubs of interest to him are positioned at A, B, and C, as pictured below. He starts at the entrance way O at the far end of the gym, walks directly toward A, followed by B and C, and then exits the gym through the exit P at the opposite end.

a. Find the resultant vector of all his movement.

b. How far did he walk during this sign-up adventure?



81. Torque. *Torque* is the tendency for an arm to rotate about a pivot point. If a force $\mathbf{F}$ is applied at an angle θ to turn an arm of length L, as pictured below, then the magnitude of the torque $= L|\mathbf{F}|\sin\theta$.



Assume that a force of 45 N is applied to a bar 0.2 meter wide on a sewer shut-off valve at an angle 85°. What is the magnitude of the torque in N-m?

82. Torque. You walk through a swinging mall door to enter a department store. You exert a force of 40 N applied perpendicular to the door. The door is 0.85 meter wide. Assuming that you pushed the door at its edge and the hinge is the pivot point, find the magnitude of the torque.

83. Torque. You walk through a swinging mall door to enter a department store. You exert a force of 40 N applied at an angle 110° to the door. The door is 0.85 meter wide. Assuming that you pushed the door at its edge and the hinge is the pivot point, find the magnitude of the torque.

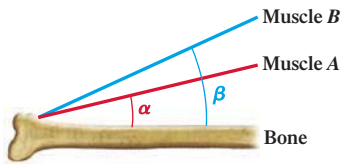
84. Torque. Suppose that within the context of Exercises 82 and 83, the magnitude of the torque turned out to be 0 N-m. When can this occur?

85. Resultant Force. A person is walking two dogs fastened to separate leashes that meet in a connective hub, leading to a single leash that she is holding. Dog 1 applies a force N 60° W with a magnitude of 8, and Dog 2 applies a force of N 45° E with a magnitude of 6. Find the magnitude and direction of the force $\mathbf{w}$ that the walker applies to the leash in order to counterbalance the total force exerted by the dogs.

86. Resultant Force. A person is walking three dogs fastened to separate leashes that meet in a connective hub, leading to a single leash that she is holding. Dog 1 applies a force N 60° W with a magnitude of 8, Dog 2 applies a force of N 45° E with a magnitude of 6, and Dog 3 moves directly N with a magnitude of 12. Find the magnitude and direction of the force $\mathbf{w}$ that the walker applies to the leash in order to counterbalance the total force exerted by the dogs.

For Exercises 87 and 88, refer to the following:

Muscle A and muscle B are attached to a bone as indicated in the figure below. Muscle A exerts a force on the bone at angle α, while muscle B exerts a force on the bone at angle β.



87. Health/Medicine. Assume muscle A exerts a force of 900 N on the bone at angle $\alpha = 8°$, while muscle B exerts a force of 750 N on the bone at angle $\beta = 33°$. Find the resultant force and the angle of the force due to muscle A and muscle B on the bone.

88. Health/Medicine. Assume muscle A exerts a force of 1000 N on the bone at angle $\alpha = 9°$, while muscle B exerts a force of 820 N on the bone at angle $\beta = 38°$. Find the resultant force and the angle of the force due to muscle A and muscle B on the bone.

CATCH THE MISTAKE

In Exercises 89 and 90, explain the mistake that is made.

89. Find the magnitude of the vector $\langle -2, -8 \rangle$.

Solution:

Factor the -1.	$-\langle 2, 8 \rangle$
Find the magnitude of $\langle 2,8 \rangle$.	$\lvert\langle 2, 8\rangle\rvert = \sqrt{2^2 + 8^2}$ $= \sqrt{68} = 2\sqrt{17}$
Write the magnitude of $\langle -2,-8 \rangle$.	$\lvert\langle -2, -8\rangle\rvert = -2\sqrt{17}$

This is incorrect. What mistake was made?

90. Find the direction angle of the vector $\langle -2, -8 \rangle$.

Solution:

Write the formula for the direction angle of $\langle a, b \rangle$.	$\tan\theta = \dfrac{b}{a}$
Let $a = -2$ and $b = -8$.	$\tan\theta = \dfrac{-8}{-2}$
Apply the inverse tangent function.	$\theta = \tan^{-1}4$
Evaluate with a calculator.	$\theta = 76°$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 91–94, determine whether each statment is true or false.

91. The magnitude of the vector **i** is the imaginary number i.

92. The arrow components of equal vectors must coincide.

93. The magnitude of a vector is always greater than or equal to the magnitude of its horizontal component.

94. The magnitude of a vector is always greater than or equal to the magnitude of its vertical component.

95. Would a scalar or a vector represent the following? *The car is driving 72 miles per hour due east (90° with respect to north).*

96. Would a scalar or vector represent the following? *The granite has a mass of 131 kilograms.*

97. Find the magnitude of the vector $\langle -a, b \rangle$ if $a > 0$ and $b > 0$.

98. Find the direction angle of the vector $\langle -a, b \rangle$ if $a > 0$ and $b > 0$.

CHALLENGE

99. Show that if **u** is a unit vector in the direction of **v**, then $\mathbf{v} = |\mathbf{u}|\,\mathbf{u}$.

100. Show that if $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then (a, b) lies on the unit circle.

101. A vector **u** is a *linear combination* of **p** and **q** if there exist constants c_1 and c_2 such that $\mathbf{u} = c_1\mathbf{p} + c_2\mathbf{q}$. Show that $\langle -6, 4 \rangle$ is a linear combination of $\langle -8, 4 \rangle$ and $\langle 1, -1 \rangle$.

102. Show that $\left\langle -\frac{2}{9}a, \frac{8}{9}b \right\rangle$ is a linear combination of $\langle a, 3b \rangle$ and $\langle -a, -b \rangle$, for any real constants a and b.

103. Prove that $\mathbf{u} + 3(2\mathbf{v} - \mathbf{u}) = 6\mathbf{v} - 2\mathbf{u}$, showing carefully how all relevant properties and definitions enter the proof.

104. Let $\mathbf{u} = \langle 2a, a \rangle$, $\mathbf{v} = \langle -a, -2a \rangle$. Compute $\left| \dfrac{2\mathbf{u}}{|\mathbf{v}|} - \dfrac{3\mathbf{v}}{|\mathbf{u}|} \right|$.

TECHNOLOGY

For Exercises 105–110, refer to the following:

Vectors can be represented as column matrices. For example, the vector $\mathbf{u} = \langle 3, -4 \rangle$ can be represented as a 2×1 column matrix $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$. With a TI-83 calculator, vectors can be entered as matrices in two ways, directly or via $\boxed{\text{MATRIX}}$.

Directly:

```
[[3][-4]]
              [[3 ]
               [-4]]
```

Matrix:

```
[A]
              [[3 ]
               [-4]]
```

Use a calculator to perform the vector operation given $\mathbf{u} = \langle 8, -5 \rangle$ and $\mathbf{v} = \langle -7, 11 \rangle$.

105. $\mathbf{u} + 3\mathbf{v}$

106. $-9(\mathbf{u} - 2\mathbf{v})$

Use a calculator to find a unit vector in the direction of the given vector.

107. $\mathbf{u} = \langle 10, -24 \rangle$

108. $\mathbf{u} = \langle -9, -40 \rangle$

Use the graphing calculator $\boxed{\text{SUM}}$ command to find the magnitude of the given vector. Also, find the direction angle to the nearest degree.

109. $\langle -33, 180 \rangle$

110. $\left\langle -20, -30\sqrt{5} \right\rangle$

PREVIEW TO CALCULUS

There is a branch of calculus devoted to the study of vector-valued functions; these are functions that map real numbers onto vectors. For example, $\mathbf{v}(t) = \langle t, 2t \rangle$.

111. Find the magnitude of the vector-valued function $\mathbf{v}(t) = \langle \cos t, \sin t \rangle$.

112. Find the direction of the vector-valued function $\mathbf{v}(t) = \langle -3t, -4t \rangle$.

The difference quotient for the vector-valued function $\mathbf{v}(t)$ is defined as $\dfrac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h}$. In Exercises 113 and 114, find the difference quotient of the vector-valued function.

113. $\mathbf{v}(t) = \langle t, t^2 \rangle$

114. $\mathbf{v}(t) = \langle t^2 + 1, t^3 \rangle$

SECTION 7.2 THE DOT PRODUCT

SKILLS OBJECTIVES

- Find the dot product of two vectors.
- Use the dot product to find the angle between two vectors.
- Determine whether two vectors are parallel or perpendicular.
- Use the dot product to calculate the amount of work associated with a physical problem.

CONCEPTUAL OBJECTIVE

- Understand the difference between scalar vector multiplication and the dot product of two vectors.

The Dot Product

With two-dimensional vectors, there are two types of multiplication defined for vectors: scalar multiplication and the dot product. Scalar multiplication (which we already demonstrated in Section 7.1) is multiplication of a scalar by a vector; the result is a vector. Now we discuss the *dot product* of two vectors. In this case, there are two important things to note: (1) The dot product of two vectors is defined only if the vectors have the same number of components and (2) if the dot product does exist, then the result is a scalar.

DOT PRODUCT

The **dot product** of two vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ is given by

$$\mathbf{u} \cdot \mathbf{v} = ac + bd$$

$\mathbf{u} \cdot \mathbf{v}$ is pronounced "u dot v."

EXAMPLE 1 Finding the Dot Product of Two Vectors

Find the dot product $\langle -7, 3\rangle \cdot \langle 2, 5\rangle$.

Solution:

Sum the products of the first components and the products of the second components.

$$\langle -7, 3\rangle \cdot \langle 2, 5\rangle = (-7)(2) + (3)(5)$$

Simplify.

$$= -14 + 15$$

$$= \boxed{1}$$

Study Tip

The dot product of two vectors is a scalar.

■ **Answer:** -9

■ **YOUR TURN** Find the dot product $\langle 6, 1\rangle \cdot \langle -2, 3\rangle$.

The following box summarizes the properties of the dot product:

PROPERTIES OF THE DOT PRODUCT

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
3. $\mathbf{0} \cdot \mathbf{u} = 0$
4. $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
5. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
6. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

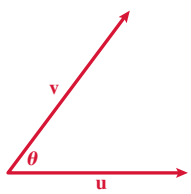
These properties are verified in the exercises.

Angle Between Two Vectors

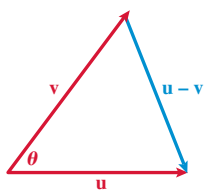
We can use the properties of the dot product to develop an equation that relates the angle between two vectors and the dot product of the vectors.

WORDS | **MATH**

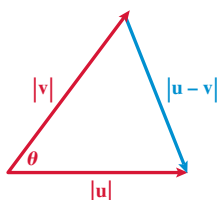
Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors with the same initial point, and let θ be the angle between them.



The vector $\mathbf{u} - \mathbf{v}$ is opposite angle θ.



A triangle is formed with side lengths equal to the magnitudes of the three vectors.



Apply the Law of Cosines. $\boxed{|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta}$

Use properties of the dot product to rewrite the left side of equation.

Property (2): $|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v})\cdot(\mathbf{u} - \mathbf{v})$

Property (6): $= \mathbf{u}\cdot(\mathbf{u} - \mathbf{v}) - \mathbf{v}\cdot(\mathbf{u} - \mathbf{v})$

Property (6): $= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$

Property (2): $= |\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + |\mathbf{v}|^2$

Property (1): $\boxed{= |\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2}$

Substitute this last expression for the left side of the original Law of Cosines equation. $|\mathbf{u}|^2 - 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$

Simplify. $-2(\mathbf{u}\cdot\mathbf{v}) = -2|\mathbf{u}||\mathbf{v}|\cos\theta$

Isolate $\cos\theta$. $\boxed{\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}}$

Notice that **u** and **v** have to be nonzero vectors, since we divided by them in the last step.

Angle between two vectors

If θ is the angle between two nonzero vectors **u** and **v**, where $0° \le \theta \le 180°$, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Study Tip

The angle between two vectors $\theta = \cos^{-1}\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$ is an angle between 0° and 180° (the range of the inverse cosine function).

In the Cartesian plane, there are two angles between two vectors, θ and $360° - \theta$. **We assume that θ is the "smaller" angle.**

EXAMPLE 2 Finding the Angle Between Two Vectors

Find the angle between $\langle 2, -3\rangle$ and $\langle -4, 3\rangle$.

Solution:

Let $\mathbf{u} = \langle 2, -3\rangle$ and $\mathbf{v} = \langle -4, 3\rangle$.

Step 1 Find $\mathbf{u}\cdot\mathbf{v}$.

$$\mathbf{u}\cdot\mathbf{v} = \langle 2, -3\rangle\cdot\langle -4, 3\rangle = (2)(-4) + (-3)(3) = -17$$

Step 2 Find $|\mathbf{u}|$.

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

Step 3 Find $|\mathbf{v}|$.

$$|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$$

Step 4 Find θ.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{-17}{5\sqrt{13}}$$

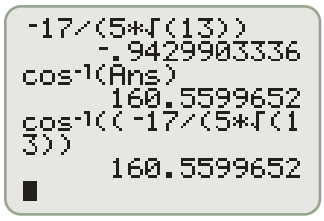
Approximate θ with a calculator.

$$\theta = \cos^{-1}\left(-\frac{17}{5\sqrt{13}}\right) \approx 160.559965°$$

$$\boxed{\theta \approx 161°}$$

Technology Tip

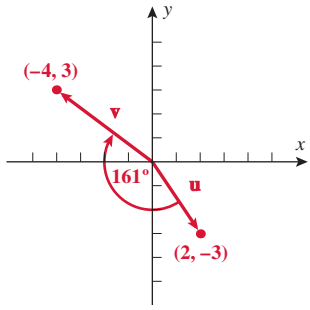
Use a TI calculator to find θ, $\cos\theta = \dfrac{-17}{5\sqrt{13}}$.



STEP 5 Draw a picture to confirm the answer.

Draw the vectors $\langle 2, -3\rangle$ and $\langle -4, 3\rangle$.

161° appears to be correct.

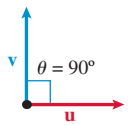


■ **Answer:** 38°

■ **YOUR TURN** Find the angle between $\langle 1, 5\rangle$ and $\langle -2, 4\rangle$.

When two vectors are **parallel**, the angle between them is 0° or 180°.

When two vectors are **perpendicular (orthogonal)**, the angle between them is 90°.



Note: We did not include 270° because the angle $0° \le \theta \le 180°$ between two vectors is taken to be the smaller angle.

WORDS	MATH
When two vectors **u** and **v** are perpendicular, $\theta = 90°$.	$\cos 90° = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert}$
Substitute $\cos 90° = 0$.	$0 = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert}$
Therefore, the dot product of **u** and **v** must be zero.	$\mathbf{u}\cdot\mathbf{v} = 0$

ORTHOGONAL VECTORS

Two vectors **u** and **v** are **orthogonal** (perpendicular) if and only if their dot product is zero.

$$\mathbf{u}\cdot\mathbf{v} = 0$$

EXAMPLE 3 Determining Whether Vectors Are Orthogonal

Determine whether each pair of vectors is orthogonal.

a. $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle 3, 2 \rangle$ **b.** $\mathbf{u} = \langle -7, -3 \rangle$ and $\mathbf{v} = \langle 7, 3 \rangle$

Solution (a):

Find the dot product $\mathbf{u} \cdot \mathbf{v}$. $\mathbf{u} \cdot \mathbf{v} = (2)(3) + (-3)(2)$

Simplify. $\mathbf{u} \cdot \mathbf{v} = 0$

Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, since $\mathbf{u} \cdot \mathbf{v} = 0$.

Solution (b):

Find the dot product $\mathbf{u} \cdot \mathbf{v}$. $\mathbf{u} \cdot \mathbf{v} = (-7)(7) + (-3)(3)$

Simplify. $\mathbf{u} \cdot \mathbf{v} = -58$

Vectors $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal, since $\mathbf{u} \cdot \mathbf{v} \neq 0$.

Work

If you had to carry barbells with weights or pillows for 1 mile, which would you choose? You would probably pick the pillows over the barbell with weights, because the pillows are lighter. It requires less work to carry the pillows than it does to carry the weights. If asked to carry either of them 1 mile or 10 miles, you would probably pick 1 mile, because it's a shorter distance and requires less work. **Work** is done when a *force causes an object to move a certain distance.*

The simplest case is when the force is in the same direction as the displacement—for example, a stagecoach (the horses pull with a force in the same direction). In this case the work is defined as the magnitude of the force times the magnitude of the displacement, distance d.

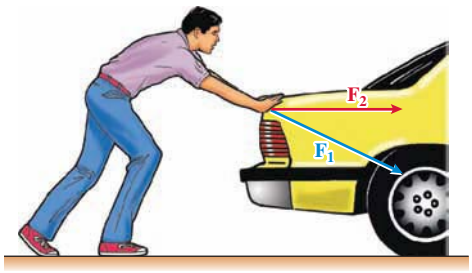
$$W = |\mathbf{F}|\, d$$

Notice that the magnitude of the force is a scalar, the distance d is a scalar, and hence the product is a scalar.

If the horses pull with a force of 1000 pounds and they move the stagecoach 100 feet, the work done by the force is

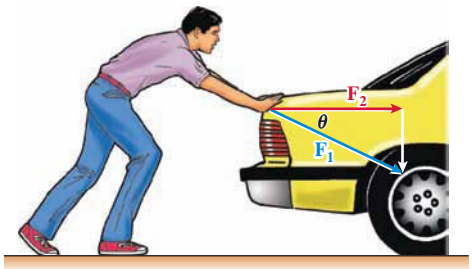
$$W = (1000 \text{ lb})(100 \text{ ft}) = 100{,}000 \text{ ft-lb}$$

In many physical applications, however, the force is not in the same direction as the displacement, and hence vectors (not just their magnitudes) are required.



We often want to know how much of a force is applied in a certain direction. For example, when your car runs out of gasoline and you try to push it, some of the force vector $\mathbf{F}_1$ you generate from pushing translates into the horizontal component $\mathbf{F}_2$; hence, the car moves horizontally.

If we let θ be the angle between the vectors $\mathbf{F}_1$ and $\mathbf{F}_2$, then the horizontal component of $\mathbf{F}_1$ is $\mathbf{F}_2$ where $|F_2| = |F_1| \cos\theta$.



If the man in the picture pushes at an angle of 25° with a force of 150 pounds, then the horizontal component of the force vector $\mathbf{F}_1$ is

$$(150\text{ lb})(\cos 25°) \approx \boxed{136\text{ lb}}$$

WORDS	MATH
To develop a generalized formula when the force exerted and the displacement are not in the same direction, we start with the formula for the angle between two vectors.	$\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert}$
We then isolate the dot product $\mathbf{u}\cdot\mathbf{v}$.	$\mathbf{u}\cdot\mathbf{v} = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta$
Let $\mathbf{u} = \mathbf{F}$ and $\mathbf{v} = \mathbf{d}$.	$W = \mathbf{F}\cdot\mathbf{d} = \lvert\mathbf{F}\rvert\lvert\mathbf{d}\rvert\cos\theta = \underbrace{\lvert\mathbf{F}\rvert\cos\theta}_{\text{magnitude of force in direction of displacement}} \cdot \underbrace{\lvert\mathbf{d}\rvert}_{\text{distance}}$

WORK

If an object is moved from point A to point B by a constant force, then the work associated with this displacement is

$$W = \mathbf{F}\cdot\mathbf{d}$$

where $\mathbf{d}$ is the displacement vector and $\mathbf{F}$ is the force vector.

Work is typically expressed in one of two units:

SYSTEM	FORCE	DISTANCE	WORK
U.S. customary	pound	foot	ft-lb
SI	newton	meter	N-m

EXAMPLE 4 Calculating Work

How much work is done when a force (in pounds) $\mathbf{F} = \langle 2, 4 \rangle$ moves an object from (0, 0) to (5, 9) (the distance is in feet)?

Solution:

Find the displacement vector **d**.	$\mathbf{d} = \langle 5, 9 \rangle$
Apply the work formula, $W = \mathbf{F} \cdot \mathbf{d}$.	$W = \langle 2, 4 \rangle \cdot \langle 5, 9 \rangle$
Calculate the dot product.	$W = (2)(5) + (4)(9)$
Simplify.	$\boxed{W = 46 \text{ ft-lb}}$

YOUR TURN How much work is done when a force (in newtons) $\mathbf{F} = \langle 1, 3 \rangle$ moves an object from (0, 0) to (4, 7) (the distance is in meters)?

Answer: 25 N-m

SECTION 7.2 SUMMARY

In this section, we defined the dot product as a form of multiplication of two vectors. A scalar times a vector results in a vector, whereas the dot product of two vectors is a scalar.

$$\langle a, b \rangle \cdot \langle c, d \rangle = ac + bd$$

We developed a formula that determines the angle θ between two vectors **u** and **v**.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Orthogonal (perpendicular) vectors have an angle of 90° between them, and consequently, the dot product of two orthogonal vectors is equal to zero. Work is the result of a force displacing an object. When the force and displacement are in the same direction, the work is equal to the product of the magnitude of the force and the distance (magnitude of the displacement). When the force and displacement are not in the same direction, work is the dot product of the force vector and displacement vector, $\mathbf{W} = \mathbf{F} \cdot \mathbf{d}$.

SECTION 7.2 EXERCISES

SKILLS

In Exercises 1–12, find the indicated dot product.

1. $\langle 4, -2 \rangle \cdot \langle 3, 5 \rangle$

2. $\langle 7, 8 \rangle \cdot \langle 2, -1 \rangle$

3. $\langle -5, 6 \rangle \cdot \langle 3, 2 \rangle$

4. $\langle 6, -3 \rangle \cdot \langle 2, 1 \rangle$

5. $\langle -7, -4 \rangle \cdot \langle -2, -7 \rangle$

6. $\langle 5, -2 \rangle \cdot \langle -1, -1 \rangle$

7. $\langle \sqrt{3}, -2 \rangle \cdot \langle 3\sqrt{3}, -1 \rangle$

8. $\langle 4\sqrt{2}, \sqrt{7} \rangle \cdot \langle -\sqrt{2}, -\sqrt{7} \rangle$

9. $\langle 5, a \rangle \cdot \langle -3a, 2 \rangle$

10. $\langle 4x, 3y \rangle \cdot \langle 2y, -5x \rangle$

11. $\langle 0.8, -0.5 \rangle \cdot \langle 2, 6 \rangle$

12. $\langle -18, 3 \rangle \cdot \langle 10, -300 \rangle$

In Exercises 13–24, find the angle (round to the nearest degree) between each pair of vectors.

13. $\langle -4, 3\rangle$ and $\langle -5, -9\rangle$

14. $\langle 2, -4\rangle$ and $\langle 4, -1\rangle$

15. $\langle -2, -3\rangle$ and $\langle -3, 4\rangle$

16. $\langle 6, 5\rangle$ and $\langle 3, -2\rangle$

17. $\langle -4, 6\rangle$ and $\langle -6, 8\rangle$

18. $\langle 1, 5\rangle$ and $\langle -3, -2\rangle$

19. $\left\langle -2, 2\sqrt{3}\right\rangle$ and $\left\langle -\sqrt{3}, 1\right\rangle$

20. $\left\langle -3\sqrt{3}, -3\right\rangle$ and $\left\langle -2\sqrt{3}, 2\right\rangle$

21. $\left\langle -5\sqrt{3}, -5\right\rangle$ and $\left\langle \sqrt{2}, -\sqrt{2}\right\rangle$

22. $\left\langle -5, -5\sqrt{3}\right\rangle$ and $\left\langle 2, -\sqrt{2}\right\rangle$

23. $\langle 4, 6\rangle$ and $\langle -6, -9\rangle$

24. $\langle 2, 8\rangle$ and $\langle -12, 3\rangle$

In Exercises 25–36, determine whether each pair of vectors is orthogonal.

25. $\langle -6, 8\rangle$ and $\langle -8, 6\rangle$

26. $\langle 5, -2\rangle$ and $\langle -5, 2\rangle$

27. $\langle 6, -4\rangle$ and $\langle -6, -9\rangle$

28. $\langle 8, 3\rangle$ and $\langle -6, 16\rangle$

29. $\langle 0.8, 4\rangle$ and $\langle 3, -6\rangle$

30. $\langle -7, 3\rangle$ and $\left\langle \frac{1}{7}, -\frac{1}{3}\right\rangle$

31. $\langle 5, -0.4\rangle$ and $\langle 1.6, 20\rangle$

32. $\langle 12, 9\rangle$ and $\langle 3, -4\rangle$

33. $\left\langle \sqrt{3}, \sqrt{6}\right\rangle$ and $\left\langle -\sqrt{2}, 1\right\rangle$

34. $\left\langle \sqrt{7}, -\sqrt{3}\right\rangle$ and $\langle 3, 7\rangle$

35. $\left\langle \frac{4}{3}, \frac{8}{15}\right\rangle$ and $\left\langle -\frac{1}{12}, \frac{5}{24}\right\rangle$

36. $\left\langle \frac{5}{6}, \frac{6}{7}\right\rangle$ and $\left\langle \frac{36}{25}, -\frac{49}{36}\right\rangle$

APPLICATIONS

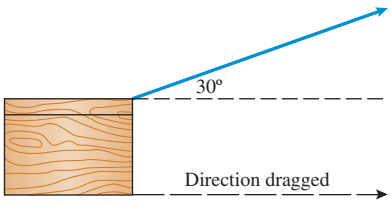
37. Lifting Weights. How much work does it take to lift 100 pounds vertically 4 feet?

38. Lifting Weights. How much work does it take to lift 150 pounds vertically 3.5 feet?

39. Raising Wrecks. How much work is done by a crane to lift a 2-ton car to a level of 20 feet?

40. Raising Wrecks. How much work is done by a crane to lift a 2.5-ton car to a level of 25 feet?

41. Work. To slide a crate across the floor, a force of 50 pounds at a 30° angle is needed. How much work is done if the crate is dragged 30 feet?



42. Work. To slide a crate across the floor, a force of 800 pounds at a 20° angle is needed. How much work is done if the crate is dragged 50 feet?

43. Close a Door. A sliding door is closed by pulling a cord with a constant force of 35 pounds at a constant angle of 45°. The door is moved 6 feet to close it. How much work is done?

44. Close a Door. A sliding door is closed by pulling a cord with a constant force of 45 pounds at a constant angle of 55°. The door is moved 6 feet to close it. How much work is done?

45. Braking Power. A car that weighs 2500 pounds is parked on a hill in San Francisco with a slant of 40° from the horizontal. How much force will keep it from rolling down the hill?

46. Towing Power. A car that weighs 2500 pounds is parked on a hill in San Francisco with a slant of 40° from the horizontal. A tow truck has to remove the car from its parking spot and move it 120 feet up the hill. How much work is required?

47. Towing Power. A semitrailer truck that weighs 40,000 pounds is parked on a hill in San Francisco with a slant of 10° from the horizontal. A tow truck has to remove the truck from its parking spot and move it 100 feet up the hill. How much work is required?

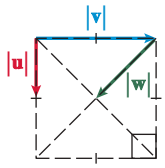
48. Braking Power. A truck that weighs 40,000 pounds is parked on a hill in San Francisco with a slant of 10° from the horizontal. How much force will keep it from rolling down the hill?

49. Business. Suppose that $\mathbf{u} = \langle 2000, 5000\rangle$ represents the number of units of battery A and B, respectively, produced by a company and $\mathbf{v} = \langle 8.40, 6.50\rangle$ represents the price (in dollars) of a 10-pack of battery A and B, respectively. Compute and interpret $\mathbf{u} \cdot \mathbf{v}$.

50. Demographics. Suppose that $\mathbf{u} = \langle 120, 80\rangle$ represents the number of males and females in a high school class, and $\mathbf{v} = \langle 7.2, 5.3\rangle$ represents the average number of minutes it takes a male and female, respectively, to register. Compute and interpret $\mathbf{u} \cdot \mathbf{v}$.

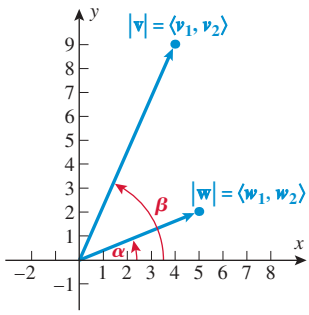
51. Geometry. Use vector methods to show that the diagonals of a rhombus are perpendicular to each other.

52. Geometry. Let **u** be a unit vector, and consider the following diagram:



Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.

53. Geometry. Consider the following diagram:



a. Compute $\cos\beta$, $\sin\beta$, $\cos\alpha$, and $\sin\alpha$.

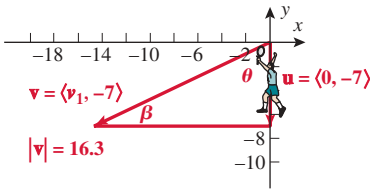
b. Use (a) to show that $\cos(\alpha - \beta) = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\sqrt{\mathbf{v} \cdot \mathbf{v}}\sqrt{\mathbf{w} \cdot \mathbf{w}}}$.

54. Geometry. Consider the diagram in Exercise 53.

a. Compute $\cos\beta$, $\sin\beta$, $\cos\alpha$, and $\sin\alpha$.

b. Use (a) to show that $\cos(\alpha + \beta) = \dfrac{\mathbf{v} \cdot \langle w_1, -w_2 \rangle}{\sqrt{\mathbf{v} \cdot \mathbf{v}}\sqrt{\mathbf{w} \cdot \mathbf{w}}}$.

55. Tennis. A player hits an overhead smash at full arm extension at the top of his racquet, which is 7 feet from the ground. The ball travels 16.3 feet (ignore the effects of gravity). Consult the following diagram:



a. Determine v_1.

b. Find the angle θ with which the player hits this smash.

56. Tennis. In Exercise 55, use the dot product to determine the angle β with which the ball hits the ground.

57. Optimization. Let $\mathbf{u} = \langle a, b \rangle$ be a given vector and suppose that the head of $\mathbf{n} = \langle n_1, n_2 \rangle$ lies on the circle $x^2 + y^2 = r^2$. Find the vector **n** such that $\mathbf{u} \cdot \mathbf{n}$ is as big as possible. Find the actual value of $\mathbf{u} \cdot \mathbf{n}$ in this case.

58. Optimization. Let $\mathbf{u} = \langle a, b \rangle$ be a given vector and suppose that the head of $\mathbf{n} = \langle n_1, n_2 \rangle$ lies on the circle $x^2 + y^2 = r^2$. Find the vector **n** such that $\mathbf{u} \cdot \mathbf{n}$ is as small as possible. Find the actual value of $\mathbf{u} \cdot \mathbf{n}$ in this case.

59. Pursuit Theory. Assume that the head of **u** is restricted so that its tail is at the origin and its head is on the unit circle in quadrant II or quadrant III. A vector **v** has its tail at the origin and its head must lie on the line $y = 2 - x$ in quadrant I. Find the least value of $\mathbf{u} \cdot \mathbf{v}$.

60. Pursuit Theory. Assume that the head of **u** is restricted so that its tail is at the origin and its head is on the unit circle in quadrant I or quadrant IV. A vector **v** has its tail at the origin and its head must lie on the line $y = 2 - x$ in quadrant I. Find the largest value of $\mathbf{u} \cdot \mathbf{v}$.

CATCH THE MISTAKE

In Exercises 61 and 62, explain the mistake that is made.

61. Find the dot product $\langle -3, 2 \rangle \cdot \langle 2, 5 \rangle$.

Solution:

Multiply component by component. $\langle -3, 2 \rangle \cdot \langle 2, 5 \rangle = \langle (-3)(2), (2)(5) \rangle$

Simplify. $\langle -3, 2 \rangle \cdot \langle 2, 5 \rangle = \langle -6, 10 \rangle$

This is incorrect. What mistake was made?

62. Find the dot product $\langle 11, 12 \rangle \cdot \langle -2, 3 \rangle$.

Solution:

Multiply the outer and inner components.

$$\langle 11, 12 \rangle \cdot \langle -2, 3 \rangle = (11)(3) + (12)(-2)$$

Simplify. $\langle 11, 12 \rangle \cdot \langle -2, 3 \rangle = 9$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 63–66, determine whether each statement is true or false.

63. A dot product of two vectors is a vector.

64. A dot product of two vectors is a scalar.

65. Orthogonal vectors have a dot product equal to zero.

66. If the dot product of two nonzero vectors is equal to zero, then the vectors must be perpendicular.

For Exercises 67 and 68, refer to the following to find the dot product:

The dot product of vectors with n components is

$\langle a_1, a_2, \ldots, a_n\rangle \cdot \langle b_1, b_2, \ldots, b_n\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$

67. $\langle 3, 7, -5\rangle \cdot \langle -2, 4, 1\rangle$

68. $\langle 1, 0, -2, 3\rangle \cdot \langle 5, 2, 3, 1\rangle$

In Exercises 69–72, given $u = \langle a, b\rangle$ and $v = \langle c, d\rangle$, show that the following properties are true:

69. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

70. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$

71. $\mathbf{0}\cdot\mathbf{u} = 0$

72. $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$, k is a scalar

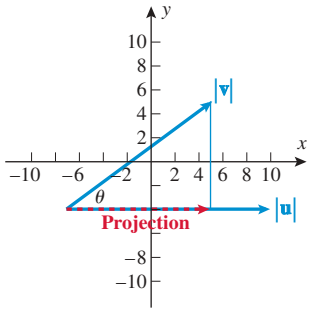
CHALLENGE

73. Show that $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$.

74. Show that $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2(\mathbf{u}\cdot\mathbf{v})$.

75. The *projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ is defined by $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\right)\mathbf{u}$.

This vector is depicted below. Heuristically, this is the "shadow" of $\mathbf{v}$ on $\mathbf{u}$.



a. Compute $\text{proj}_{\mathbf{u}}\,2\mathbf{u}$.

b. What is $\text{proj}_{\mathbf{u}}\,c\mathbf{u}$ for any $c > 0$?

76. **a.** Compute $\text{proj}_{\mathbf{u}}\,2\mathbf{u}$.

b. What is $\text{proj}_{\mathbf{u}}\,c\mathbf{u}$ for any $c > 0$?

77. Suppose that you are given a vector $\mathbf{u}$. For what vectors $\mathbf{v}$ does $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \mathbf{0}$?

78. True or false: $\text{proj}_{\mathbf{u}}(\mathbf{v} + \mathbf{w}) = \text{proj}_{\mathbf{u}}\,\mathbf{v} + \text{proj}_{\mathbf{u}}\,\mathbf{w}$.

79. If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, determine the maximum and minimum value of $(-2\mathbf{u})\cdot(3\mathbf{v})$.

80. Assume that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \dfrac{\pi}{3}$. Show that

$$\frac{(\mathbf{u}\cdot\mathbf{v})\mathbf{u}}{|\mathbf{v}|} - \frac{(\mathbf{v}\cdot\mathbf{u})\mathbf{v}}{|\mathbf{u}|} = \frac{|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}}{2}.$$

TECHNOLOGY

For Exercises 81 and 82, find the indicated dot product with a calculator.

81. $\langle -11, 34\rangle \cdot \langle 15, -27\rangle$

82. $\langle 23, -350\rangle \cdot \langle 45, 202\rangle$

83. A rectangle has sides with lengths 18 units and 11 units. Find the angle to one decimal place between the diagonal and the side with length of 18 units. *Hint:* Set up a rectangular coordinate system, and use vectors $\langle 18, 0\rangle$ to represent the side of length 18 units and $\langle 18, 11\rangle$ to represent the diagonal.

84. The definition of a dot product and the formula to find the angle between two vectors can be extended and applied to vectors with more than two components. A rectangular box has sides with lengths 12 feet, 7 feet, and 9 feet. Find the angle, to the nearest degree, between the diagonal and the side with length 7 feet.

Use the graphing calculator $\boxed{\text{SUM}}$ command to find the angle (round to the nearest degree) between each pair of vectors.

85. $\langle -25, 42\rangle, \langle 10, 35\rangle$

86. $\langle -12, 9\rangle, \langle -21, -13\rangle$

PREVIEW TO CALCULUS

There is a branch of calculus devoted to the study of vector-valued functions; these are functions that map real numbers onto vectors. For example, $\mathbf{v}(t) = \langle t, 2t \rangle$.

87. Calculate the dot product of the vector-valued functions $\mathbf{u}(t) = \langle 2t, t^2 \rangle$ and $\mathbf{v}(t) = \langle t, -3t \rangle$.

88. Calculate the dot product of the vector-valued functions $\mathbf{u}(t) = \langle \cos t, \sin t \rangle$ and $\mathbf{v}(t) = \langle \cos t, -\sin t \rangle$.

89. Find the angle between the vector-valued functions $\mathbf{u}(t) = \langle \sin t, \cos t \rangle$ and $\mathbf{v}(t) = \langle \csc t, -\cos t \rangle$ when $t = \frac{\pi}{6}$.

90. Find the values of t that make the vector-valued functions $\mathbf{u}(t) = \langle \sin t, \sin t \rangle$ and $\mathbf{v}(t) = \langle \cos t, -\sin t \rangle$ orthogonal.

SECTION 7.3 POLAR (TRIGONOMETRIC) FORM OF COMPLEX NUMBERS

SKILLS OBJECTIVES

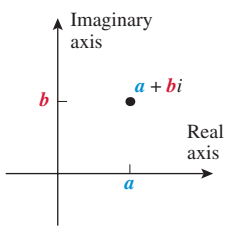
- Graph a point in the complex plane.
- Convert complex numbers from rectangular form to polar form.
- Convert complex numbers from polar form to rectangular form.

CONCEPTUAL OBJECTIVES

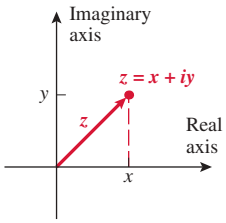
- Understand that a complex number can be represented either in rectangular or polar form.
- Relate the horizontal axis in the complex plane to the real component of a complex number.
- Relate the vertical axis in the complex plane to the imaginary component of a complex number.

Complex Numbers in Rectangular Form

We are already familiar with the **rectangular coordinate system**, where the horizontal axis is called the x-axis and the vertical axis is called the y-axis. In our study of complex numbers, we refer to the **standard (rectangular) form** as $a + bi$, where a represents the real part and b represents the imaginary part. If we let the horizontal axis be the **real axis** and the vertical axis be the **imaginary axis**, the result is the **complex plane**. The point $a + bi$ is located in the complex plane by finding the coordinates (a, b).



When $b = 0$, the result is a real number, and therefore any numbers along the horizontal axis are real numbers. When $a = 0$, the result is an imaginary number, so any numbers along the vertical axis are imaginary numbers.



The variable z is often used to represent a complex number: $z = x + iy$. Complex numbers are analogous to vectors. Suppose we define a vector $\mathbf{z} = \langle x, y \rangle$, whose initial point is the origin and whose terminal point is (x, y); then the magnitude of that vector is $|\mathbf{z}| = \sqrt{x^2 + y^2}$. Similarly, the magnitude, or *modulus*, of a complex number is defined like the magnitude of a position vector in the xy-plane, as the distance from the origin $(0, 0)$ to the point (x, y) in the complex plane.

Technology Tip

To use a TI calculator to find the modulus of a complex number, press [MATH] [▶] [CPX] [▼] [5: ABS (] and enter the complex number.

DEFINITION **Modulus of a Complex Number**

The **modulus**, or magnitude, of a complex number $z = x + iy$ is the distance from the origin to the point (x, y) in the complex plane given by

$$|z| = \sqrt{x^2 + y^2}$$

Recall that a complex number $z = x + iy$ has a complex conjugate $\bar{z} = x - iy$. The bar above a complex number denotes its conjugate. Notice that

$$z\bar{z} = (x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2$$

and therefore the modulus can also be written as

$$\boxed{|z| = \sqrt{z\bar{z}}}$$

Technology Tip

Find the modulus of $z = -3 + 2i$.

[MATH] [▶] [CPX] [▼] [5: ABS (] [ENTER] [(−)] [3] [+] [2] [2nd] [·] [)] [ENTER]

```
abs(-3+2i)
        3.605551275
√(13)
        3.605551275
```

EXAMPLE 1 **Finding the Modulus of a Complex Number**

Find the modulus of $z = -3 + 2i$.

COMMON MISTAKE

Including the i in the imaginary part.

CORRECT

Let $x = -3$ and $y = 2$ in $|z| = \sqrt{x^2 + y^2}$.

$$|-3 + 2i| = \sqrt{(-3)^2 + 2^2}$$

Eliminate the parentheses.

$$|-3 + 2i| = \sqrt{9 + 4}$$

Simplify.

$$\boxed{|z| = |-3 + 2i| = \sqrt{13}}$$

INCORRECT

Let $x = -3$ and $y = 2i$ **ERROR**

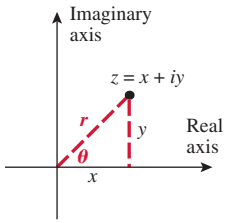
$$|-3 + 2i| = \sqrt{(-3)^2 + (2i)^2}$$

The i is not included in the formula. Only the imaginary part (the coefficient of i) is used.

■ **YOUR TURN** Find the modulus of $z = 2 - 5i$.

■ **Answer:** $|z| = |2 - 5i| = \sqrt{29}$

Complex Numbers in Polar Form



We say that a complex number $z = x + iy$ is in *rectangular* form because it is located at the point (x, y), which is expressed in rectangular coordinates, in the complex plane. Another convenient way of expressing complex numbers is in *polar* form. Recall from our study of vectors (Section 7.1) that vectors have both magnitude and a direction angle. The same is true of numbers in the complex plane. Let r represent the magnitude, or distance from the origin to the point (x, y), and θ represent the direction angle; then we have the following relationships:

$$r = \sqrt{x^2 + y^2}$$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Isolating x and y in the sinusoidal functions, we find

$$x = r\cos\theta \qquad y = r\sin\theta$$

When we use these expressions for x and y, a complex number can be written in *polar* form.

$$z = x + yi = (r\cos\theta) + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

Polar (Trigonometric) Form of Complex Numbers

The following expression is the **polar form** of a complex number:

$$z = r(\cos\theta + i\sin\theta)$$

where r represents the **modulus** (magnitude) of the complex number and θ represents the **argument** of z.

The following is standard notation for modulus and argument:

$$r = \operatorname{mod} z = |z| \qquad \text{and} \qquad \theta = \operatorname{Arg} z, \qquad 0 \leq \theta < 2\pi \qquad \text{or} \qquad 0° \leq \theta < 360°$$

Converting Complex Numbers Between Rectangular and Polar Forms

We can convert back and forth between rectangular and polar (trigonometric) forms of complex numbers using the modulus and trigonometric ratios.

$$r = \sqrt{x^2 + y^2} \qquad \sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

Converting Complex Numbers from Rectangular Form to Polar Form

Step 1: Plot the point $z = x + iy$ in the complex plane (note the quadrant).

Step 2: Find r. Use $r = \sqrt{x^2 + y^2}$.

Step 3: Find θ. Apply $\tan\theta = \dfrac{y}{x}$, $x \neq 0$, where θ is in the quadrant found in Step 1.

Step 4: Write the complex number in polar form: $z = r(\cos\theta + i\sin\theta)$.

Notice that imaginary numbers, $z = bi$, lie on the imaginary axis. Therefore, $\theta = 90°$ if $b > 0$ and $\theta = 270°$ if $b < 0$.

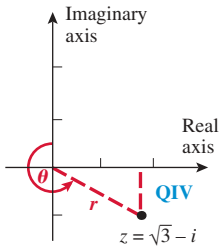
EXAMPLE 2 Converting from Rectangular to Polar Form

Express the complex number $z = \sqrt{3} - i$ in polar form.

Solution:

STEP 1 Plot the point.

The point lies in **quadrant IV**.



STEP 2 Find r.

Let $x = \sqrt{3}$ and $y = -1$ in $r = \sqrt{x^2 + y^2}$. $\quad r = \sqrt{(\sqrt{3})^2 + (-1)^2}$

Eliminate the parentheses. $\quad r = \sqrt{3 + 1}$

Simplify. $\quad \boxed{r = 2}$

STEP 3 Find θ.

Let $x = \sqrt{3}$ and $y = -1$ in $\tan\theta = \dfrac{y}{x}$. $\quad \tan\theta = -\dfrac{1}{\sqrt{3}}$

Solve for θ. $\quad \theta = \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$

Find the reference angle. $\quad$ reference angle $= \dfrac{\pi}{6}$

The complex number lies in quadrant IV. $\quad \boxed{\theta = \dfrac{11\pi}{6}}$

STEP 4 Write the complex number in polar form. $z = r(\cos\theta + i\sin\theta)$

$$\boxed{z = 2\left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right]}$$

Note: An alternative form is in degrees: $z = 2(\cos 330° + i\sin 330°)$.

■ **YOUR TURN** Express the complex number $z = 1 - i\sqrt{3}$ in polar form.

Technology Tip

Express the complex number $z = \sqrt{3} - i$ in polar form.

Method I: Use $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ to find the reference angle for θ, which is in QIV.

```
tan⁻¹(1/√(3)
                30
360−Ans
               330
```

Method II: Use the [angle (] feature on the calculator to find θ. You still have to find the actual angle in QIV. Press

[MATH] [▶] [CPX] [▼] [4: angle (] [ENTER] [2nd] [x²] [3] [)] [−] [2nd] [·] [)] [ENTER].

```
angle(√(3)−i)
               -30
360+Ans
               330
```

■ **Answer:** $z = 2\left[\cos\left(\dfrac{5\pi}{3}\right) + i\sin\left(\dfrac{5\pi}{3}\right)\right]$ or $2(\cos 300° + i\sin 300°)$

You must be very careful in converting from rectangular to polar form. Remember that the inverse tangent function is a one-to-one function and will yield values in quadrants I and IV. If the point lies in quadrant II or III, add 180° to the angle found through the inverse tangent function.

EXAMPLE 3 Converting from Rectangular to Polar Form

COMMON MISTAKE

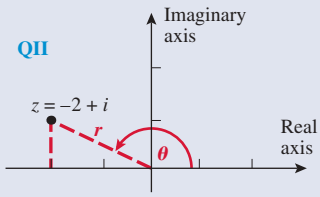
Forgetting to confirm the quadrant, which results in using the reference angle instead of the actual angle.

Express the complex number $z = -2 + i$ in polar form.

CORRECT

Step 1: Plot the point.

The point lies in **quadrant II**.



Step 2: Find r.

Let $x = -2$ and $y = 1$ in.
$r = \sqrt{x^2 + y^2}$.

$$r = \sqrt{(-2)^2 + 1^2}$$

Simplify.

$$\boxed{r = \sqrt{5}}$$

Step 3: Find θ.

Let $x = -2$ and $y = 1$ in.

$$\tan\theta = \frac{y}{x}.$$

$$\tan\theta = -\frac{1}{2}$$

$$\theta = \tan^{-1}\left(-\frac{1}{2}\right)$$

$$= -26.565°$$

The complex number lies in quadrant II.

$$\boxed{\theta = -26.6° + 180° = 153.4°}$$

Step 4: Write the complex number in polar form $z = r(\cos\theta + i\sin\theta)$.

$$\boxed{z = \sqrt{5}(\cos 153.4° + i\sin 153.4°)}$$

INCORRECT

Evaluate the inverse function with a calculator.

$$\theta = \tan^{-1}\left(-\frac{1}{2}\right) = -26.565°$$

Write the complex number in polar form.

$$z = r(\cos\theta + i\sin\theta)$$
$$z = \sqrt{5}[\cos(-26.6°) + i\sin(-26.6°)]$$

Note: $\theta = -26.565°$ lies in quadrant IV, whereas the original point lies in quadrant II. Therefore, we should have added 180° to θ in order to arrive at a point in quadrant II.

YOUR TURN Express the complex number $z = -1 + 2i$ in polar form.

Technology Tip

Express the complex number $z = -2 + i$ in polar form.

```
abs(-2+i)
        2.236067977
√(5)
        2.236067977
angle(-2+i)
        153.4349488
```

Answer:
$z = \sqrt{5}(\cos 116.6° + i\sin 116.6°)$

To convert from polar to rectangular form, simply evaluate the trigonometric functions.

Technology Tip

Express $z = 4(\cos 120° + i \sin 120°)$ in rectangular form.

```
4cos(120)
                -2
4sin(120)
      3.464101615
2√(3)
      3.464101615
```

■ **Answer:** $z = -\sqrt{3} - i$

EXAMPLE 4 Converting from Polar to Rectangular Form

Express $z = 4(\cos 120° + i \sin 120°)$ in rectangular form.

Solution:

Evaluate the trigonometric functions exactly.

$$z = 4\left(\underbrace{\cos 120°}_{-\frac{1}{2}} + i\underbrace{\sin 120°}_{\frac{\sqrt{3}}{2}}\right)$$

Distribute the 4.

$$z = 4\left(-\frac{1}{2}\right) + 4\left(\frac{\sqrt{3}}{2}\right)i$$

Simplify.

$$\boxed{z = -2 + 2\sqrt{3}i}$$

■ **YOUR TURN** Express $z = 2(\cos 210° + i \sin 210°)$ in rectangular form.

Technology Tip

Express $z = 4(\cos 109° + i \sin 109°)$ in rectangular form.

3 (cos − 109) + 2nd · sin 109)) MATH ▶ CPX ▼ 6: ▶ Rect ENTER ENTER

```
3(cos(109)+isin(
109))▸Rect
  -.9767+2.8366i
```

■ **Answer:** $z = -5.5904 - 4.2127i$

EXAMPLE 5 Using a Calculator to Convert from Polar to Rectangular Form

Express $z = 3(\cos 109° + i \sin 109°)$ in rectangular form. Round to four decimal places.

Solution:

Use a calculator to evaluate the trigonometric functions.

$$z = 3\left(\underbrace{\cos 109°}_{-0.325568} + i\underbrace{\sin 109°}_{0.945519}\right)$$

Simplify.

$$\boxed{z = -0.9767 + 2.8366i}$$

■ **YOUR TURN** Express $z = 7(\cos 217° + i \sin 217°)$ in rectangular form. Round to four decimal places.

SECTION 7.3 SUMMARY

In the complex plane, the horizontal axis is the real axis and the vertical axis is the imaginary axis. We can express complex numbers in either rectangular or polar form:

rectangular form: $z = x + iy$

or

polar form: $z = r(\cos\theta + i \sin\theta)$

The modulus of a complex number, $z = x + iy$, is given by

$$|z| = \sqrt{x^2 + y^2}$$

To convert from rectangular to polar form, we use the relationships

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \tan\theta = \frac{y}{x},\ x \neq 0 \text{ and } 0 \leq \theta < 2\pi$$

It is important to note in which quadrant the point lies. To convert from polar to rectangular form, simply evaluate the trigonometric functions.

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

SECTION 7.3 EXERCISES

SKILLS

In Exercises 1–8, graph each complex number in the complex plane.

1. $7 + 8i$ **2.** $3 + 5i$ **3.** $-2 - 4i$ **4.** $-3 - 2i$

5. 2 **6.** 7 **7.** $-3i$ **8.** $-5i$

In Exercises 9–24, express each complex number in polar form.

9. $1 - i$ **10.** $2 + 2i$ **11.** $1 + \sqrt{3}i$ **12.** $-3 - \sqrt{3}i$

13. $-4 + 4i$ **14.** $\sqrt{5} - \sqrt{5}i$ **15.** $\sqrt{3} - 3i$ **16.** $-\sqrt{3} + i$

17. $3 + 0i$ **18.** $-2 + 0i$ **19.** $-\frac{1}{2} - \frac{1}{2}i$ **20.** $\frac{1}{6} - \frac{1}{6}i$

21. $-\sqrt{6} - \sqrt{6}i$ **22.** $\frac{1}{3} - \frac{1}{3}i$ **23.** $-5 + 5i$ **24.** $3 + 3i$

In Exercises 25–40, use a calculator to express each complex number in polar form.

25. $3 - 7i$ **26.** $2 + 3i$ **27.** $-6 + 5i$ **28.** $-4 - 3i$

29. $-5 + 12i$ **30.** $24 + 7i$ **31.** $8 - 6i$ **32.** $-3 + 4i$

33. $-\frac{1}{2} + \frac{3}{4}i$ **34.** $-\frac{5}{8} - \frac{11}{4}i$ **35.** $5.1 + 2.3i$ **36.** $1.8 - 0.9i$

37. $-2\sqrt{3} - \sqrt{5}i$ **38.** $-\frac{4\sqrt{5}}{3} + \frac{\sqrt{5}}{2}i$ **39.** $4.02 - 2.11i$ **40.** $1.78 - 0.12i$

In Exercises 41–52, express each complex number in rectangular form.

41. $5(\cos 180° + i\sin 180°)$ **42.** $2(\cos 135° + i\sin 135°)$ **43.** $2(\cos 315° + i\sin 315°)$

44. $3(\cos 270° + i\sin 270°)$ **45.** $-4(\cos 60° + i\sin 60°)$ **46.** $-4(\cos 210° + i\sin 210°)$

47. $\sqrt{3}(\cos 150° + i\sin 150°)$ **48.** $\sqrt{3}(\cos 330° + i\sin 330°)$ **49.** $\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

50. $2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$ **51.** $6\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$ **52.** $4\left[\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right]$

In Exercises 53–64, use a calculator to express each complex number in rectangular form.

53. $5(\cos 295° + i\sin 295°)$ **54.** $4(\cos 35° + i\sin 35°)$ **55.** $3(\cos 100° + i\sin 100°)$

56. $6(\cos 250° + i\sin 250°)$ **57.** $-7(\cos 140° + i\sin 140°)$ **58.** $-5(\cos 320° + i\sin 320°)$

59. $3\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$ **60.** $2\left[\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)\right]$ **61.** $-2\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right]$

62. $-4\left[\cos\left(\frac{15\pi}{11}\right) + i\sin\left(\frac{15\pi}{11}\right)\right]$ **63.** $-5\left[\cos\left(\frac{4\pi}{9}\right) + i\sin\left(\frac{4\pi}{9}\right)\right]$ **64.** $6\left[\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)\right]$

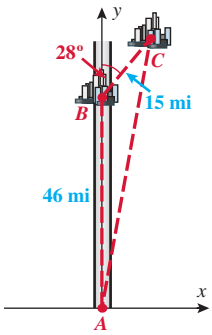
APPLICATIONS

65. Road Construction. Engineers are planning the construction of a bypass in a north–south highway to connect cities B and C.

a. What is the distance from A to C?

b. Write the vector AC as a complex number in polar form. (Use degrees for the angle.)

c. What is the angle BAC?

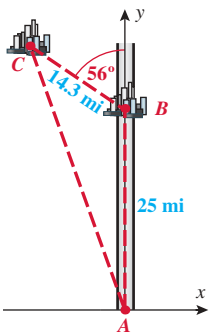


66. Road Construction. Engineers are planning the construction of a bypass in a north–south highway to connect cities B and C.

a. What is the distance from A to C?

b. Write the vector AC as a complex number in polar form. (Use degrees for the angle.)

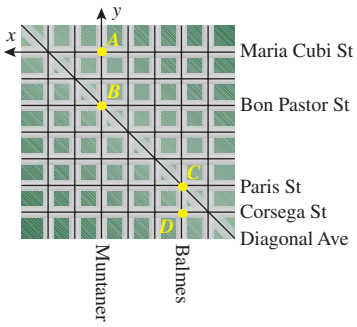
c. What is the angle BAC?



67. City Map Barcelona. The city of Barcelona is crossed by the Diagonal Avenue as shown in the map.

a. Which complex numbers, in rectangular form, represent the street segment A-B, B-C, and C-D?

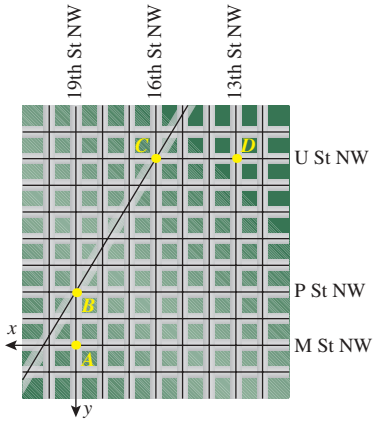
b. Which complex number, in polar form, represents A-D?



68. City Map Washington, D.C. A simplified map of Washington, D.C., is shown below.

a. Which complex numbers, in rectangular form, represent the street segment A-B, B-C, and C-D?

b. Which complex number, in polar form, represents A-D?



For Exercises 69 and 70, refer to the following:

In the design of AC circuits, the voltage across a resistance is regarded as a real number. When the voltage goes across an inductor or a capacitor, it is considered an imaginary number: positive ($I > 0$) in the inductor case and negative ($I < 0$) in the capacitor case. The impedance results from the combination of the voltages in the circuit and is given by the formula

$$z = |z|(\cos\theta + i\sin\theta), \text{ where } |z| = \sqrt{R^2 + I^2}$$
$$\text{and } \theta = \tan^{-1}\left(\frac{I}{R}\right)$$

where z is impedance, R is resistance, and I is inductance.

69. AC Circuits. Find the impedance of a circuit with resistance 4 ohms and inductor of 6 ohms. Write your answer in polar form.

70. AC Circuits. Find the impedance of a circuit with resistance 7 ohms and capacitor of 5 ohms. Write your answer in polar form.

CATCH THE MISTAKE

In Exercises 71 and 72, explain the mistake that is made.

71. Express $z = -3 - 8i$ in polar form.

Solution:

Find r. $\quad r = \sqrt{x^2 + y^2} = \sqrt{9 + 64} = \sqrt{73}$

Find θ. $\quad \tan\theta = \dfrac{8}{3}$

$$\theta = \tan^{-1}\left(\frac{8}{3}\right) = 69.44°$$

Write the complex number in polar form.

$$z = \sqrt{73}(\cos 69.44° + i \sin 69.44°)$$

This is incorrect. What mistake was made?

72. Express $z = -3 + 8i$ in polar form.

Solution:

Find r. $\quad r = \sqrt{x^2 + y^2} = \sqrt{9 + 64} = \sqrt{73}$

Find θ. $\quad \tan\theta = -\dfrac{8}{3}$

$$\theta = \tan^{-1}\left(-\frac{8}{3}\right) = -69.44°$$

Write the complex number in polar form.

$$z = \sqrt{73}[\cos(-69.44°) + i \sin(-69.44°)]$$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 73–76, determine whether each statement is true or false.

73. In the complex plane, any point that lies along the horizontal axis is a real number.

74. In the complex plane, any point that lies along the vertical axis is an imaginary number.

75. The modulus of z and the modulus of $\bar{z}$ are equal.

76. The argument of z and the argument of $\bar{z}$ are equal.

77. Find the argument of $z = a$, where a is a positive real number.

78. Find the argument of $z = bi$, where b is a positive real number.

79. Find the modulus of $z = bi$, where b is a negative real number.

80. Find the modulus of $z = a$, where a is a negative real number.

In Exercises 81 and 82, use a calculator to express the complex number in polar form.

81. $a - 2ai$, where $a > 0$

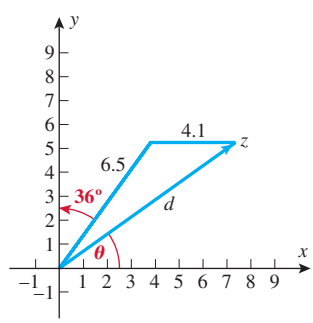
82. $-3a - 4ai$, where $a > 0$

CHALLENGE

83. Suppose that a complex number z lies on the circle $x^2 + y^2 = \pi^2$. If $\cos\left(\dfrac{\theta}{2}\right) = \dfrac{1}{2}$ and $\sin\theta < 0$, find the rectangular form of z.

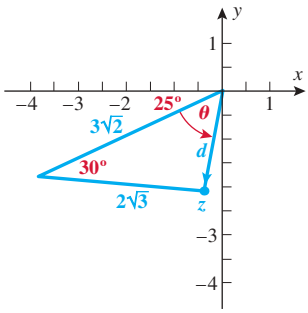
84. Suppose that a complex number z lies on the circle $x^2 + y^2 = 8$. If $\sin\left(\dfrac{\theta}{2}\right) = -\dfrac{\sqrt{3}}{2}$ and $\cos\theta < 0$, find the rectangular form of z.

85. Consider the following diagram:



Find z in trigonometric form.
Hint: Use the Law of Cosines.

86. Consider the following diagram:



Find z in trigonometric form.
Hint: Use the Law of Cosines.

87. Consider the complex number in polar form $z = r(\cos\theta + i\sin\theta)$. What is the polar form of $-z$?

88. Consider the complex number in polar form $z = r(\cos\theta + i\sin\theta)$. What is the polar form of $\bar{z}$?

TECHNOLOGY

Graphing calculators are able to convert complex numbers from rectangular to polar form using the $\boxed{\text{ABS}}$ command to find the modulus and the angle command to find the argument.

89. Find abs$(1 + i)$. Find angle$(1 + i)$. Write $1 + i$ in polar form.

90. Find abs$(1 - i)$. Find angle$(1 - i)$. Write $1 - i$ in polar form.

A second way of using a graphing calculator to convert between rectangular and polar coordinates is with the $\boxed{\text{Pol}}$ and $\boxed{\text{Rec}}$ commands.

91. Find Pol(2, 1). Write $2 + i$ in polar form.

92. Find Rec(345°). Write $3(\cos 45° + i\sin 45°)$ in rectangular form.

Another way of using a graphing calculator to represent complex numbers in rectangular form is to enter the real and imaginary parts as a list of two numbers and use the $\boxed{\text{SUM}}$ command to find the modulus.

93. Write $28 - 21i$ in polar form using the $\boxed{\text{SUM}}$ command to find its modulus, and round the angle to the nearest degree.

94. Write $-\sqrt{21} + 10i$ in polar form using the $\boxed{\text{SUM}}$ command to find its modulus, and round the angle to the nearest degree.

PREVIEW TO CALCULUS

In Exercises 95–98, refer to the following:

The use of a different system of coordinates simplifies many mathematical expressions and some calculations are performed in an easier way. The rectangular coordinates (x, y) are transformed into polar coordinates by the equations

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2}\,(r \neq 0)$ and $\tan\theta = \dfrac{y}{x}\,(x \neq 0)$. In polar coordinates, the equation of the unit circle $x^2 + y^2 = 1$ is just $r = 1$. In calculus, we use polar coordinates extensively. Transform the rectangular equation to polar form.

95. $x^2 + y^2 = 25$

96. $x^2 + y^2 = 4x$

97. $y^2 - 2y = -x^2$

98. $(x^2 + y^2)^2 - 16(x^2 - y^2) = 0$

SECTION 7.4 PRODUCTS, QUOTIENTS, POWERS, AND ROOTS OF COMPLEX NUMBERS

SKILLS OBJECTIVES

- Find the product of two complex numbers.
- Find the quotient of two complex numbers.
- Raise a complex number to an integer power.
- Find the nth roots of a complex number.
- Solve a polynomial equation by finding complex roots.

CONCEPTUAL OBJECTIVES

- Derive the identities for products and quotients of complex numbers.
- Relate De Moivre's theorem (the power rule) for complex numbers to the product rule for complex numbers.

In this section, we will multiply complex numbers, divide complex numbers, raise complex numbers to powers, and find roots of complex numbers.

Products of Complex Numbers

First, we will derive a formula for the product of two complex numbers.

WORDS	MATH
Start with two complex numbers z_1 and z_2.	$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
Multiply z_1 and z_2.	$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$
Use the FOIL method to multiply the expressions in parentheses.	$z_1z_2 = r_1r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \underbrace{i^2}_{-1}\sin\theta_1\sin\theta_2)$
Group the real parts and the imaginary parts.	$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$
Apply the cosine and sine sum identities.	$z_1z_2 = r_1r_2\left[\underbrace{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)}_{\cos(\theta_1+\theta_2)} + i\underbrace{(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)}_{\sin(\theta_1+\theta_2)}\right]$
Simplify.	$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

PRODUCT OF TWO COMPLEX NUMBERS

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. The complex product z_1z_2 is given by

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

In other words, *when multiplying two complex numbers, multiply the magnitudes and add the arguments.*

Study Tip

When two complex numbers are multiplied, the magnitudes are multiplied and the arguments are added.

EXAMPLE 1 Multiplying Complex Numbers

Find the product of $z_1 = 3(\cos 35° + i\sin 35°)$ and $z_2 = 2(\cos 10° + i\sin 10°)$.

Solution:

Set up the product.	$z_1z_2 = 3(\cos 35° + i\sin 35°)\cdot 2(\cos 10° + i\sin 10°)$
Multiply the magnitudes and add the arguments.	$z_1z_2 = 3\cdot 2[\cos(35° + 10°) + i\sin(35° + 10°)]$

Simplify. $z_1z_2 = 6(\cos 45° + i\sin 45°)$

The product is in polar form. To express the product in rectangular form, evaluate the trigonometric functions. $z_1z_2 = 6\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 3\sqrt{2} + 3i\sqrt{2}$

Product in polar form: $\boxed{z_1z_2 = 6(\cos 45° + i\sin 45°)} = 6\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

Product in rectangular form: $\boxed{z_1z_2 = 3\sqrt{2} + 3i\sqrt{2}}$

■ **Answer:**
$z_1z_2 = 10(\cos 120° + i\sin 120°)$ or
$z_1z_2 = -5 + 5i\sqrt{3}$

■ **YOUR TURN** Find the product of $z_1 = 2(\cos 55° + i\sin 55°)$ and $z_2 = 5(\cos 65° + i\sin 65°)$. Express the answer in both polar and rectangular form.

Quotients of Complex Numbers

We now derive a formula for the quotient of two complex numbers.

Words	**Math**
Start with two complex numbers z_1 and z_2.	$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
Divide z_1 by z_2.	$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \left(\frac{r_1}{r_2}\right)\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right)$
Multiply the numerator and the denominator of the second expression in parentheses by the conjugate of the denominator, $\cos\theta_2 - i\sin\theta_2$.	$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right)\left(\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}\right)$
Use the FOIL method to multiply the expressions in parentheses in the last two expressions.	$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)\left(\frac{\cos\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i\sin\theta_2\cos\theta_1}{\cos^2\theta_2 - i^2\sin^2\theta_2}\right)$
Substitute $i^2 = -1$ and group the real parts and the imaginary parts.	$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}{\underbrace{\cos^2\theta_2 + \sin^2\theta_2}_{1}}\right]$
Simplify.	$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)]$
Use the cosine and sine difference identities.	$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)\underbrace{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}_{\cos(\theta_1 - \theta_2)} + i\underbrace{(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)}_{\sin(\theta_1 - \theta_2)}$
Simplify.	$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

It is important to notice that the argument difference is the argument of the numerator minus the argument of the denominator.

QUOTIENT OF TWO COMPLEX NUMBERS

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. The complex quotient $\frac{z_1}{z_2}$ is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

In other words, *when dividing two complex numbers, divide the magnitudes and subtract the arguments. It is important to note that the argument difference is the argument of the complex number in the numerator minus the argument of the complex number in the denominator.*

EXAMPLE 2 Dividing Complex Numbers

Let $z_1 = 6(\cos 125° + i\sin 125°)$ and $z_2 = 3(\cos 65° + i\sin 65°)$. Find $\frac{z_1}{z_2}$.

Solution:

Set up the quotient.
$$\frac{z_1}{z_2} = \frac{6(\cos 125° + i\sin 125°)}{3(\cos 65° + i\sin 65°)}$$

Divide the magnitudes and subtract the arguments.
$$\frac{z_1}{z_2} = \frac{6}{3}[\cos(125° - 65°) + i\sin(125° - 65°)]$$

Simplify.
$$\frac{z_1}{z_2} = 2(\cos 60° + i\sin 60°)$$

The quotient is in polar form. To express the quotient in rectangular form, evaluate the trigonometric functions.
$$\frac{z_1}{z_2} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

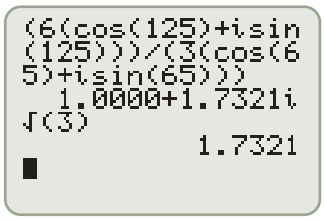
Polar form:
$$\boxed{\frac{z_1}{z_2} = 2(\cos 60° + i\sin 60°)}$$

Rectangular form:
$$\boxed{\frac{z_1}{z_2} = 1 + i\sqrt{3}}$$

■ **YOUR TURN** Let $z_1 = 10(\cos 275° + i\sin 275°)$ and $z_2 = 5(\cos 65° + i\sin 65°)$. Find $\frac{z_1}{z_2}$. Express the answers in both polar and rectangular form.

Technology Tip

Let $z_1 = 6(\cos 125° + i\sin 125°)$ and $z_2 = 3(\cos 65° + i\sin 65°)$. Find $\frac{z_1}{z_2}$. Be sure to include parentheses for z_1 and z_2.



■ **Answer:**
$\frac{z_1}{z_2} = 2(\cos 210° + i\sin 210°)$ or
$\frac{z_1}{z_2} = -\sqrt{3} - i$

When multiplying or dividing complex numbers, we have considered only those values of θ such that $0° \le \theta < 360°$. When the value of θ is negative or greater than or equal to 360°, find the coterminal angle in the interval [0°, 360°).

Powers of Complex Numbers

Raising a number to a positive integer power is the same as multiplying that number by itself repeated times.

$$x^3 = x \cdot x \cdot x \qquad (a + b)^2 = (a + b)(a + b)$$

Therefore, raising a complex number to a power that is a positive integer is the same as multiplying the complex number by itself multiple times. Let us illustrate this with the complex number $z = r(\cos\theta + i\sin\theta)$, which we will raise to positive integer powers (n).

WORDS	MATH
Take the case $n = 2$.	$z^2 = [r(\cos\theta + i\sin\theta)][r(\cos\theta + i\sin\theta)]$
Apply the product rule (multiply the magnitudes and add the arguments).	$z^2 = r^2[\cos(2\theta) + i\sin(2\theta)]$
Take the case $n = 3$.	$z^3 = z^2 z = \{r^2[\cos(2\theta) + i\sin(2\theta)]\}[r(\cos\theta + i\sin\theta)]$
Apply the product rule (multiply the magnitudes and add the arguments).	$z^3 = r^3[\cos(3\theta) + i\sin(3\theta)]$
Take the case $n = 4$.	$z^4 = z^3 z = \{r^3[\cos(3\theta) + i\sin(3\theta)]\}[r(\cos\theta + i\sin\theta)]$
Apply the product rule (multiply the magnitudes and add the arguments).	$z^4 = r^4[\cos(4\theta) + i\sin(4\theta)]$
The pattern observed for any n is	$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$

Although we will not prove this generalized representation of a complex number raised to a power, it was proved by Abraham De Moivre, and hence its name.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

when n is a positive integer ($n \geq 1$).

In other words, when raising a complex number to a power n, raise the magnitude to the same power n and multiply the argument by n.

Although De Moivre's theorem was proved for all real numbers n, we will only use it for positive integer values of n and their reciprocals (nth roots). This is a very powerful theorem. For example, if asked to find $(\sqrt{3} + i)^{10}$, you have two choices: (1) Multiply out the expression algebraically, which we will call the long way or (2) convert to polar coordinates and use De Moivre's theorem, which we will call the short way. We will use De Moivre's theorem.

EXAMPLE 3 Finding a Power of a Complex Number

Find $(\sqrt{3} + i)^{10}$ and express the answer in rectangular form.

Solution:

STEP 1 Convert to polar form. $(\sqrt{3} + i)^{10} = [2(\cos 30° + i \sin 30°)]^{10}$

STEP 2 Apply De Moivre's theorem with $n = 10$.

$$(\sqrt{3} + i)^{10} = [2(\cos 30° + i \sin 30°)]^{10}$$
$$= 2^{10}[\cos(10 \cdot 30°) + i \sin(10 \cdot 30°)]$$

STEP 3 Simplify.

$$(\sqrt{3} + i)^{10} = 2^{10}(\cos 300° + i \sin 300°)$$

Evaluate 2^{10} and the sine and cosine functions.

$$= 1024\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

Simplify.

$$\boxed{= 512 - 512i\sqrt{3}}$$

■ **YOUR TURN** Find $(1 + i\sqrt{3})^{10}$ and express the answer in rectangular form.

Technology Tip

Find $(\sqrt{3} + i)^{10}$ and express the answer in rectangular form.

```
(√(3)+i)^10
512-886.8100135i
(2(cos(30)+isin(
30)))^10
512-886.8100135i
```

■ **Answer:** $-512 - 512i\sqrt{3}$

Roots of Complex Numbers

De Moivre's theorem is the basis for the *nth root theorem*. Before we proceed, let us motivate it with a problem: Solve $x^3 - 1 = 0$. Recall that a polynomial of degree n has n solutions (roots in the complex number system). So the polynomial $P(x) = x^3 - 1$ is of degree 3 and has three solutions (roots). We can solve it algebraically.

WORDS	MATH
List the potential rational roots of the polynomial $P(x) = x^3 - 1$.	$x = \pm 1$
Use synthetic division to test $x = 1$.	$\begin{array}{r\|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 1 & 1 \\ \hline & \underbrace{1 \quad 1 \quad 1}_{x^2+x+1} & & & \boxed{0} \end{array}$
Since $x = 1$ is a zero, then the polynomial can be written as a product of the linear factor $(x - 1)$ and a quadratic factor.	$P(x) = (x - 1)(x^2 + x + 1)$
Use the quadratic formula on $x^2 + x + 1 = 0$ to solve for x.	$x = \dfrac{-1 \pm \sqrt{1 - 4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2} = -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$
So the three solutions to the equation $x^3 - 1 = 0$ are	$\boxed{x = 1, x = -\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}, \text{ and } x = -\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}}$.

An alternative approach to solving $x^3 - 1 = 0$ is to use the *nth root theorem* to find the additional complex cube roots of 1.

Derivation of the *n*th Root Theorem

WORDS	MATH
Let z and w be complex numbers such that w is the nth root of z.	$w = z^{1/n}$ or $w = \sqrt[n]{z}$, where n is a positive integer
Raise both sides of the equation to the nth power.	$w^n = z$
Let $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\alpha + i\sin\alpha)$.	$[s(\cos\alpha + i\sin\alpha)]^n = r(\cos\theta + i\sin\theta)$
Apply De Moivre's theorem to the left side of the equation.	$s^n[\cos(n\alpha) + i\sin(n\alpha)] = r(\cos\theta + i\sin\theta)$
For these two expressions to be equal, their magnitudes must be equal and their angles must be coterminal.	$s^n = r$ and $n\alpha = \theta + 2k\pi$, where k is any integer
Solve for s and α.	$s = r^{1/n}$ and $\alpha = \dfrac{\theta + 2k\pi}{n}$
Substitute $s = r^{1/n}$ and $\alpha = \dfrac{\theta + 2k\pi}{n}$ into $w = z^{1/n}$.	$z^{1/n} = r^{1/n}\left[\cos\left(\dfrac{\theta + 2k\pi}{n}\right) + i\sin\left(\dfrac{\theta + 2k\pi}{n}\right)\right]$

Notice that when $k = n$, the arguments $\dfrac{\theta}{n} + 2\pi$ and $\dfrac{\theta}{n}$ are coterminal. Therefore, to get distinct roots, let $k = 0, 1, \ldots, n - 1$. If we let z be a given complex number and w be any complex number that satisfies the relationship $z^{1/n} = w$ or $z = w^n$, where $n \geq 2$, then we say that w is a **complex *n*th root** of z.

*n*TH ROOT THEOREM

The ***n*th roots** of the complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right] \qquad \theta \text{ in radians}$$

or

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{k\cdot 360^\circ}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k\cdot 360^\circ}{n}\right)\right] \qquad \theta \text{ in degrees}$$

where $k = 0, 1, 2, \ldots, n - 1$.

EXAMPLE 4 Finding Roots of Complex Numbers

Find the three distinct cube roots of $-4 - 4i\sqrt{3}$, and plot the roots in the complex plane.

Solution:

STEP 1 Write $-4 - 4i\sqrt{3}$ in polar form. $\qquad 8(\cos 240^\circ + i\sin 240^\circ)$

STEP 2 Find the three cube roots.

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{k\cdot 360^\circ}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k\cdot 360^\circ}{n}\right)\right]$$

$$\theta = 240^\circ,\ r = 8,\ n = 3,\ k = 0, 1, 2$$

$k = 0$: $$w_0 = 8^{1/3}\left[\cos\left(\frac{240^\circ}{3} + \frac{0\cdot 360^\circ}{3}\right) + i\sin\left(\frac{240^\circ}{3} + \frac{0\cdot 360^\circ}{3}\right)\right]$$

Simplify. $$\boxed{w_0 = 2(\cos 80^\circ + i\sin 80^\circ)}$$

$k = 1$: $$w_1 = 8^{1/3}\left[\cos\left(\frac{240°}{3} + \frac{1 \cdot 360°}{3}\right) + i\sin\left(\frac{240°}{3} + \frac{1 \cdot 360°}{3}\right)\right]$$

Simplify. $$\boxed{w_1 = 2(\cos 200° + i\sin 200°)}$$

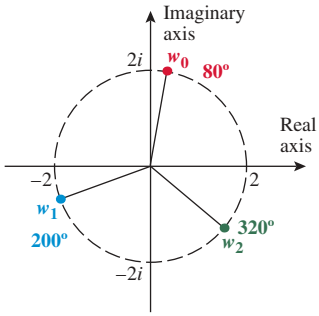
$k = 2$: $$w_2 = 8^{1/3}\left[\cos\left(\frac{240°}{3} + \frac{2 \cdot 360°}{3}\right) + i\sin\left(\frac{240°}{3} + \frac{2 \cdot 360°}{3}\right)\right]$$

Simplify. $$\boxed{w_2 = 2(\cos 320° + i\sin 320°)}$$

STEP 3 Plot the three cube roots in the complex plane.

Notice the following:

- The roots all have a magnitude of 2.
- The roots lie on a circle of radius 2.
- The roots are equally spaced around the circle (120° apart).



■ **YOUR TURN** Find the three distinct cube roots of $4 - 4i\sqrt{3}$, and plot the roots in the complex plane.

Solving Equations Using Roots of Complex Numbers

Let us return to solving the equation $x^3 - 1 = 0$. As stated, $x = 1$ is the real solution to this cubic equation. However, there are two additional (complex) solutions. Since we are finding the zeros of a third-degree polynomial, we expect three solutions. Furthermore, when complex solutions arise in finding the roots of polynomials with real coefficients, they come in conjugate pairs.

EXAMPLE 5 Solving Equations Using Complex Roots

Find all complex solutions to $x^3 - 1 = 0$.

Solution: $x^3 = 1$

STEP 1 Write 1 in polar form. $$1 = 1 + 0i = \cos 0° + i\sin 0°$$

STEP 2 Find the three cube roots of 1.

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)\right]$$

$$r = 1, \theta = 0°, n = 3, k = 0, 1, 2$$

$k = 0$: $$w_0 = 1^{1/3}\left[\cos\left(\frac{0°}{3} + \frac{0 \cdot 360°}{3}\right) + i\sin\left(\frac{0°}{3} + \frac{0 \cdot 360°}{3}\right)\right]$$

Simplify. $$w_0 = \cos 0° + i\sin 0°$$

$k = 1$: $$w_1 = 1^{1/3}\left[\cos\left(\frac{0°}{3} + \frac{1 \cdot 360°}{3}\right) + i\sin\left(\frac{0°}{3} + \frac{1 \cdot 360°}{3}\right)\right]$$

Simplify. $$w_1 = \cos 120° + i\sin 120°$$

$k = 2$: $$w_2 = 1^{1/3}\left[\cos\left(\frac{0°}{3} + \frac{2 \cdot 360°}{3}\right) + i\sin\left(\frac{0°}{3} + \frac{2 \cdot 360°}{3}\right)\right]$$

Simplify. $$w_2 = \cos 240° + i\sin 240°$$

Technology Tip

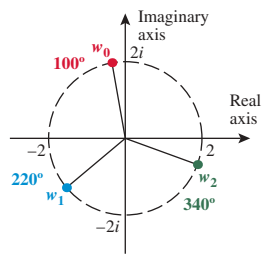
Find the three distinct roots of $-4 - 4i\sqrt{3}$.

Caution: If you use a TI calculator to find $(-4 - 4i\sqrt{3})^{1/3}$, the calculator will return only one root.

```
(-4-4i√(3))^(1/3)
        1.53-1.29i
2(cos(320)+isin(
320))
        1.53-1.29i
```

To find all three distinct roots, you need to change to polar form and apply the nth root theorem.

■ **Answer:**
$w_0 = 2(\cos 100° + i\sin 100°)$
$w_1 = 2(\cos 220° + i\sin 220°)$
$w_2 = 2(\cos 340° + i\sin 340°)$



Technology Tip

The solution to the equation $x^3 - 1 = 0$ is $x = (1 + 0i)^{1/3}$.

```
abs(1)
                1.00
angle(1)
                0.00
```

```
(cos(0)+isin(0))
                1.00
(cos(0+360/3)+is
in(0+360/3))
          -.50+.87i
```

```
cos(0)+isin(0)
                1.00
cos(120)+isin(12
0)
          -.50+.87i
```

```
(cos(0+2*360/3)+
isin(0+2*360/3))
          -.50-.87i
```

```
cos(240)+isin(24
0)
          -.50-.87i
```

Step 3 Write the roots in rectangular form.

w_0: $$w_0 = \underbrace{\cos 0^\circ}_{1} + i\underbrace{\sin 0^\circ}_{0} = 1$$

w_1: $$w_1 = \underbrace{\cos 120^\circ}_{-\frac{1}{2}} + i\underbrace{\sin 120^\circ}_{\frac{\sqrt{3}}{2}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

w_2: $$w_2 = \underbrace{\cos 240^\circ}_{-\frac{1}{2}} + i\underbrace{\sin 240^\circ}_{-\frac{\sqrt{3}}{2}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Step 4 Write the solutions to the equation $x^3 - 1 = 0$.

$$\boxed{x = 1} \qquad \boxed{x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}} \qquad \boxed{x = -\frac{1}{2} - i\frac{\sqrt{3}}{2}}$$

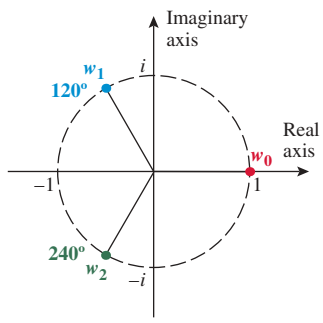
Notice that there is one real solution and there are two (nonreal) complex solutions and that the two complex solutions are complex conjugates.

It is always a good idea to check that the solutions indeed satisfy the equation. The equation $x^3 - 1 = 0$ can also be written as $x^3 = 1$, so the check in this case is to cube the three solutions and confirm that the result is 1.

$x = 1$: $$1^3 = 1$$

$x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$: $$\begin{aligned}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3 &= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\\ &= \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\\ &= \frac{1}{4} + \frac{3}{4}\\ &= 1\end{aligned}$$

$x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$: $$\begin{aligned}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 &= \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\\ &= \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\\ &= \frac{1}{4} + \frac{3}{4}\\ &= 1\end{aligned}$$



SECTION 7.4 SUMMARY

In this section, we multiplied and divided complex numbers and, using De Moivre's theorem, raised complex numbers to integer powers and found the nth roots of complex numbers, as follows:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers.

The **product** z_1z_2 is given by

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

The **quotient** $\dfrac{z_1}{z_2}$ is given by

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number. Then for a positive integer n,

z raised to a **power** n is given by

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

The ***n*th roots** of z are given by

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{k\cdot 360°}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k\cdot 360°}{n}\right)\right]$$

where θ is in degrees

or

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{k\cdot 2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k\cdot 2\pi}{n}\right)\right]$$

where θ is in radians and $k = 0, 1, 2, \ldots, n - 1$.

SECTION 7.4 EXERCISES

SKILLS

In Exercises 1–10, find the product z_1z_2 and express it in rectangular form.

1. $z_1 = 4(\cos 40° + i\sin 40°)$ and $z_2 = 3(\cos 80° + i\sin 80°)$
2. $z_1 = 2(\cos 100° + i\sin 100°)$ and $z_2 = 5(\cos 50° + i\sin 50°)$
3. $z_1 = 4(\cos 80° + i\sin 80°)$ and $z_2 = 2(\cos 145° + i\sin 145°)$
4. $z_1 = 3(\cos 130° + i\sin 130°)$ and $z_2 = 4(\cos 170° + i\sin 170°)$
5. $z_1 = 2(\cos 10° + i\sin 10°)$ and $z_2 = 4(\cos 80° + i\sin 80°)$
6. $z_1 = 3(\cos 190° + i\sin 190°)$ and $z_2 = 5(\cos 80° + i\sin 80°)$
7. $z_1 = \sqrt{3}\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$ and $z_2 = \sqrt{27}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$
8. $z_1 = \sqrt{5}\left[\cos\left(\frac{\pi}{15}\right) + i\sin\left(\frac{\pi}{15}\right)\right]$ and $z_2 = \sqrt{5}\left[\cos\left(\frac{4\pi}{15}\right) + i\sin\left(\frac{4\pi}{15}\right)\right]$
9. $z_1 = 4\left[\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right.$ and $z_2 = 3\left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right]$
10. $z_1 = 6\left[\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)\right.$ and $z_2 = 5\left[\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right]$

In Exercises 11–20, find the quotient $\frac{z_1}{z_2}$ and express it in rectangular form.

11. $z_1 = 6(\cos 100° + i\sin 100°)$ and $z_2 = 2(\cos 40° + i\sin 40°)$

12. $z_1 = 8(\cos 80° + i\sin 80°)$ and $z_2 = 2(\cos 35° + i\sin 35°)$

13. $z_1 = 10(\cos 200° + i\sin 200°)$ and $z_2 = 5(\cos 65° + i\sin 65°)$

14. $z_1 = 4(\cos 280° + i\sin 280°)$ and $z_2 = 4(\cos 55° + i\sin 55°)$

15. $z_1 = \sqrt{12}(\cos 350° + i\sin 350°)$ and $z_2 = \sqrt{3}(\cos 80° + i\sin 80°)$

16. $z_1 = \sqrt{40}(\cos 110° + i\sin 110°)$ and $z_2 = \sqrt{10}(\cos 20° + i\sin 20°)$

17. $z_1 = 9\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$ and $z_2 = 3\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$

18. $z_1 = 8\left[\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right]$ and $z_2 = 4\left[\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right]$

19. $z_1 = 45\left[\cos\left(\frac{22\pi}{15}\right) + i\sin\left(\frac{22\pi}{15}\right)\right]$ and $z_2 = 9\left[\cos\left(\frac{2\pi}{15}\right) + i\sin\left(\frac{2\pi}{15}\right)\right]$

20. $z_1 = 22\left[\cos\left(\frac{11\pi}{18}\right) + i\sin\left(\frac{11\pi}{18}\right)\right]$ and $z_2 = 11\left[\cos\left(\frac{5\pi}{18}\right) + i\sin\left(\frac{5\pi}{18}\right)\right]$

In Exercises 21–30, find the result of each expression using De Moivre's theorem. Write the answer in rectangular form.

21. $(-1 + i)^5$ **22.** $(1 - i)^4$ **23.** $(-\sqrt{3} + i)^6$ **24.** $(\sqrt{3} - i)^8$ **25.** $(1 - \sqrt{3}i)^4$

26. $(-1 + \sqrt{3}i)^5$ **27.** $(4 - 4i)^8$ **28.** $(-3 + 3i)^{10}$ **29.** $(4\sqrt{3} + 4i)^7$ **30.** $(-5 + 5\sqrt{3}i)^7$

In Exercises 31–40, find all nth roots of z. Write the answers in polar form, and plot the roots in the complex plane.

31. $2 - 2i\sqrt{3}$, $n = 2$ **32.** $2 + 2\sqrt{3}i$, $n = 2$ **33.** $\sqrt{18} - \sqrt{18}i$, $n = 2$ **34.** $-\sqrt{2} + \sqrt{2}i$, $n = 2$

35. $4 + 4\sqrt{3}i$, $n = 3$ **36.** $-\frac{27}{2} + \frac{27\sqrt{3}}{2}i$, $n = 3$ **37.** $\sqrt{3} - i$, $n = 3$ **38.** $4\sqrt{2} + 4\sqrt{2}i$, $n = 3$

39. $8\sqrt{2} - 8\sqrt{2}i$, $n = 4$ **40.** $-\sqrt{128} + \sqrt{128}i$, $n = 4$

In Exercises 41–56, find all complex solutions to the given equations.

41. $x^4 - 16 = 0$ **42.** $x^3 - 8 = 0$ **43.** $x^3 + 8 = 0$ **44.** $x^3 + 1 = 0$

45. $x^4 + 16 = 0$ **46.** $x^6 + 1 = 0$ **47.** $x^6 - 1 = 0$ **48.** $4x^2 + 1 = 0$

49. $x^2 + i = 0$ **50.** $x^2 - i = 0$ **51.** $x^4 - 2i = 0$ **52.** $x^4 + 2i = 0$

53. $x^5 + 32 = 0$ **54.** $x^5 - 32 = 0$ **55.** $x^7 - \pi^{14}i = 0$ **56.** $x^7 + \pi^{14} = 0$

APPLICATIONS

57. Complex Pentagon. When you graph the five fifth roots of $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ and connect the points, you form a pentagon. Find the roots and draw the pentagon.

58. Complex Square. When you graph the four fourth roots of $16i$ and connect the points, you form a square. Find the roots and draw the square.

59. Hexagon. Compute the six sixth roots of $\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and form a hexagon by connecting successive roots.

60. Octagon. Compute the eight eighth roots of $2i$, and form an octagon by connecting successive roots.

CATCH THE MISTAKE

In Exercises 61–64, explain the mistake that is made.

61. Let $z_1 = 6(\cos 65° + i\sin 65°)$ and $z_2 = 3(\cos 125° + i\sin 125°)$. Find $\frac{z_1}{z_2}$.

Solution:

Use the quotient formula.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Substitute values.

$$\frac{z_1}{z_2} = \frac{6}{3}[\cos(125° - 65°) + i\sin(125° - 65°)]$$

Simplify. $\frac{z_1}{z_2} = 2(\cos 60° + i\sin 60°)$

Evaluate the trigonometric functions.

$$\frac{z_1}{z_2} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

This is incorrect. What mistake was made?

62. Let $z_1 = 6(\cos 65° + i\sin 65°)$ and $z_2 = 3(\cos 125° + i\sin 125°)$. Find z_1z_2.

Solution:

Write the product.

$$z_1z_2 = 6(\cos 65° + i\sin 65°)\cdot 3(\cos 125° + i\sin 125°)$$

Multiply the magnitudes.

$$z_1z_2 = 18(\cos 65° + i\sin 65°)(\cos 125° + i\sin 125°)$$

Multiply the cosine terms and sine terms (add the arguments).

$$z_1z_2 = 18[\cos(65° + 125°) + i^2\sin(65° + 125°)]$$

Simplify ($i^2 = -1$).

$$z_1z_2 = 18(\cos 190° - \sin 190°)$$

This is incorrect. What mistake was made?

63. Find $(\sqrt{2} + i\sqrt{2})^6$.

Solution:

Raise each term to the sixth power. $(\sqrt{2})^6 + i^6(\sqrt{2})^6$

Simplify. $8 + 8i^6$

Let $i^6 = i^4 \cdot i^2 = -1$. $8 - 8 = 0$

This is incorrect. What mistake was made?

64. Find all complex solutions to $x^5 - 1 = 0$.

Solution:

Add 1 to both sides. $x^5 = 1$

Raise both sides to the fifth power. $x = 1^{1/5}$

Simplify. $x = 1$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 65–70, determine whether the statement is true or false.

65. The product of two complex numbers is a complex number.

66. The quotient of two complex numbers is a complex number.

67. There are always n distinct real solutions of the equation $x^n - a = 0$, where a is not zero.

68. There are always n distinct complex solutions of the equation $x^n - a = 0$, where a is not zero.

69. There are n distinct complex zeros of $\frac{1}{a + bi}$, where a and b are positive real numbers.

70. There exists a complex number for which there is no complex square root.

71. The distance between any consecutive pair of the n complex roots of a number is a constant.

72. If $2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$ is one of the n complex roots of a number, then n is even.

CHALLENGE

In Exercises 73–76, use the following identity:

In calculus you will see an identity called Euler's formula or identity, $e^{i\theta} = \cos\theta + i\sin\theta$. Notice that when $\theta = \pi$, the identity reduces to $e^{i\pi} + 1 = 0$, which is a beautiful identity in that it relates the five fundamental numbers (e, π, 1, i, and 0) and the fundamental operations (multiplication, addition, exponents, and equality) in mathematics.

73. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1e^{i\theta_1}$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2e^{i\theta_2}$ be two complex numbers. Use the properties of exponentials to show that $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$.

74. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1e^{i\theta_1}$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2e^{i\theta_2}$ be two complex numbers. Use the properties of exponentials to show that $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$.

75. Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Use the properties of exponents to show that $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$.

76. Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. Use the properties of exponents to show that
$$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right].$$

77. Use De Moivre's theorem to prove the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

78. Use De Moivre's theorem to derive an expression for $\sin(3\theta)$.

79. Use De Moivre's theorem to derive an expression for $\cos(3\theta)$.

80. Calculate $\dfrac{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{14}}{\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{20}}$.

81. Calculate $(1 - i)^n \cdot (1 + i)^m$, where n and m are positive integers.

82. Calculate $\dfrac{(1 + i)^n}{(1 - i)^m}$, where n and m are positive integers.

TECHNOLOGY

For Exercises 83–88, refer to the following:

According to the nth root theorem, the first of the nth roots of the complex number $z = r(\cos\theta + i\sin\theta)$ is given by

$$w_1 = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)\right], \text{ with } \theta \text{ in radians}$$

$$\text{or } w_1 = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{360°}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{360°}{n}\right)\right],$$

with θ in degrees.

Using the graphing calculator to plot the n roots of a complex number z, enter $r_1 = r$, $\theta\min = \dfrac{\theta}{n}$, $\theta\max = 2\pi + \dfrac{\theta}{n}$ or $360° + \dfrac{\theta}{n}$, $\theta\,\text{step} = \dfrac{2\pi}{n}$ or $\dfrac{360°}{n}$, xmin $= -r$, xmax $= r$, ymin $= -r$, ymax $= r$, and $\boxed{\text{MODE}}$ in radians or degrees.

83. Find the fifth roots of $\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$, and plot the roots with a calculator.

84. Find the fourth roots of $-\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}i$, and plot the roots with a calculator.

85. Find the sixth roots of $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$, and draw the complex hexagon with a calculator.

86. Find the fifth roots of $-4 + 4i$, and draw the complex pentagon with a calculator.

87. Find the cube roots of $\dfrac{27\sqrt{2}}{2} + \dfrac{27\sqrt{2}}{2}i$, and draw the complex triangle with a calculator.

88. Find the fifth roots of $8\sqrt{2}(\sqrt{3} - 1) + 8\sqrt{2}(\sqrt{3} - 1)i$, and draw the complex triangle with a calculator.

PREVIEW TO CALCULUS

In advanced calculus, complex numbers in polar form are used extensively. Use De Moivre's formula to show that

89. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

90. $\sin(2\theta) = 2\sin\theta\cos\theta$

91. $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$

92. $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

SECTION 7.5 POLAR COORDINATES AND GRAPHS OF POLAR EQUATIONS

SKILLS OBJECTIVES

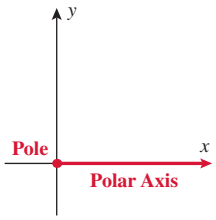
- Plot points in the polar coordinate system.
- Convert between rectangular and polar coordinates.
- Convert equations between polar form and rectangular form.
- Graph polar equations.

CONCEPTUAL OBJECTIVES

- Relate the rectangular coordinate system to the polar coordinate system.
- Classify common shapes that arise from plotting certain types of polar equations.

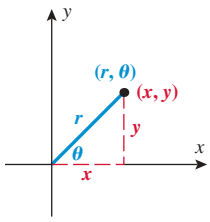
We have discussed the rectangular and the trigonometric (polar) form of complex numbers in the complex plane. We now turn our attention back to the familiar Cartesian plane, where the horizontal axis represents the x-variable, the vertical axis represents the y-variable, and points in this plane represent pairs of real numbers. It is often convenient to instead represent real-number plots in the *polar coordinate system*.

Polar Coordinates

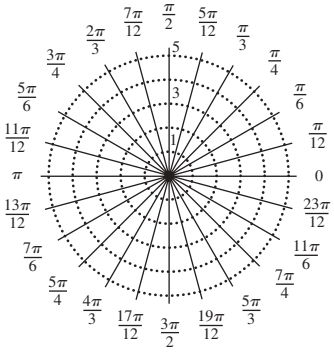


The **polar coordinate system** is anchored by a point, called the **pole** (taken to be the **origin**), and a ray with a vertex at the pole, called the **polar axis**. The polar axis is normally shown where we expect to find the positive x-axis in Cartesian coordinates.

If you align the pole with the origin on the rectangular graph and the polar axis with the positive x-axis, you can label a point either with rectangular coordinates (x, y) or with an ordered pair (r, θ) in **polar coordinates**.



Typically, polar graph paper is used that gives the angles and radii. The graph below gives the angles in radians (the angle also can be given in degrees) and shows the radii from 0 through 5.



When plotting points in the polar coordinate system, $|r|$ represents the distance from the origin to the point. The following procedure guides us in plotting points in the polar coordinate system.

POINT-PLOTTING POLAR COORDINATES

To plot a point (r, θ):

1. Start on the polar axis and rotate the terminal side of an angle to the value θ.
2. If $r > 0$, the point is r units from the origin in the *same direction* of the terminal side of θ.
3. If $r < 0$, the point is $|r|$ units from the origin in the *opposite direction* of the terminal side of θ.

EXAMPLE 1 Plotting Points in the Polar Coordinate System

Plot the points in a polar coordinate system.

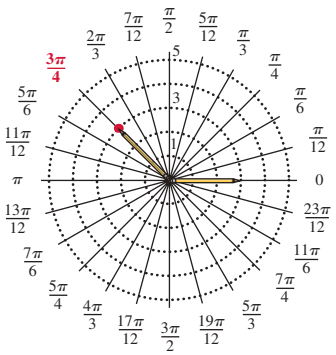
a. $\left(3, \dfrac{3\pi}{4}\right)$ **b.** $(-2, 60°)$

Solution (a):

Start by placing a pencil along the polar axis.

Rotate the pencil to the angle $\dfrac{3\pi}{4}$.

Go out (in the direction of the pencil) three units.

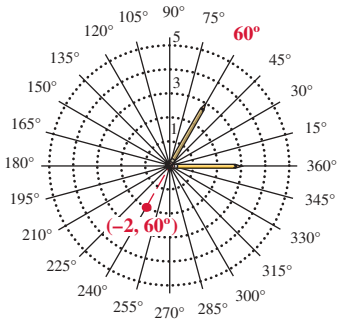


Solution (b):

Start by placing a pencil along the polar axis.

Rotate the pencil to the angle 60°.

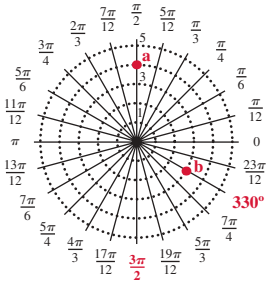
Go out (opposite the direction of the pencil) two units.



■ **YOUR TURN** Plot the points in the polar coordinate system.

a. $\left(-4, \dfrac{3\pi}{2}\right)$ **b.** $(3, 330°)$

■ **Answer:**

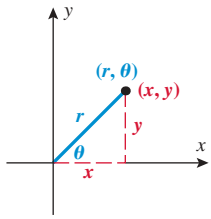


In polar form it is important to note that (r, θ), the name of the point, is not unique, whereas in rectangular form (x, y) it is unique. For example, $(2, 30°) = (-2, 210°)$.

Converting Between Polar and Rectangular Coordinates

The relationships between polar and rectangular coordinates are the familiar ones:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0) \qquad r^2 = x^2 + y^2$$



Converting Between Polar and Rectangular Coordinates

From	To	Identities
Polar (r, θ)	Rectangular (x, y)	$x = r\cos\theta \qquad y = r\sin\theta$
Rectangular (x, y)	Polar (r, θ)	$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$ Make sure that θ is in the correct quadrant.

EXAMPLE 2 Converting Between Polar and Rectangular Coordinates

a. Convert $(-1, \sqrt{3})$ to polar coordinates.

b. Convert $(6\sqrt{2}, 135°)$ to rectangular coordinates.

Solution (a): $(-1, \sqrt{3})$ lies in quadrant II.

Identify x and y. $\quad x = -1, \quad y = \sqrt{3}$

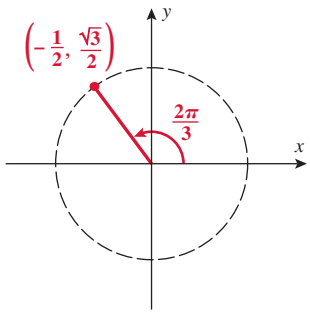
Find r. $\quad r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

Find θ. $\quad \tan\theta = \frac{\sqrt{3}}{-1} \quad$ (θ lies in quadrant II)

Identify θ from the unit circle. $\quad \theta = \frac{2\pi}{3}$

Write the point in polar coordinates. $\quad \boxed{\left(2, \frac{2\pi}{3}\right)}$

Note: Other polar coordinates like $\left(2, -\frac{4\pi}{3}\right)$ and $\left(-2, \frac{5\pi}{3}\right)$ also correspond to the point $(-1, \sqrt{3})$.



Solution (b): $(6\sqrt{2}, 135°)$ lies in quadrant II.

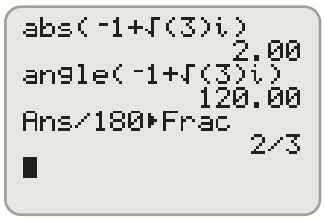
Identify r and θ. $\quad r = 6\sqrt{2} \qquad \theta = 135°$

Find x. $\quad x = r\cos\theta = 6\sqrt{2}\cos 135° = 6\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = -6$

Find y. $\quad y = r\sin\theta = 6\sqrt{2}\sin 135° = 6\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 6$

Write the point in rectangular coordinates. $\quad \boxed{(-6, 6)}$

Technology Tip



The solution is $(2, \frac{2}{3}\pi)$.

Graphs of Polar Equations

We are familiar with equations in rectangular form such as

$$y = 3x + 5 \quad \text{(line)} \qquad y = x^2 + 2 \quad \text{(parabola)} \qquad x^2 + y^2 = 9 \quad \text{(circle)}$$

We now discuss equations in polar form (known as **polar equations**) such as

$$r = 5\theta \qquad r = 2\cos\theta \qquad r = \sin(5\theta)$$

which you will learn to recognize in this section as typical equations whose plots are some general shapes.

Our first example deals with two of the simplest forms of polar equations: when r or θ is constant. The results are a circle centered at the origin and a line that passes through the origin, respectively.

EXAMPLE 3 Graphing a Polar Equation of the Form r = Constant or θ = Constant

Graph the polar equations.

a. $r = 3$ **b.** $\theta = \dfrac{\pi}{4}$

Solution (a): Constant value of r

Approach 1 (polar coordinates): $r = 3$ (θ can take on any value)

Plot points for arbitrary θ and $r = 3$.

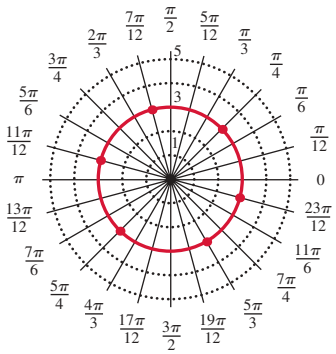
Connect the points; a circle with radius 3.

Approach 2 (rectangular coordinates): $r = 3$

Square both sides. $r^2 = 9$

Remember that in rectangular coordinates $r^2 = x^2 + y^2$. $x^2 + y^2 = 3^2$

This is a circle, centered at the origin, with radius 3.



Solution (b): Constant value of θ

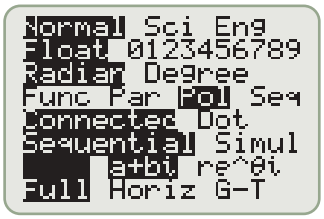
Approach 1: $\theta = \dfrac{\pi}{4}$ (r can take on any value, positive or negative.)

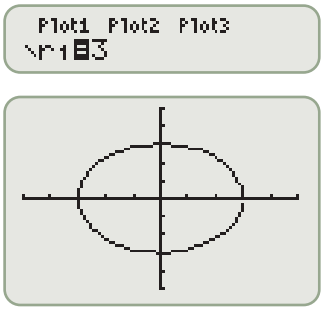
Plot points for $\theta = \dfrac{\pi}{4}$ at several arbitrary values of r.

Connect the points. The result is a line passing through the origin with slope $= 1 \left[m = \tan\left(\dfrac{\pi}{4}\right)\right]$.

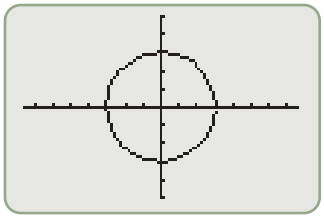
Technology Tip

Graph the polar equation $r = 3$. To use the calculator, set it in Radian mode and Pol mode.





With the window setting, the circle is displayed as an oval. To display the circle properly, press ZOOM ▼ 5:ZSquare ENTER.



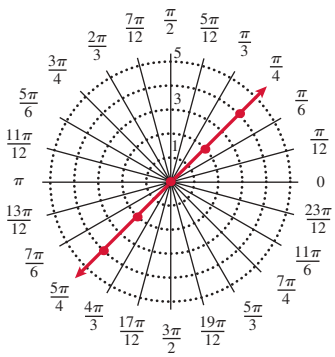
Approach 2: $\theta = \dfrac{\pi}{4}$

Take the tangent of both sides. $\tan\theta = \underbrace{\tan\left(\dfrac{\pi}{4}\right)}_{1}$

Use the identity $\tan\theta = \dfrac{y}{x}$. $\dfrac{y}{x} = 1$

Multiply by x. $y = x$

The result is a line passing through the origin with slope = 1.



Rectangular equations that depend on varying (not constant) values of x or y can be graphed by point-plotting (making a table and plotting the points). We will use this same procedure for graphing polar equations that depend on varying (not constant) values of r or θ.

EXAMPLE 4 Graphing a Polar Equation of the Form $r = c \cdot \cos\theta$ or $r = c \cdot \sin\theta$

Graph $r = 4\cos\theta$.

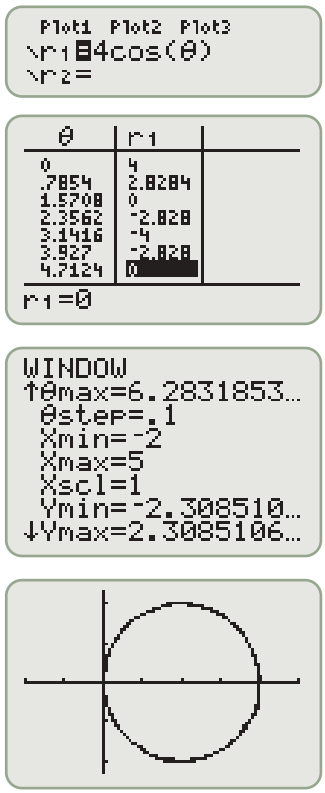
Solution:

STEP 1 Make a table and find several key values.

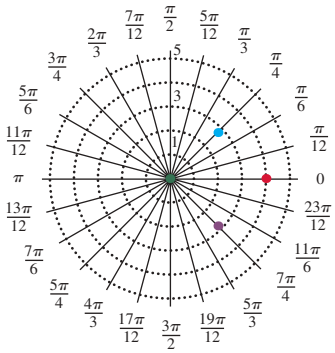
θ	$r = 4\cos\theta$	(r, θ)
0	$4(1) = 4$	$(4, 0)$
$\dfrac{\pi}{4}$	$4\left(\dfrac{\sqrt{2}}{2}\right) \approx 2.8$	$\left(2.8, \dfrac{\pi}{4}\right)$
$\dfrac{\pi}{2}$	$4(0) = 0$	$\left(0, \dfrac{\pi}{2}\right)$
$\dfrac{3\pi}{4}$	$4\left(-\dfrac{\sqrt{2}}{2}\right) \approx -2.8$	$\left(-2.8, \dfrac{3\pi}{4}\right)$
π	$4(-1) = -4$	$(-4, \pi)$
$\dfrac{5\pi}{4}$	$4\left(-\dfrac{\sqrt{2}}{2}\right) \approx -2.8$	$\left(-2.8, \dfrac{5\pi}{4}\right)$
$\dfrac{3\pi}{2}$	$4(0) = 0$	$\left(0, \dfrac{3\pi}{2}\right)$
$\dfrac{7\pi}{4}$	$4\left(\dfrac{\sqrt{2}}{2}\right) \approx 2.8$	$\left(2.8, \dfrac{7\pi}{4}\right)$
2π	$4(1) = 4$	$(4, 2\pi)$

Technology Tip

Graph $r = 4\cos\theta$.

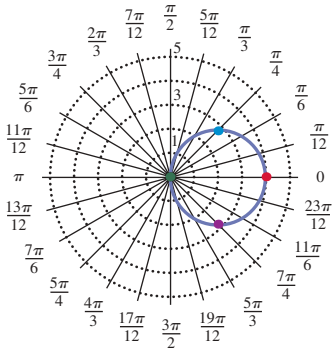


STEP 2 Plot the points in polar coordinates.

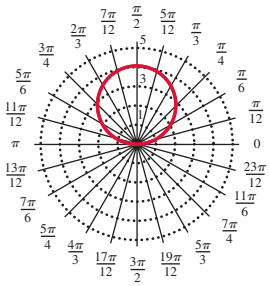


STEP 3 Connect the points with a smooth curve.

Notice that **(4, 0)** and **(−4, π)** correspond to the same point. There is no need to continue with angles beyond π, because the result would be to go around the same circle again.



■ **Answer:**



■ **YOUR TURN** Graph $r = 4\sin\theta$.

Study Tip

Graphs of $r = a\sin\theta$ and $r = a\cos\theta$ are circles.

Compare the result of Example 4, the graph of $r = 4\cos\theta$, with the result of the Your Turn, the graph of $r = 4\sin\theta$. Notice that they are 90° out of phase (we simply rotate one graph 90° about the pole to get the other graph).

In general, graphs of polar equations of the form $r = a\sin\theta$ and $r = a\cos\theta$ are circles.

WORDS	MATH	
Start with the polar form.	$r = a\sin\theta$	$r = a\cos\theta$
Apply trigonometric ratios: $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$.	$r = a\frac{y}{r}$	$r = a\frac{x}{r}$
Multiply the equations by r.	$r^2 = ay$	$r^2 = ax$
Let $r^2 = x^2 + y^2$.	$x^2 + y^2 = ay$	$x^2 + y^2 = ax$
Group x terms together and y terms together.	$x^2 + (y^2 - ay) = 0$	$(x^2 - ax) + y^2 = 0$
	$x^2 + \left[y^2 - ay + \left(\frac{a}{2}\right)^2\right] = \left(\frac{a}{2}\right)^2$	$\left[x^2 - ax + \left(\frac{a}{2}\right)^2\right] + y^2 = \left(\frac{a}{2}\right)^2$
Complete the square on the expressions in parentheses.	$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2$	$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$
Identify the center and radius.	Center: $\left(0, \frac{a}{2}\right)$ Radius: $\frac{a}{2}$	Center: $\left(\frac{a}{2}, 0\right)$ Radius: $\frac{a}{2}$

EXAMPLE 5 Graphing a Polar Equation of the Form $r = c \cdot \sin(2\theta)$ or $r = c \cdot \cos(2\theta)$

Graph $r = 5\sin(2\theta)$.

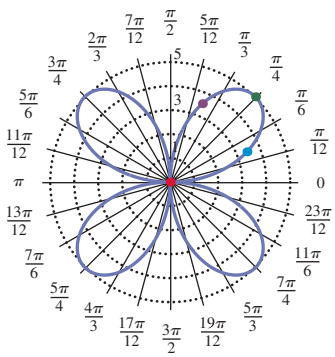
Solution:

STEP 1 Make a table and find key values. Since the argument of the sine function is doubled, the period is halved. Therefore, instead of steps of $\frac{\pi}{4}$, take steps of $\frac{\pi}{8}$.

θ	$r = 5\sin(2\theta)$	(r, θ)
0	$5(0) = 0$	$(0, 0)$
$\frac{\pi}{8}$	$5\left(\frac{\sqrt{2}}{2}\right) \approx 3.5$	$\left(3.5, \frac{\pi}{8}\right)$
$\frac{\pi}{4}$	$5(1) = 5$	$\left(5, \frac{\pi}{4}\right)$
$\frac{3\pi}{8}$	$5\left(\frac{\sqrt{2}}{2}\right) \approx 3.5$	$\left(3.5, \frac{3\pi}{8}\right)$
$\frac{\pi}{2}$	$5(0) = 0$	$\left(0, \frac{\pi}{2}\right)$

STEP 2 Label the polar coordinates.

The values in the table represent what happens in quadrant I. The same pattern repeats in the other three quadrants. The result is a **four-leaved rose**.

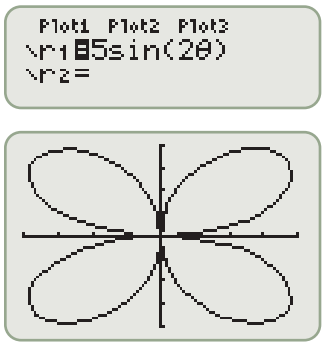


STEP 3 Connect the points with smooth curves.

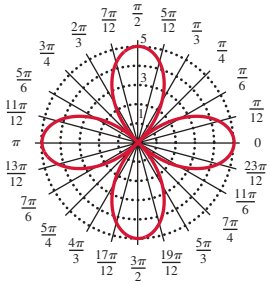
■ **YOUR TURN** Graph $r = 5\cos(2\theta)$.

Technology Tip

Graph $r = 5\sin(2\theta)$.



■ **Answer:**



Compare the result of Example 5, the graph of $r = 5\sin(2\theta)$, with the result of the Your Turn, the graph of $r = 5\cos(2\theta)$. Notice that they are 45° out of phase (we rotate one graph 45° about the pole to get the other graph).

In general, for $r = a\sin(n\theta)$ or $r = a\cos(n\theta)$, the graph has n leaves (petals) if n is odd and $2n$ leaves (petals) if n is even. As a increases, the leaves (petals) get longer.

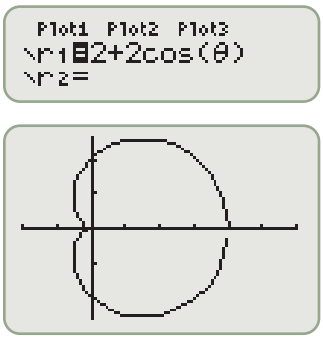
The next class of graphs are called **limaçons**, which have equations of the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. When $a = b$, the result is a **cardioid** (heart shape).

Study Tip

Graphs of $r = a\sin(n\theta)$ and $r = a\cos(n\theta)$ are roses with n leaves if n is odd and $2n$ leaves if n is even.

Technology Tip

Graph $r = 2 + 2\cos\theta$.



EXAMPLE 6 The Cardioid as a Polar Equation

Graph $r = 2 + 2\cos\theta$.

Solution:

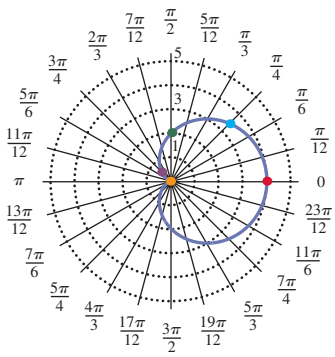
STEP 1 Make a table and find key values.

This behavior repeats in quadrant III and quadrant IV, because the cosine function has corresponding values in quadrant I and quadrant IV and in quadrant II and quadrant III.

θ	$r = 2 + 2\cos\theta$	(r, θ)
0	$2 + 2(1) = 4$	$(4, 0)$
$\frac{\pi}{4}$	$2 + 2\left(\frac{\sqrt{2}}{2}\right) = 3.4$	$\left(3.4, \frac{\pi}{4}\right)$
$\frac{\pi}{2}$	$2 + 2(0) = 2$	$\left(2, \frac{\pi}{2}\right)$
$\frac{3\pi}{4}$	$2 + 2\left(-\frac{\sqrt{2}}{2}\right) \approx 0.6$	$\left(0.6, \frac{3\pi}{4}\right)$
π	$2 + 2(-1) = 0$	$(0, \pi)$

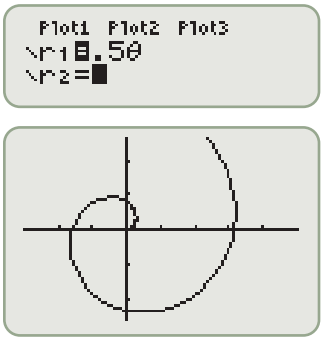
STEP 2 Plot the points in polar coordinates.

STEP 3 Connect the points with a smooth curve. The curve is a *cardioid*, a term formed from Greek roots meaning "heart-shaped."



Technology Tip

Graph $r = 0.5\theta$.



EXAMPLE 7 Graphing a Polar Equation of the Form $r = c \cdot \theta$

Graph $r = 0.5\theta$.

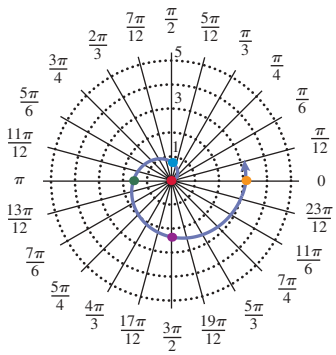
Solution:

STEP 1 Make a table and find key values.

θ	$r = 0.5\theta$	(r, θ)
0	$0.5(0) = 0$	$(0, 0)$
$\frac{\pi}{2}$	$0.5\left(\frac{\pi}{2}\right) = 0.8$	$\left(0.8, \frac{\pi}{2}\right)$
π	$0.5(\pi) = 1.6$	$(1.6, \pi)$
$\frac{3\pi}{2}$	$0.5\left(\frac{3\pi}{2}\right) = 2.4$	$\left(2.4, \frac{3\pi}{2}\right)$
2π	$0.5(2\pi) = 3.1$	$(3.1, 2\pi)$

STEP 2 Plot the points in polar coordinates.

STEP 3 Connect the points with a smooth curve. The curve is a *spiral*.



EXAMPLE 8 Graphing a Polar Equation of the Form $r^2 = c \cdot \sin(2\theta)$ or $r^2 = c \cdot \cos(2\theta)$

Graph $r^2 = 4\cos(2\theta)$.

Solution:

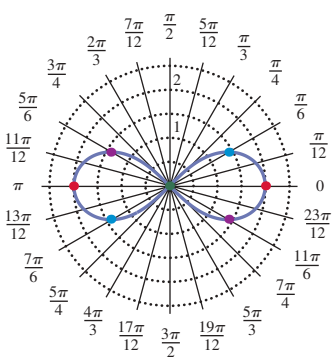
STEP 1 Make a table and find key values.

Solving for r yields $r = \pm 2\sqrt{\cos(2\theta)}$. All coordinates $(-r, \theta)$ can be expressed as $(r, \theta + \pi)$. The following table does not have values for $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, because the corresponding values of $\cos(2\theta)$ are negative and hence r is an imaginary number. The table also does not have values for $\theta > \pi$, because $2\theta > 2\pi$ and the corresponding points are repeated.

θ	$\cos(2\theta)$	$r = \pm 2\sqrt{\cos(2\theta)}$	(r, θ)
0	1	$r = \pm 2$	$(2, 0)$ and $(-2, 0) = (2, \pi)$
$\frac{\pi}{6}$	0.5	$r = \pm 1.4$	$\left(1.4, \frac{\pi}{6}\right)$ and $\left(-1.4, \frac{\pi}{6}\right) = \left(1.4, \frac{7\pi}{6}\right)$
$\frac{\pi}{4}$	0	$r = 0$	$\left(0, \frac{\pi}{4}\right)$
$\frac{3\pi}{4}$	0	$r = 0$	$\left(0, \frac{3\pi}{4}\right)$
$\frac{5\pi}{6}$	0.5	$r = \pm 1.4$	$\left(1.4, \frac{5\pi}{6}\right)$ and $\left(-1.4, \frac{5\pi}{6}\right) = \left(1.4, \frac{11\pi}{6}\right)$
π	1	$r = \pm 2$	$(2, \pi)$ and $(-2, \pi) = (2, 2\pi)$

STEP 2 Plot the points in polar coordinates.

STEP 3 Connect the points with a smooth curve. The resulting curve is known as a *lemniscate*.

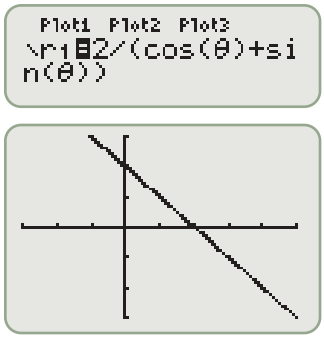


Converting Equations Between Polar and Rectangular Form

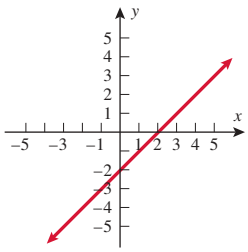
It is not always advantageous to plot an equation in the form in which it is given. It is sometimes easier to first convert to rectangular form and then plot. For example, to plot $r = \dfrac{2}{\cos\theta + \sin\theta}$, we could make a table of values. However, as you will see in Example 9, it is much easier to convert this equation to rectangular coordinates.

Technology Tip

Graph $r = \dfrac{2}{\cos\theta + \sin\theta}$.



■ **Answer:** $y = x - 2$



EXAMPLE 9 Converting an Equation from Polar Form to Rectangular Form

Graph $r = \dfrac{2}{\cos\theta + \sin\theta}$.

Solution:

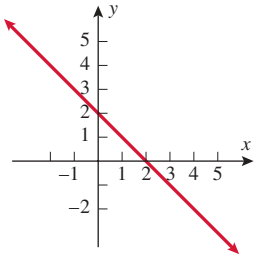
Multiply the equation by $\cos\theta + \sin\theta$. $\quad r(\cos\theta + \sin\theta) = 2$

Eliminate parentheses. $\quad r\cos\theta + r\sin\theta = 2$

Convert the result to rectangular form. $\quad \underbrace{r\cos\theta}_{x} + \underbrace{r\sin\theta}_{y} = 2$

Simplify. The result is a straight line. $\quad \boxed{y = -x + 2}$

Graph the line.



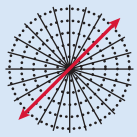
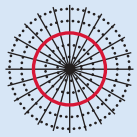
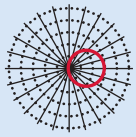
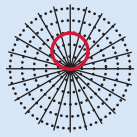
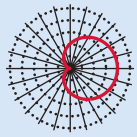
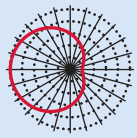
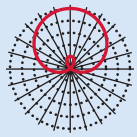
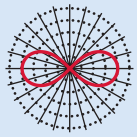
■ **YOUR TURN** Graph $r = \dfrac{2}{\cos\theta - \sin\theta}$.

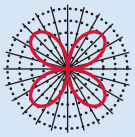
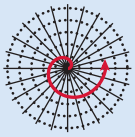
SECTION 7.5 SUMMARY

Graph polar coordinates (r, θ) in the polar coordinate system first by rotating a ray to get the terminal side of the angle. Then if r is positive, go out r units from the origin in the direction of the terminal side. If r is negative, go out $|r|$ units in the opposite direction of the terminal side. Conversions between polar and rectangular forms are given by

FROM	TO	IDENTITIES
Polar (r, θ)	Rectangular (x, y)	$x = r\cos\theta \qquad y = r\sin\theta$
Rectangular (x, y)	Polar (r, θ)	$r = \sqrt{x^2 + y^2} \qquad \tan\theta = \dfrac{y}{x},\ x \neq 0$ Be careful to note the proper quadrant for θ.

We can graph polar equations by point-plotting. Common shapes that arise are given in the following table. Sine and cosine curves have the same shapes (just rotated). If more than one equation is given, then the top equation corresponds to the actual graph. In this table, a and b are assumed to be positive.

Classification	Description	Polar Equations	Graph
Line	Radial line	$\theta = a$	
Circle	Circle centered at the origin	$r = a$	
Circle	Circle that touches the pole and whose center is on the polar axis	$r = a\cos\theta$	
Circle	Circle that touches the pole and whose center is on the line $\theta = \frac{\pi}{2}$	$r = a\sin\theta$	
Limaçon	Cardioid **a = b**	$r = a + b\cos\theta$ $r = a + b\sin\theta$	
Limaçon	Without inner loop $a > b$	$r = -a - b\cos\theta$ $r = a + b\sin\theta$	
Limaçon	With inner loop $a < b$	$r = a + b\sin\theta$ $r = a + b\cos\theta$	
Lemniscate		$r^2 = a^2\cos(2\theta)$ $r^2 = a^2\sin(2\theta)$	
Rose	Three* rose petals	$r = a\sin(3\theta)$ $r = a\cos(3\theta)$	

CLASSIFICATION	DESCRIPTION	POLAR EQUATIONS	GRAPH
Rose	Four*/rose petals	$r = a\sin(2\theta)$ $r = a\cos(2\theta)$	
Spiral		$r = a\theta$	

*In the argument $n\theta$, if n is odd, there are n petals (leaves), and if n is even, there are $2n$ petals (leaves).

SECTION 7.5 EXERCISES

SKILLS

In Exercises 1–10, plot each indicated point in a polar coordinate system.

1. $\left(3, \frac{5\pi}{6}\right)$ **2.** $\left(2, \frac{5\pi}{4}\right)$ **3.** $\left(4, \frac{11\pi}{6}\right)$ **4.** $\left(1, \frac{2\pi}{3}\right)$ **5.** $\left(-2, \frac{\pi}{6}\right)$

6. $\left(-4, \frac{7\pi}{4}\right)$ **7.** $(-4, 270°)$ **8.** $(3, 135°)$ **9.** $(4, 225°)$ **10.** $(-2, 60°)$

In Exercises 11–20, convert each point to exact polar coordinates. Assume that $0 \le \theta < 2\pi$.

11. $(2, 2\sqrt{3})$ **12.** $(3, -3)$ **13.** $(-1, -\sqrt{3})$ **14.** $(6, 6\sqrt{3})$ **15.** $(-4, 4)$

16. $(0, \sqrt{2})$ **17.** $(3, 0)$ **18.** $(-7, -7)$ **19.** $(-\sqrt{3}, -1)$ **20.** $(2\sqrt{3}, -2)$

In Exercises 21–30, convert each point to exact rectangular coordinates.

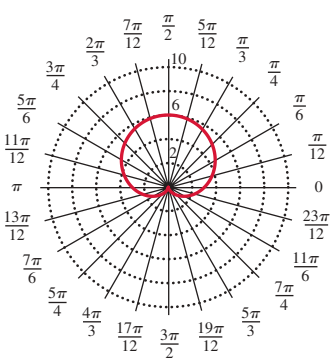
21. $\left(4, \frac{5\pi}{3}\right)$ **22.** $\left(2, \frac{3\pi}{4}\right)$ **23.** $\left(-1, \frac{5\pi}{6}\right)$ **24.** $\left(-2, \frac{7\pi}{4}\right)$ **25.** $\left(0, \frac{11\pi}{6}\right)$

26. $(6, 0)$ **27.** $(2, 240°)$ **28.** $(-3, 150°)$ **29.** $(-1, 135°)$ **30.** $(5, 315°)$

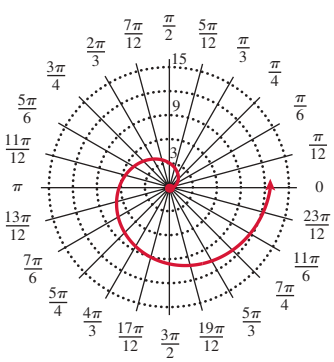
In Exercises 31–34, match the polar graphs with their corresponding equations.

31. $r = 4\cos\theta$ **32.** $r = 2\theta$ **33.** $r = 3 + 3\sin\theta$ **34.** $r = 3\sin(2\theta)$

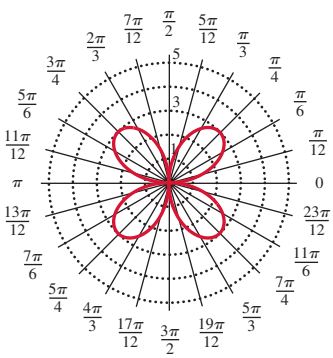
a.



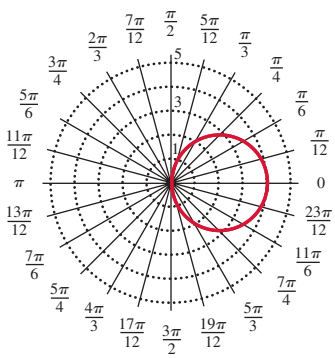
b.



c.



d.



In Exercises 35–50, graph each equation.

35. $r = 5$

36. $\theta = -\dfrac{\pi}{3}$

37. $r = 2\cos\theta$

38. $r = 3\sin\theta$

39. $r = 4\sin(2\theta)$

40. $r = 5\cos(2\theta)$

41. $r = 3\sin(3\theta)$

42. $r = 4\cos(3\theta)$

43. $r^2 = 9\cos(2\theta)$

44. $r^2 = 16\sin(2\theta)$

45. $r = -2\cos\theta$

46. $r = -3\sin(3\theta)$

47. $r = 4\theta$

48. $r = -2\theta$

49. $r = -3 + 2\cos\theta$

50. $r = 2 + 3\sin\theta$

In Exercises 51–54, convert the equation from polar to rectangular form. Identify the resulting equation as a line, parabola, or circle.

51. $r(\sin\theta + 2\cos\theta) = 1$

52. $r(\sin\theta - 3\cos\theta) = 2$

53. $r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta = 8$

54. $r^2\cos^2\theta - r\sin\theta = -2$

In Exercises 55–60, graph the polar equation.

55. $r = -\frac{1}{3}\theta$

56. $r = \frac{1}{4}\theta$

57. $r = 4\sin(5\theta)$

58. $r = -3\cos(4\theta)$

59. $r = -2 - 3\cos\theta$

60. $r = 4 - 3\sin\theta$

APPLICATIONS

61. Halley's Comet. Halley's comet travels an elliptical path that can be modeled with the polar equation $r = \dfrac{0.587(1 + 0.967)}{1 - 0.967\cos\theta}$. Sketch the graph of the path of Halley's comet.

62. Dwarf Planet Pluto. The planet Pluto travels in an elliptical orbit that can be modeled with the polar equation $r = \dfrac{29.62(1 + 0.249)}{1 - 0.249\cos\theta}$. Sketch the graph of Pluto's orbit.

For Exercises 63 and 64, refer to the following:

Spirals are seen in nature, as in the swirl of a pine cone; they are also used in machinery to convert motions. An Archimedes spiral has the general equation $r = a\theta$. A more general form for the equation of a spiral is $r = a\theta^{1/n}$, where n is a constant that determines how tightly the spiral is wrapped.

63. Archimedes Spiral. Compare the Archimedes spiral $r = \theta$ with the spiral $r = \theta^{1/2}$ by graphing both on the same polar graph.

64. Archimedes Spiral. Compare the Archimedes spiral $r = \theta$ with the spiral $r = \theta^{4/3}$ by graphing both on the same polar graph.

For Exercises 65 and 66, refer to the following:

The *lemniscate motion* occurs naturally in the flapping of birds' wings. The bird's vertical lift and wing sweep create the distinctive figure-eight pattern. The patterns vary with the different wing profiles.

65. Flapping Wings of Birds. Compare the following two possible lemniscate patterns by graphing them on the same polar graph: $r^2 = 4\cos(2\theta)$ and $r^2 = \frac{1}{4}\cos(2\theta)$.

66. Flapping Wings of Birds. Compare the following two possible lemniscate patterns by graphing them on the same polar graph: $r^2 = 4\cos(2\theta)$ and $r^2 = 4\cos(2\theta + 2)$.

For Exercises 67 and 68, refer to the following:

Many microphone manufacturers advertise that their microphones' exceptional pickup capabilities isolate the sound source and minimize background noise. These microphones are described as cardioid microphones because of the pattern formed by the range of the pickup.

67. Cardioid Pickup Pattern. Graph the cardioid curve $r = 2 + 2\sin\theta$ to see what the range looks like.

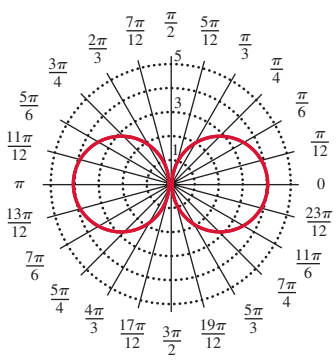
68. Cardioid Pickup Pattern. Graph the cardioid curve $r = -4 - 4\sin\theta$ to see what the range looks like.

For Exercises 69 and 70, refer to the following:

The sword artistry of the Samurai is legendary in Japanese folklore and myth. The elegance with which a samurai could wield a sword rivals the grace exhibited by modern figure skaters. In more modern times, such legends have been rendered digitally in many different video games (e.g., *Onimusha*). In order to make the characters realistically move across the screen, and in particular, wield various sword motions true to the legends, trigonometric functions are extensively used in constructing the underlying graphics module. One famous movement is a figure eight, swept out with two hands on the sword. The actual path of the tip of the blade as the movement progresses in this figure-eight motion depends essentially on the length L of the sword and the speed with which it is swept out. Such a path is modeled using a polar equation of the form

$$r^2\theta = L\cos(A\theta) \text{ or } r^2\theta = L\sin(A\theta),\ \theta_1 \le \theta \le \theta_2$$

whose graphs are called *lemniscates*.



69. Video Games. Graph the following equations:

a. $r^2\theta = 5\cos\theta, 0 \le \theta \le 2\pi$

b. $r^2\theta = 5\cos(2\theta), 0 \le \theta \le \pi$

c. $r^2\theta = 5\cos(4\theta), 0 \le \theta \le \dfrac{\pi}{2}$

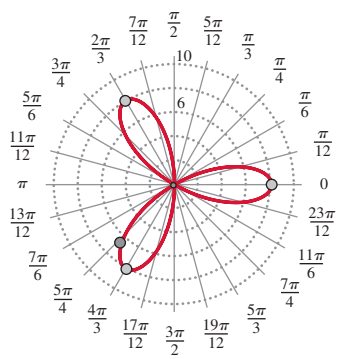
What do you notice about all of these graphs? Suppose that the movement of the tip of the sword in a game is governed by these graphs. Describe what happens if you change the domain in (b) and (c) to $0 \le \theta \le 2\pi$.

70. Video Games. Write a polar equation that would describe the motion of a sword 12 units long that makes 8 complete motions in $[0, 2\pi]$.

71. Home Improvement. The owner of a garden maze has decided to replace the square central plot of the maze, which is 60 feet $\times$ 60 feet, with a section comprised of a spiral to make the participants literally feel as though they are going in circles as they reach the tall slide in the center of the maze (that leads to the exit). Assume that the center of the maze is at the origin. If the walkway must be 3 feet wide to accommodate a participant, and the bushes on either side of the walkway are 1.5 feet thick, how many times can the spiral wrap around before the center is reached? Graph the resulting spiral.

72. Home Improvement. Consider the garden maze in Exercise 71, but now suppose that the owner wants the spiral to wrap around exactly 3 times. How wide can the walkway then be throughout the spiral? Graph it.

73. Magnetic Pendulum. A magnetic bob is affixed to an arm of length L, which is fastened to a pivot point. Three magnets of equal strength are positioned on a plane 8 inches from the center; one is placed on the x-axis, one at 120° with respect to the positive x-axis, and the other at 240° with respect to the positive x-axis. The path swept out is a three-petal rose, as shown below:



a. Find the equation of this path.

b. How many times does the path retrace itself on the interval $[0, 100\pi]$?

74. Magnetic Pendulum. In reference to the context of Exercise 73, now position 8 magnets, each 8 units from the origin and at the vertices of a regular octagon, one being on the x-axis. Assume that the path of the pendulum is an eight-petal rose.

a. Find the equation of this path.

b. Graph this equation.

CATCH THE MISTAKE

In Exercises 75 and 76, explain the mistake that is made.

75. Convert $(-2, -2)$ to polar coordinates.

Solution:

Label x and y. $x = -2, y = -2$

Find r. $r = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

Find θ. $\tan\theta = \dfrac{-2}{-2} = 1$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4}$$

Write the point in polar coordinates. $\left(2\sqrt{2}, \dfrac{\pi}{4}\right)$

This is incorrect. What mistake was made?

76. Convert $\left(-\sqrt{3}, 1\right)$ to polar coordinates.

Solution:

Label x and y. $x = -\sqrt{3}, y = 1$

Find r. $r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = \sqrt{4} = 2$

Find θ. $\tan\theta = \dfrac{1}{-\sqrt{3}} = -\dfrac{1}{\sqrt{3}}$

$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{4}$$

Write the point in polar coordinates. $\left(2, -\dfrac{\pi}{4}\right)$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 77 and 78, determine whether each statement is true or false.

77. All cardioids are limaçons, but not all limaçons are cardioids.

78. All limaçons are cardioids, but not all cardioids are limaçons.

79. Find the polar equation that is equivalent to a vertical line, $x = a$.

80. Find the polar equation that is equivalent to a horizontal line, $y = b$.

81. Give another pair of polar coordinates for the point (a, θ).

82. Convert $(-a, b)$ to polar coordinates. Assume that $a > 0, b > 0$.

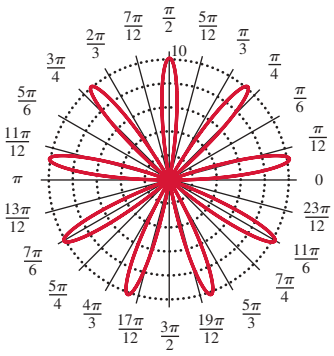
CHALLENGE

83. Determine the values of θ at which $r = 4\cos\theta$ and $r\cos\theta = 1$ intersect. Graph both equations.

84. Find the Cartesian equation for $r = a\sin\theta + b\cos\theta$, where a and b are positive. Identify the type of graph.

85. Find the Cartesian equation for $r = \dfrac{a\sin(2\theta)}{\cos^3\theta - \sin^3\theta}$.

86. Identify an equation for the following graph:



87. Consider the equation $r = 2a\cos(\theta - b)$. Sketch the graph for various values of a and b, and then give a general description of the graph.

88. Consider the equation $r = a\sin(b\theta)$, where $a, b > 0$. Determine the smallest number M for which the graph starts to repeat.

TECHNOLOGY

89. Given $r = \cos\left(\frac{\theta}{2}\right)$, find the θ-intervals for the inner loop above the x-axis.

90. Given $r = 2\cos\left(\frac{3\theta}{2}\right)$, find the θ-intervals for the petal in the first quadrant.

91. Given $r = 1 + 3\cos\theta$, find the θ-intervals for the inner loop.

92. Given $r = 1 + \sin(2\theta)$ and $r = 1 - \cos(2\theta)$, find all points of intersection.

93. Given $r = 2 + \sin(4\theta)$ and $r = 1$, find the angles of all points of intersection.

94. Given $r = 2 - \cos(3\theta)$ and $r = 1.5$, find the angles of all points of intersection.

PREVIEW TO CALCULUS

In calculus, when we need to find the area enclosed by two polar curves, the first step consists of finding the points where the curves coincide.

In Exercises 95–98, find the points of intersection of the given curves.

95. $r = 4\sin\theta$ and $r = 4\cos\theta$

96. $r = \cos\theta$ and $r = 2 + 3\cos\theta$

97. $r = 1 - \sin\theta$ and $r = 1 + \cos\theta$

98. $r = 1 + \sin\theta$ and $r = 1 + \cos\theta$

CHAPTER 7 INQUIRY-BASED LEARNING PROJECT

When it comes to raising complex numbers to integer powers, you can do it the Brut force way (i.e., "the long way") or you can take advantage of De Moivre's theorem (i.e., "the short way").

Let us start with the complex number $z = 1 + i$ (rectangular coordinates) or in polar (trigonometric) form can be written as $z = \sqrt{2}\,(\cos 45° + i \sin 45°) = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right]$

Brut Force

$n = 1$: $(1 + i)^1 =$ _____

$n = 2$: $(1 + i)^2 =$ _____

$n = 3$: $(1 + i)^3 =$ _____

$n = 4$: $(1 + i)^4 =$ _____

De Moivre's Theorem: $z^n = r^n[\cos(n\theta) + i \sin(n\theta)]$

$n = 1$: $z^1 = r[\cos(\theta) + i \sin(\theta)] = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right] =$ _____

$n = 2$: $z^2 = r^2[\cos(2\theta) + i \sin(2\theta)] = (\sqrt{2})^2\left[\cos\left(2 \cdot \frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right] =$ _____

$n = 3$: $z^3 = r^3[\cos(3\theta) + i \sin(3\theta)] = (\sqrt{2})^3\left[\cos\left(3 \cdot \frac{\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)\right] =$ _____

$n = 4$: $z^4 = r^4[\cos(4\theta) + i \sin(4\theta)] = (\sqrt{2})^4\left[\cos\left(4 \cdot \frac{\pi}{4}\right) + i \sin\left(4 \cdot \frac{\pi}{4}\right)\right] =$ _____

At what n would you use De Moivre's theorem?

MODELING OUR WORLD

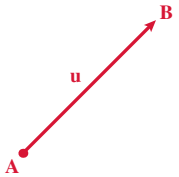
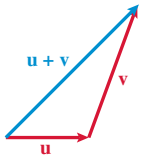
In the summer of 2008, the average price of unleaded gasoline in the United States topped $4 per gallon. This meant that considering hybrid alternative automobiles was no longer just a "green" pursuit for the environmentally conscious, but an attractive financial option. When hybrids were first introduced, gasoline was roughly $3 per gallon and hybrid owners had to drive three to five years (depending on how many miles per year they drove) before their savings at the pump equaled the initial additional cost of the purchase. With gasoline approaching the $4 per gallon price, the rising price of gasoline allows a consumer to make up the initial additional cost of a hybrid automobile in as little as one to three years, depending on driving habits.

The following table illustrates the approximate gross vehicle weight of both a large SUV and a small hybrid and the approximate fuel economy rates in miles per gallon:

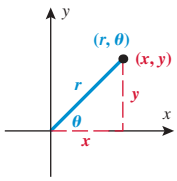
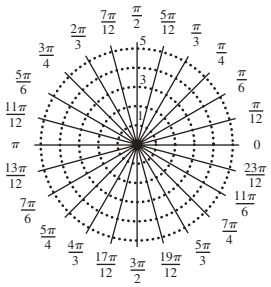
Automobile	Weight	MPG
Ford Expedition	7100 lb	18
Toyota Prius	2800 lb	45

Recall that the amount of work to push an object that weighs F pounds a distance of d feet along a horizontal is $W = \boldsymbol{F \cdot d}$.

1. Calculate how much work it would take to move a Ford Expedition 100 feet.
2. Calculate how much work it would take to move a Toyota Prius 100 feet.
3. Compare the values you calculated in Questions 1 and 2. What is the ratio of work to move the Expedition to work required to move the Prius?
4. Compare the result in Question 3 with the ratio of fuel economy (mpg) for these two vehicles. What can you conclude about the relationship between weight of an automobile and fuel economy?
5. Calculate the work required to move both the Ford Expedition and Toyota Prius 100 feet along an incline that makes a 45° angle with the ground (horizontal).
6. Based on your results in Question 5, do you expect the fuel economy ratios to be the same in this inclined scenario compared with the horizontal? In other words, should consumers in Florida (flat) be guided by the same "numbers" as consumers in the Appalachian Mountains (North Carolina)?

SECTION	CONCEPT	KEY IDEAS/FORMULAS
7.1	**Vectors**	Vector $\mathbf{u}$ or $\overrightarrow{AB}$ 
	Magnitude and direction of vectors	$\lvert\mathbf{u}\rvert = \sqrt{a^2 + b^2}$ $\tan\theta = \dfrac{b}{a}$ Magnitude (length of a vector): $\mathbf{u} = \langle a, b\rangle$ Geometric: tail-to-tip  $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$ $\mathbf{u} + \mathbf{v} = \langle a + c, b + d\rangle$
	Vector operations	Scalar multiplication: $k\langle a, b\rangle = \langle ka, kb\rangle$
	Horizontal and vertical components of a vector	Horizontal component: $a = \lvert\mathbf{u}\rvert\cos\theta$ Vertical component: $b = \lvert\mathbf{u}\rvert\sin\theta$
	Unit vectors	$\mathbf{u} = \dfrac{\mathbf{v}}{\lvert\mathbf{v}\rvert}$
	Resultant vectors	■ Resultant velocities ■ Resultant forces
7.2	**The dot product**	■ The product of a scalar and a vector is a vector. ■ The dot product of two vectors is a scalar.
	The dot product	$\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$ $\mathbf{u}\cdot\mathbf{v} = ac + bd$
	Angle between two vectors	If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, where $0° \le \theta \le 180°$, then $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert}$ Orthogonal (perpendicular) vectors: $\mathbf{u}\cdot\mathbf{v} = 0$
	Work	When force and displacement are in the same direction: $W = \lvert\mathbf{F}\rvert\lvert\mathbf{d}\rvert$. When force and displacement are not in the same direction: $W = \mathbf{F}\cdot\mathbf{d}$.

Section	Concept	Key Ideas/Formulas
7.3	**Polar (trigonometric) form of complex numbers**	
	Complex numbers in rectangular form	The **modulus**, or magnitude, of a complex number $z = x + iy$ is the distance from the origin to the point (x, y) in the complex plane given by $$\lvert z\rvert = \sqrt{x^2 + y^2}$$
	Complex numbers in polar form	The **polar form** of a complex number is $$z = r(\cos\theta + i\sin\theta)$$ where r represents the **modulus** (magnitude) of the complex number and θ represents the **argument** of z. **Converting complex numbers between rectangular and polar forms** Step 1: Plot the point $z = x + yi$ in the complex plane (note the quadrant). Step 2: Find r. Use $r = \sqrt{x^2 + y^2}$. Step 3: Find θ. Use $\tan\theta = \frac{y}{x}$, $x \neq 0$, where θ is in the quadrant found in Step 1.
7.4	**Products, quotients, powers, and roots of complex numbers**	
	Products of complex numbers	Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. The product z_1z_2 is given by $$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$ Multiply the magnitudes and add the arguments.
	Quotients of complex numbers	Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. The quotient $\frac{z_1}{z_2}$ is given by $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$ Divide the magnitudes and subtract the arguments.
	Powers of complex numbers	**De Moivre's theorem** If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$, $n \geq 1$, where n is an integer.
	Roots of complex numbers	The ***n*th roots** of the complex number $z = r(\cos\theta + i\sin\theta)$ are given by $$w_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)\right]$$ θ in degrees, where $k = 0, 1, 2, \ldots, n - 1$.

SECTION	CONCEPT	KEY IDEAS/FORMULAS
7.5	**Polar coordinates and graphs of polar equations**	
	Polar coordinates	To plot a point (r, θ): ■ Start on the polar axis and rotate a ray to form the terminal side of an angle θ. ■ If $r > 0$, the point is r units from the origin in the *same direction* as the terminal side of θ. ■ If $r < 0$, the point is $\lvert r \rvert$ units from the origin in the *opposite direction* of the terminal side of θ. 
	Converting between polar and rectangular coordinates	From polar (r, θ) to rectangular (x, y): $x = r\cos\theta \qquad y = r\sin\theta$ From rectangular (x, y) to polar (r, θ): $r = \sqrt{x^2 + y^2} \qquad \tan\theta = \dfrac{y}{x} \quad x \neq 0$
	Graphs of polar equations	Radial line, circle, spiral, rose petals, lemniscate, and limaçon

CHAPTER REVIEW

7.1 Vectors

Find the magnitude of vector $\overrightarrow{AB}$.

1. $A = (4, -3)$ and $B = (-8, 2)$
2. $A = (-2, 11)$ and $B = (2, 8)$
3. $A = (0, -3)$ and $B = (5, 9)$
4. $A = (3, -11)$ and $B = (9, -3)$

Find the magnitude and direction angle of the given vector.

5. $\mathbf{u} = \langle -10, 24 \rangle$
6. $\mathbf{u} = \langle -5, -12 \rangle$
7. $\mathbf{u} = \langle 16, -12 \rangle$
8. $\mathbf{u} = \langle 0, 3 \rangle$

Perform the vector operation, given that $\mathbf{u} = \langle 7, -2 \rangle$ and $\mathbf{v} = \langle -4, 5 \rangle$.

9. $2\mathbf{u} + 3\mathbf{v}$
10. $\mathbf{u} - \mathbf{v}$
11. $6\mathbf{u} + \mathbf{v}$
12. $-3(\mathbf{u} + 2\mathbf{v})$

Find the vector, given its magnitude and direction angle.

13. $|\mathbf{u}| = 10, \theta = 75°$
14. $|\mathbf{u}| = 8, \theta = 225°$
15. $|\mathbf{u}| = 12, \theta = 105°$
16. $|\mathbf{u}| = 20, \theta = 15°$

Find a unit vector in the direction of the given vector.

17. $\mathbf{v} = \langle \sqrt{6}, -\sqrt{6} \rangle$
18. $\mathbf{v} = \langle -11, 60 \rangle$

Perform the indicated vector operation.

19. $(3\mathbf{i} - 4\mathbf{j}) + (2\mathbf{i} + 5\mathbf{j})$
20. $(-6\mathbf{i} + \mathbf{j}) - (9\mathbf{i} - \mathbf{j})$

7.2 The Dot Product

Find the indicated dot product.

21. $\langle 6, -3 \rangle \cdot \langle 1, 4 \rangle$
22. $\langle -6, 5 \rangle \cdot \langle -4, 2 \rangle$
23. $\langle 3, 3 \rangle \cdot \langle 3, -6 \rangle$
24. $\langle -2, -8 \rangle \cdot \langle -1, 1 \rangle$
25. $\langle 0, 8 \rangle \cdot \langle 1, 2 \rangle$
26. $\langle 4, -3 \rangle \cdot \langle -1, 0 \rangle$

Find the angle (round to the nearest degree) between each pair of vectors.

27. $\langle 3, 4 \rangle$ and $\langle -5, 12 \rangle$
28. $\langle -4, 5 \rangle$ and $\langle 5, -4 \rangle$
29. $\langle 1, \sqrt{2} \rangle$ and $\langle -1, 3\sqrt{2} \rangle$
30. $\langle 7, -24 \rangle$ and $\langle -6, 8 \rangle$
31. $\langle 3, 5 \rangle$ and $\langle -4, -4 \rangle$
32. $\langle -1, 6 \rangle$ and $\langle 2, -2 \rangle$

Determine whether each pair of vectors is orthogonal.

33. $\langle 8, 3 \rangle$ and $\langle -3, 12 \rangle$
34. $\langle -6, 2 \rangle$ and $\langle 4, 12 \rangle$
35. $\langle 5, -6 \rangle$ and $\langle -12, -10 \rangle$
36. $\langle 1, 1 \rangle$ and $\langle -4, 4 \rangle$
37. $\langle 0, 4 \rangle$ and $\langle 0, -4 \rangle$
38. $\langle -7, 2 \rangle$ and $\left\langle \frac{1}{7}, -\frac{1}{2} \right\rangle$
39. $\langle 6z, a - b \rangle$ and $\langle a + b, -6z \rangle$
40. $\langle a - b, -1 \rangle$ and $\langle a + b, a^2 - b^2 \rangle$

7.3 Polar (Trigonometric) Form of Complex Numbers

Graph each complex number in the complex plane.

41. $-6 + 2i$
42. $5i$

Express each complex number in polar form.

43. $\sqrt{2} - \sqrt{2}i$
44. $\sqrt{3} + i$
45. $-8i$
46. $-8 - 8i$

With a calculator, express each complex number in polar form.

47. $-60 + 11i$
48. $9 - 40i$
49. $15 + 8i$
50. $-10 - 24i$

Express each complex number in rectangular form.

51. $6(\cos 300° + i \sin 300°)$
52. $4(\cos 210° + i \sin 210°)$
53. $\sqrt{2}(\cos 135° + i \sin 135°)$
54. $4(\cos 150° + i \sin 150°)$

With a calculator, express each complex number in rectangular form.

55. $4(\cos 200° + i \sin 200°)$
56. $3(\cos 350° + i \sin 350°)$

7.4 Products, Quotients, Powers, and Roots of Complex Numbers

Find the product $z_1 z_2$.

57. $z_1 = 3(\cos 200° + i \sin 200°)$ and $z_2 = 4(\cos 70° + i \sin 70°)$
58. $z_1 = 3(\cos 20° + i \sin 20°)$ and $z_2 = 4(\cos 220° + i \sin 220°)$
59. $z_1 = 7(\cos 100° + i \sin 100°)$ and $z_2 = 3(\cos 140° + i \sin 140°)$
60. $z_1 = (\cos 290° + i \sin 290°)$ and $z_2 = 4(\cos 40° + i \sin 40°)$

Find the quotient $\frac{z_1}{z_2}$.

61. $z_1 = \sqrt{6}(\cos 200° + i \sin 200°)$ and $z_2 = \sqrt{6}(\cos 50° + i \sin 50°)$
62. $z_1 = 18(\cos 190° + i \sin 190°)$ and $z_2 = 2(\cos 100° + i \sin 100°)$
63. $z_1 = 24(\cos 290° + i \sin 290°)$ and $z_2 = 4(\cos 110° + i \sin 110°)$
64. $z_1 = \sqrt{200}(\cos 93° + i \sin 93°)$ and $z_2 = \sqrt{2}(\cos 48° + i \sin 48°)$

Find the result of each expression using De Moivre's theorem. Write the answer in rectangular form.

65. $(3 + 3i)^4$
66. $(3 + \sqrt{3}i)^4$
67. $(1 + \sqrt{3}i)^5$
68. $(-2 - 2i)^7$

Find all nth roots of z. Write the answers in polar form, and plot the roots in the complex plane.

69. $2 + 2\sqrt{3}i, n = 2$

70. $-8 + 8\sqrt{3}i, n = 4$

71. $-256, n = 4$

72. $-18i, n = 2$

Find all complex solutions to the given equations.

73. $x^3 + 216 = 0$

74. $x^4 - 1 = 0$

75. $x^4 + 1 = 0$

76. $x^3 - 125 = 0$

7.5 Polar Coordinates and Graphs of Polar Equations

Convert each point to exact polar coordinates (assuming that $0 \le \theta < 2\pi$), and then graph the point in the polar coordinate system.

77. $(-2, 2)$

78. $(4, -4\sqrt{3})$

79. $(-5\sqrt{3}, -5)$

80. $(\sqrt{3}, \sqrt{3})$

81. $(0, -2)$

82. $(11, 0)$

Convert each polar point to exact rectangular coordinates.

83. $\left(-3, \frac{5\pi}{3}\right)$

84. $\left(4, \frac{5\pi}{4}\right)$

85. $\left(2, \frac{\pi}{3}\right)$

86. $\left(6, \frac{7\pi}{6}\right)$

87. $\left(1, \frac{4\pi}{3}\right)$

88. $\left(-3, \frac{7\pi}{4}\right)$

Graph each equation.

89. $r = 4\cos(2\theta)$

90. $r = \sin(3\theta)$

91. $r = -\theta$

92. $r = 4 - 3\sin\theta$

Technology Exercises

Section 7.1

With the graphing calculator $\boxed{\text{SUM}}$ command, find the magnitude of the given vector. Also, find the direction angle to the nearest degree.

93. $\langle 25, -60\rangle$

94. $\langle -70, 10\sqrt{15}\rangle$

Section 7.2

With the graphing calculator $\boxed{\text{SUM}}$ command, find the angle (round to the nearest degree) between each pair of vectors.

95. $\langle 14, 37\rangle, \langle 9, -26\rangle$

96. $\langle -23, -8\rangle, \langle 18, -32\rangle$

Section 7.3

Another way of using a graphing calculator to represent complex numbers in rectangular form is to enter the real and imaginary parts as a list of two numbers and use the $\boxed{\text{SUM}}$ command to find the modulus.

97. Write $-\sqrt{23} - 11i$ in polar form using the $\boxed{\text{SUM}}$ command to find its modulus, and round the angle to the nearest degree.

98. Write $11 + \sqrt{23}i$ in polar form using the $\boxed{\text{SUM}}$ command to find its modulus, and round the angle to the nearest degree.

Section 7.4

99. Find the fourth roots of $-8 + 8\sqrt{3}i$, and draw the complex rectangle with the calculator.

100. Find the fourth roots of $8\sqrt{3} + 8i$, and draw the complex rectangle with the calculator.

Section 7.5

101. Given $r = 1 - 2\sin(3\theta)$, find the angles of all points of intersection (where $r = 0$).

102. Given $r = 1 + 2\cos(3\theta)$, find the angles of all points of intersection (where $r = 0$).

1. Find the magnitude and direction angle of the vector $\mathbf{u} = \langle -5, 12 \rangle$.

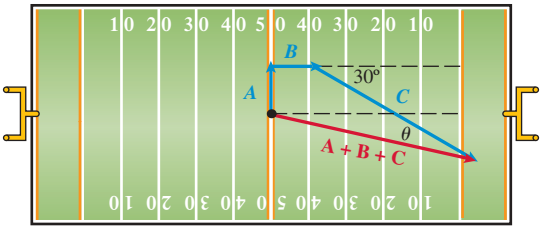
2. Find a unit vector pointing in the same direction as $\mathbf{v} = \langle -3, -4 \rangle$.

3. Perform the indicated operations:

a. $2\langle -1, 4 \rangle - 3\langle 4, 1 \rangle$

b. $\langle -7, -1 \rangle \cdot \langle 2, 2 \rangle$

4. In a post pattern in football, the receiver in motion runs past the quarterback parallel to the line of scrimmage (A), runs 12 yards perpendicular to the line of scrimmage (B), and then cuts toward the goal post (C).



A receiver runs the post pattern. If the magnitudes of the vectors are $|A| = 3$ yd, $|B| = 12$ yd, and $|C| = 18$ yd, find the magnitude of the resultant vector $\mathbf{A} + \mathbf{B} + \mathbf{C}$ and the direction angle θ.

5. Find the dot product $\langle 4, -51 \rangle \cdot \langle -2, -\frac{1}{3} \rangle$.

6. If the dot product $\langle a, -2a \rangle \cdot \langle 4, 5 \rangle = 18$, find the value of a.

For Exercises 7 and 8, use the complex number $z = 16(\cos 120° + i \sin 120°)$.

7. Find z^4.

8. Find the four distinct fourth roots of z.

9. Convert the point $(3, 210°)$ to rectangular coordinates.

10. Convert the polar point $\left(4, \frac{5\pi}{4}\right)$ to rectangular coordinates.

11. Convert the point $(30, -15)$ to polar coordinates.

12. Graph $r = 6\sin(2\theta)$.

13. Graph $r^2 = 9\cos(2\theta)$.

14. Find x such that $\langle x, 1 \rangle$ is perpendicular to $3\mathbf{i} - 4\mathbf{j}$.

15. Prove that $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$.

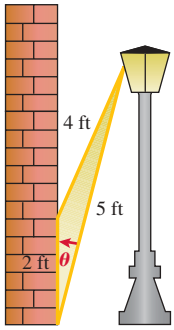
16. Construct a unit vector in the opposite direction of $\langle 3, 5 \rangle$.

17. Compute $\mathbf{u} \cdot \mathbf{v}$ if $|\mathbf{u}| = 4$, $|\mathbf{v}| = 10$, and $\theta = \frac{2\pi}{3}$.

18. Determine whether $\mathbf{u}$ and $\mathbf{v}$ are parallel, perpendicular, or neither: $\mathbf{u} = \langle \sin\theta, \cos\theta \rangle$, and $\mathbf{v} = \langle -\cos\theta, \sin\theta \rangle$.

19. Find the magnitude of $-\mathbf{i} - \mathbf{j}$.

20. Determine θ, when a streetlight is formed as follows:



21. Two tugboats pull a cruiser off the port of Miami. The first one pulls with a force of 25,000 pounds and the second one pulls with a force of 27,000 pounds. If the angle between the lines connecting the cruiser with the tugboats is 25°, what is the resultant force vector of the two tugboats?

22. True or false: If $\mathbf{u} + \mathbf{v}$ is perpendicular to $\mathbf{u} - \mathbf{v}$, then $|\mathbf{u}| = |\mathbf{v}|$.

23. Solve $z^4 + 256i = 0$.

24. Convert to a Cartesian equation: $r^2 = \tan\theta$.

25. With the graphing calculator $\boxed{\text{SUM}}$ command, find the angle (round to the nearest degree) between each pair of vectors: $\langle -8, -11 \rangle$ and $\langle -16, 26 \rangle$.

26. Find the fourth roots of $-8\sqrt{3} - 8i$, and draw the complex rectangle with the calculator.

1. Given $f(x) = x^2 - 4$ and $g(x) = \dfrac{1}{\sqrt{3x+5}}$, find $(f \circ g)(x)$ and the domain of f, g, and $f \circ g$.

2. Determine whether the function $f(x) = |x^3|$ is even, odd, or neither.

3. Find the quadratic function whose graph has a vertex at $(-1, 2)$ and that passes through the point $(2, -1)$. Express the quadratic function in both standard and general forms.

4. For the polynomial function
$f(x) = x^5 - 4x^4 + x^3 + 10x^2 - 4x - 8$

a. List each real zero and its multiplicity.

b. Determine whether the graph touches or crosses at each x-intercept.

5. Find all vertical and horizontal or slant asymptotes (if any) in the following:

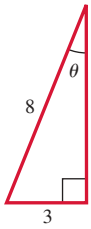
$$f(x) = \frac{x^3 - 3x^2 + 2x - 1}{x^2 - 2x + 1}$$

6. How much money should be invested today in a money market account that pays 1.4% a year compounded continuously if you desire \$5000 in 8 years?

7. Write the exponential equation $\sqrt[4]{625} = 5$ in its equivalent logarithmic form.

8. What is the radian measure of an angle of 305°? Express your answer in terms of π.

9. Find all trigonometric functions of the angle θ. Rationalize any denominators containing radicals that you encounter in your answers.



10. Solve the triangle $\alpha = 30°$, $\beta = 30°$, and $c = 4$ in.

11. Solve the triangle $a = 14.2$ m, $b = 16.5$ m, and $\gamma = 50°$.

12. Find the exact value of each trigonometric function:

a. $\sin\left(\dfrac{3\pi}{2}\right)$

b. $\cos 0$

c. $\tan\left(\dfrac{5\pi}{4}\right)$

d. $\cot\left(\dfrac{11\pi}{6}\right)$

e. $\sec\left(\dfrac{2\pi}{3}\right)$

f. $\csc\left(\dfrac{5\pi}{6}\right)$

13. State the amplitude, period, phase shift, and vertical shift of the function $y = 4 - \frac{1}{3}\sin(4x - \pi)$.

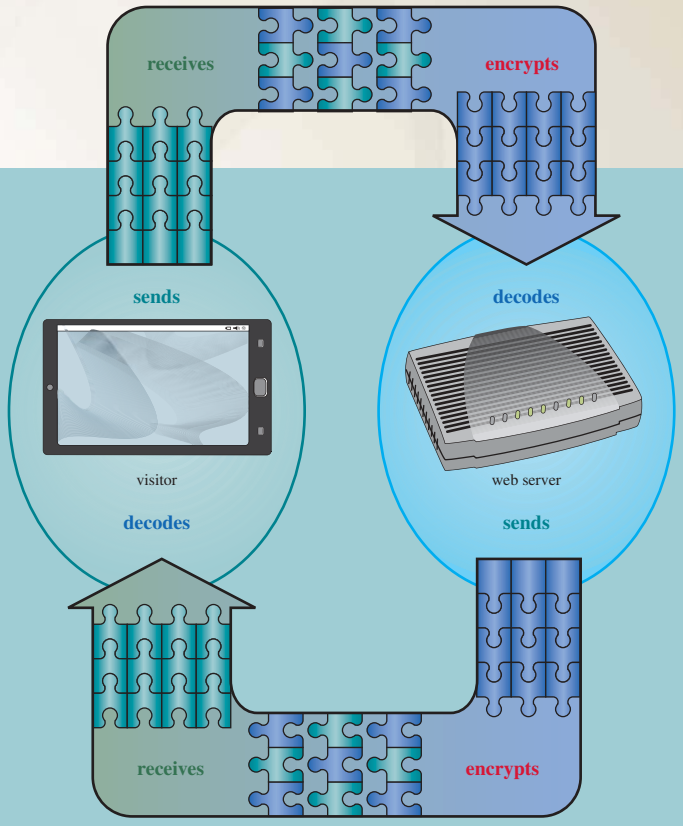
14. If $\sin x = \dfrac{1}{\sqrt{3}}$ and $\cos x < 0$, find $\cos(2x)$.

15. Find the exact value of $\tan\left[\cos^{-1}\left(-\frac{3}{5}\right) + \sin^{-1}\left(\frac{1}{2}\right)\right]$.

16. Find $(1 + \sqrt{3}i)^8$. Express the answer in rectangular form.

8

Systems of Linear Equations and Inequalities



Cryptography is the practice and study of encryption and decryption—encoding data so that it can be decoded only by specific individuals. In other words, it turns a message into gibberish so that only the person who has the deciphering tools can turn that gibberish back into the original message. ATM cards, online shopping sites, and secure military communications all depend on coding and decoding of information. Matrices are used extensively in cryptography. A *matrix* is used as the "key" to encode the data, and then its *inverse matrix* is used as the key to decode the data.*

*Section 8.4, Exercises 87–92.

IN THIS CHAPTER we will solve systems of linear equations using the elimination and substitution methods. We will then solve systems of linear equations using matrices three different ways: using augmented matrices (Gauss–Jordan elimination), matrix algebra (inverse matrices), and determinants (Cramer's rule). We will then discuss an application of systems of linear equations that is useful in calculus called partial–fraction decomposition. Finally, we will solve systems of linear inequalities.

SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

8.1 Systems of Linear Equations in Two Variables	8.2 Systems of Linear Equations in Three Variables	8.3 Systems of Linear Equations and Matrices	8.4 Matrix Algebra	8.5 The Determinant of a Square Matrix and Cramer's Rule	8.6 Partial Fractions	8.7 Systems of Linear Inequalities in Two Variables
• Solving Systems of Linear Equations in Two Variables • Three Methods and Three Types of Solutions	• Solving Systems of Linear Equations in Three Variables • Types of Solutions	• Matrices • Augmented Matrices • Row Operations on a Matrix • Row–Echelon Form of a Matrix • Gaussian Elimination with Back-Substitution • Gauss–Jordan Elimination • Inconsistent and Dependent Systems	• Equality of Matrices • Matrix Addition and Subtraction • Scalar and Matrix Multiplication • Matrix Equations • Finding the Inverse of a Square Matrix • Solving Systems of Linear Equations Using Matrix Algebra and Inverses of Square Matrices	• Determinant of a 2×2 Matrix • Determinant of an $n \times n$ Matrix • Cramer's Rule: Systems of Linear Equations in Two Variables • Cramer's Rule: Systems of Linear Equations in Three Variables	• Performing Partial Fraction Decomposition	• Linear Inequalities in Two Variables • Systems of Linear Inequalities in Two Variables • The Linear Programming Model

LEARNING OBJECTIVES

- Solve systems of linear equations in two variables using elimination and substitution methods.
- Solve systems of linear equations in three variables using elimination and substitution methods.
- Use Gauss–Jordan elimination (augmented matrices) to solve systems of linear equations in more than two variables.
- Use matrix algebra and inverse matrices to solve systems of linear equations.
- Use Cramer's rule to solve systems of linear equations.
- Perform partial–fraction decomposition on rational expressions.
- Solve systems of linear inequalities in two variables.

SECTION 8.1

SYSTEMS OF LINEAR EQUATIONS IN TWO VARIABLES

SKILLS OBJECTIVES

- Solve systems of linear equations in two variables using the substitution method.
- Solve systems of linear equations in two variables using the elimination method.
- Solve systems of linear equations in two variables by graphing.
- Solve applications involving systems of linear equations.

CONCEPTUAL OBJECTIVES

- Understand that a system of linear equations has either one solution, no solution, or infinitely many solutions.
- Visualize two lines that intersect at one point, no points (parallel lines), or infinitely many points (same line).

Solving Systems of Linear Equations in Two Variables

Overview

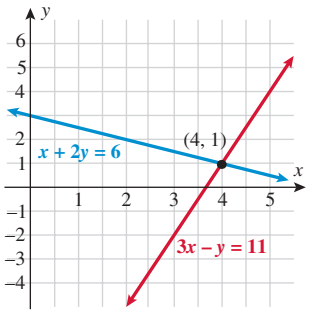
A linear equation in two variables is given in standard form by

$$Ax + By = C$$

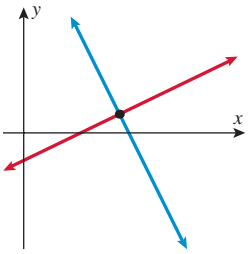
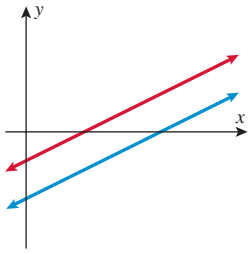
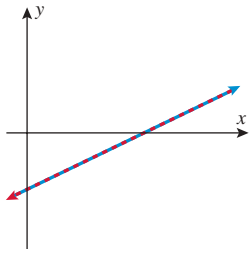
and the graph of this linear equation is a line, provided that A and B are not both equal to zero. In this section, we discuss **systems of linear equations**, which can be thought of as simultaneous equations. To **solve** a system of linear equations in two variables means to find the solution that satisfies *both* equations. Suppose we are given the following system of equations:

$$\begin{aligned} x + 2y &= 6 \\ 3x - y &= 11 \end{aligned}$$

We can interpret the solution to this system of equations both algebraically and graphically.

	ALGEBRAIC	GRAPHICAL
Solution	$x = 4$ and $y = 1$	(4, 1)
Check	**Equation 1** $x + 2y = 6$, $(4) + 2(1) = 6$ ✓ **Equation 2** $3x - y = 11$, $3(4) - 1 = 11$ ✓	
Interpretation	$x = 4$ and $y = 1$ satisfy both equations.	The point (4, 1) lies on both lines.

This particular example had *one solution*. There are systems of equations that have *no solution* or *infinitely many solutions*. We give these systems special names: **independent**, **inconsistent**, and **dependent**, respectively.

Independent System	Inconsistent System	Dependent System
One solution	No solution	Infinitely many solutions
		
Lines have different slopes.	Lines are parallel (same slope and different y-intercepts).	Lines coincide (same slope and same y-intercept).

In this section, we discuss three methods for solving systems of two linear equations in two variables: *substitution*, *elimination*, and *graphing*. We use the algebraic methods—substitution and elimination—to find solutions exactly; we then look at a graphical interpretation of the solution (two lines that intersect at one point, parallel lines, or coinciding lines).

We will illustrate each method with the same example given earlier:

$$x + 2y = 6 \qquad \text{Equation (1)}$$
$$3x - y = 11 \qquad \text{Equation (2)}$$

Substitution Method

The following box summarizes the substitution method for solving systems of two linear equations in two variables:

Substitution Method

Step 1: Solve one of the equations for one variable in terms of the other variable.

Equation (2): $y = 3x - 11$

Step 2: Substitute the expression found in Step 1 into the *other* equation. The result is an equation in one variable.

Equation (1): $x + 2(3x - 11) = 6$

Step 3: Solve the equation obtained in Step 2.

$$x + 6x - 22 = 6$$
$$7x = 28$$
$$\boxed{x = 4}$$

Step 4: Back-substitute the value found in Step 3 into the expression found in Step 1.

$$y = 3(4) - 11$$
$$\boxed{y = 1}$$

Step 5: Check that the solution satisfies *both* equations. Substitute (4, 1) into both equations.

Equation (1): $x + 2y = 6$
$(4) + 2(1) = 6$ ✓

Equation (2): $3x - y = 11$
$3(4) - 1 = 11$ ✓

EXAMPLE 1 Determining by Substitution That a System Has One Solution

Use the substitution method to solve the following system of linear equations:

$$x + y = 8 \qquad \text{Equation (1)}$$
$$3x - y = 4 \qquad \text{Equation (2)}$$

Solution:

STEP 1 Solve Equation (2) for y in terms of x. $\quad y = 3x - 4$

STEP 2 Substitute $y = 3x - 4$ into Equation (1). $\quad x + (3x - 4) = 8$

STEP 3 Solve for x.

$$x + 3x - 4 = 8$$
$$4x = 12$$
$$\boxed{x = 3}$$

STEP 4 Back-substitute $x = 3$ into Equation (1). $\quad 3 + y = 8$

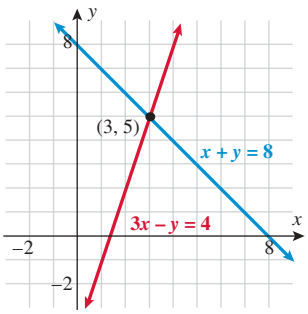
$$\boxed{y = 5}$$

STEP 5 Check that (3, 5) satisfies *both* equations.

Equation (1): $\quad x + y = 8$
$\quad 3 + 5 = 8$

Equation (2): $\quad 3x - y = 4$
$\quad 3(3) - 5 = 4$

Note: The graphs of the two equations are two lines that intersect at the point (3, 5).



EXAMPLE 2 Determining by Substitution That a System Has No Solution

Use the substitution method to solve the following system of linear equations:

$$x - y = 2 \qquad \text{Equation (1)}$$
$$2x - 2y = 10 \qquad \text{Equation (2)}$$

Solution:

STEP 1 Solve Equation (1) for y in terms of x. $\quad y = x - 2$

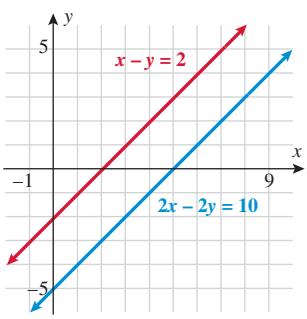
STEP 2 Substitute $y = x - 2$ into Equation (2). $\quad 2x - 2(x - 2) = 10$

STEP 3 Solve for x.

$$2x - 2x + 4 = 10$$
$$4 = 10$$

$4 = 10$ is never true, so this is called an inconsistent system. There is $\boxed{\text{no solution}}$ to this system of linear equations.

Note: The graphs of the two equations are parallel lines.



EXAMPLE 3 Determining by Substitution That a System Has Infinitely Many Solutions

Use the substitution method to solve the following system of linear equations:

$$x - y = 2 \qquad \text{Equation (1)}$$
$$-x + y = -2 \qquad \text{Equation (2)}$$

Solution:

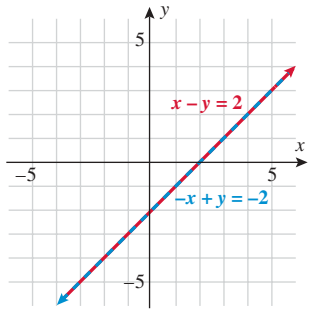
STEP 1 Solve Equation (1) for y in terms of x. $\quad y = x - 2$

STEP 2 Substitute $y = x - 2$ into Equation (2). $\quad -x + (x - 2) = -2$

STEP 3 Solve for x.

$$-x + x - 2 = -2$$
$$-2 = -2$$

$-2 = -2$ is always true, so this is called a dependent system. Notice, for instance, that the points (2, 0), (4, 2), and (7, 5) all satisfy both equations. In fact, there are $\boxed{\text{infinitely many solutions}}$ to this system of linear equations. All solutions are in the form (x, y), where $\boxed{y = x - 2}$. (The graphs of these two equations are the same line.) If we let $x = a$, then $y = a - 2$. In other words, all of the points $(a, a - 2)$ where a is any real number are solutions to this system of linear equations.



YOUR TURN Use the substitution method to solve each system of linear equations.

a. $2x + y = 3$, $4x + 2y = 4$ **b.** $x - y = 2$, $4x - 3y = 10$ **c.** $x + 2y = 1$, $2x + 4y = 2$

Answer: a. no solution **b.** (4, 2) **c.** infinitely many solutions where $y = -\frac{1}{2}x + \frac{1}{2}$ or $\left(a, \frac{1 - a}{2}\right)$.

Elimination Method

We now turn our attention to another method, *elimination*, which is often preferred over substitution and will later be used in higher order systems. In a system of two linear equations in two variables, the equations can be combined, resulting in a third equation in one variable, thus *eliminating* one of the variables. The following is an example of when elimination would be preferred because the y terms sum to zero when the two equations are added together:

$$\begin{array}{r} 2x - y = 5 \\ -x + y = -2 \\ \hline x \quad = 3 \end{array}$$

When you cannot eliminate a variable simply by *adding* the two equations, multiply one equation by a constant that will cause the coefficients of some variable in the two equations to match and be opposite in sign.

The following box summarizes the *elimination method*, also called the *addition method*, for solving systems of two linear equations in two variables using the same example given earlier:

$$x + 2y = 6 \qquad \text{Equation (1)}$$
$$3x - y = 11 \qquad \text{Equation (2)}$$

ELIMINATION METHOD

Step 1*: **Multiply** the coefficients of one (or both) of the equations so that one of the variables will be eliminated when the two equations are added.

Multiply Equation (2) by 2:
$6x - 2y = 22$

Step 2: **Eliminate** one of the variables by adding the equation found in Step 1 to the *other* original equation. The result is an equation in one variable.

$$\begin{array}{r} x + 2y = 6 \\ 6x - 2y = 22 \\ \hline 7x \quad = 28 \end{array}$$

Step 3: **Solve** the equation obtained in Step 2.

$$7x = 28$$
$$\boxed{x = 4}$$

Step 4: **Back-substitute** the value found in Step 3 into either of the two original equations.

$$(4) + 2y = 6$$
$$2y = 2$$
$$\boxed{y = 1}$$

Step 5: **Check** that the solution satisfies *both* equations. Substitute $\boxed{(4, 1)}$ into both equations.

Equation (1):
$x + 2y = 6$
$(4) + 2(1) = 6$ ✓

Equation (2):
$3x - y = 11$
$3(4) - 1 = 11$ ✓

*Step 1 is not necessary in cases where a pair of corresponding terms already sum to zero.

EXAMPLE 4 Applying the Elimination Method When One Variable Is Eliminated by Adding the Two Original Equations

Use the elimination method to solve the following system of linear equations:

$$2x - y = -5 \qquad \text{Equation (1)}$$
$$4x + y = 11 \qquad \text{Equation (2)}$$

Solution:

STEP 1 Not necessary.

STEP 2 Eliminate y by adding Equation (1) to Equation (2).

$$\begin{array}{r} 2x - y = -5 \\ 4x + y = 11 \\ \hline 6x \quad = 6 \end{array}$$

STEP 3 Solve for x.

$$\boxed{x = 1}$$

STEP 4 Back-substitute $x = 1$ into Equation (2). Solve for y.

$$4(1) + y = 11$$
$$\boxed{y = 7}$$

STEP 5 Check that (1, 7) satisfies both equations.

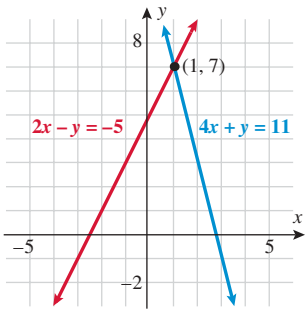
Equation (1): $2x - y = -5$
$2(1) - (7) = -5$ ✓

Equation (2): $4x + y = 11$
$4(1) + (7) = 11$ ✓

Note: The graphs of the two given equations correspond to two lines that intersect at the point (1, 7).

Study Tip

You can eliminate one variable from the system by addition when (1) the coefficients are equal and (2) the signs are opposite.



In Example 4, we eliminated the variable y simply by adding the two equations. Sometimes it is necessary to multiply one (Example 5) or both (Example 6) equations by constants prior to adding.

EXAMPLE 5 Applying the Elimination Method When Multiplying One Equation by a Constant Is Necessary

Use the elimination method to solve the following system of linear equations:

$$-4x + 3y = 23 \qquad \text{Equation (1)}$$
$$12x + 5y = 1 \qquad \text{Equation (2)}$$

Solution:

STEP 1 Multiply Equation (1) by 3. $\quad -12x + 9y = 69$

STEP 2 Eliminate x by adding the modified Equation (1) to Equation (2).

$$\begin{array}{r} -12x + 9y = 69 \\ 12x + 5y = 1 \\ \hline 14y = 70 \end{array}$$

STEP 3 Solve for y. $\quad \boxed{y = 5}$

STEP 4 Back-substitute $y = 5$ into Equation (2). $\quad 12x + 5(5) = 1$

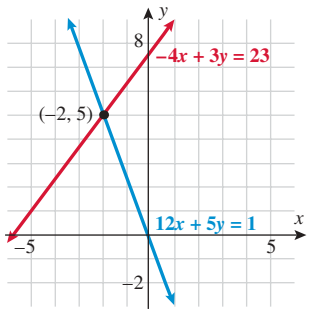
Solve for x.

$$\begin{aligned} 12x + 25 &= 1 \\ 12x &= -24 \\ \boxed{x = -2} \end{aligned}$$

STEP 5 Check that $(-2, 5)$ satisfies both equations.

Equation (1): $\quad -4(-2) + 3(5) = 23$
$\quad 8 + 15 = 23$ ✓

Equation (2): $\quad 12(-2) + 5(5) = 1$
$\quad -24 + 25 = 1$ ✓



Note: The graphs of the two given equations correspond to two lines that intersect at the point $(-2, 5)$.

Study Tip

Be sure to multiply the **entire** equation by the constant.

In Example 5, we eliminated x simply by multiplying the first equation by a constant and adding the result to the second equation. In order to eliminate either of the variables in Example 6, we will have to multiply *both* equations by constants prior to adding.

EXAMPLE 6 Applying the Elimination Method When Multiplying Both Equations by Constants Is Necessary

Use the elimination method to solve the following system of linear equations:

$$3x + 2y = 1 \qquad \text{Equation (1)}$$
$$5x + 7y = 9 \qquad \text{Equation (2)}$$

Solution:

STEP 1 Multiply Equation (1) by 5 and Equation (2) by -3.

$$\begin{aligned} 15x + 10y &= 5 \\ -15x - 21y &= -27 \end{aligned}$$

STEP 2 Eliminate x by adding the modified Equation (1) to the modified Equation (2).

$$\begin{aligned} 15x + 10y &= 5 \\ -15x - 21y &= -27 \\ \hline -11y &= -22 \end{aligned}$$

STEP 3 Solve for y.

$$\boxed{y = 2}$$

STEP 4 Back-substitute $y = 2$ into Equation (1). Solve for x.

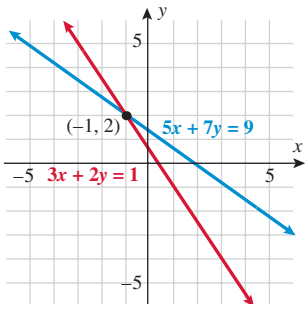
$$\begin{aligned} 3x + 2(2) &= 1 \\ 3x &= -3 \\ \boxed{x = -1} \end{aligned}$$

STEP 5 Check that $(-1, 2)$ satisfies both equations.

Equation (1):
$$\begin{aligned} 3x + 2y &= 1 \\ 3(-1) + 2(2) &= 1 \checkmark \end{aligned}$$

Equation (2):
$$\begin{aligned} 5x + 7y &= 9 \\ 5(-1) + 7(2) &= 9 \checkmark \end{aligned}$$

Note: The graphs of the two given equations correspond to two lines that intersect at the point $(-1, 2)$.



Notice in Example 6 that we could have also eliminated y by multiplying the first equation by 7 and the second equation by -2. Typically, the choice is dictated by which approach will keep the coefficients as simple as possible. In the event that the original coefficients contain fractions or decimals, first rewrite the equations in standard form with integer coefficients and then make the decision.

EXAMPLE 7 Determining by the Elimination Method That a System Has No Solution

Use the elimination method to solve the following system of linear equations:

$$-x + y = 7 \qquad \text{Equation (1)}$$
$$2x - 2y = 4 \qquad \text{Equation (2)}$$

Solution:

STEP 1 Multiply Equation (1) by 2.

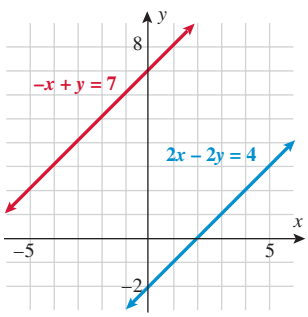
$$-2x + 2y = 14$$

STEP 2 Eliminate y by adding the modified Equation (1) found in Step 1 to Equation (2).

$$\begin{aligned} -2x + 2y &= 14 \\ 2x - 2y &= 4 \\ \hline 0 &= 18 \end{aligned}$$

This system is inconsistent since $0 = 18$ is never true. Therefore, there are no values of x and y that satisfy both equations. We say that there is $\boxed{\text{no solution}}$ to this system of linear equations.

Note: The graphs of the two equations are two parallel lines.



EXAMPLE 8 Determining by the Elimination Method That a System Has Infinitely Many Solutions

Use the elimination method to solve the following system of linear equations:

$$\begin{aligned} 7x + y &= 2 && \text{Equation (1)} \\ -14x - 2y &= -4 && \text{Equation (2)} \end{aligned}$$

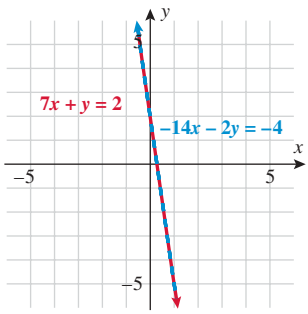
Solution:

STEP 1 Multiply Equation (1) by 2.

$$14x + 2y = 4$$

STEP 2 Add the modified Equation (1) found in Step 1 to Equation (2).

$$\begin{aligned} 14x + 2y &= 4 \\ -14x - 2y &= -4 \\ \hline 0 &= 0 \end{aligned}$$



This system is dependent since $0 = 0$ is always true. We say that there are $\boxed{\text{infinitely many solutions}}$ to this system of linear equations of the form $\boxed{y = -7x + 2}$ and these can be represented by the points $(a, 2 - 7a)$.

Note: The graphs of the two equations are the same line.

YOUR TURN Apply the elimination method to solve each system of linear equations.

a. $\begin{aligned} 2x + 3y &= 1 \\ 4x - 3y &= -7 \end{aligned}$ **b.** $\begin{aligned} x - 5y &= 2 \\ -10x + 50y &= -20 \end{aligned}$ **c.** $\begin{aligned} x - y &= 14 \\ -x + y &= 9 \end{aligned}$

Study Tip

Systems of linear equations in two variables have either one solution, no solution, or infinitely many solutions.

Answer:
a. $(-1, 1)$
b. infinitely many solutions of the form $y = \frac{1}{5}x - \frac{2}{5}$ or $\left(a, \frac{a-2}{5}\right)$.
c. no solution

Graphing Method

A third way to solve a system of linear equations in two variables is to graph the two lines. If the two lines intersect, then the point of intersection is the solution. Graphing is the most labor-intensive method for solving systems of linear equations in two variables. The graphing method is typically not used to solve systems of linear equations when an exact solution is desired. Instead, it is used to interpret or confirm the solution(s) found by the other two methods (substitution and elimination). If you are using a graphing calculator, however, you will get as accurate an answer using the graphing method as you will when applying the other methods.

The following box summarizes the graphing method for solving systems of linear equations in two variables using the same example given earlier:

$$\begin{aligned} x + 2y &= 6 && \text{Equation (1)} \\ 3x - y &= 11 && \text{Equation (2)} \end{aligned}$$

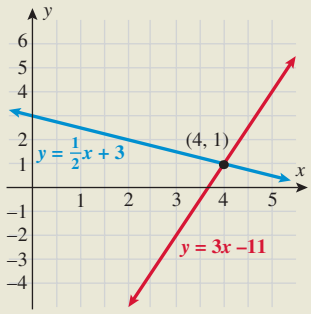
GRAPHING METHOD

Step 1*: Write the equations in slope–intercept form.

Equation (1): $y = -\frac{1}{2}x + 3$

Equation (2): $y = 3x - 11$

Step 2: Graph the two lines.



Step 3: Identify the point of intersection.

$\boxed{(4, 1)}$

Step 4: Check that the solution satisfies *both* equations.

Equation (1):
$x + 2y = 6$
$(4) + 2(1) = 6$ ✓

Equation (2):
$3x - y = 11$
$3(4) - 1 = 11$ ✓

*Step 1 is not necessary when the lines are already in slope–intercept form.

EXAMPLE 9 Determining by Graphing That a System Has One Solution

Use graphing to solve the following system of linear equations:

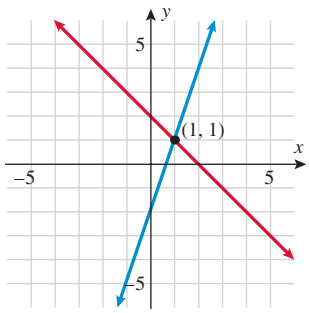
$$x + y = 2 \quad \text{Equation (1)}$$
$$3x - y = 2 \quad \text{Equation (2)}$$

Solution:

STEP 1 Write each equation in slope–intercept form.

$y = -x + 2$ Equation (1)
$y = 3x - 2$ Equation (2)

STEP 2 Plot both lines on the same graph.



STEP 3 Identify the point of intersection.

$\boxed{(1, 1)}$

STEP 4 Check that the point (1, 1) satisfies both equations.

$x + y = 2$
$1 + 1 = 2$ ✓ Equation (1)

$3x - y = 2$
$3(1) - (1) = 2$ ✓ Equation (2)

Note: There is one solution, because the two lines intersect at one point.

EXAMPLE 10 Determining by Graphing That a System Has No Solution

Use graphing to solve the following system of linear equations:

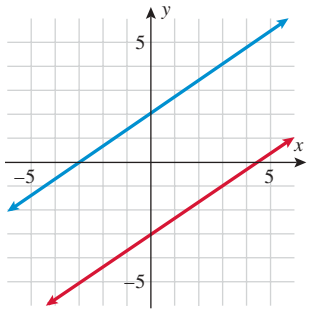
$$2x - 3y = 9 \quad \text{Equation (1)}$$
$$-4x + 6y = 12 \quad \text{Equation (2)}$$

Solution:

STEP 1 Write each equation in slope–intercept form.

$$y = \frac{2}{3}x - 3 \quad \text{Equation (1)}$$
$$y = \frac{2}{3}x + 2 \quad \text{Equation (2)}$$

STEP 2 Plot both lines on the same graph.

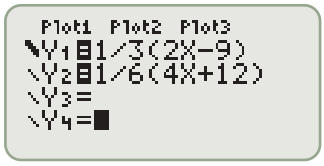


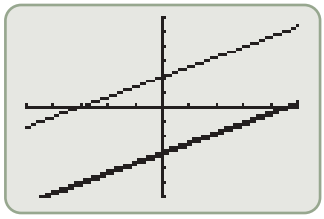
STEP 3 Identify the point of intersection. None

The two lines are parallel because they have the same slope, but different y-intercepts. For this reason there is $\boxed{\text{no solution}}$—two parallel lines do not intersect.

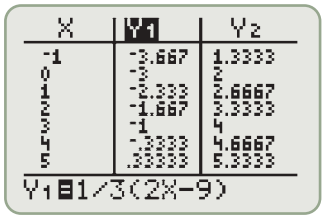
Technology Tip

First solve each equation for y. The graphs of $Y_1 = \frac{1}{3}(2x - 9)$ and $Y_2 = \frac{1}{6}(4x + 12)$ are shown.





The graphs and table show that there is no solution to the system.



X	Y1	Y2
-1	-3.667	1.3333
0	-3	2
1	-2.333	2.6667
2	-1.667	3.3333
3	-1	4
4	-.3333	4.6667
5	.33333	5.3333

Y1=1/3(2X-9)

EXAMPLE 11 Determining by Graphing That a System Has Infinitely Many Solutions

Use graphing to solve the following system of linear equations:

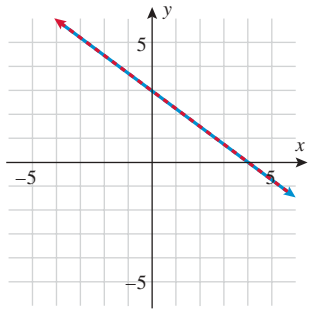
$$3x + 4y = 12 \quad \text{Equation (1)}$$
$$\frac{3}{4}x + y = 3 \quad \text{Equation (2)}$$

Solution:

STEP 1 Write each equation in slope–intercept form.

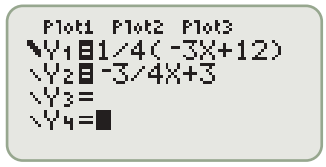
$$y = -\frac{3}{4}x + 3 \qquad y = -\frac{3}{4}x + 3$$

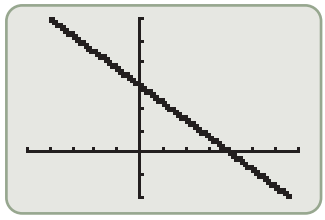
STEP 2 Plot both lines on the same graph.



Technology Tip

Solve for y in each equation first. The graphs of $Y_1 = \frac{1}{4}(-3x + 12)$ and $Y_2 = -\frac{3}{4}x + 3$ are shown.





Both lines are on the same graph.

STEP 3 Identify the point of intersection. Infinitely many points

There are $\boxed{\text{infinitely many solutions, } y = -\frac{3}{4}x + 3}$, since the two lines are identical and coincide. The points that lie along the line are $\left(a, -\frac{3}{4}a + 3\right)$.

■ **YOUR TURN** Utilize graphing to solve each system of linear equations.

a. $x - 2y = 1$, $2x - 4y = 2$ **b.** $x - 2y = 1$, $2x + y = 7$ **c.** $2x + y = 3$, $2x + y = 7$

■ **Answer:**
a. infinitely many solutions of the form $y = \frac{1}{2}x - \frac{1}{2}$
b. (3, 1)
c. no solution

Three Methods and Three Types of Solutions

Given any system of two linear equations in two variables, any of the three methods (substitution, elimination, or graphing) can be utilized. If you find that it is easy to eliminate a variable by adding multiples of the two equations, then elimination is the preferred choice. If you do not see an obvious elimination, then solve the system by substitution. For exact solutions, choose one of these two algebraic methods. You should typically use graphing to confirm the solution(s) you have found by applying the other two methods or when you are using a graphing utility.

EXAMPLE 12 Identifying Which Method to Use

State which of the two algebraic methods (elimination or substitution) would be the preferred method to solve each system of linear equations.

a. $x - 2y = 1$, $-x + y = 2$ **b.** $x = 2y - 1$, $2x - y = 4$ **c.** $7x - 20y = 1$, $5x + 3y = 18$

Solution:

a. Elimination: Because the x variable is eliminated when the two equations are added.

b. Substitution: Because the first equation is easily substituted into the second equation (for x).

c. Either: There is no preferred method, as both elimination and substitution require substantial work.

Regardless of which method is used to solve systems of two linear equations in two variables, in general, we can summarize the three types of solutions both algebraically and graphically.

THREE TYPES OF SOLUTIONS TO SYSTEMS OF LINEAR EQUATIONS

NUMBER OF SOLUTIONS	GRAPHICAL INTERPRETATION
One solution	The two lines intersect at one point.
No solution	The two lines are parallel. (*Same* slope/*different* y-intercepts.)
Infinitely many solutions	The two lines coincide. (*Same* slope/*same* y-intercept.)

Applications

Suppose you have two job offers that require sales. One pays a higher base, while the other pays a higher commission. Which job do you take?

EXAMPLE 13 Deciding Which Job to Take

Suppose that upon graduation you are offered a job selling biomolecular devices to laboratories studying DNA. The Beckman-Coulter Company offers you a job selling its DNA sequencer with an annual base salary of $20,000 plus 5% commission on total sales. The MJ Research Corporation offers you a job selling its PCR Machine that makes copies of DNA with an annual base salary of $30,000 plus 3% commission on sales. Determine what the total sales would have to be to make the Beckman-Coulter job the better offer.

Solution:

STEP 1 Identify the question.

When would these two jobs have equal compensations?

STEP 2 Make notes.

Beckman-Coulter salary	20,000 + 5% of sales
MJ Research salary	30,000 + 3% of sales

STEP 3 Set up the equations.

Let x = total sales and y = compensation.

Equation (1)	Beckman-Coulter:	$y = 20{,}000 + 0.05x$
Equation (2)	MJ Research:	$y = 30{,}000 + 0.03x$

STEP 4 Solve the system of equations.
*Substitution method**

Substitute Equation (1) into Equation (2). $20{,}000 + 0.05x = 30{,}000 + 0.03x$

Solve for x.

$$0.02x = 10{,}000$$
$$x = 500{,}000$$

If you make $500,000 worth of sales per year, the jobs will yield equal compensations. If you sell less than $500,000, the MJ Research job is the better offer, and more than $\boxed{\$500{,}000}$, the Beckman-Coulter job is the better offer.

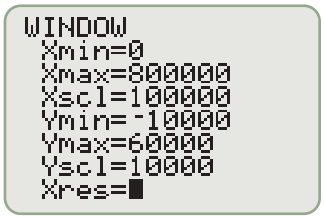
*The elimination method could also have been used.

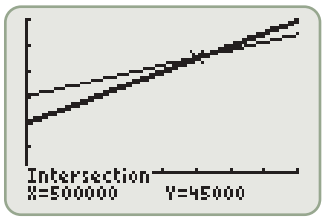
STEP 5 Check the solution.

Equation (1)	Beckman-Coulter:	$y = 20{,}000 + 0.05(500{,}000) = \$45{,}000$
Equation (2)	MJ Research:	$y = 30{,}000 + 0.03(500{,}000) = \$45{,}000$

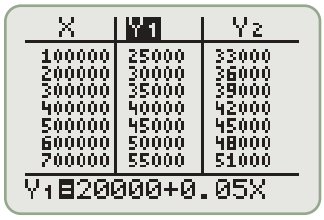
Technology Tip

The graphs of $Y_1 = 20{,}000 + 0.05x$ and $Y_2 = 30{,}000 + 0.03x$ are shown.





The graphs and table support the solution to the system.



X	Y1	Y2
100000	25000	33000
200000	30000	36000
300000	35000	39000
400000	40000	42000
500000	45000	45000
600000	50000	48000
700000	55000	51000

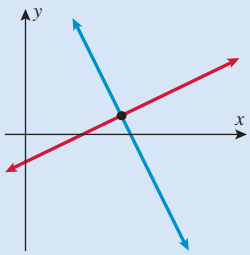
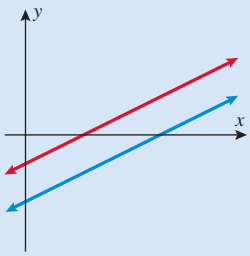
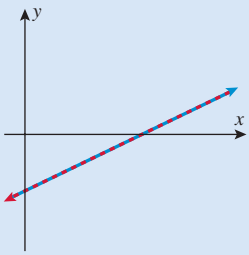
Y1=20000+0.05X

SECTION 8.1 SUMMARY

In this section, we discussed two algebraic techniques for solving systems of two linear equations in two variables:

- Substitution method
- Elimination method

The algebraic methods are preferred for exact solutions, and the graphing method is typically used to give a visual interpretation and confirmation of the solution. There are three types of solutions to systems of two linear equations in two variables: one solution, no solution, or infinitely many solutions.

INDEPENDENT SYSTEM	INCONSISTENT SYSTEM	DEPENDENT SYSTEM
One solution	No solution	Infinitely many solutions
		
Lines have different slopes.	Lines are parallel (same slope and different y-intercepts).	Lines coincide (same slope and same y-intercept).

SECTION 8.1 EXERCISES

SKILLS

In Exercises 1–20, solve each system of linear equations by substitution.

1. $x + y = 7$
$x - y = 9$

2. $x - y = -10$
$x + y = 4$

3. $2x - y = 3$
$x - 3y = 4$

4. $4x + 3y = 3$
$2x + y = 1$

5. $3x + y = 5$
$2x - 5y = -8$

6. $6x - y = -15$
$2x - 4y = -16$

7. $2u + 5v = 7$
$3u - v = 5$

8. $m - 2n = 4$
$3m + 2n = 1$

9. $2x + y = 7$
$-2x - y = 5$

10. $3x - y = 2$
$3x - y = 4$

11. $4r - s = 1$
$8r - 2s = 2$

12. $-3p + q = -4$
$6p - 2q = 8$

13. $5r - 3s = 15$
$-10r + 6s = -30$

14. $-5p - 3q = -1$
$10p + 6q = 2$

15. $2x - 3y = -7$
$3x + 7y = 24$

16. $4x - 5y = -7$
$3x + 8y = 30$

17. $\frac{1}{3}x - \frac{1}{4}y = 0$
$-\frac{2}{3}x + \frac{3}{4}y = 2$

18. $\frac{1}{5}x + \frac{2}{3}y = 10$
$-\frac{1}{2}x - \frac{1}{6}y = -7$

19. $-3.9x + 4.2y = 15.3$
$-5.4x + 7.9y = 16.7$

20. $6.3x - 7.4y = 18.6$
$2.4x + 3.5y = 10.2$

In Exercises 21–40, solve each system of linear equations by elimination.

21. $x - y = -3$
$x + y = 7$

22. $x - y = -10$
$x + y = 8$

23. $5x + 3y = -3$
$3x - 3y = -21$

24. $-2x + 3y = 1$
$2x - y = 7$

25. $2x - 7y = 4$
$5x + 7y = 3$

26. $3x + 2y = 6$
$-3x + 6y = 18$

27. $2x + 5y = 7$
$3x - 10y = 5$

28. $6x - 2y = 3$
$-3x + 2y = -2$

29. $2x + 5y = 5$
$-4x - 10y = -10$

30. $11x + 3y = 3$
$22x + 6y = 6$

31. $3x - 2y = 12$
$4x + 3y = 16$

32. $5x - 2y = 7$
$3x + 5y = 29$

33. $6x - 3y = -15$
$7x + 2y = -12$

34. $7x - 4y = -1$
$3x - 5y = 16$

35. $4x - 5y = 22$
$3x + 4y = 1$

36. $6x - 5y = 32$
$2x - 6y = 2$

37. $\frac{1}{3}x + \frac{1}{2}y = 1$
$\frac{1}{5}x + \frac{7}{2}y = 2$

38. $\frac{1}{2}x - \frac{1}{3}y = 0$
$\frac{3}{2}x + \frac{1}{2}y = \frac{3}{4}$

39. $3.4x + 1.7y = 8.33$
$-2.7x - 7.8y = 15.96$

40. $-0.04x + 1.12y = 9.815$
$2.79x + 1.19y = -0.165$

In Exercises 41–44, match the systems of equations with the graphs.

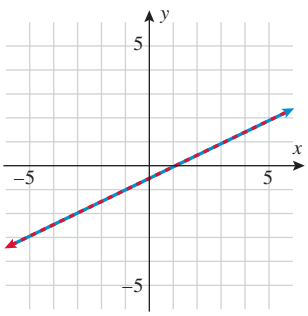
41. $3x - y = 1$
$3x + y = 5$

42. $-x + 2y = -1$
$2x + y = 7$

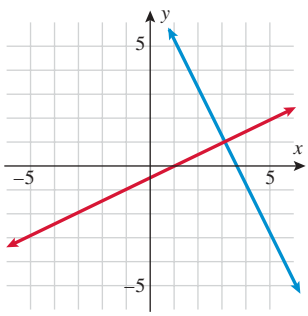
43. $2x + y = 3$
$2x + y = 7$

44. $x - 2y = 1$
$2x - 4y = 2$

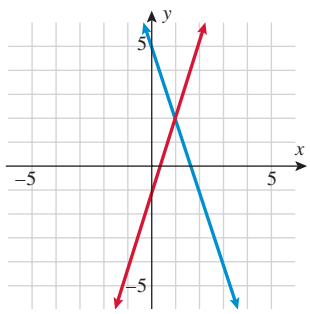
a.



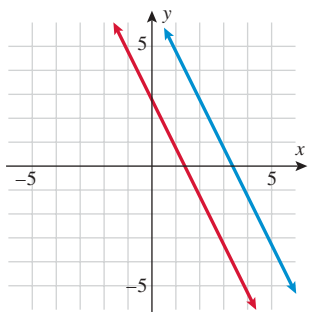
b.



c.



d.



In Exercises 45–52, solve each system of linear equations by graphing.

45. $y = -x$
$y = x$

46. $x - 3y = 0$
$x + 3y = 0$

47. $2x + y = -3$
$x + y = -2$

48. $x - 2y = -1$
$-x - y = -5$

49. $\frac{1}{2}x - \frac{2}{3}y = 4$
$\frac{1}{4}x - y = 6$

50. $\frac{1}{5}x - \frac{5}{2}y = 10$
$\frac{1}{15}x - \frac{5}{6}y = \frac{10}{3}$

51. $1.6x - y = 4.8$
$-0.8x + 0.5y = 1.5$

52. $1.1x - 2.2y = 3.3$
$-3.3x + 6.6y = -6.6$

In Exercises 53–58, use any method to solve each system of linear equations.

53. $x - y = 2$
$x + y = 4$

54. $-0.5x + 0.3y = 0.8$
$-1.5x + 0.9y = 2.4$

55. $x - y = 1$
$x + y = 1$

56. $x + y = 2$
$x - y = -2$

57. $0.02x + 0.05y = 1.25$
$-0.06x - 0.15y = -3.75$

58. $x - y = 2$
$x + y = -2$

APPLICATIONS

59. Environment. Approximately 2 million dry erase markers are disposed of by teachers each year. Traditional dry erase markers are toxic and nonbiodegradable. EcoSmart World sells AusPens, which are dry erase markers that are recyclable, refillable, and nontoxic. The markers are available in a kit containing x AusPens and x refill ink bottles, each a different color. One refill ink bottle can refill up to 40 AusPens, and one kit is equivalent to y dry erase markers. If one kit is equivalent to 246 traditional dry erase markers, find the number of AusPens that are in each kit.

60. Pharmacy. A pharmacy technician receives an order for 454 grams of a 3% zinc oxide cream. If the pharmacy has 1% and 10% zinc oxide creams in stock, how much of each should be mixed to fill the order?

61. Mixture. In chemistry lab, Stephanie has to make a 37 milliliter solution that is 12% HCl. All that is in the lab is 8% and 15% HCl. How many milliliters of each solution should she use to obtain the desired mix?

62. Mixture. A mechanic has 340 gallons of gasoline and 10 gallons of oil to make gas/oil mixtures. He wants one mixture to be 4% oil and the other mixture to be 2.5% oil. If he wants to use all of the gas and oil, how many gallons of gas and oil are in each of the resulting mixtures?

63. Salary Comparison. Upon graduation with a degree in management of information systems (MIS), you decide to work for a company that buys data from states' departments of motor vehicles and sells to banks and car dealerships customized reports detailing how many cars at each dealership are financed through particular banks. Autocount Corporation offers you a $15,000 base salary and 10% commission on your total annual sales. Polk Corporation offers you a base salary of $30,000 plus a 5% commission on your total annual sales. How many total sales would you have to make per year to earn more money at Autocount?

64. Salary Comparison. Two types of residential real estate agents are those who sell existing houses (resale) and those who sell new homes for developers. Resale of existing homes typically earns 6% commission on every sale, and representing developers in selling new homes typically earns a base salary of $15,000 per year plus an additional 1.5% commission, because agents are required to work 5 days a week on site in a new development. Find the total value (dollars) an agent would have to sell per year to make more money in resale than in new homes?

65. Gas Mileage. A Honda Accord gets approximately 26 mpg on the highway and 19 mpg in the city. You drove 349.5 miles on a full tank (16 gallons) of gasoline. Approximately how many miles did you drive in the city and how many on the highway?

66. Wireless Plans. AT&T is offering a 600-minute peak plan with free mobile-to-mobile and weekend minutes at $59 per month plus $0.13 per minute for every minute over 600. The next plan up is the 800-minute plan that costs $79 per month. You think you may go over 600 minutes, but are not sure you need 800 minutes. How many minutes would you have to talk for the 800-minute plan to be the better deal?

67. Distance/Rate/Time. A direct flight on Delta Air Lines from Atlanta to Paris is 4000 miles and takes approximately 8 hours going east (Atlanta to Paris) and 10 hours going west (Paris to Atlanta). Although the plane averages the same airspeed, there is a headwind while traveling west and a tailwind while traveling east, resulting in different air speeds. What is the average air speed of the plane, and what is the average wind speed?

68. Distance/Rate/Time. A private pilot flies a Cessna 172 on a trip that is 500 miles each way. It takes her approximately 3 hours to get there and 4 hours to return. What is the approximate average air speed of the Cessna, and what is the approximate wind speed?

69. Investment Portfolio. Leticia has been tracking two volatile stocks. Stock A over the last year has increased 10%, and stock B has increased 14% (using a simple interest model). She has $10,000 to invest and would like to split it between these two stocks. If the stocks continue to perform at the same rate, how much should she invest in each for one year to result in a balance of $11,260?

70. Investment Portfolio. Toby split his savings into two different investments, one earning 5% and the other earning 7%. He put twice as much in the investment earning the higher rate. In one year, he earned $665 in interest. How much money did he invest in each account?

71. Break-Even Analysis. A company produces CD players for a unit cost of $15 per CD player. The company has fixed costs of $120. If each CD player can be sold for $30, how many CD players must be sold to break even? Determine the cost equation first. Next, determine the revenue equation. Use the two equations you have found to determine the break-even point.

72. Managing a Lemonade Stand. An elementary-school-age child wants to have a lemonade stand. She would sell each glass of lemonade for $0.25. She has determined that each glass of lemonade costs about $0.10 to make (for lemons and sugar). It costs her $15.00 for materials to make the lemonade stand. How many glasses of lemonade must she sell to break even?

73. Meal Cost. An airline is deciding which meals to buy from its provider. If the airline orders the same number of meals of types I and II totaling 150 meals, the cost is \$1275; if they order 60% of type I and 40% of type II, the cost is \$1260. What is the cost of each type of meal?

74. Meal Cost. In a school district, the board of education has decided on two menus to serve in the school cafeterias. The annual budget for the meal plan is \$1.2 million and one of the menus is 5% more expensive than the other. What is the annual cost of each menu? Round your answer to the nearest integer.

75. Population. The U.S. Census Bureau reports that Florida's population in the year 2008 was 18,328,340 habitants. The number of females exceeded the number of males by 329,910. What is the number of habitants, by gender, in Florida in 2008?

76. Population. According to the U.S. Census Bureau, in 2000, the U.S. population was 281,420,906 habitants. Some projections indicate that by 2020 there will be 341,250,007 habitants. The number of senior citizens will increase 30%, while the number of citizens under the age of 65 will increase 20%. Find the number of senior citizens and nonsenior citizens in the year 2000. Round your answer to the nearest integer.

CATCH THE MISTAKE

In Exercises 77–80, explain the mistake that is made.

77. Solve the system of equations by elimination.

$$\begin{aligned} 2x + y &= -3 \\ 3x + y &= 8 \end{aligned}$$

Solution:

Multiply Equation (1) by -1. $\quad 2x - y = -3$

Add the result to Equation (2).

$$\begin{array}{r} 2x - y = -3 \\ 3x + y = 8 \\ \hline 5x \quad = 5 \end{array}$$

Solve for x. $\quad x = 1$

Substitute $x = 1$ into Equation (2). $\quad 3(1) + y = 8$

$$y = 5$$

The answer (1, 5) is incorrect. What mistake was made?

78. Solve the system of equations by elimination.

$$\begin{aligned} 4x - y &= 12 \\ 4x - y &= 24 \end{aligned}$$

Solution:

Multiply Equation (1) by -1. $\quad -4x + y = -12$

Add the result to Equation (2).

$$\begin{array}{r} -4x + y = -12 \\ 4x - y = 24 \\ \hline 0 = 12 \end{array}$$

Answer: Infinitely many solutions.

This is incorrect. What mistake was made?

79. Solve the system of equations by substitution.

$$\begin{aligned} x + 3y &= -4 \\ -x + 2y &= -6 \end{aligned}$$

Solution:

Solve Equation (1) for x. $\quad x = -3y - 4$

Substitute $x = -3y - 4$ into Equation (2). $\quad -(-3y - 4) + 2y = -6$

Solve for y.

$$\begin{aligned} 3y - 4 + 2y &= -6 \\ 5y &= -2 \\ y &= -\frac{2}{5} \end{aligned}$$

Substitute $y = -\frac{2}{5}$ into Equation (1). $\quad x + 3\left(-\frac{2}{5}\right) = -4$

Solve for x. $\quad x = -\frac{14}{5}$

The answer $\left(-\frac{2}{5}, -\frac{14}{5}\right)$ is incorrect. What mistake was made?

80. Solve the system of equations by graphing.

$$\begin{aligned} 2x + 3y &= 5 \\ 4x + 6y &= 10 \end{aligned}$$

Solution:

Write both equations in slope–intercept form.

$$y = -\frac{2}{3}x + \frac{5}{3}$$

$$y = -\frac{2}{3}x + \frac{5}{3}$$

Since these lines have the same slope, they are parallel lines.

Parallel lines do not intersect, so there is no solution.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 81–84, determine whether each statement is true or false on the *xy*-plane.

81. A system of equations represented by a graph of two lines with the same slope always has no solution.

82. A system of equations represented by a graph of two lines with slopes that are negative reciprocals always has one solution.

83. If two lines do not have exactly one point of intersection, then they must be parallel.

84. The system of equations, $Ax - By = 1$ and $-Ax + By = -1$, has no solution.

85. The point $(2, -3)$ is a solution to the system of equations

$$\begin{aligned} Ax + By &= -29 \\ Ax - By &= 13 \end{aligned}$$

Find A and B.

86. If you graph the lines

$$\begin{aligned} x - 50y &= 100 \\ x - 48y &= -98 \end{aligned}$$

they appear to be parallel lines. However, there is a unique solution. Explain how this might be possible.

CHALLENGE

87. Energy Drinks. A nutritionist wishes to market a new vitamin-enriched fruit drink and is preparing two versions of it to distribute at a local health club. She has 100 cups of pineapple juice and 4 cups of super vitamin-enriched pomegranate concentrate. One version of the drink is to contain 2% pomegranate and the other version 4% pomegranate. How much of each drink can she create if drinks are 1 cup and she uses all of the ingredients?

88. Easter Eggs. A family is coloring Easter eggs and wants to make 2 shades of purple, "light purple" and "deep purple." They have 30 tablespoons of deep red solution and 2 tablespoons of blue solution. If "light purple" consists of 2% blue solution and "deep purple" consists of 10% blue solution, how much of each version of purple solution can be created?

89. The line $y = mx + b$ connects the points $(-2, 4)$ and $(4, -2)$. Find the values of m and b.

90. Find b and c such that the parabola $y = x^2 + bx + c$ goes through the points $(2, 7)$ and $(-6, 7)$.

91. Find b and c such that the parabola $y = bx^2 + bx + c$ goes through the points $(4, 46)$ and $(-2, 10)$.

92. The system of equations

$$\begin{aligned} x^2 + y^2 &= 4 \\ x^2 - y^2 &= 2 \end{aligned}$$

can be solved by a change of variables. Taking $u = x^2$ and $v = y^2$, we can transform the system into

$$\begin{aligned} u + v &= 4 \\ u - v &= 2 \end{aligned}$$

Find the solutions of the original system.

93. The system of equations

$$\begin{aligned} x^2 + 2y^2 &= 11 \\ 4x^2 + y^2 &= 16 \end{aligned}$$

can be solved by a change of variables. Taking $u = x^2$ and $v = y^2$, we can transform the system into

$$\begin{aligned} u + 2v &= 11 \\ 4u + v &= 16 \end{aligned}$$

Find the solutions of the original system.

94. The parabola $y = bx^2 - 2x - a$ goes through the points $(-2, a)$ and $(-1, b - 2)$. Find a and b.

TECHNOLOGY

95. Apply a graphing utility to graph the two equations $y = -1.25x + 17.5$ and $y = 2.3x - 14.1$. Approximate the solution to this system of linear equations.

96. Apply a graphing utility to graph the two equations $y = 14.76x + 19.43$ and $y = 2.76x + 5.22$. Approximate the solution to this system of linear equations.

97. Apply a graphing utility to graph the two equations $23x + 15y = 7$ and $46x + 30y = 14$. Approximate the solution to this system of linear equations.

98. Apply a graphing utility to graph the two equations $-3x + 7y = 2$ and $6x - 14y = 3$. Approximate the solution to this system of linear equations.

99. Apply a graphing utility to graph the two equations $\frac{1}{3}x - \frac{5}{12}y = \frac{5}{6}$ and $\frac{3}{7}x + \frac{1}{14}y = \frac{29}{28}$. Approximate the solution to this system of linear equations.

100. Apply a graphing utility to graph the two equations $\frac{5}{9}x + \frac{11}{13}y = 2$ and $\frac{3}{4}x + \frac{5}{7}y = \frac{13}{14}$. Approximate the solution to this system of linear equations.

PREVIEW TO CALCULUS

For Exercises 101–104, refer to the following:

In calculus, when integrating rational functions, we decompose the function into partial fractions. This technique involves the solution of systems of equations. For example, suppose

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2} = \frac{A(x + 2) + B(x - 1)}{(x - 1)(x + 2)}$$

and we want to find A and B such that $1 = A(x + 2) + B(x - 1)$, which is equivalent to $1 = (A + B)x + (2A - B)$. From this equation, we obtain the system of equations

$$\begin{aligned} A + B &= 0 \\ 2A - B &= 1 \end{aligned}$$

which solution is $\left(\frac{1}{3}, -\frac{1}{3}\right)$.

Find the values of A and B that make each equation true.

101. $x + 5 = A(x + 2) + B(x - 4)$

102. $6x = A(x + 1) + B(x - 2)$

103. $x + 1 = A(x + 2) + B(x - 3)$

104. $5 = A(x - 2) + B(2x + 1)$

SECTION 8.2 SYSTEMS OF LINEAR EQUATIONS IN THREE VARIABLES

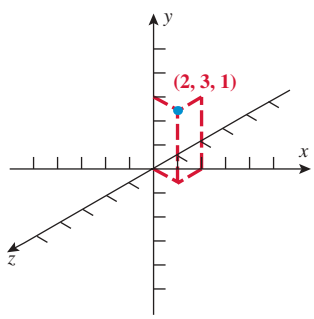
SKILLS OBJECTIVES

- Solve systems of linear equations in three variables using a combination of both the elimination method and the substitution method.
- Solve application problems using systems of linear equations in three variables.

CONCEPTUAL OBJECTIVES

- Understand that a graph of a linear equation in three variables corresponds to a plane.
- Identify three types of solutions: one solution (point), no solution, or infinitely many solutions (a single line in three-dimensional space or a plane).

Solving Systems of Linear Equations in Three Variables



In Section 8.1, we solved systems of two linear equations in two variables. Graphs of linear equations in two variables correspond to lines. Now we turn our attention to linear equations in *three* variables. A **linear equation in three variables**, x, y, and z, is given by

$$Ax + By + Cz = D$$

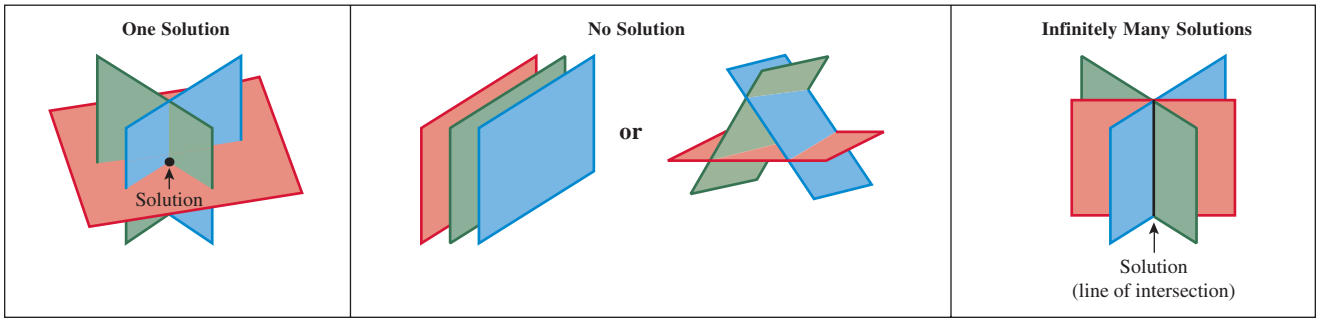
where A, B, C, and D are real numbers that are not all equal to zero. All three variables have degree equal to one, which is why this is called a linear equation in three variables. The graph of any equation in three variables requires a three-dimensional coordinate system.

The x-axis, y-axis, and z-axis are each perpendicular to the other two. For the three-dimensional coordinate system on the right, a point $(x, y, z) = (2, 3, 1)$ is found by starting at the origin, moving two units to the right, three units up, and one unit out toward you.

Study Tip

If all three planes are coplaner (the same plane) there are infinitely many solutions.

In two variables, the graph of a linear equation is a line. In three variables, however, the graph of a linear equation is a **plane**. A plane can be thought of as an infinite sheet of paper. When solving systems of linear equations in three variables, we find one of three possibilities: one solution, no solution, or infinitely many solutions.



There are many ways to solve systems of linear equations in more than two variables. One method is to combine the elimination and substitution methods, which will be discussed in this section. Other methods involve matrices, which will be discussed in Sections 8.3–8.5. We now outline a procedure for solving systems of linear equations in three variables, which can be extended to solve systems of more than three variables. Solutions are sometimes given as ordered triples of the form (x, y, z).

SOLVING SYSTEMS OF LINEAR EQUATIONS IN THREE VARIABLES USING ELIMINATION AND SUBSTITUTION

Step 1: Reduce the system of three equations in three variables to two equations in two (of the same) variables by applying elimination.

Step 2: Solve the resulting system of two linear equations in two variables by applying elimination or substitution.

Step 3: Substitute the solutions in Step 2 into *any* one of the original equations and solve for the third variable.

Step 4: Check that the solution satisfies *all* three original equations.

EXAMPLE 1 Solving a System of Linear Equations in Three Variables

Solve the system:

$$\begin{aligned} 2x + y + 8z &= -1 && \text{Equation (1)} \\ x - y + z &= -2 && \text{Equation (2)} \\ 3x - 2y - 2z &= 2 && \text{Equation (3)} \end{aligned}$$

Solution:

Inspecting the three equations, we see that y is easily eliminated when Equations (1) and (2) are added, because the coefficients of y, $+1$ and -1, are equal in magnitude and opposite in sign. We can also eliminate y from Equation (3) by adding Equation (3) to *either* 2 times Equation (1) *or* -2 times Equation (2). Therefore, our plan of attack is to eliminate y from the system of equations, so the result will be two equations in two variables x and z.

STEP 1 Eliminate y in Equation (1) and Equation (2).

Equation (1): $2x + y + 8z = -1$
Equation (2): $x - y + z = -2$
Add. $\mathbf{3x + 9z = -3}$

Eliminate y in Equation (2) and Equation (3).

Multiply Equation (2) by -2. $-2x + 2y - 2z = 4$
Equation (3): $3x - 2y - 2z = 2$
Add. $\mathbf{x - 4z = 6}$

STEP 2 Solve the system of two linear equations in two variables.

$$\mathbf{3x + 9z = -3}$$
$$\mathbf{x - 4z = 6}$$

Substitution* method: $x = 4z + 6$ $\quad 3(4z + 6) + 9z = -3$

Distribute. $\quad 12z + 18 + 9z = -3$

Combine like terms. $\quad 21z = -21$

Solve for z. $\quad \boxed{z = -1}$

Substitute $z = -1$ into $x = 4z + 6$. $\quad x = 4(-1) + 6 = 2$

$\boxed{x = 2}$ and $\boxed{z = -1}$ are the solutions to the system of two equations.

STEP 3 Substitute $x = 2$ and $z = -1$ into any one of the three original equations and solve for y.

Substitute $x = 2$ and $z = -1$ into Equation (2). $\quad 2 - y - 1 = -2$

Solve for y. $\quad \boxed{y = 3}$

STEP 4 Check that $x = 2$, $y = 3$, and $z = -1$ satisfy all three equations.

Equation (1): $2(2) + 3 + 8(-1) = 4 + 3 - 8 = -1$

Equation (2): $2 - 3 - 1 = -2$

Equation (3): $3(2) - 2(3) - 2(-1) = 6 - 6 + 2 = 2$

The solution is $\boxed{x = 2, y = 3, z = -1, \text{ or } (2, 3, -1)}$.

*Elimination method could also be used.

YOUR TURN Solve the system:

$$2x - y + 3z = -1$$
$$x + y - z = 0$$
$$3x + 3y - 2z = 1$$

Study Tip

First eliminate the *same* variable from two different pairs of equations.

Answer:
$x = -1, y = 2, z = 1$

In Example 1 and the Your Turn, the variable y was eliminated by adding the first and second equations. In practice, any of the three variables can be eliminated, but typically we select the most convenient variable to eliminate. If a variable is missing from one of the equations (has a coefficient of 0), then we eliminate that variable from the other two equations.

EXAMPLE 2 Solving a System of Linear Equations in Three Variables When One Variable Is Missing

Solve the system:

$$\begin{aligned} x \quad\quad + z &= 1 && \text{Equation (1)} \\ 2x + y - z &= -3 && \text{Equation (2)} \\ x + 2y - z &= -1 && \text{Equation (3)} \end{aligned}$$

Solution:

Since y is missing from Equation (1), y is the variable to be eliminated in Equation (2) and Equation (3).

STEP 1 Eliminate y.

Multiply Equation (2) by -2. $\quad -4x - 2y + 2z = 6$

Equation (3): $\quad x + 2y - z = -1$

Add. $\quad -3x + z = 5$

STEP 2 Solve the system of two equations. $\quad x + z = 1$

Equation (1) and the resulting equation in Step 1. $\quad -3x + z = 5$

Multiply the second equation by (-1) and add it to the first equation.

$$\begin{aligned} x + z &= 1 \\ 3x - z &= -5 \\ \hline 4x \quad &= -4 \end{aligned}$$

Solve for x. $\quad x = -1$

Substitute $x = -1$ into Equation (1). $\quad -1 + z = 1$

Solve for z. $\quad z = 2$

STEP 3 Substitute $x = -1$ and $z = 2$ into one of the original equations [Equation (2) or Equation (3)] and solve for y.

Substitute $x = -1$ and $z = 2$ into $x + 2y - z = -1$. $\quad (-1) + 2y - 2 = -1$

Gather like terms. $\quad 2y = 2$

Solve for y. $\quad y = 1$

STEP 4 Check that $x = -1$, $y = 1$, and $z = 2$ satisfy all three equations.

Equation (1): $(-1) + 2 = 1$

Equation (2): $2(-1) + (1) - (2) = -3$

Equation (3): $(-1) + 2(1) - (2) = -1$

The solution is $\boxed{x = -1, y = 1, z = 2}$.

■ **Answer:** $x = 1, y = 2, z = -3$

■ **YOUR TURN** Solve the system:

$$\begin{aligned} x + y + z &= 0 \\ 2x \quad\quad + z &= -1 \\ x - y - z &= 2 \end{aligned}$$

Types of Solutions

Systems of linear equations in three variables have three possible solutions: one solution, infinitely many solutions, or no solution. Examples 1 and 2 each had one solution. Examples 3 and 4 illustrate systems with infinitely many solutions and no solution, respectively.

EXAMPLE 3 A Dependent System of Linear Equations in Three Variables (Infinitely Many Solutions)

Solve the system:

$$\begin{aligned} 2x + y - z &= 4 && \text{Equation (1)} \\ x + y &= 2 && \text{Equation (2)} \\ 3x + 2y - z &= 6 && \text{Equation (3)} \end{aligned}$$

Solution:

Since z is missing from Equation (2), z is the variable to be eliminated from Equation (1) and Equation (3).

STEP 1 Eliminate z.

Multiply Equation (1) by (-1). $\quad -2x - y + z = -4$

Equation (3): $\quad 3x + 2y - z = 6$

Add. $\quad x + y = 2$

STEP 2 Solve the system of two equations:

Equation (2) and the resulting equation in Step 1.

$$\begin{aligned} x + y &= 2 \\ x + y &= 2 \end{aligned}$$

Multiply the first equation by (-1) and add it to the second equation.

$$\begin{aligned} -x - y &= -2 \\ x + y &= 2 \\ \hline 0 &= 0 \end{aligned}$$

This statement is always true; therefore, there are $\boxed{\text{infinitely many solutions}}$. The original system has been reduced to a system of two identical linear equations. Therefore, the equations are dependent (share infinitely many solutions). Typically, to define those infinitely many solutions, we let $z = a$, where a stands for any real number, and then find x and y in terms of a. The resulting ordered triple showing the three variables in terms of a is called a **parametric representation** of a line in three dimensions.

STEP 3 Let $\boxed{z = a}$ and find x and y in terms of a.

Solve Equation (2) for y. $\quad y = 2 - x$

Let $y = 2 - x$ and $z = a$ in Equation (1). $\quad 2x + (2 - x) - a = 4$

Solve for x. $\quad 2x + 2 - x - a = 4$

$$x - a = 2$$

$$\boxed{x = a + 2}$$

Let $\boxed{x = a + 2}$ in Equation (2). $\quad (a + 2) + y = 2$

Solve for y. $\quad \boxed{y = -a}$

The infinitely many solutions are written as $\boxed{(a + 2, -a, a)}$.

STEP 4 Check that $x = a + 2$, $y = -a$, and $z = a$ satisfy all three equations.

Equation (1): $2(a + 2) + (-a) - a = 2a + 4 - a - a = 4$ ✓

Equation (2): $(a + 2) + (-a) = a + 2 - a = 2$ ✓

Equation (3): $3(a + 2) + 2(-a) - a = 3a + 6 - 2a - a = 6$ ✓

■ **Answer:** $(a - 1, a + 1, a)$

■ **YOUR TURN** Solve the system:

$$\begin{aligned} x + y - 2z &= 0 \\ x - z &= -1 \\ x - 2y + z &= -3 \end{aligned}$$

EXAMPLE 4 An Inconsistent System of Linear Equations in Three Variables (No Solution)

Solve the system:

$$\begin{aligned} x + 2y - z &= 3 && \text{Equation (1)} \\ 2x + y + 2z &= -1 && \text{Equation (2)} \\ -2x - 4y + 2z &= 5 && \text{Equation (3)} \end{aligned}$$

Solution:

STEP 1 Eliminate x.

Multiply Equation (1) by -2. $\quad -2x - 4y + 2z = -6$

Equation (2): $\quad 2x + y + 2z = -1$

Add. $\quad \mathbf{-3y + 4z = -7}$

Equation (2): $\quad 2x + y + 2z = -1$

Equation (3): $\quad -2x - 4y + 2z = 5$

Add. $\quad \mathbf{-3y + 4z = 4}$

STEP 2 Solve the system of two equations:

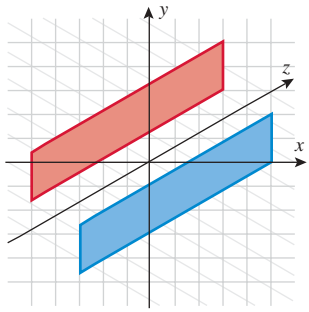
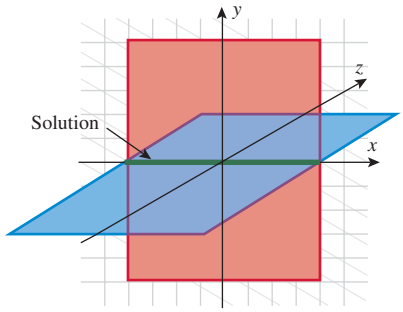
$$\begin{aligned} \mathbf{-3y + 4z} &\mathbf{= -7} \\ \mathbf{-3y + 4z} &\mathbf{= 4} \end{aligned}$$

Multiply the top equation by (-1) and add it to the second equation.

$$\begin{aligned} 3y - 4z &= 7 \\ -3y + 4z &= 4 \\ \hline 0 &= 11 \end{aligned}$$

This is a contradiction, or inconsistent statement, and therefore, there is $\boxed{\text{no solution}}$.

So far in this section, we have discussed only systems of *three* linear equations in *three* variables. What happens if we have a system of *two* linear equations in *three* variables? The two linear equations in three variables will always correspond to two planes in three dimensions. The possibilities are no solution (the two planes are parallel) or infinitely many solutions (the two planes intersect in a line or two planes are coplanar).

No Solution	Infinitely Many Solutions (line)
	

EXAMPLE 5 Solving a System of Two Linear Equations in Three Variables

Solve the system of linear equations: $x - y + z = 7$ Equation (1)

$x + y + 2z = 2$ Equation (2)

Solution:

Eliminate y by adding the two equations.

$$\begin{array}{r} x - y + z = 7 \\ x + y + 2z = 2 \\ \hline 2x \quad + 3z = 9 \end{array}$$

Therefore, Equation (1) and Equation (2) are both true if $2x + 3z = 9$. Since we know there is a solution, it must be a line. To define the line of intersection, we again turn to parametric representation.

Let $\boxed{z = a}$, where a is any real number. $2x + 3a = 9$

Solve for x. $\boxed{x = \frac{9}{2} - \frac{3}{2}a}$

Substitute $z = a$ and $x = \frac{9}{2} - \frac{3}{2}a$ into Equation (1). $\left(\frac{9}{2} - \frac{3}{2}a\right) - y + a = 7$

Solve for y. $\boxed{y = -\frac{1}{2}a - \frac{5}{2}}$

The solution is the line in three dimensions given by $\boxed{\left(\frac{9}{2} - \frac{3}{2}a, -\frac{1}{2}a - \frac{5}{2}, a\right)}$, where a is any real number.

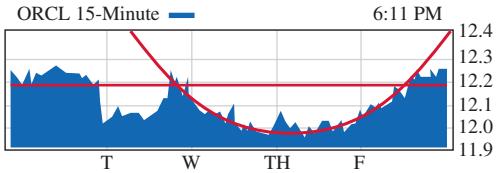
Note: Every real number a corresponds to a point on the line of intersection.

a	$\left(\frac{9}{2} - \frac{3}{2}a, -\frac{1}{2}a - \frac{5}{2}, a\right)$
-1	$(6, -2, -1)$
0	$\left(\frac{9}{2}, -\frac{5}{2}, 0\right)$
1	$(3, -3, 1)$

Modeling with a System of Three Linear Equations

Many times in the real world we see a relationship that looks like a particular function such as a quadratic function and we know particular data points, but we do not know the function. We start with the general function, fit the curve to particular data points, and solve a system of linear equations to determine the specific function parameters.

Suppose you want to model a stock price as a function of time and based on the data you feel a quadratic model would be the best fit.



Therefore, the model is given by

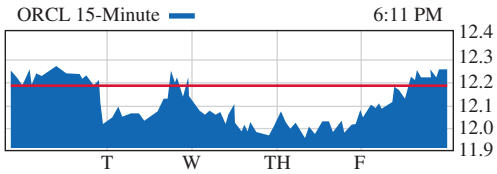
$$P(t) = at^2 + bt + c$$

where $P(t)$ is the price of the stock at time t. If we have data corresponding to three distinct points $[t, P(t)]$, the result is a system of three linear equations in three variables a, b, and c. We can solve the resulting system of linear equations, which determines the coefficients a, b, and c of the quadratic model for stock price.

EXAMPLE 6 Stock Value

The Oracle Corporation's stock (ORCL) over 3 days (Wednesday, October 13, to Friday, October 15, 2004) can be approximately modeled by a quadratic function: $f(t) = at^2 + bt + c$. If Wednesday corresponds to $t = 1$, where t is in days, then the following data points approximately correspond to the stock value:

t	$f(t)$	DAYS
1	\$12.20	Wednesday
2	\$12.00	Thursday
3	\$12.20	Friday



Determine the function that models this behavior.

Solution:

Substitute the points (1, 12.20), (2, 12.00), and (3, 12.20) into $f(t) = at^2 + bt + c$.

$$a(1)^2 + b(1) + c = 12.20$$
$$a(2)^2 + b(2) + c = 12.00$$
$$a(3)^2 + b(3) + c = 12.20$$

Simplify to a system of three equations in three variables (a, b, and c).

$$\begin{aligned} a + b + c &= 12.20 && \text{Equation (1)} \\ 4a + 2b + c &= 12.00 && \text{Equation (2)} \\ 9a + 3b + c &= 12.20 && \text{Equation (3)} \end{aligned}$$

Solve for a, b, and c by applying the technique of this section.

STEP 1 Eliminate c.

Multiply Equation (1) by (-1).	$-a - b - c = -12.20$
Equation (2):	$4a + 2b + c = 12.20$
Add.	$\mathbf{3a + b = -0.20}$
Multiply Equation (1) by -1.	$-a - b - c = -12.20$
Equation (3):	$9a + 3b + c = 12.20$
Add.	$\mathbf{8a + 2b = 0}$

STEP 2 Solve the system of two equations.

$$\begin{aligned} 3a + b &= -0.20 \\ 8a + 2b &= 0 \end{aligned}$$

Multiply the first equation by -2 and add to the second equation.

$$\begin{aligned} -6a - 2b &= 0.40 \\ 8a + 2b &= 0 \end{aligned}$$

Add. $2a = 0.4$

Solve for a. $a = 0.2$

Substitute $a = 0.2$ into $8a + 2b = 0$. $8(0.2) + 2b = 0$

Simplify. $2b = -1.6$

Solve for b. $b = -0.8$

STEP 3 Substitute $a = 0.2$ and $b = -0.8$ into one of the original three equations.

Substitute $a = 0.2$ and $b = -0.8$ into $a + b + c = 12.20$. $0.2 - 0.8 + c = 12.20$

Gather like terms. $-0.6 + c = 12.20$

Solve for c. $c = 12.80$

STEP 4 Check that $a = 0.2$, $b = -0.8$, and $c = 12.80$ satisfy all three equations.

Equation (1): $a + b + c = 0.2 - 0.8 + 12.8 = 12.20$

Equation (2): $4a + 2b + c = 4(0.2) + 2(-0.8) + 12.80$
$= 0.8 - 1.6 + 12.8 = 12.00$

Equation (3): $9a + 3b + c = 9(0.2) + 3(-0.8) + 12.80$
$= 1.8 - 2.4 + 12.8 = 12.20$

The model is given by $\boxed{f(t) = 0.2t^2 - 0.8t + 12.80}$.

SECTION 8.2 SUMMARY

Graphs of linear equations in *two* variables are *lines*, whereas graphs of linear equations in *three* variables are *planes*. Systems of linear equations in three variables have one of three solutions:

- One solution (the intersection point of the three planes)
- No solution (no intersection of all three planes)
- Infinitely many solutions (planes intersect along a line)

When the solution to a system of three linear equations is a line in three dimensions, we use parametric representation to express the solution.

SECTION 8.2 EXERCISES

SKILLS

In Exercises 1–32, solve each system of linear equations.

1. $$\begin{aligned} x - y + z &= 6 \\ -x + y + z &= 3 \\ -x - y - z &= 0 \end{aligned}$$

2. $$\begin{aligned} -x - y + z &= -1 \\ -x + y - z &= 3 \\ x - y - z &= 5 \end{aligned}$$

3. $$\begin{aligned} x + y - z &= 2 \\ -x - y - z &= -3 \\ -x + y - z &= 6 \end{aligned}$$

4. $$\begin{aligned} x + y + z &= -1 \\ -x + y - z &= 3 \\ -x - y + z &= 8 \end{aligned}$$

5. $$\begin{aligned} -x + y - z &= -1 \\ x - y - z &= 3 \\ x + y - z &= 9 \end{aligned}$$

6. $$\begin{aligned} x - y - z &= 2 \\ -x - y + z &= 4 \\ -x + y - z &= 6 \end{aligned}$$

7. $$\begin{aligned} 2x - 3y + 4z &= -3 \\ -x + y + 2z &= 1 \\ 5x - 2y - 3z &= 7 \end{aligned}$$

8. $$\begin{aligned} x - 2y + z &= 0 \\ -2x + y - z &= -5 \\ 13x + 7y + 5z &= 6 \end{aligned}$$

9. $\begin{aligned} 3y - 4x + 5z &= 2 \\ 2x - 3y - 2z &= -3 \\ 3z + 4y - 2x &= 1 \end{aligned}$

10. $\begin{aligned} 2y + z - x &= 5 \\ 2x + 3z - 2y &= 0 \\ -2z + y - 4x &= 3 \end{aligned}$

11. $\begin{aligned} x - y + z &= -1 \\ y - z &= -1 \\ -x + y + z &= 1 \end{aligned}$

12. $\begin{aligned} -y + z &= 1 \\ x - y + z &= -1 \\ x - y - z &= -1 \end{aligned}$

13. $\begin{aligned} 3x - 2y - 3z &= -1 \\ x - y + z &= -4 \\ 2x + 3y + 5z &= 14 \end{aligned}$

14. $\begin{aligned} 3x - y + z &= 2 \\ x - 2y + 3z &= 1 \\ 2x + y - 3z &= -1 \end{aligned}$

15. $\begin{aligned} -3x - y - z &= 2 \\ x + 2y - 3z &= 4 \\ 2x - y + 4z &= 6 \end{aligned}$

16. $\begin{aligned} 2x - 3y + z &= 1 \\ x + 4y - 2z &= 2 \\ 3x - y + 4z &= -3 \end{aligned}$

17. $\begin{aligned} 3x + 2y + z &= 4 \\ -4x - 3y - z &= -15 \\ x - 2y + 3z &= 12 \end{aligned}$

18. $\begin{aligned} 3x - y + 4z &= 13 \\ -4x - 3y - z &= -15 \\ x - 2y + 3z &= 12 \end{aligned}$

19. $\begin{aligned} -x + 2y + z &= -2 \\ 3x - 2y + z &= 4 \\ 2x - 4y - 2z &= 4 \end{aligned}$

20. $\begin{aligned} 2x - y &= 1 \\ -x + z &= -2 \\ -2x + y &= -1 \end{aligned}$

21. $\begin{aligned} x - z - y &= 10 \\ 2x - 3y + z &= -11 \\ y - x + z &= -10 \end{aligned}$

22. $\begin{aligned} 2x + z + y &= -3 \\ 2y - z + x &= 0 \\ x + y + 2z &= 5 \end{aligned}$

23. $\begin{aligned} 3x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 + x_3 &= -3 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$

24. $\begin{aligned} 2x_1 + x_2 + x_3 &= -1 \\ x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 - x_3 &= 1 \end{aligned}$

25. $\begin{aligned} 2x + 5y &= 9 \\ x + 2y - z &= 3 \\ -3x - 4y + 7z &= 1 \end{aligned}$

26. $\begin{aligned} x - 2y + 3z &= 1 \\ -2x + 7y - 9z &= 4 \\ x + z &= 9 \end{aligned}$

27. $\begin{aligned} 2x_1 - x_2 + x_3 &= 3 \\ x_1 - x_2 + x_3 &= 2 \\ -2x_1 + 2x_2 - 2x_3 &= -4 \end{aligned}$

28. $\begin{aligned} x_1 - x_2 - 2x_3 &= 0 \\ -2x_1 + 5x_2 + 10x_3 &= -3 \\ 3x_1 + x_2 &= 0 \end{aligned}$

29. $\begin{aligned} 2x + y - z &= 2 \\ x - y - z &= 6 \end{aligned}$

30. $\begin{aligned} 3x + y - z &= 0 \\ x + y + 7z &= 4 \end{aligned}$

31. $\begin{aligned} 4x + 3y - 3z &= 5 \\ 6x + 2z &= 10 \end{aligned}$

32. $\begin{aligned} x + 2y + 4z &= 12 \\ -3x - 4y + 7z &= 21 \end{aligned}$

APPLICATIONS

33. Business. A small company has an assembly line that produces three types of widgets. The basic widget is sold for \$10 per unit, the midprice widget for \$12 per unit, and the top-of-the-line widget for \$15 per unit. The assembly line has a daily capacity of producing 300 widgets that may be sold for a total of \$3700. Find the quantity of each type of widget produced on a day when the number of basic widgets and top-of-the-line widgets is the same.

34. Business. A small company has an assembly line that produces three types of widgets. The basic widget is sold for \$10 per unit, the midprice widget for \$12 per unit, and the top-of-the-line widget for \$15 per unit. The assembly line has a daily capacity of producing 325 widgets that may be sold for a total of \$3825. Find the quantity of each type of widget produced on a day when twice as many basic widgets as top-of-the-line widgets are produced.

Exercises 35 and 36 rely on a selection of Subway sandwiches whose nutrition information is given in the table.

Suppose you are going to eat only Subway sandwiches for a week (seven days) for lunch and dinner (total of 14 meals).

SANDWICH	CALORIES	FAT (GRAMS)
Mediterranean chicken	350	18
Six-inch tuna	430	19
Six-inch roast beef	290	5

www.subway.com

35. Diet. Your goal is a total of 4840 calories and 190 grams of fat. How many of each sandwich would you eat that week to obtain this goal?

36. Diet. Your goal is a total of 4380 calories and 123 grams of fat. How many of each sandwich would you eat that week to obtain this goal?

Exercises 37 and 38 involve vertical motion and the effect of gravity on an object.

Because of gravity, an object that is projected upward will eventually reach a maximum height and then fall to the ground. The equation that determines the height h of a projectile t seconds after it is shot upward is given by

$$h = \frac{1}{2}at^2 + v_0t + h_0$$

where a is the acceleration due to gravity, h_0 is the initial height of the object at time $t = 0$, and v_0 is the initial velocity of the object at time $t = 0$. Note that a projectile follows the path of a parabola opening down, so $a < 0$.

37. Vertical Motion. An object is thrown upward and the following table depicts the height of the ball t seconds after the projectile is released. Find the initial height, initial velocity, and acceleration due to gravity.

t SECONDS	HEIGHT (FEET)
1	36
2	40
3	12

38. Vertical Motion. An object is thrown upward and the following table depicts the height of the ball t seconds after the projectile is released. Find the initial height, initial velocity, and acceleration due to gravity.

t Seconds	Height (feet)
1	84
2	136
3	156

39. Data Curve-Fitting. The number of minutes that an average person of age x spends driving a car can be modeled by a quadratic function $y = ax^2 + bx + c$, where $a < 0$ and $18 \le x \le 65$. The following table gives the average number of minutes per day that a person spends driving a car. Determine the quadratic function that models this quantity.

Age	Average Daily Minutes Driving
20	30
40	60
60	40

40. Data Curve-Fitting. The average age when a woman gets married began increasing during the last century. In 1930 the average age was 18.6, in 1950 the average age was 20.2, and in 2002 the average age was 25.3. Find a quadratic function $y = ax^2 + bx + c$, where $a > 0$ and $18 < y < 35$, that models the average age y when a woman gets married as a function of the year x ($x = 0$ corresponds to 1930). What will the average age be in 2010?

41. Money. Tara and Lamar decide to place \$20,000 of their savings into investments. They put some in a money market account earning 3% interest, some in a mutual fund that has been averaging 7% a year, and some in a stock that rose 10% last year. If they put \$6000 more in the money market than in the mutual fund and the mutual fund and stocks experience the same growth the next year as they did the previous year, they will earn \$1180 in a year. How much money did Tara and Lamar put in each of the three investments?

42. Money. Tara talks Lamar into putting less money in the money market and more money in the stock (see Exercise 41). They place \$20,000 of their savings into investments. They put some in a money market account earning 3% interest, some in a mutual fund that has been averaging 7% a year, and some in a stock that rose 10% last year. If they put \$6000 more in the stock than in the mutual fund and the mutual fund and stock experience the same growth the next year as they did the previous year, they will earn \$1680 in a year. How much money did Tara and Lamar put in each of the three investments?

43. Ski Production. A company produces three types of skis: regular model, trick ski, and slalom ski. They need to fill a customer order of 110 pairs of skis. There are two major production divisions within the company: labor and finishing. Each regular model of skis requires 2 hours of labor and 1 hour of finishing. Each trick ski model requires 3 hours of labor and 2 hours of finishing. Finally, each slalom ski model requires 3 hours of labor and 5 hours of finishing. Suppose the company has only 297 labor hours and 202 finishing hours. How many of each type ski can be made under these restrictions?

44. Automobile Production. An automobile manufacturing company produces three types of automobiles: compact car, intermediate, and luxury model. The company has the capability of producing 500 automobiles. Suppose that each compact-model car requires 200 units of steel and 30 units of rubber, each intermediate model requires 300 units of steel and 20 units of rubber, and each luxury model requires 250 units of steel and 45 units of rubber. The number of units of steel available is 128,750, and the number of units of rubber available is 15,625. How many of each type of automobile can be produced with these restraints?

45. Computer versus Man. *The Seattle Times* reported a story on November 18, 2006, about a game of Scrabble played between a human and a computer. The best Scrabble player in the United States was pitted against a computer program designed to play the game. Remarkably, the human beat the computer in the best of two out of three games competition. The total points scored by both computer and the man for all three games was 2591. The difference between the first game's total and second game's total was 62 points. The difference between the first game's total and the third game's total was only 2 points. Determine the total number of points scored by both computer and the man for each of the three contests.

46. Brain versus Computer. Can the human brain perform more calculations per second than a supercomputer? The calculating speed of the three top supercomputers, IBM's Blue Gene/L, IBM's BGW, and IBM's ASC Purple, has been determined. The speed of IBM's Blue Gene/L is 245 teraflops more than that of IBM's BGW. The computing speed of IBM's BGW is 22 teraflops more than that of IBM's ASC Purple. The combined speed of all three top supercomputers is 568 teraflops. Determine the computing speed (in teraflops) of each supercomputer. A **teraflop** is a measure of a computer's speed and can be expressed as 1 trillion floating-point operations per second. By comparison, it is estimated that the human brain can perform 10 quadrillion calculations per second.

47. Production. A factory manufactures three types of golf balls: Eagle, Birdie, and Bogey. The daily production is 10,000 balls. The number of Eagle and Birdie balls combined equals the number of Bogey balls produced. If the factory makes three times more Birdie than Eagle balls, find the daily production of each type of ball.

48. Pizza. Three-cheese pizzas are made with a mixture of three types of cheese. The cost of a pizza containing 2 parts of each cheese is \$2.40. A pizza made with 2 parts of cheese A, 1 part of cheese B, and 2 parts of cheese C costs \$2.20, while a pizza made with 2 parts of cheese A, 2 parts of cheese B, and 3 parts of cheese C costs \$2.70. Determine the cost, per part, of each cheese.

49. TV Commercials. A TV station sells intervals of time for commercials of 10 seconds for \$100, 20 seconds for \$180, and 40 seconds for \$320. It has 2 minutes for publicity during a game with a total revenue of \$1060 for six commercials shown. Find the number of commercials of each length sold by the TV station if there are twice as many 10 second commercials as 40 second commercials.

50. Airline. A commercial plane has 270 seats divided into three classes: first class, business, and coach. The first-class seats are a third of the business-class seats. There are 250 more coach seats than first-class seats. Find the number of seats of each class in the airplane.

CATCH THE MISTAKE

In Exercises 51 and 52, explain the mistake that is made.

51. Solve the system of equations.

Equation (1): $2x - y + z = 2$

Equation (2): $x - y = 1$

Equation (3): $x + z = 1$

Solution:

Equation (2): $x - y = 1$

Equation (3): $x + z = 1$

Add Equation (2) and Equation (3). $-y + z = 2$

Multiply Equation (1) by (-1). $-2x + y - z = -2$

Add. $-2x = 0$

Solve for x. $x = 0$

Substitute $x = 0$ into Equation (2). $0 - y = 1$

Solve for y. $y = -1$

Substitute $x = 0$ into Equation (3). $0 + z = 1$

Solve for z. $z = 1$

The answer is $x = 0$, $y = -1$, and $z = 1$.

This is incorrect. Although $x = 0$, $y = -1$, and $z = 1$ does satisfy the three original equations, it is only one of infinitely many solutions. What mistake was made?

52. Solve the system of equations.

Equation (1): $x + 3y + 2z = 4$

Equation (2): $3x + 10y + 9z = 17$

Equation (3): $2x + 7y + 7z = 17$

Solution:

Multiply Equation (1) by -3. $-3x - 9y - 6z = -12$

Equation (2): $3x + 10y + 9z = 17$

Add. $y + 3z = 5$

Multiply Equation (1) by -2. $-2x - 6y - 4z = -8$

Equation (3): $2x + 7y + 7z = 17$

Add. $y + 3z = 9$

Solve the system of two equations. $y + 3z = 5$

$y + 3z = 9$

Infinitely many solutions.

Let $z = a$, then $y = 5 - 3a$.

Substitute $z = a$ and $y = 5 - 3a$ into Equation (1). $x + 3y + 2z = 4$

$x + 3(5 - 3a) + 2a = 4$

Eliminate parentheses. $x + 15 - 9a + 2a = 4$

Solve for x. $x = 7a - 11$

The answer is $x = 7a - 11$, $y = 5 - 3a$, and $z = a$.

This is incorrect. There is no solution. What mistake was made?

CONCEPTUAL

In Exercises 53–56, determine whether each statement is true or false.

53. A system of linear equations that has more variables than equations cannot have a unique solution.

54. A system of linear equations that has the same number of equations as variables always has a unique solution.

55. The linear equation $Ax + By = C$ always represents a straight line.

56. If the system of linear equations

$$\begin{aligned} x + 2y + 3z &= a \\ 2x + 3y + z &= b \\ 3x + y + 2z &= c \end{aligned}$$

has a unique solution $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$, then the system of equations

$$\begin{aligned} x + 2y + 3z &= 2a \\ 2x + 3y + z &= 2b \\ 3x + y + 2z &= 2c \end{aligned}$$

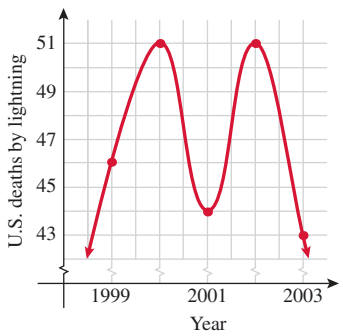
has a unique solution $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

57. The circle given by the equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(-2, 4)$, $(1, 1)$, and $(-2, -2)$. Find a, b, and c.

58. The circle given by the equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(0, 7)$, $(6, 1)$, and $(5, 4)$. Find a, b, and c.

CHALLENGE

59. A fourth-degree polynomial, $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, with $a < 0$, can be used to represent the following data on the number of deaths per year due to lightning strikes. Assume 1999 corresponds to $x = -2$ and 2003 corresponds to $x = 2$. Use the data to determine a, b, c, d, and e.



60. A copy machine accepts nickels, dimes, and quarters. After 1 hour, there are 30 coins total and their value is \$4.60. If there are four more quarters than nickels, how many nickels, quarters, and dimes are in the machine?

In Exercises 61–64, solve the system of linear equations.

61.
$$\begin{aligned} 2y + z &= 3 \\ 4x - z &= -3 \\ 7x - 3y - 3z &= 2 \\ x - y - z &= -2 \end{aligned}$$

62.
$$\begin{aligned} -2x - y + 2z &= 3 \\ 3x - 4z &= 2 \\ 2x + y &= -1 \\ -x + y - z &= -8 \end{aligned}$$

63.
$$\begin{aligned} 3x_1 - 2x_2 + x_3 + 2x_4 &= -2 \\ -x_1 + 3x_2 + 4x_3 + 3x_4 &= 4 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 + 3x_2 + x_3 + 2x_4 &= -1 \end{aligned}$$

64.
$$\begin{aligned} 5x_1 + 3x_2 + 8x_3 + x_4 &= 1 \\ x_1 + 2x_2 + 5x_3 + 2x_4 &= 3 \\ 4x_1 + x_3 - 2x_4 &= -3 \\ x_2 + x_3 + x_4 &= 0 \end{aligned}$$

65. Find the values of A, B, C, and D such that the following equation is true:

$$x^3 + x^2 + 2x + 3 = (Ax + B)\left(x^2 + 3\right) + (Cx + D)\left(x^2 + 2\right)$$

66. Find the values of A, B, C, D, and E such that the following equation is true:

$$Ax^3(x + 1) + Bx^2(x + 1) + Cx(x + 1) + D(x + 1) + Ex^4 = 4x^4 + x + 1$$

TECHNOLOGY

In Exercises 67 and 68, employ a graphing calculator to solve the system of linear equations (most graphing calculators have the capability of solving linear systems with the user entering the coefficients).

The rref under the MATRIX menu will be used to solve the system of equations by entering the coefficients of x, y, z, and the constant.

2nd MATRIX ▶ MATH ▼ B:rref(ENTER 2nd [2nd [2 , 1 , 8 , (−) 1 2nd] 2nd [1 , (−) 1 , 1 , (−) 2 2nd] 2nd [3 , (−) 2 , (−) 2 , 2 2nd] 2nd]) ENTER

```
rref([[2,1,8,-1]
[1,-1,1,-2][3,-2
,-2,2]])
      [[1  0  0  2 ]
       [0  1  0  3 ]
       [0  0  1  -1]]
```

First row gives $x = 2$, second row gives $y = 3$, and third row gives $z = -1$.

Note: The TI function rref stands for reduced row echelon form.

67.
$$\begin{aligned} x - z - y &= 10 \\ 2x - 3y + z &= -11 \\ y - x + z &= -10 \end{aligned}$$

68.
$$\begin{aligned} 2x + z + y &= -3 \\ 2y - z + x &= 0 \\ x + y + 2z &= 5 \end{aligned}$$

69. Some graphing calculators and graphing utilities have the ability to graph in three dimensions (3D) as opposed to the traditional two dimensions (2D). The line must be given in the form $z = ax + by + c$. Rewrite the system of equations in Exercise 67 in this form and graph the three lines in 3D. What is the point of intersection? Compare that with your answer in Exercise 67.

70. Some graphing calculators and graphing utilities have the ability to graph in three dimensions (3D) as opposed to the traditional two dimensions (2D). The line must be given in the form $z = ax + by + c$. Rewrite the system of equations in Exercise 68 in this form and graph the three lines in 3D. What is the point of intersection? Compare that with your answer in Exercise 68.

In Exercises 71 and 72, employ a graphing calculator to solve the system of equations.

71.
$$\begin{aligned} 0.2x - 0.7y + 0.8z &= 11.2 \\ -1.2x + 0.3y - 1.5z &= 0 \\ 0.8x - 0.1y + 2.1z &= 6.4 \end{aligned}$$

72.
$$\begin{aligned} 1.8x - 0.5y + 2.4z &= 1.6 \\ 0.3x \qquad\quad - 0.6z &= 0.2 \end{aligned}$$

PREVIEW TO CALCULUS

In calculus, when integrating rational functions, we decompose the function into partial fractions. This technique involves the solution of systems of equations.

In Exercises 73–76, find the values of A, B, and C that make each equation true.

73. $5x^2 + 6x + 2 = A(x^2 + 2x + 5) + (Bx + C)(x + 2)$

74. $2x^2 - 3x + 2 = A(x^2 + 1) + (Bx + C)x$

75. $3x + 8 = A(x^2 + 5x + 6) + B(x^2 + 3x) + C(x^2 + 2x)$

76. $x^2 + x + 1 = A(x^2 + 5x + 6) + B(x^2 + 4x + 3) + C(x^2 + 3x + 2)$

SECTION 8.3

SYSTEMS OF LINEAR EQUATIONS AND MATRICES

SKILLS OBJECTIVES

- Write a system of linear equations as an augmented matrix.
- Perform row operations on an augmented matrix.
- Write a matrix in row–echelon form.
- Solve systems of linear equations using Gaussian elimination with back-substitution.
- Write a matrix in reduced row–echelon form.
- Solve systems of linear equations using Gauss–Jordan elimination.

CONCEPTUAL OBJECTIVES

- Visualize an augmented matrix as a system of linear equations.
- Understand that solving systems with augmented matrices is equivalent to solving by the method of elimination.
- Recognize matrices that correspond to inconsistent and dependent systems.

Matrices

Some information is best displayed in a table. For example, the number of calories burned per half hour of exercise depends on the person's weight, as illustrated in the following table. Note that the rows correspond to activities and the columns correspond to weight.

Activity	127–137 lb	160–170 lb	180–200 lb
Walking/4 mph	156	183	204
Volleyball	267	315	348
Jogging/5 mph	276	345	381

Another example is the driving distance in miles from cities in Arizona (columns) to cities outside the state (rows).

City	Flagstaff	Phoenix	Tucson	Yuma
Albuquerque, NM	325	465	440	650
Las Vegas, NV	250	300	415	295
Los Angeles, CA	470	375	490	285

If we selected only the numbers in each of the preceding tables and placed brackets around them, the result would be a *matrix*.

$$\text{Calories: } \begin{bmatrix} 156 & 183 & 204 \\ 267 & 315 & 348 \\ 276 & 345 & 381 \end{bmatrix} \qquad \text{Miles: } \begin{bmatrix} 325 & 465 & 440 & 650 \\ 250 & 300 & 415 & 295 \\ 470 & 375 & 490 & 285 \end{bmatrix}$$

A ***matrix*** is a rectangular array of numbers written within brackets.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Each number a_{ij} in the matrix is called an **entry** (or **element**) of the matrix. The first subscript i is the **row index**, and the second subscript j is the **column index**. This matrix contains m rows and n columns, and is said to be of **order** $m \times n$.

Study Tip

The order of a matrix is always given as the number of rows by the number of columns.

When the number of rows equals the number of columns (i.e., when $m = n$), the matrix is a **square matrix** of order n. In a square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are the **main diagonal** entries.

The matrix

$$A_{4\times 3} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & a_{32} & * \\ * & * & * \end{bmatrix}$$

has order (dimensions) 4×3, since there are four rows and three columns. The entry a_{32} is in the third row and second column.

EXAMPLE 1 Finding the Order of a Matrix

Determine the order of each matrix given.

a. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & -2 & 5 \\ -1 & 3 & 4 \end{bmatrix}$ **c.** $\begin{bmatrix} -2 & 5 & 4 \\ 1 & -\frac{1}{3} & 0 \\ 3 & 8 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 4 & 9 & -\frac{1}{2} & 3 \end{bmatrix}$ **e.** $\begin{bmatrix} 3 & -2 \\ 5 & 1 \\ 0 & -\frac{2}{3} \\ 7 & 6 \end{bmatrix}$

Solution:

a. This matrix has **2** rows and **2** columns, so the order of the matrix is $\boxed{2 \times 2}$.

b. This matrix has **2** rows and **3** columns, so the order of the matrix is $\boxed{2 \times 3}$.

c. This matrix has **3** rows and **3** columns, so the order of the matrix is $\boxed{3 \times 3}$ or 3 since it is a square matrix.

d. This matrix has **1** row and **4** columns, so the order of the matrix is $\boxed{1 \times 4}$.

e. This matrix has **4** rows and **2** columns, so the order of the matrix is $\boxed{4 \times 2}$.

A matrix with only one column is called a **column matrix**, and a matrix that has only one row is called a **row matrix**. Notice that in Example 1 the matrices given in parts (a) and (c) are square matrices and the matrix given in part (d) is a row matrix.

You can use matrices as a shorthand way of writing systems of linear equations. There are two ways we can represent systems of linear equations with matrices: as *augmented matrices* or with *matrix equations*. In this section, we will discuss *augmented matrices* and solve systems of linear equations using two methods: *Gaussian elimination with back-substitution* and *Gauss–Jordan elimination*.

Augmented Matrices

A **coefficient matrix** is a matrix whose elements are the coefficients of a system of linear equations. A particular type of matrix that is used in representing a system of linear equations is an **augmented matrix**. It resembles a coefficient matrix with an additional vertical line and column of numbers, hence the name *augmented*. The following table illustrates examples of augmented matrices that represent systems of linear equations:

System of Linear Equations	Augmented Matrix
$3x + 4y = 1$ $x - 2y = 7$	$\left[\begin{array}{rr\|r} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$
$x - y + z = 2$ $2x + 2y - 3z = -3$ $x + y + z = 6$	$\left[\begin{array}{rrr\|r} 1 & -1 & 1 & 2 \\ 2 & 2 & -3 & -3 \\ 1 & 1 & 1 & 6 \end{array}\right]$
$x + y + z = 0$ $3x + 2y - z = 2$	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 0 \\ 3 & 2 & -1 & 2 \end{array}\right]$

Note the following:

- Each row represents an equation.
- The vertical line represents the equal sign.
- The first column represents the coefficients of the variable x.
- The second column represents the coefficients of the variable y.
- The third column (in the second and third systems) represents the coefficients of the variable z.
- The coefficients of the variables are on the left of the equal sign (vertical line) and the constants are on the right.
- Any variable that does not appear in an equation has an implied coefficient of 0.

EXAMPLE 2 Writing a System of Linear Equations as an Augmented Matrix

Write each system of linear equations as an augmented matrix.

a. $\begin{aligned} 2x - y &= 5 \\ -x + 2y &= 3 \end{aligned}$

b. $\begin{aligned} 3x - 2y + 4z &= 5 \\ y - 3z &= -2 \\ 7x - z &= 1 \end{aligned}$

c. $\begin{aligned} x_1 - x_2 + 2x_3 - 3 &= 0 \\ x_1 + x_2 - 3x_3 + 5 &= 0 \\ x_1 - x_2 + x_3 - 2 &= 0 \end{aligned}$

Solution:

a.

$$\left[\begin{array}{rr|r} 2 & -1 & 5 \\ -1 & 2 & 3 \end{array}\right]$$

b. Note that all missing terms have a 0 coefficient.

$$\begin{aligned} 3x - 2y + 4z &= 5 \\ 0x + y - 3z &= -2 \\ 7x + 0y - z &= 1 \end{aligned}$$

$$\left[\begin{array}{rrr|r} 3 & -2 & 4 & 5 \\ 0 & 1 & -3 & -2 \\ 7 & 0 & -1 & 1 \end{array}\right]$$

c. Write the constants on the right side of the vertical line in the matrix.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_1 + x_2 - 3x_3 &= -5 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 1 & 1 & -3 & -5 \\ 1 & -1 & 1 & 2 \end{array}\right]$$

■ **YOUR TURN** Write each system of linear equations as an augmented matrix.

a. $\begin{aligned} 2x + y - 3 &= 0 \\ x - y &= 5 \end{aligned}$

b. $\begin{aligned} y - x + z &= 7 \\ x - y - z &= 2 \\ z - y &= -1 \end{aligned}$

■ **Answer:**

a. $\left[\begin{array}{rr|r} 2 & 1 & 3 \\ 1 & -1 & 5 \end{array}\right]$

b. $\left[\begin{array}{rrr|r} -1 & 1 & 1 & 7 \\ 1 & -1 & -1 & 2 \\ 0 & -1 & 1 & -1 \end{array}\right]$

Row Operations on a Matrix

Row operations on a matrix are used to solve a system of linear equations when the system is written as an augmented matrix. Recall from the elimination method in Sections 8.1 and 8.2 that we could interchange equations, multiply an entire equation by a nonzero constant, and add a multiple of one equation to another equation to produce equivalent systems. Because each row in a matrix represents an equation, the operations that produced equivalent systems of equations that were used in the elimination method will also produce equivalent augmented matrices.

Study Tip

Each missing term in an equation of the system of linear equations is represented with a zero in the augmented matrix.

ROW OPERATIONS

The following operations on an augmented matrix will yield an equivalent matrix:

1. Interchange any two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another row.

The following symbols describe these row operations:

1. $R_i \leftrightarrow R_j$ — Interchange row i with row j.
2. $cR_i \rightarrow R_i$ — Multiply row i by the constant c.
3. $cR_i + R_j \rightarrow R_j$ — Multiply row i by the constant c and add to row j, writing the results in row j.

EXAMPLE 3 Applying a Row Operation to an Augmented Matrix

For each matrix, perform the given operation.

a. $\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 2 & 1 \end{array}\right]$ $R_1 \leftrightarrow R_2$

b. $\left[\begin{array}{ccc|c} -1 & 0 & 1 & -2 \\ 3 & -1 & 2 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right]$ $2R_3 \rightarrow R_3$

c. $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 2 \\ 0 & 1 & 2 & 3 & 5 \end{array}\right]$ $R_1 - 2R_2 \rightarrow R_1$

Solution:

a. Interchange the first row with the second row.

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 2 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \boxed{\left[\begin{array}{cc|c} 0 & 2 & 1 \\ 2 & -1 & 3 \end{array}\right]}$$

b. Multiply the third row by 2.

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -2 \\ 3 & -1 & 2 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right] \quad 2R_3 \rightarrow R_3 \quad \boxed{\left[\begin{array}{ccc|c} -1 & 0 & 1 & -2 \\ 3 & -1 & 2 & 3 \\ 0 & 2 & 6 & 2 \end{array}\right]}$$

c. From row 1 subtract 2 times row 2, and write the answer in row 1. Note that finding row 1 minus 2 times row 2 is the same as adding row 1 to the product of -2 with row 2.

$$R_1 - 2R_2 \rightarrow R_1 \quad \left[\begin{array}{cccc|c} 1-2(0) & 2-2(1) & 0-2(2) & 2-2(3) & 2-2(5) \\ 0 & 1 & 2 & 3 & 5 \end{array}\right]$$

$$\boxed{\left[\begin{array}{cccc|c} 1 & 0 & -4 & -4 & -8 \\ 0 & 1 & 2 & 3 & 5 \end{array}\right]}$$

■ **YOUR TURN** Perform the operation $R_1 + 2R_3 \rightarrow R_1$ on the matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

■ **Answer:**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Row–Echelon Form of a Matrix

We can solve systems of linear equations using augmented matrices with two procedures: *Gaussian elimination with back-substitution*, which uses row operations to transform a matrix into *row–echelon form*, and *Gauss–Jordan elimination*, which uses row operations to transform a matrix into *reduced row–echelon form*.

Row–Echelon Form

A matrix is in **row–echelon** form if it has all three of the following properties:

1. Any rows consisting entirely of 0s are at the bottom of the matrix.
2. For each row that does not consist entirely of 0s, the first (leftmost) nonzero entry is 1 (called the leading 1).
3. For two successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

Reduced Row–Echelon Form

If a matrix in row–echelon form has the following additional property, then the matrix is in **reduced row–echelon form**:

4. Every column containing a leading 1 has zeros in every position above and below the leading 1.

EXAMPLE 4 Determining Whether a Matrix Is in Row–Echelon Form

Determine whether each matrix is in row–echelon form. If it is in row–echelon form, determine whether it is in reduced row–echelon form.

a. $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ **c.** $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 5 \end{array}\right]$

d. $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 3 & 1 \end{array}\right]$ **e.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{array}\right]$ **f.** $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -3 \end{array}\right]$

Solution:

The matrices in (a), (b), (c), and (e) are in row–echelon form. The matrix in (d) is not in row–echelon form, by condition 2; the leading nonzero entry is not a 1 in each row. If the "3" were a "1," the matrix would be in reduced row–echelon form. The matrix in (f) is not in row–echelon form, because of condition 3; the leading 1 in row 2 is not to the left of the leading 1 in row 3. The matrices in (c) and (e) are in reduced row–echelon form, because in the columns containing the leading 1s there are zeros in every position above and below the leading 1.

Gaussian Elimination with Back-Substitution

Gaussian elimination with back-substitution is a method that uses row operations to transform an augmented matrix into row–echelon form and then uses back-substitution to find the solution to the system of linear equations.

GAUSSIAN ELIMINATION WITH BACK-SUBSTITUTION

Step 1: Write the system of linear equations as an augmented matrix.
Step 2: Use row operations to rewrite the augmented matrix in row–echelon form.
Step 3: Write the system of linear equations that corresponds to the matrix in row–echelon form found in Step 2.
Step 4: Use the system of linear equations found in Step 3 together with back-substitution to find the solution of the system.

Study Tip

For row–echelon form, get 1s along the main diagonal and 0s below these 1s.

The order in which we perform row operations is important. You should move from left to right. Here is an example of Step 2 in the procedure:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & * & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Matrices are not typically used for systems of linear equations in two variables because the methods from Section 8.1 (substitution and elimination) are more efficient. Example 5 illustrates this procedure with a simple system of linear equations in two variables.

EXAMPLE 5 Using Gaussian Elimination with Back-Substitution to Solve a System of Two Linear Equations in Two Variables

Apply Gaussian elimination with back-substitution to solve the system of linear equations.

$$\begin{aligned} 2x + y &= -8 \\ x + 3y &= 6 \end{aligned}$$

Solution:

STEP 1 Write the system of linear equations as an augmented matrix. $\left[\begin{array}{cc|c} 2 & 1 & -8 \\ 1 & 3 & 6 \end{array}\right]$

STEP 2 Use row operations to rewrite the matrix in row–echelon form.

Get a 1 in the top left. Interchange rows 1 and 2.

$$\left[\begin{array}{cc|c} 2 & 1 & -8 \\ 1 & 3 & 6 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & 1 & -8 \end{array}\right]$$

Get a 0 below the leading 1 in row 1.

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & 1 & -8 \end{array}\right] \quad R_2 - 2R_1 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -5 & -20 \end{array}\right]$$

Get a leading 1 in row 2. Make the "-5" a "1" by dividing by -5. Dividing by -5 is the same as multiplying by its reciprocal $-\frac{1}{5}$.

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -5 & -20 \end{array}\right] \quad -\tfrac{1}{5}R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 1 & 4 \end{array}\right]$$

The resulting matrix is in row–echelon form.

STEP 3 Write the system of linear equations corresponding to the row–echelon form of the matrix resulting in Step 2.

$$\left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 1 & 4 \end{array}\right] \rightarrow \begin{aligned} x + 3y &= 6 \\ y &= 4 \end{aligned}$$

STEP 4 Use back-substitution to find the solution to the system.

Let $y = 4$ in the first equation $x + 3y = 6$. $\qquad x + 3(4) = 6$

Solve for x. $\qquad x = -6$

The solution to the system of linear equations is $\boxed{x = -6, y = 4}$.

EXAMPLE 6 Using Gaussian Elimination with Back-Substitution to Solve a System of Three Linear Equations in Three Variables

Use Gaussian elimination with back-substitution to solve the system of linear equations.

$$\begin{aligned} 2x + y + 8z &= -1 \\ x - y + z &= -2 \\ 3x - 2y - 2z &= 2 \end{aligned}$$

Solution:

STEP 1 Write the system of linear equations as an augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 8 & -1 \\ 1 & -1 & 1 & -2 \\ 3 & -2 & -2 & 2 \end{array}\right]$$

STEP 2 Use row operations to rewrite the matrix in row–echelon form.

Get a 1 in the top left. Interchange rows 1 and 2.

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & -2 \\ 2 & 1 & 8 & -1 \\ 3 & -2 & -2 & 2 \end{array}\right]$$

Get 0s below the leading 1 in row 1.

$$R_2 - 2R_1 \rightarrow R_2 \quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & -2 \\ 0 & 3 & 6 & 3 \\ 3 & -2 & -2 & 2 \end{array}\right]$$

$$R_3 - 3R_1 \rightarrow R_3 \quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 1 & -5 & 8 \end{array}\right]$$

Get a leading 1 in row 2. Make the "3" a "1" by dividing by 3.

$$\tfrac{1}{3}R_2 \rightarrow R_2 \quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & -5 & 8 \end{array}\right]$$

Get a zero below the leading 1 in row 2.

$$R_3 - R_2 \rightarrow R_3 \quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -7 & 7 \end{array}\right]$$

Get a leading 1 in row 3. Make the "-7" a "1" by dividing by -7.

$$-\tfrac{1}{7}R_3 \rightarrow R_3 \quad \left[\begin{array}{rrr|r} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

STEP 3 Write the system of linear equations corresponding to the row–echelon form of the matrix resulting in Step 2.

$$\begin{aligned} x - y + z &= -2 \\ y + 2z &= 1 \\ z &= -1 \end{aligned}$$

STEP 4 Use back-substitution to find the solution to the system.

Let $\boxed{z = -1}$ in the second equation $y + 2z = 1$. $\quad y + 2(-1) = 1$

Solve for y. $\quad \boxed{y = 3}$

Let $y = 3$ and $z = -1$ in the first equation $x - y + z = -2$. $\quad x - (3) + (-1) = -2$

Solve for x. $\quad \boxed{x = 2}$

The solution to the system of linear equations is $\boxed{x = 2,\ y = 3, \text{ and } z = -1}$.

■ **Answer:** $x = -1$, $y = 2$, $z = 1$

■ **YOUR TURN** Use Gaussian elimination with back-substitution to solve the system of linear equations.

$$\begin{aligned} x + y - z &= 0 \\ 2x + y + z &= 1 \\ 2x - y + 3z &= -1 \end{aligned}$$

Gauss–Jordan Elimination

In Gaussian elimination with back-substitution, we used row operations to rewrite the matrix in an equivalent row–echelon form. If we continue using row operations until the matrix is in *reduced* row–echelon form, this eliminates the need for back-substitution, and we call this process *Gauss–Jordan elimination*.

Gauss–Jordan elimination

Step 1: Write the system of linear equations as an augmented matrix.
Step 2: Use row operations to rewrite the augmented matrix in *reduced* row–echelon form.
Step 3: Write the system of linear equations that corresponds to the matrix in reduced row–echelon form found in Step 2. The result is the solution to the system.

The order in which we perform row operations is important. You should move from left to right. Think of this process as climbing *down* a set of stairs first and then back up the stairs second. On the way *down* the stairs always use operations with rows *above* where you currently are, and on the way back *up* the stairs always use rows *below* where you currently are.

Study Tip

For reduced row–echelon form, get 1s along the main diagonal and 0s above and below these 1s.

Down the stairs:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & * & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Up the stairs:

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

EXAMPLE 7 Using Gauss–Jordan Elimination to Solve a System of Linear Equations in Three Variables

Apply Gauss–Jordan elimination to solve the system of linear equations.

$$\begin{aligned} x - y + 2z &= -1 \\ 3x + 2y - 6z &= 1 \\ 2x + 3y + 4z &= 8 \end{aligned}$$

Solution:

STEP 1 Write the system as an augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 3 & 2 & -6 & 1 \\ 2 & 3 & 4 & 8 \end{array}\right]$$

STEP 2 Utilize row operations to rewrite the matrix in reduced row–echelon form.

There is already a 1 in the first row/first column.

Get 0s below the leading 1 in row 1.

$$\begin{array}{r} R_2 - 3R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 5 & -12 & 4 \\ 0 & 5 & 0 & 10 \end{array}\right]$$

Get a 1 in row 2/column 2.

$$R_2 \leftrightarrow R_3 \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 5 & 0 & 10 \\ 0 & 5 & -12 & 4 \end{array}\right]$$

$$\tfrac{1}{5}R_2 \to R_2 \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & -12 & 4 \end{array}\right]$$

Get a 0 in row 3/column 2.

$$R_3 - 5R_2 \to R_3 \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -12 & -6 \end{array}\right]$$

Get a 1 in row 3/column 3.

$$-\tfrac{1}{12}R_3 \to R_3 \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Now, go back up the stairs.

Get 0s above the 1 in row 3/column 3.

$$R_1 - 2R_3 \to R_1 \left[\begin{array}{rrr|r} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Get a 0 in row 1/column 2.

$$R_1 + R_2 \to R_1 \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

STEP 3 Identify the solution.

$$\boxed{x = 0,\ y = 2,\ z = \tfrac{1}{2}}$$

■ **Answer:** $x = -1$, $y = 2$, $z = 3$

■ **YOUR TURN** Use an augmented matrix and Gauss–Jordan elimination to solve the system of equations.

$$\begin{aligned} x + y - z &= -2 \\ 3x + y - z &= -4 \\ 2x - 2y + 3z &= 3 \end{aligned}$$

EXAMPLE 8 Solving a System of Four Linear Equations in Four Variables

Solve the system of equations with Gauss–Jordan elimination.

$$\begin{aligned} x_1 + x_2 - x_3 + 3x_4 &= 3 \\ 3x_2 \qquad - 2x_4 &= 4 \\ 2x_1 \qquad - 3x_3 \qquad &= -1 \\ 2x_1 \qquad\qquad + 4x_4 &= -6 \end{aligned}$$

Solution:

STEP 1 Write the system as an augmented matrix.

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 3 \\ 0 & 3 & 0 & -2 & 4 \\ 2 & 0 & -3 & 0 & -1 \\ 2 & 0 & 0 & 4 & -6 \end{array}\right]$$

STEP 2 Use row operations to rewrite the matrix in reduced row–echelon form.

There is already a 1 in the first row/first column.

Get 0s below the 1 in row 1/column 1.

$$\begin{array}{r} \\ \\ R_3 - 2R_1 \to R_3 \\ R_4 - 2R_1 \to R_4 \end{array} \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 3 \\ 0 & 3 & 0 & -2 & 4 \\ 0 & -2 & -1 & -6 & -7 \\ 0 & -2 & 2 & -2 & -12 \end{array}\right]$$

Get a 1 in row 2/column 2.

$$R_2 \leftrightarrow R_4 \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 3 \\ 0 & -2 & 2 & -2 & -12 \\ 0 & -2 & -1 & -6 & -7 \\ 0 & 3 & 0 & -2 & 4 \end{array}\right]$$

$$-\tfrac{1}{2}R_2 \leftrightarrow R_2 \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & -2 & -1 & -6 & -7 \\ 0 & 3 & 0 & -2 & 4 \end{array}\right]$$

Get 0s below the 1 in row 2/column 2.

$$\begin{array}{r} \\ R_3 + 2R_2 \to R_3 \\ R_4 - 3R_2 \to R_4 \\ \\ \end{array} \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & 0 & -3 & -4 & 5 \\ 0 & 0 & 3 & -5 & -14 \end{array}\right]$$

Get a 1 in row 3/column 3.

$$-\tfrac{1}{3}R_3 \to R_3 \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{5}{3} \\ 0 & 0 & 3 & -5 & -14 \end{array}\right]$$

Get a 0 in row 4/column 3.

$$R_4 - 3R_3 \to R_4 \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & -9 & -9 \end{array}\right]$$

Study Tip

Careful attention should be paid to order of terms, and zeros should be used for missing terms.

Get a 1 in row 4/column 4.

$$-\tfrac{1}{9}R_4 \to R_4 \quad \left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 3 \\ 0 & 1 & -1 & 1 & 6 \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Now go back up the stairs.

Get 0s above the 1 in row 4/column 4.

$$\begin{array}{l} R_3 - \tfrac{4}{3}R_4 \to R_3 \\ R_2 - R_4 \to R_2 \\ R_1 - 3R_4 \to R_1 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Get 0s above the 1 in row 3/column 3.

$$\begin{array}{l} R_2 + R_3 \to R_2 \\ R_1 + R_3 \to R_1 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Get a 0 in row 1/column 2.

$$R_1 - R_2 \to R_1 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

STEP 3 Identify the solution.

$$\boxed{x_1 = -5,\ x_2 = 2,\ x_3 = -3,\ x_4 = 1}$$

Inconsistent and Dependent Systems

Recall from Section 8.1 that systems of linear equations can be independent, inconsistent, or dependent systems and therefore have *one solution*, *no solution*, or *infinitely many solutions*. All of the systems we have solved so far in this section have been independent systems (unique solution). When solving a system of linear equations using Gaussian elimination or Gauss–Jordan elimination, the following will indicate the three possible types of solutions.

System	Type of Solution	Matrix During Gauss–Jordan Elimination	Example
Independent	One (unique) solution	Diagonal entries are all 1s, and the 0s occupy all other coefficient positions.	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$ or $\begin{array}{rcr} x &=& 1 \\ y &=& -3 \\ z &=& 2 \end{array}$
Inconsistent	No solution	One row will have only zero entries for coefficients and a nonzero entry for the constant.	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 2 \end{array}\right]$ or $\begin{array}{rcr} x &=& 1 \\ y &=& -3 \\ 0 &=& 2 \end{array}$
Dependent	Infinitely many solutions	One row will be entirely 0s when the number of equations equals the number of variables.	$\left[\begin{array}{ccc\|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ or $\begin{array}{rcr} x - 2z &=& 1 \\ y + z &=& -3 \\ 0 &=& 0 \end{array}$

EXAMPLE 9 Determining That a System Is Inconsistent: No Solution

Solve the system of equations.

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + y + 2z &= -1 \\ -2x - 4y + 2z &= 5 \end{aligned}$$

Solution:

STEP 1 Write the system of equations as an augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 2 & 1 & 2 & -1 \\ -2 & -4 & 2 & 5 \end{array}\right]$$

STEP 2 Apply row operations to rewrite the matrix in row–echelon form.

Get 0s below the 1 in column 1.

$$\begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -3 & 4 & -7 \\ 0 & 0 & 0 & 11 \end{array}\right]$$

There is no need to continue because row 3 is a contradiction.

$$0x + 0y + 0z = 11 \text{ or } 0 = 11$$

Since this is inconsistent, there is *no solution* to this system of equations.

EXAMPLE 10 Determining That a System Is Dependent: Infinitely Many Solutions

Solve the system of equations.

$$\begin{aligned} x + z &= 3 \\ 2x + y + 4z &= 8 \\ 3x + y + 5z &= 11 \end{aligned}$$

Solution:

STEP 1 Write the system of equations as an augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 3 \\ 2 & 1 & 4 & 8 \\ 3 & 1 & 5 & 11 \end{array}\right]$$

STEP 2 Use row operations to rewrite the matrix in reduced row–echelon form.

Get the 0s below the 1 in column 1.

$$\begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

Get a 0 in row 3/column 2.

$$R_3 - R_2 \to R_3 \left[\begin{array}{rrr|r} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in reduced row–echelon form. This matrix corresponds to a dependent system of linear equations and has infinitely many solutions.

STEP 3 Write the augmented matrix as a system of linear equations.

$$\begin{aligned} x + z &= 3 \\ y + 2z &= 2 \end{aligned}$$

Let $z = a$, where a is any real number, and substitute this into the two equations.

We find that $x = 3 - a$ and $y = 2 - 2a$. The general solution is $\boxed{x = 3 - a,\ y = 2 - 2a,\ z = a}$ for a any real number. Note that (2, 0, 1) and (3, 2, 0) are particular solutions when $a = 1$ and $a = 0$, respectively.

Study Tip

In a system with three variables, say, x, y, and z, we typically let $z = a$ (where a is called a parameter) and then solve for x and y in terms of a.

A common mistake that is made is to identify a unique solution as no solution when one of the variables is equal to zero. For example, what is the difference between the following two matrices?

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 3 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

The first matrix has a *unique solution*, whereas the second matrix has *no solution*. The third row of the first matrix corresponds to the equation $3z = 0$, which implies that $z = 0$. The third row of the second matrix corresponds to the equation $0x + 0y + 0z = 3$ or $0 = 3$, which is inconsistent, and therefore the system has no solution.

EXAMPLE 11 Determining That a System Is Dependent: Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{aligned} 2x + y + z &= 8 \\ x + y - z &= -3 \end{aligned}$$

Solution:

STEP 1 Write the system of equations as an augmented matrix. $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 8 \\ 1 & 1 & -1 & -3 \end{array}\right]$

STEP 2 Use row operations to rewrite the matrix in reduced row–echelon form.

Get a 1 in row 1/column 1. $R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 2 & 1 & 1 & 8 \end{array}\right]$

Get a 0 in row 2/column 1. $R_2 - 2R_1 \rightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & -1 & 3 & 14 \end{array}\right]$

Get a 1 in row 2/column 2. $-R_2 \rightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & -3 & -14 \end{array}\right]$

Get a 0 in row 1/column 2. $R_1 - R_2 \rightarrow R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 11 \\ 0 & 1 & -3 & -14 \end{array}\right]$

This matrix is in reduced row–echelon form.

STEP 3 Identify the solution.

$$\begin{aligned} x + 2z &= 11 \\ y - 3z &= -14 \end{aligned}$$

Let $z = a$, where a is any real number. Substituting $z = a$ into these two equations gives the infinitely many solutions $\boxed{x = 11 - 2a,\ y = 3a - 14,\ z = a}$.

■ **YOUR TURN** Solve the system of equations using an augmented matrix.

$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - z &= 2 \end{aligned}$$

■ **Answer:** $x = 3a + 2$, $y = -4a - 2$, $z = a$, where a is any real number.

In Example 10 there were three equations and three unknowns (x, y, and z). In Example 11 there were two equations and three unknowns. Whenever there are more unknowns than equations, the system is dependent, that is, infinitely many solutions.

Applications

Remember Jared who lost all that weight eating at Subway and is still keeping it off 10 years later? He ate Subway sandwiches for lunch and dinner for one year and lost 235 pounds! The following table gives nutritional information for Subway's 6-inch sandwiches advertised with 6 grams of fat or less.

IVAN CHAVEZ/©AP/Wide World Photos

Sandwich	Calories	Fat (g)	Carbohydrates (g)	Protein (g)
Veggie Delight	350	18	17	36
Oven-roasted chicken breast	430	19	46	20
Ham (Black Forest without cheese)	290	5	45	19

EXAMPLE 12 Subway Diet

Suppose you are going to eat only Subway 6-inch sandwiches for a week (seven days) for both lunch and dinner (total of 14 meals). If your goal is to eat 388 grams of protein and 4900 calories in those 14 sandwiches, how many of each sandwich should you eat that week?

Solution:

Step 1 Determine the system of linear equations.

Let three variables represent the number of each type of sandwich you eat in a week.

x = number of Veggie Delight sandwiches
y = number of chicken breast sandwiches
z = number of ham sandwiches

The total number of sandwiches eaten is 14. $x + y + z = 14$

The total number of calories consumed is 4900. $350x + 430y + 290z = 4900$

The total number of grams of protein consumed is 388. $36x + 20y + 19z = 388$

Write an augmented matrix representing this system of linear equations.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 14 \\ 350 & 430 & 290 & 4900 \\ 36 & 20 & 19 & 388 \end{array}\right]$$

Step 2 Utilize row operations to rewrite the matrix in reduced row–echelon form.

$$\begin{array}{l} R_2 - 350R_1 \to R_2 \\ R_3 - 36R_1 \to R_3 \end{array} \quad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 14 \\ 0 & 80 & -60 & 0 \\ 0 & -16 & -17 & -116 \end{array}\right]$$

$$\tfrac{1}{80}R_2 \to R_2 \quad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 14 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & -16 & -17 & -116 \end{array}\right]$$

$$R_3 + 16R_2 \to R_3 \quad \left[\begin{array}{rrr|r} 1 & 1 & 1 & 14 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & -29 & -116 \end{array}\right]$$

$$-\tfrac{1}{29}R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 14 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{l} R_2 + \tfrac{3}{4}R_3 \to R_2 \\ R_1 - R_3 \to R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$R_1 - R_2 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

STEP 3 Identify the solution. $\boxed{x = 7, y = 3, z = 4}$

You should eat $\boxed{\text{7 Veggie Delights, 3 oven-roasted chicken breast, and 4 ham sandwiches}}$.

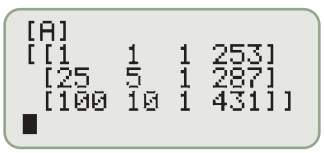
Technology Tip

Enter the matrix into the TI or graphing calculator. Press [2nd] [*MATRIX*]. Use [▶] [*EDIT*] [ENTER] [ENTER].

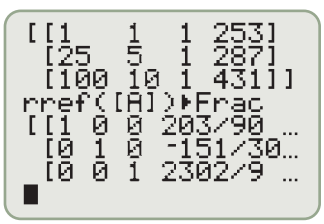
Now enter the size by typing the number of rows first and the number of columns second. Enter the elements of the matrix one row at a time by pressing the number and [ENTER] each time.

When done, press [2nd] [*QUIT*].

To show the matrix A, press [2nd] [*MATRIX*] [ENTER] [ENTER].



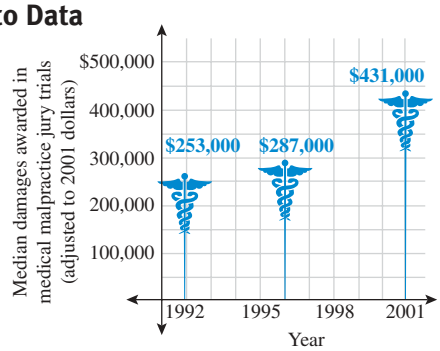
To find the reduced row–echelon form of the original matrix directly use [*rref*] (matrix) command. Press [2nd] [*MATRIX*] [▶] [*MATH*] [▼] [*B:rref(*] [ENTER] [2nd] [*MATRIX*] [ENTER] [)] [▶] [*MATH*] [*1:Frac*] [ENTER].



The solution to the system is $a = \frac{203}{90}$, $b = -\frac{151}{30}$, $c = \frac{2302}{9}$.

EXAMPLE 13 Fitting a Curve to Data

The amount of money awarded in medical malpractice suits is rising. This can be modeled with a quadratic function $y = at^2 + bt + c$, where $t > 0$ and $a > 0$. Determine a quadratic function that passes through the three points shown on the graph. Based on this trend, how much money will be spent on malpractice in 2011?



Solution:

Let 1991 correspond to $t = 0$ and y represent the number of dollars awarded for malpractice suits. The following data are reflected in the illustration above:

YEAR	t	y (THOUSANDS OF DOLLARS)	(t, y)
1992	1	253	(1, 253)
1996	5	287	(5, 287)
2001	10	431	(10, 431)

Substitute the three points (1, 253), (5, 287), and (10, 431) into the general quadratic equation: $y = at^2 + bt + c$.

POINT	$y = at^2 + bt + c$	SYSTEM OF EQUATIONS
(1, 253)	$253 = a(1)^2 + b(1) + c$	$a + b + c = 253$
(5, 287)	$287 = a(5)^2 + b(5) + c$	$25a + 5b + c = 287$
(10, 431)	$431 = a(10)^2 + b(10) + c$	$100a + 10b + c = 431$

STEP 1 Write this system of linear equations as an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 253 \\ 25 & 5 & 1 & 287 \\ 100 & 10 & 1 & 431 \end{array}\right]$$

STEP 2 Apply row operations to rewrite the matrix in reduced row–echelon form.

$$R_2 - 25R_1 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 253 \\ 0 & -20 & -24 & -6038 \\ 100 & 10 & 1 & 431 \end{array}\right]$$

$$R_3 - 100R_1 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 253 \\ 0 & -20 & -24 & -6038 \\ 0 & -90 & -99 & -24{,}869 \end{array}\right]$$

$$-\tfrac{1}{20}R_2 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 253 \\ 0 & 1 & \frac{6}{5} & \frac{3019}{10} \\ 0 & -90 & -99 & -24{,}869 \end{array}\right]$$

$$R_3 + 90R_2 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 253 \\ 0 & 1 & \frac{6}{5} & \frac{3019}{10} \\ 0 & 0 & 9 & 2302 \end{array}\right]$$

$$\tfrac{1}{9}R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 253 \\ 0 & 1 & \frac{6}{5} & \frac{3019}{10} \\ 0 & 0 & 1 & \frac{2302}{9} \end{array}\right]$$

$$R_2 - \tfrac{6}{5}R_3 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 253 \\ 0 & 1 & 0 & -\frac{151}{30} \\ 0 & 0 & 1 & \frac{2302}{9} \end{array}\right]$$

$$R_1 - R_3 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{25}{9} \\ 0 & 1 & 0 & -\frac{151}{30} \\ 0 & 0 & 1 & \frac{2302}{9} \end{array}\right]$$

$$R_1 - R_2 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{203}{90} \\ 0 & 1 & 0 & -\frac{151}{30} \\ 0 & 0 & 1 & \frac{2302}{9} \end{array}\right]$$

STEP 3 Identify the solution.

$$a = \frac{203}{90}, \quad b = -\frac{151}{30}, \quad c = \frac{2302}{9}$$

Substituting $a = \frac{203}{90}$, $b = -\frac{151}{30}$, $c = \frac{2302}{9}$ into $y = at^2 + bt + c$, we find that the thousands of dollars spent on malpractice suits as a function of year is given by

$$\boxed{y = \frac{203}{90}t^2 - \frac{151}{30}t + \frac{2302}{9}} \qquad \text{1991 is } t = 0$$

Notice that all three points lie on this curve.

For 2011, we let $t = 20$, which results in approximately $\boxed{\$1.06\text{ M}}$ in malpractice.

SECTION 8.3 SUMMARY

In this section, we used augmented matrices to represent a system of linear equations.

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \Leftrightarrow \left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Any missing terms correspond to a 0 in the matrix. A matrix is in **row–echelon** form if it has all three of the following properties:

1. Any rows consisting entirely of 0s are at the bottom of the matrix.
2. For each row that does not consist entirely of 0s, the first (leftmost) nonzero entry is 1 (called the leading 1).
3. For two successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

If a matrix in row–echelon form has the following additional property, then the matrix is in **reduced row–echelon form**:

4. Every column containing a leading 1 has zeros in every position above and below the leading 1.

The two methods used for solving systems of linear equations represented as augmented matrices are Gaussian elimination with back-substitution and Gauss–Jordan elimination. In both cases, we represent the system of linear equations as an augmented matrix and then use row operations to rewrite in row–echelon form. With Gaussian elimination we then stop and perform back-substitution to solve the system, and with Gauss–Jordan elimination we continue with row operations until the matrix is in reduced row–echelon form and then identify the solution to the system.

SECTION 8.3 EXERCISES

SKILLS

In Exercises 1–6, determine the order of each matrix.

1. $\begin{bmatrix} -1 & 3 & 4 \\ 2 & 7 & 9 \end{bmatrix}$ **2.** $\begin{bmatrix} 0 & 1 \\ 3 & 9 \\ 7 & 8 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ **4.** $\begin{bmatrix} 3 \\ 7 \\ -1 \\ 10 \end{bmatrix}$ **5.** $[0]$ **6.** $\begin{bmatrix} -1 & 3 & 6 & 8 \\ 2 & 9 & 7 & 3 \\ 5 & 4 & -2 & -10 \\ 6 & 3 & 1 & 5 \end{bmatrix}$

In Exercises 7–14, write the augmented matrix for each system of linear equations.

7. $\begin{aligned} 3x - 2y &= 7 \\ -4x + 6y &= -3 \end{aligned}$

8. $\begin{aligned} -x + y &= 2 \\ x - y &= -4 \end{aligned}$

9. $\begin{aligned} 2x - 3y + 4z &= -3 \\ -x + y + 2z &= 1 \\ 5x - 2y - 3z &= 7 \end{aligned}$

10. $\begin{aligned} x - 2y + z &= 0 \\ -2x + y - z &= -5 \\ 13x + 7y + 5z &= 6 \end{aligned}$

11. $\begin{aligned} x + y &= 3 \\ x - z &= 2 \\ y + z &= 5 \end{aligned}$

12. $\begin{aligned} x - y &= -4 \\ y + z &= 3 \end{aligned}$

13. $\begin{aligned} 3y - 4x + 5z - 2 &= 0 \\ 2x - 3y - 2z &= -3 \\ 3z + 4y - 2x - 1 &= 0 \end{aligned}$

14. $\begin{aligned} 2y + z - x - 3 &= 2 \\ 2x + 3z - 2y &= 0 \\ -2z + y - 4x - 3 &= 0 \end{aligned}$

In Exercises 15–20, write the system of linear equations represented by the augmented matrix. Utilize the variables x, y, and z.

15. $\left[\begin{array}{cc|c} -3 & 7 & 2 \\ 1 & 5 & 8 \end{array}\right]$

16. $\left[\begin{array}{ccc|c} -1 & 2 & 4 & 4 \\ 7 & 9 & 3 & -3 \\ 4 & 6 & -5 & 8 \end{array}\right]$

17. $\left[\begin{array}{ccc|c} -1 & 0 & 0 & 4 \\ 7 & 9 & 3 & -3 \\ 4 & 6 & -5 & 8 \end{array}\right]$

18. $\left[\begin{array}{ccc|c} 2 & 3 & -4 & 6 \\ 7 & -1 & 5 & 9 \end{array}\right]$

19. $\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right]$

20. $\left[\begin{array}{ccc|c} 3 & 0 & 5 & 1 \\ 0 & -4 & 7 & -3 \\ 2 & -1 & 0 & 8 \end{array}\right]$

In Exercises 21–30, indicate whether each matrix is in row–echelon form. If it is, determine whether it is in reduced row–echelon form.

21. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & 1 & 2 \end{array}\right]$

22. $\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 0 & 2 \end{array}\right]$

23. $\left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 3 & 14 \end{array}\right]$

24. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 3 & 14 \end{array}\right]$

25. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right]$

26. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

27. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

28. $\left[\begin{array}{ccc|c} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 5 \end{array}\right]$

29. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

30. $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 31–40, perform the indicated row operations on each augmented matrix.

31. $\left[\begin{array}{cc|c} 1 & -2 & -3 \\ 2 & 3 & -1 \end{array}\right]$ $R_2 - 2R_1 \to R_2$

32. $\left[\begin{array}{cc|c} 2 & -3 & -4 \\ 1 & 2 & 5 \end{array}\right]$ $R_1 \leftrightarrow R_2$

33. $\left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 2 & 1 & -3 & 6 \\ 3 & -2 & 5 & -8 \end{array}\right]$ $R_2 - 2R_1 \to R_2$

34. $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -2 & 6 \\ -3 & 0 & -1 & -5 \end{array}\right]$ $R_3 + 3R_1 \to R_3$

35. $\left[\begin{array}{cccc|c} 1 & -2 & 5 & -1 & 2 \\ 0 & 3 & 0 & -1 & -2 \\ 0 & -2 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 & -6 \end{array}\right]$ $R_3 + R_2 \to R_2$

36. $\left[\begin{array}{cccc|c} 1 & 0 & 5 & -10 & 15 \\ 0 & 1 & 2 & -3 & 4 \\ 0 & 2 & -3 & 0 & -1 \\ 0 & 0 & 1 & -1 & -3 \end{array}\right]$ $R_2 - \frac{1}{2}R_3 \to R_3$

37. $\left[\begin{array}{cccc|c} 1 & 0 & 5 & -10 & -5 \\ 0 & 1 & 2 & -3 & -2 \\ 0 & 2 & -3 & 0 & -1 \\ 0 & -3 & 2 & -1 & -3 \end{array}\right]$ $R_3 - 2R_2 \to R_3$, $R_4 + 3R_2 \to R_4$

38. $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$ $R_2 - 2R_3 \to R_2$, $R_1 - 4R_3 \to R_1$

39. $\left[\begin{array}{cccc|c} 1 & 0 & 4 & 8 & 3 \\ 0 & 1 & 2 & -3 & -2 \\ 0 & 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$ $R_3 - 6R_4 \to R_3$, $R_2 + 3R_4 \to R_2$, $R_1 - 8R_4 \to R_1$

40. $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 5 & 2 \\ 0 & 1 & 2 & 3 & -5 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$ $R_3 + 2R_4 \to R_3$, $R_2 - 3R_4 \to R_2$, $R_1 - 5R_4 \to R_1$

In Exercises 41–50, use row operations to transform each matrix to reduced row–echelon form.

41. $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 3 & 2 \end{array}\right]$

42. $\left[\begin{array}{cc|c} 1 & -1 & 3 \\ -3 & 2 & 2 \end{array}\right]$

43. $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \end{array}\right]$

44. $\left[\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{array}\right]$

45. $\left[\begin{array}{ccc|c} 3 & -2 & -3 & -1 \\ 1 & -1 & 1 & -4 \\ 2 & 3 & 5 & 14 \end{array}\right]$

46. $\left[\begin{array}{ccc|c} 3 & -1 & 1 & 2 \\ 1 & -2 & 3 & 1 \\ 2 & 1 & -3 & -1 \end{array}\right]$

47. $\left[\begin{array}{ccc|c} 2 & 1 & -6 & 4 \\ 1 & -2 & 2 & -3 \end{array}\right]$

48. $\left[\begin{array}{ccc|c} -3 & -1 & 2 & -1 \\ -1 & -2 & 1 & -3 \end{array}\right]$

49. $\left[\begin{array}{ccc|c} -1 & 2 & 1 & -2 \\ 3 & -2 & 1 & 4 \\ 2 & -4 & -2 & 4 \end{array}\right]$

50. $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 0 & 1 & -2 \\ -2 & 1 & 0 & -1 \end{array}\right]$

In Exercises 51–70, solve the system of linear equations using Gaussian elimination with back-substitution.

51. $\begin{aligned} 2x + 3y &= 1 \\ x + y &= -2 \end{aligned}$

52. $\begin{aligned} 3x + 2y &= 11 \\ x - y &= 12 \end{aligned}$

53. $\begin{aligned} -x + 2y &= 3 \\ 2x - 4y &= -6 \end{aligned}$

54. $\begin{aligned} 3x - y &= -1 \\ 2y + 6x &= 2 \end{aligned}$

55. $\begin{aligned} \tfrac{2}{3}x + \tfrac{1}{3}y &= \tfrac{8}{9} \\ \tfrac{1}{2}x + \tfrac{1}{4}y &= \tfrac{3}{4} \end{aligned}$

56. $\begin{aligned} 0.4x - 0.5y &= 2.08 \\ -0.3x + 0.7y &= 1.88 \end{aligned}$

57. $\begin{aligned} x - z - y &= 10 \\ 2x - 3y + z &= -11 \\ y - x + z &= -10 \end{aligned}$

58. $\begin{aligned} 2x + z + y &= -3 \\ 2y - z + x &= 0 \\ x + y + 2z &= 5 \end{aligned}$

59. $\begin{aligned} 3x_1 + x_2 - x_3 &= 1 \\ x_1 - x_2 + x_3 &= -3 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$

60. $\begin{aligned} 2x_1 + x_2 + x_3 &= -1 \\ x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 - x_3 &= 1 \end{aligned}$

61. $\begin{aligned} 2x + 5y &= 9 \\ x + 2y - z &= 3 \\ -3x - 4y + 7z &= 1 \end{aligned}$

62. $\begin{aligned} x - 2y + 3z &= 1 \\ -2x + 7y - 9z &= 4 \\ x + z &= 9 \end{aligned}$

63. $\begin{aligned} 2x_1 - x_2 + x_3 &= 3 \\ x_1 - x_2 + x_3 &= 2 \\ -2x_1 + 2x_2 - 2x_3 &= -4 \end{aligned}$

64. $\begin{aligned} x_1 - x_2 - 2x_3 &= 0 \\ -2x_1 + 5x_2 + 10x_3 &= -3 \\ 3x_1 + x_2 &= 0 \end{aligned}$

65. $\begin{aligned} 2x + y - z &= 2 \\ x - y - z &= 6 \end{aligned}$

66. $\begin{aligned} 3x + y - z &= 0 \\ x + y + 7z &= 4 \end{aligned}$

67. $\begin{aligned} 2y + z &= 3 \\ 4x - z &= -3 \\ 7x - 3y - 3z &= 2 \\ x - y - z &= -2 \end{aligned}$

68. $\begin{aligned} -2x - y + 2z &= 3 \\ 3x - 4z &= 2 \\ 2x + y &= -1 \\ -x + y - z &= -8 \end{aligned}$

69. $\begin{aligned} 3x_1 - 2x_2 + x_3 + 2x_4 &= -2 \\ -x_1 + 3x_2 + 4x_3 + 3x_4 &= 4 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 + 3x_2 + x_3 + 2x_4 &= -1 \end{aligned}$

70. $\begin{aligned} 5x_1 + 3x_2 + 8x_3 + x_4 &= 1 \\ x_1 + 2x_2 + 5x_3 + 2x_4 &= 3 \\ 4x_1 + x_3 - 2x_4 &= -3 \\ x_2 + x_3 + x_4 &= 0 \end{aligned}$

In Exercises 71–86, solve the system of linear equations using Gauss–Jordan elimination.

71. $\begin{aligned} x + 3y &= -5 \\ -2x - y &= 0 \end{aligned}$

72. $\begin{aligned} 5x - 4y &= 31 \\ 3x + 7y &= -19 \end{aligned}$

73. $\begin{aligned} x + y &= 4 \\ -3x - 3y &= 10 \end{aligned}$

74. $\begin{aligned} 3x - 4y &= 12 \\ -6x + 8y &= -24 \end{aligned}$

75. $\begin{aligned} x - 2y + 3z &= 5 \\ 3x + 6y - 4z &= -12 \\ -x - 4y + 6z &= 16 \end{aligned}$

76. $\begin{aligned} x + 2y - z &= 6 \\ 2x - y + 3z &= -13 \\ 3x - 2y + 3z &= -16 \end{aligned}$

77. $\begin{aligned} x + y + z &= 3 \\ x - z &= 1 \\ y - z &= -4 \end{aligned}$

78. $\begin{aligned} x - 2y + 4z &= 2 \\ 2x - 3y - 2z &= -3 \\ \tfrac{1}{2}x + \tfrac{1}{4}y + z &= -2 \end{aligned}$

79. $\begin{aligned} x + 2y + z &= 3 \\ 2x - y + 3z &= 7 \\ 3x + y + 4z &= 5 \end{aligned}$

80. $\begin{aligned} x + 2y + z &= 3 \\ 2x - y + 3z &= 7 \\ 3x + y + 4z &= 10 \end{aligned}$

81. $\begin{aligned} 3x - y + z &= 8 \\ x + y - 2z &= 4 \end{aligned}$

82. $\begin{aligned} x - 2y + 3z &= 10 \\ -3x + z &= 9 \end{aligned}$

83. $\begin{aligned} 4x - 2y + 5z &= 20 \\ x + 3y - 2z &= 6 \end{aligned}$

84. $\begin{aligned} y + z &= 4 \\ x + y &= 8 \end{aligned}$

85. $\begin{aligned} x - y - z - w &= 1 \\ 2x + y + z + 2w &= 3 \\ x - 2y - 2z - 3w &= 0 \\ 3x - 4y + z + 5w &= -3 \end{aligned}$

86. $\begin{aligned} x - 3y + 3z - 2w &= 4 \\ x + 2y - z &= -3 \\ x + 3z + 2w &= 3 \\ y + z + 5w &= 6 \end{aligned}$

APPLICATIONS

87. Astronomy. Astronomers have determined the number of stars in a small region of the universe to be 2,880,968 classified as red dwarfs, yellow, and blue stars. For every blue star there are 120 red dwarfs; for every red dwarf there are 3000 yellow stars. Determine the number of stars by type in that region of the universe.

88. Orange Juice. Orange juice producers use three varieties of oranges: Hamlin, Valencia, and navel. They want to make a juice mixture to sell at \$3.00 per gallon. The price per gallon of each variety of juice is \$2.50, \$3.40, and \$2.80, respectively. To maintain their quality standards, they use the same amount of Valencia and navel oranges. Determine the quantity of each juice used to produce 1 gallon of mixture.

Exercises 89 and 90 rely on a selection of Subway sandwiches whose nutrition information is given in the table below. Suppose you are going to eat only Subway sandwiches for a week (seven days) for lunch and dinner (a total of 14 meals).

Sandwich	Calories	Fat (g)	Carbohydrates (g)	Protein (g)
Mediterranean chicken	350	18	17	36
6-inch tuna	430	19	46	20
6-inch roast beef	290	5	45	19
Turkey–bacon wrap	430	27	20	34

www.subway.com

89. Diet. Your goal is a low-fat diet consisting of 526 grams of carbohydrates, 168 grams of fat, and 332 grams of protein. How many of each sandwich would you eat that week to obtain this goal?

90. Diet. Your goal is a low-carb diet consisting of 5180 calories, 335 grams of carbohydrates, and 263 grams of fat. How many of each sandwich would you eat that week to obtain this goal?

Exercises 91 and 92 involve vertical motion and the effect of gravity on an object.

Because of gravity, an object that is projected upward will eventually reach a maximum height and then fall to the ground. The equation that relates the height h of a projectile t seconds after it is projected upward is given by

$$h = \frac{1}{2}at^2 + v_0t + h_0$$

where a is the acceleration due to gravity, h_0 is the initial height of the object at time $t = 0$, and v_0 is the initial velocity of the object at time $t = 0$. Note that a projectile follows the path of a parabola opening down, so $a < 0$.

91. Vertical Motion. An object is thrown upward, and the table below depicts the height of the ball t seconds after the projectile is released. Find the initial height, initial velocity, and acceleration due to gravity.

t (seconds)	Height (feet)
1	34
2	36
3	6

92. Vertical Motion. An object is thrown upward, and the table below depicts the height of the ball t seconds after the projectile is released. Find the initial height, initial velocity, and acceleration due to gravity.

t (seconds)	Height (feet)
1	54
2	66
3	46

93. Data Curve-Fitting. The average number of minutes that a person spends driving a car can be modeled by a quadratic function $y = ax^2 + bx + c$, where $a < 0$ and $15 < x < 65$. The table below gives the average number of minutes a day that a person spends driving a car. Determine a quadratic function that models this quantity.

Age	Average Daily Minutes Driving
16	25
40	64
65	40

94. Data Curve-Fitting. The average age when a woman gets married has been increasing during the last century. In 1920 the average age was 18.4, in 1960 the average age was 20.3, and in 2002 the average age was 25.30. Find a quadratic function $y = ax^2 + bx + c$, where $a > 0$ and $18 < x < 35$, that models the average age y when a woman gets married as a function of the year x ($x = 0$ corresponds to 1920). What will the average age be in 2010?

95. Chemistry/Pharmacy. A pharmacy receives an order for 100 milliliters of 5% hydrogen peroxide solution. The pharmacy has a 1.5% and a 30% solution on hand. A technician will mix the 1.5% and 30% solutions to make the 5% solution. How much of the 1.5% and 30% solutions, respectively, will be needed to fill this order? Round to the nearest milliliter.

96. Chemistry/Pharmacy. A pharmacy receives an order for 60 grams of a 0.7% hydrocortisone cream. The pharmacy has 1% and 0.5% hydrocortisone creams as well as a Eucerin cream for use as a base (0% hydrocortisone). The technician must use twice as much 0.5% hydrocortisone cream than the Eucerin base. How much of the 1% and 0.5% hydrocortisone creams and Eucerin cream are needed to fill this order?

97. Business. A small company has an assembly line that produces three types of widgets. The basic widget is sold for \$12 per unit, the midprice widget for \$15 per unit, and the top-of-the-line widget for \$18 per unit. The assembly line has a daily capacity of producing 375 widgets that may be sold for a total of \$5250. Find the quantity of each type of widget produced on a day when twice as many basic widgets as midprice widgets are produced.

98. Business. A small company has an assembly line that produces three types of widgets. The basic widget is sold for \$10 per unit, the midprice widget for \$12 per unit, and the top-of-the-line widget for \$15 per unit. The assembly line has a daily capacity of producing 350 widgets that may be sold for a total of \$4600. Find the quantity of each type of widget produced on a day when twice as many top-of-the-line widgets as basic widgets are produced.

99. Money. Gary and Ginger decide to place \$10,000 of their savings into investments. They put some in a money market account earning 3% interest, some in a mutual fund that has been averaging 7% a year, and some in a stock that rose 10% last year. If they put \$3000 more in the money market than in the mutual fund and the mutual fund and stocks have the same growth in the next year as they did in the previous year, they will earn \$540 in a year. How much money did they put in each of the three investments?

100. Money. Ginger talks Gary into putting less money in the money market and more money in the stock (see Exercise 99). They place \$10,000 of their savings into investments. They put some in a money market account earning 3% interest, some in a mutual fund that has been averaging 7% a year, and some in a stock that rose 10% last year. If they put \$3000 more in the stock than in the mutual fund and the mutual fund and stock have the same growth in the next year as they did in the previous year, they will earn \$840 in a year. How much money did they put in each of the three investments?

101. Manufacturing. A company produces three products x, y, and z. Each item of product x requires 20 units of steel, 2 units of plastic, and 1 unit of glass. Each item of product y requires 25 units of steel, 5 units of plastic, and no units of glass. Each item of product z requires 150 units of steel, 10 units of plastic, and 0.5 units of glass. The available amounts of steel, plastic, and glass are 2400, 310, and 28, respectively. How many items of each type can the company produce and utilize all the available raw materials?

102. Geometry. Find the values of a, b, and c such that the graph of the quadratic function $y = ax^2 + bx + c$ passes through the points $(1, 5)$, $(-2, -10)$, and $(0, 4)$.

103. Ticket Sales. One hundred students decide to buy tickets to a football game. There are three types of tickets: general admission, reserved, and end zone. Each general admission ticket costs \$20, each reserved ticket costs \$40, and each end zone ticket costs \$15. The students spend a total of \$2375 for all the tickets. There are five more reserved tickets than general admission tickets, and 20 more end zone tickets than general admission tickets. How many of each type of ticket were purchased by the students?

104. Exercise and Nutrition. Ann would like to exercise one hour per day to burn calories and lose weight. She would like to engage in three activities: walking, step-up exercise, and weight training. She knows she can burn 85 calories walking at a certain pace in 15 minutes, 45 calories doing the step-up exercise in 10 minutes, and 137 calories by weight training for 20 minutes.

a. Determine the number of calories per minute she can burn doing each activity.

b. Suppose she has time to exercise for only one hour (60 minutes). She sets a goal of burning 358 calories in one hour and would like to weight train twice as long as walking. How many minutes must she engage in each exercise to burn the required number of calories in one hour?

105. Geometry. The circle given by the equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(4, 4)$, $(-3, -1)$, and $(1, -3)$. Find a, b, and c.

106. Geometry. The circle given by the equation $x^2 + y^2 + ax + by + c = 0$ passes through the points $(0, 7)$, $(6, 1)$, and $(5, 4)$. Find a, b, and c.

CATCH THE MISTAKE

In Exercises 107–110, explain the mistake that is made.

107. Solve the system of equations using an augmented matrix.

$$\begin{aligned} y - x + z &= 2 \\ x - 2z + y &= -3 \\ x + y + z &= 6 \end{aligned}$$

Solution:

Step 1: Write as an augmented matrix. $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & -2 & 1 & -3 \\ 1 & 1 & 1 & 6 \end{array}\right]$

Step 2: Reduce the matrix using Gaussian elimination. $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & -6 \end{array}\right]$

Step 3: Identify the solution. Row 3 is inconsistent, so there is no solution.

This is incorrect. The correct answer is $x = 1$, $y = 2$, $z = 3$. What mistake was made?

108. Perform the indicated row operations on the matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & -3 & 1 & 4 \\ 3 & 1 & 2 & -6 \end{array}\right]$$

a. $R_2 - 2R_1 \rightarrow R_2$
b. $R_3 - 3R_1 \rightarrow R_3$

Solution:

a. $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & -3 & 1 & 4 \\ 3 & 1 & 2 & -6 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & -3 & 1 & 4 \\ 0 & 1 & 2 & -6 \end{array}\right]$

This is incorrect. What mistake was made?

109. Solve the system of equations using an augmented matrix.

$$\begin{aligned} 3x - 2y + z &= -1 \\ x + y - z &= 3 \\ 2x - y + 3z &= 0 \end{aligned}$$

Solution:

Step 1: Write the system as an augmented matrix. $\left[\begin{array}{ccc|c} 3 & -2 & 1 & -1 \\ 1 & 1 & -1 & 3 \\ 2 & -1 & 3 & 0 \end{array}\right]$

Step 2: Reduce the matrix using Gaussian elimination. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Step 3: Identify the answer: Row 3 is inconsistent $1 = 0$, therefore there is no solution.

This is incorrect. What mistake was made?

110. Solve the system of equations using an augmented matrix.

$$\begin{aligned} x + 3y + 2z &= 4 \\ 3x + 10y + 9z &= 17 \\ 2x + 7y + 7z &= 17 \end{aligned}$$

Solution:

Step 1: Write the system as an augmented matrix. $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 3 & 10 & 9 & 17 \\ 2 & 7 & 7 & 17 \end{array}\right]$

Step 2: Reduce the matrix using Gaussian elimination. $\left[\begin{array}{ccc|c} 1 & 0 & -7 & -11 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{array}\right]$

Step 3: Identify the answer: Infinitely many solutions.

$$\begin{aligned} x &= 7t - 11 \\ y &= -3t + 5 \\ z &= t \end{aligned}$$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 111–118, determine whether each of the following statements is true or false:

111. A system of equations represented by a nonsquare coefficient matrix cannot have a unique solution.

112. The procedure for Gaussian elimination can be used only for a system of linear equations represented by a square matrix.

113. A system of linear equations represented by a square coefficient matrix that has a unique solution has a reduced matrix with 1s along the main diagonal and 0s above and below the 1s.

114. A system of linear equations represented by a square coefficient matrix with an all-zero row has infinitely many solutions.

115. When a system of linear equations is represented by a square augmented matrix, the system of equations always has a unique solution.

116. Gauss–Jordan elimination produces a matrix in reduced row–echelon form.

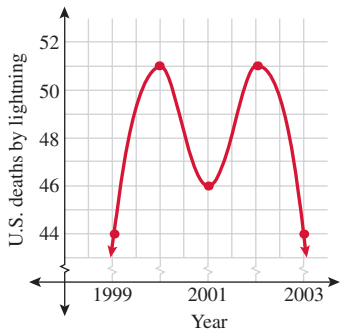
117. An inconsistent system of linear equations has infinitely many solutions.

118. Every system of linear equations with a unique solution is represented by an augmented matrix of order $n \times (n + 1)$. (Assume no two rows are identical.)

CHALLENGE

119. A fourth-degree polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + k$, with $a < 0$, can be used to represent the data on the number of deaths per year due to lightning strikes (assume 1999 corresponds to $x = 0$).

Use the data below to determine a, b, c, d, and k.



120. A copy machine accepts nickels, dimes, and quarters. After one hour, it holds 30 coins total, and their value is \$4.60. How many nickels, quarters, and dimes are in the machine?

121. A ferry goes down a river from city A to city B in 5 hours. The return trip takes 7 hours. How long will a canoe take to make the trip from A to B if it moves at the river speed?

122. Solve the system of equations.

$$\frac{3}{x} - \frac{4}{y} + \frac{6}{z} = 1$$

$$\frac{9}{x} + \frac{8}{y} - \frac{12}{z} = 3$$

$$\frac{9}{x} - \frac{4}{y} + \frac{12}{z} = 4$$

123. The sides of a triangle are formed by the lines $x - y = -3$, $3x + 4y = 5$, and $6x + y = 17$. Find the vertices of the triangle.

124. A winery has three barrels, A, B, and C, containing mixtures of three different wines, w_1, w_2, and w_3. In barrel A, the wines are in the ratio 1:2:3. In barrel B, the wines are in the ratio 3:5:7. In barrel C, the wines are in the ratio 3:7:9. How much wine must be taken from each barrel to get a mixture containing 17 liters of w_1, 35 liters of w_2, and 47 liters of w_3?

TECHNOLOGY

125. In Exercise 57, you were asked to solve this system of equations using an augmented matrix.

$$\begin{aligned} x - z - y &= 10 \\ 2x - 3y + z &= -11 \\ y - x + z &= -10 \end{aligned}$$

A graphing calculator or graphing utility can be used to solve systems of linear equations by entering the coefficients of the matrix. Solve this system and confirm your answer with the calculator's answer.

126. In Exercise 58, you were asked to solve this system of equations using an augmented matrix.

$$\begin{aligned} 2x + z + y &= -3 \\ 2y - z + x &= 0 \\ x + y + 2z &= 5 \end{aligned}$$

A graphing calculator or graphing utility can be use to solve systems of linear equations by entering the coefficients of the matrix. Solve this system and confirm your answer with the calculator's answer.

In Exercises 127 and 128, you are asked to model a set of three points with a quadratic function $y = ax^2 + bx + c$ and determine the quadratic function.

a. Set up a system of equations, use a graphing utility or graphing calculator to solve the system by entering the coefficients of the augmented matrix.

b. Use the graphing calculator commands $\boxed{\text{STAT}}$ $\boxed{\text{QuadReg}}$ to model the data using a quadratic function. Round your answers to two decimal places.

127. $(-6, -8)$, $(2, 7)$, $(7, 1)$

128. $(-9, 20)$, $(2, -18)$, $(11, 16)$

PREVIEW TO CALCULUS

In calculus, when solving systems of linear differential equations with initial conditions, the solution of a system of linear equations is required. In Exercises 129–132, solve each system of equations.

129. $\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 5c_2 &= -3 \end{aligned}$

130. $\begin{aligned} 3c_1 + 3c_2 &= 0 \\ 2c_1 + 3c_2 &= 0 \end{aligned}$

131. $\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0 \\ 2c_1 \qquad - 2c_3 &= 2 \\ c_1 - c_2 + c_3 &= 6 \end{aligned}$

132. $\begin{aligned} c_1 + c_4 &= 1 \\ c_3 &= 1 \\ c_2 + 3c_4 &= 1 \\ c_1 - 2c_3 &= 1 \end{aligned}$

SECTION 8.4 MATRIX ALGEBRA

SKILLS OBJECTIVES

- Use equality of matrices.
- Add and subtract matrices.
- Perform scalar multiplication.
- Multiply two matrices.
- Write a system of linear equations as a matrix equation.
- Find the inverse of a square matrix.
- Solve systems of linear equations using inverse matrices.

CONCEPTUAL OBJECTIVES

- Understand what is meant by equal matrices.
- Understand why multiplication of some matrices is undefined.
- Realize that matrix multiplication is *not* commutative.
- Visualize a system of linear equations as a matrix equation.
- Understand that only a square matrix can have an inverse.
- Realize that not every square matrix has an inverse.

Equality of Matrices

In Section 8.3, we defined a matrix with m rows and n columns to have order $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Capital letters are used to represent (or name) a matrix, and lowercase letters are used to represent the entries (elements) of the matrix. The subscripts are used to denote the location (row/column) of each entry. The order of a matrix is often written as a subscript of the matrix name: $A_{m \times n}$. Other words like "size" and "dimension" are used as synonyms of "order." Matrices are a convenient way to represent data.

There is an entire field of study called **matrix algebra** that treats matrices similarly to functions and variables in traditional algebra. This section serves as an introduction to matrix algebra. It is important to pay special attention to the *order* of a matrix, because it determines whether certain operations are defined.

Two matrices are equal if and only if they have the same order, $m \times n$, and all of their corresponding entries are equal.

DEFINITION **Equality of Matrices**

Two matrices, A and B, are **equal**, written as $A = B$, if and only if *both* of the following are true:

- A and B have the same order $m \times n$.
- Every pair of corresponding entries is equal: $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

EXAMPLE 1 **Equality of Matrices**

Referring to the definition of equality of matrices, find the indicated entries.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & -7 & 1 \\ 0 & 5 & -3 \\ -1 & 8 & 9 \end{bmatrix}$$

Find the main diagonal entries: a_{11}, a_{22}, and a_{33}.

Solution:

Since the matrices are equal, their corresponding entries are equal. $\boxed{a_{11} = 2}$ $\boxed{a_{22} = 5}$ $\boxed{a_{33} = 9}$

Matrix Addition and Subtraction

Two matrices, A and B, can be added or subtracted only if they have the *same order*. Suppose A and B are both of order $m \times n$; then the *sum* $A + B$ is found by adding corresponding entries, or taking $a_{ij} + b_{ij}$. The *difference* $A - B$ is found by subtracting the entries in B from the corresponding entries in A, or finding $a_{ij} - b_{ij}$.

DEFINITION **Matrix Addition and Matrix Subtraction**

If A is an $m \times n$ matrix and B is an $m \times n$ matrix, then their **sum** $A + B$ is an $m \times n$ matrix whose entries are given by

$$a_{ij} + b_{ij}$$

and their **difference** $A - B$ is an $m \times n$ matrix whose entries are given by

$$a_{ij} - b_{ij}$$

EXAMPLE 2 Adding and Subtracting Matrices

Given that $A = \begin{bmatrix} -1 & 3 & 4 \\ -5 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -3 \\ 0 & -5 & 4 \end{bmatrix}$, find:

a. $A + B$ **b.** $A - B$

Solution:

Since $A_{2\times 3}$ and $B_{2\times 3}$ have the same order, they can be added or subtracted.

a. Write the sum. $A + B = \begin{bmatrix} -1 & 3 & 4 \\ -5 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -3 \\ 0 & -5 & 4 \end{bmatrix}$

Add the corresponding entries. $= \begin{bmatrix} -1+2 & 3+1 & 4+(-3) \\ -5+0 & 2+(-5) & 0+4 \end{bmatrix}$

Simplify. $= \boxed{\begin{bmatrix} 1 & 4 & 1 \\ -5 & -3 & 4 \end{bmatrix}}$

b. Write the difference. $A - B = \begin{bmatrix} -1 & 3 & 4 \\ -5 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -3 \\ 0 & -5 & 4 \end{bmatrix}$

Subtract the corresponding entries. $= \begin{bmatrix} -1-2 & 3-1 & 4-(-3) \\ -5-0 & 2-(-5) & 0-4 \end{bmatrix}$

Simplify. $= \boxed{\begin{bmatrix} -3 & 2 & 7 \\ -5 & 7 & -4 \end{bmatrix}}$

YOUR TURN Perform the indicated matrix operations, if possible.

$A = \begin{bmatrix} -4 & 0 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ -4 & 0 \end{bmatrix}$ $C = [2 \quad 9 \quad 5 \quad -1]$ $D = \begin{bmatrix} 0 \\ -3 \\ 4 \\ 2 \end{bmatrix}$

a. $B - A$ **b.** $C + D$ **c.** $A + B$ **d.** $A + D$

Technology Tip

Enter the matrices as A and B.

```
[A]
      [[-1 3 4]
       [-5 2 0]]
[B]
      [[2 1  -3]
       [0 -5 4 ]]
```

Now enter $[A] + [B]$ and $[A] - [B]$.

```
[A]+[B]
      [[1  4  1]
       [-5 -3 4]]
[A]-[B]
      [[-3 2 7 ]
       [-5 7 -4]]
```

Answer:

a. $\begin{bmatrix} 6 & 3 \\ -5 & -2 \end{bmatrix}$ **b.** not defined

c. $\begin{bmatrix} -2 & 3 \\ -3 & 2 \end{bmatrix}$ **d.** not defined

It is important to note that only matrices of the same order can be added or subtracted. For example, if $A = \begin{bmatrix} -1 & 3 & 4 \\ -5 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 \\ 12 & 1 \end{bmatrix}$, the sum and difference of these matrices are undefined because $A_{2\times 3}$ and $B_{2\times 2}$ do not have the same order.

Study Tip

Only matrices of the same order can be added or subtracted.

A matrix whose entries are all equal to 0 is called a **zero matrix**, denoted **0**. The following are examples of zero matrices:

2×2 square zero matrix	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
3×2 zero matrix	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
1×4 zero matrix	$[0 \quad 0 \quad 0 \quad 0]$

If A, an $m \times n$ matrix, is added to the $m \times n$ zero matrix, the result is A.

$$A + \mathbf{0} = A$$

For example,

$$\begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

Because of this result, an $m \times n$ zero matrix is called the **additive identity** for $m \times n$ matrices. Similarly, for any matrix A, there exists an **additive inverse**, $-A$, such that each entry of $-A$ is the negative of the corresponding entry of A.

For example, $A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$ and $-A = \begin{bmatrix} -1 & 3 \\ -2 & -5 \end{bmatrix}$, and adding these two matrices results in a zero matrix: $A + (-A) = \mathbf{0}$.

The same properties that hold for adding real numbers also hold for adding matrices, provided that addition of matrices is defined.

PROPERTIES OF MATRIX ADDITION

If A, B, and C are all $m \times n$ matrices and $\mathbf{0}$ is the $m \times n$ zero matrix, then the following are true:

Commutative property:	$A + B = B + A$
Associative property:	$(A + B) + C = A + (B + C)$
Additive identity property:	$A + \mathbf{0} = A$
Additive inverse property:	$A + (-A) = \mathbf{0}$

Scalar and Matrix Multiplication

There are two types of multiplication involving matrices: *scalar multiplication* and *matrix multiplication*. A **scalar** is any real number. *Scalar multiplication* is the multiplication of a matrix by a scalar, or real number, and is defined for all matrices. *Matrix multiplication* is the multiplication of two matrices and is defined only for certain pairs of matrices, depending on the order of each matrix.

Scalar Multiplication

To multiply a matrix A by a scalar k, multiply every entry in A by k.

$$3\begin{bmatrix} -1 & 0 & 4 \\ 7 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 3(-1) & 3(0) & 3(4) \\ 3(7) & 3(5) & 3(-2) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 12 \\ 21 & 15 & -6 \end{bmatrix}$$

Here, the scalar is $k = 3$.

DEFINITION **Scalar Multiplication**

If A is an $m \times n$ matrix and k is any real number, then their product kA is an $m \times n$ matrix whose entries are given by

$$ka_{ij}$$

In other words, every entry a_{ij} of A is multiplied by k.

In general, uppercase letters are used to denote a matrix and lowercase letters are used to denote scalars. Notice that the elements of each matrix are also represented with lowercase letters, since they are real numbers.

EXAMPLE 3 Multiplying a Matrix by a Scalar

Given that $A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, perform:

a. $2A$ **b.** $-3B$ **c.** $2A - 3B$

Solution (a):

Write the scalar multiplication. $2A = 2\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

Multiply all entries of A by 2. $2A = \begin{bmatrix} 2(-1) & 2(2) \\ 2(-3) & 2(4) \end{bmatrix}$

Simplify. $\boxed{2A = \begin{bmatrix} -2 & 4 \\ -6 & 8 \end{bmatrix}}$

Solution (b):

Write the scalar multiplication. $-3B = -3\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

Multiply all entries of B by -3. $-3B = \begin{bmatrix} -3(0) & -3(1) \\ -3(-2) & -3(3) \end{bmatrix}$

Simplify. $\boxed{-3B = \begin{bmatrix} 0 & -3 \\ 6 & -9 \end{bmatrix}}$

Solution (c):

Add the results of parts (a) and (b). $2A - 3B = 2A + (-3B)$

$$2A - 3B = \begin{bmatrix} -2 & 4 \\ -6 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 6 & -9 \end{bmatrix}$$

Add the corresponding entries. $2A - 3B = \begin{bmatrix} -2 + 0 & 4 + (-3) \\ -6 + 6 & 8 + (-9) \end{bmatrix}$

Simplify. $\boxed{2A - 3B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}}$

■ **YOUR TURN** For the matrices A and B given in Example 3, find $-5A + 2B$.

Technology Tip

Enter matrices as A and B.

```
[A]
          [[-1 2]
           [-3 4]]
[B]
          [[0  1]
           [-2 3]]
```

Now enter $2A$, $-3B$, $2A - 3B$.

```
2[A]
          [[-2 4]
           [-6 8]]
-3[B]
          [[0 -3]
           [6 -9]]
```

```
2[A]-3[B]
          [[-2 1 ]
           [0  -1]]
```

■ **Answer:**
$-5A + 2B = \begin{bmatrix} 5 & -8 \\ 11 & -14 \end{bmatrix}$

Matrix Multiplication

Scalar multiplication is straightforward in that it is defined for all matrices and is performed by multiplying every entry in the matrix by the scalar. Addition of matrices is also an entry-by-entry operation. *Matrix multiplication*, on the other hand, is not as straightforward in that we *do not multiply the corresponding entries* and it is not defined for all matrices. Matrices are multiplied using a row-by-column method.

Study Tip

When we multiply matrices, we *do not* multiply corresponding entries.

Study Tip

For the product AB of two matrices A and B to be defined, the number of columns in the first matrix must equal the number of rows in the second matrix.

Before we even try to find the product AB of two matrices A and B, we first have to determine whether the product is defined. For the product AB to exist, **the number of columns in the first matrix A must equal the number of rows in the second matrix B**. In other words, if the matrix $A_{m\times n}$ has m rows and n columns and the matrix $B_{n\times p}$ has n rows and p columns, then the product $(AB)_{m\times p}$ is defined and has m rows and p columns.

Matrix:	A	B	AB
Order:	$m \times n$	$n \times p$	$m \times p$

Equal

Order of AB

EXAMPLE 4 Determining Whether the Product of Two Matrices Is Defined

Given the matrices

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 5 & -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 7 \\ 4 & 9 \end{bmatrix} \quad C = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \quad D = [-3 \quad -2]$$

state whether each of the following products exists. If the product exists, state the order of the product matrix.

a. AB **b.** AC **c.** BC **d.** CD **e.** DC

Solution:

Label the order of each matrix: $A_{2\times 3}$, $B_{3\times 2}$, $C_{2\times 2}$, and $D_{1\times 2}$.

a. AB is defined, because A has 3 columns and B has 3 rows. $A_{2\times 3}B_{3\times 2}$
AB is order $\boxed{2 \times 2}$. $(AB)_{2\times 2}$

b. AC is $\boxed{\text{not defined}}$, because A has 3 columns and C has 2 rows.

c. BC is defined, because B has 2 columns and C has 2 rows. $B_{3\times 2}C_{2\times 2}$
BC is order $\boxed{3 \times 2}$. $(BC)_{3\times 2}$

d. CD is $\boxed{\text{not defined}}$, because C has 2 columns and D has 1 row.

e. DC is defined, because D has 2 columns and C has 2 rows. $D_{1\times 2}C_{2\times 2}$
DC is order $\boxed{1 \times 2}$. $(DC)_{1\times 2}$

Notice that in part (d) we found that CD is not defined, but in part (e) we found that DC is defined. **Matrix multiplication is not commutative**. Therefore, the order in which matrices are multiplied is important in determining whether the product is defined or undefined. For the product of two matrices to exist, the number of *columns* in the *first* matrix A must equal the number of *rows* in the *second* matrix B.

YOUR TURN For the matrices given in Example 4, state whether the following products exist. If the product exists, state the order of the product matrix.

a. DA **b.** CB **c.** BA

Answer:
a. DA exists and is order 1×3.
b. CB does not exist.
c. BA exists and is order 3×3

Now that we can determine whether a product of two matrices is defined and, if so, what the order of the resulting product is, let us turn our attention to how to multiply two matrices.

DEFINITION **Matrix Multiplication**

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their product AB is an $m \times p$ matrix whose entries are given by

$$(ab)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, the entry $(ab)_{ij}$, which is in the ith row and jth column of AB, is the sum of the products of the corresponding entries in the ith row of A and the jth column of B. Multiply *across* the row and *down* the column.

$$\begin{bmatrix} \overbrace{a_{i1} \quad a_{i2} \quad a_{i3}}^{i\text{th row of } A} \end{bmatrix} \begin{bmatrix} \overbrace{b_{1j}}^{j\text{th column of } B} \\ b_{2j} \\ b_{3j} \end{bmatrix} = \begin{bmatrix} \overbrace{(ab)_{ij}}^{\text{Element in the } i\text{th row and } j\text{th column of } AB} \end{bmatrix}$$

$$(ab)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}$$

EXAMPLE 5 **Multiplication of Two 2×2 Matrices**

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, find AB.

COMMON MISTAKE

Do not multiply entry by entry.

CORRECT

Write the product of the two matrices A and B.

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Perform the row-by-column multiplication.

$$AB = \begin{bmatrix} (1)(5) + (2)(7) & (1)(6) + (2)(8) \\ (3)(5) + (4)(7) & (3)(6) + (4)(8) \end{bmatrix}$$

Simplify.

$$\boxed{AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}}$$

INCORRECT

Multiply the corresponding entries.

ERROR

$$AB \neq \begin{bmatrix} (1)(5) & (2)(6) \\ (3)(7) & (4)(8) \end{bmatrix}$$

YOUR TURN For matrices A and B given in Example 5, find BA.

Compare the products obtained in Example 5 and the preceding Your Turn. Note that $AB \neq BA$. Therefore, there is **no commutative property for matrix multiplication**.

Technology Tip

Enter the matrices as A and B and calculate AB.

```
[A]
          [[1 2]
           [3 4]]
[B]
          [[5 6]
           [7 8]]
```

```
[A][B]
          [[19 22]
           [43 50]]
```

Answer:

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Technology Tip

Enter the matrices as A and B and calculate AB.

```
[A]
      [[-1  2  -3]
       [-2  0   4]]
[B]
       [[2   0 ]
        [1   3 ]
        [-1  -2]]
```

```
[A][B]
       [[3   12]
        [-8  -8]]
```

■ **Answer:** $AB = \begin{bmatrix} 0 & -5 \\ -1 & -7 \end{bmatrix}$

EXAMPLE 6 Multiplying Matrices

For $A = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ -1 & -2 \end{bmatrix}$, find AB.

Solution:

Since A is order 2×3 and B is order 3×2, the product AB is defined and has order 2×2.

$$A_{2\times3}B_{3\times2} = (AB)_{2\times2}$$

Write the product of the two matrices.

$$AB = \begin{bmatrix} -1 & 2 & -3 \\ -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 3 \\ -1 & -2 \end{bmatrix}$$

Perform the row-by-column multiplication.

$$AB = \begin{bmatrix} (-1)(2) + (2)(1) + (-3)(-1) & (-1)(0) + (2)(3) + (-3)(-2) \\ (-2)(2) + (0)(1) + (4)(-1) & (-2)(0) + (0)(3) + (4)(-2) \end{bmatrix}$$

Simplify.

$$\boxed{AB = \begin{bmatrix} 3 & 12 \\ -8 & -8 \end{bmatrix}}$$

■ **YOUR TURN** For $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 0 & -2 \end{bmatrix}$, find AB.

EXAMPLE 7 Multiplying Matrices

For $A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 1 \\ -3 & -1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$, find AB.

Solution:

Since A is order 2×3 and B is order 3×3, the product AB is defined and has order 2×3.

$$A_{2\times3}B_{3\times3} = (AB)_{2\times3}$$

Write the product of the two matrices.

$$AB = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ -3 & -1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

Perform the row-by-column multiplication.

$$AB = \begin{bmatrix} (1)(-2) + (0)(-3) + (3)(0) & (1)(0) + (0)(-1) + (3)(2) & (1)(1) + (0)(4) + (3)(5) \\ (-2)(-2) + (5)(-3) + (-1)(0) & (-2)(0) + (5)(-1) + (-1)(2) & (-2)(1) + (5)(4) + (-1)(5) \end{bmatrix}$$

Simplify.

$$\boxed{AB = \begin{bmatrix} -2 & 6 & 16 \\ -11 & -7 & 13 \end{bmatrix}}$$

■ **YOUR TURN** Given $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = [4 \quad 5]$, find:

a. AB, if it exists **b.** BA, if it exists

■ **Answer: a.** $AB = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}$

b. does not exist

Although we have shown repeatedly that there is no commutative property of multiplication for matrices, matrices do have an associative property of multiplication, as well as a distributive property of multiplication similar to real numbers.

PROPERTIES OF MATRIX MULTIPLICATION

If A, B, and C are all matrices for which AB, AC, BC, $A + B$, and $B + C$ are all defined, then the following properties are true:

Associative property: $A(BC) = (AB)C$

Distributive property: $A(B + C) = AB + AC$ or $(A + B)C = AC + BC$

EXAMPLE 8 Application of Matrix Multiplication

The following table gives fuel and electric requirements per mile associated with gasoline and electric automobiles:

	NUMBER OF GALLONS/MILE	NUMBER OF kW-hr/MILE
Gas car	0.05	0
Hybrid car	0.02	0.1
Electric car	0	0.25

The following table gives an average cost for gasoline and electricity:

Cost per gallon of gasoline	\$3.00
Cost per kW-hr of electricity	\$0.05

a. Let matrix A represent the gasoline and electricity consumption and matrix B represent the costs of gasoline and electricity.
b. Find AB and describe what the entries of the product matrix represent.
c. Assume you drive 12,000 miles per year. What are the yearly costs associated with driving the three types of cars?

Solution (a):

A has order 3×2.

$$\boxed{A = \begin{bmatrix} 0.05 & 0 \\ 0.02 & 0.1 \\ 0 & 0.25 \end{bmatrix}}$$

B has order 2×1.

$$\boxed{B = \begin{bmatrix} \$3.00 \\ \$0.05 \end{bmatrix}}$$

Solution (b):

Find the order of the product matrix AB.

$$A_{3\times 2}B_{2\times 1} = (AB)_{3\times 1}$$

$$AB = \begin{bmatrix} 0.05 & 0 \\ 0.02 & 0.1 \\ 0 & 0.25 \end{bmatrix}\begin{bmatrix} \$3.00 \\ \$0.05 \end{bmatrix}$$

Calculate AB.

$$= \begin{bmatrix} (0.05)(\$3.00) + (0)(\$0.05) \\ (0.02)(\$3.00) + (0.1)(\$0.05) \\ (0)(\$3.00) + (0.25)(\$0.05) \end{bmatrix}$$

$$\boxed{AB = \begin{bmatrix} \$0.15 \\ \$0.065 \\ \$0.0125 \end{bmatrix}}$$

Technology Tip

Enter the matrices as A and B and calculate AB.

```
[A]
      [[.05 0  ]
       [.02 .1 ]
       [0   .25]]
[B]
          [[3  ]
           [.05]]
```

```
[A][B]
          [[.15  ]
           [.065 ]
           [.0125]]
```

Interpret the product matrix.

$$AB = \begin{bmatrix} \text{Cost per mile to drive the gas car} \\ \text{Cost per mile to drive the hybrid car} \\ \text{Cost per mile to drive the electric car} \end{bmatrix}$$

Solution (c):

Find $12{,}000AB$.

$$12{,}000\begin{bmatrix} \$0.15 \\ \$0.065 \\ \$0.0125 \end{bmatrix} = \begin{bmatrix} \$1800 \\ \$780 \\ \$150 \end{bmatrix}$$

	GAS/ELECTRIC COSTS PER YEAR ($)
Gas car	1800
Hybrid car	780
Electric car	150

Matrix Equations

Matrix equations are another way of writing systems of linear equations.

WORDS	MATH
Start with a matrix equation.	$\begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$
Multiply the two matrices on the left.	$\begin{bmatrix} 2x - 3y \\ x + 5y \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$
Apply equality of two matrices.	$\begin{aligned} 2x - 3y &= -7 \\ x + 5y &= 9 \end{aligned}$

Let A be a matrix with m rows and n columns, which represents the coefficients in the system. Also, let X be a column matrix of order $n \times 1$ that represents the variables in the system and let B be a column matrix of order $m \times 1$ that represents the constants in the system. Then, a system of linear equations can be written as $AX = B$.

SYSTEM OF LINEAR EQUATIONS	A	X	B	MATRIX EQUATION: $AX = B$
$\begin{aligned} 3x + 4y &= 1 \\ x - 2y &= 7 \end{aligned}$	$\begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} 1 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$
$\begin{aligned} x - y + z &= 2 \\ 2x + 2y - 3z &= -3 \\ x + y + z &= 6 \end{aligned}$	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$
$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - z &= 2 \end{aligned}$	$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

EXAMPLE 9 Writing a System of Linear Equations as a Matrix Equation

Write each system of linear equations as a matrix equation.

a. $\begin{aligned} 2x - y &= 5 \\ -x + 2y &= 3 \end{aligned}$

b. $\begin{aligned} 3x - 2y + 4z &= 5 \\ y - 3z &= -2 \\ 7x \quad - z &= 1 \end{aligned}$

c. $\begin{aligned} x_1 - x_2 + 2x_3 - 3 &= 0 \\ x_1 + x_2 - 3x_3 + 5 &= 0 \\ x_1 - x_2 + x_3 - 2 &= 0 \end{aligned}$

Solution:

a.

$$\boxed{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}}$$

b. Note that all missing terms have 0 coefficients.

$$\begin{aligned} 3x - 2y + 4z &= 5 \\ 0x + y - 3z &= -2 \\ 7x + 0y - z &= 1 \end{aligned}$$

$$\boxed{\begin{bmatrix} 3 & -2 & 4 \\ 0 & 1 & -3 \\ 7 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}$$

c. Write the constants on the right side of the equal sign.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_1 + x_2 - 3x_3 &= -5 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

$$\boxed{\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}}$$

■ **YOUR TURN** Write each system of linear equations as a matrix equation.

a. $\begin{aligned} 2x + y - 3 &= 0 \\ x - y &= 5 \end{aligned}$

b. $\begin{aligned} y - x + z &= 7 \\ x - y - z &= 2 \\ z - y &= -1 \end{aligned}$

■ **Answer:**

a. $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -1 \end{bmatrix}$

Finding the Inverse of a Square Matrix

Before we discuss solving systems of linear equations in the form $AX = B$, let us first recall how we solve $ax = b$, where a and b are real numbers (not matrices).

WORDS	MATH
Write the linear equation in one variable.	$ax = b$
Multiply both sides by a^{-1} (same as dividing by a), provided $a \neq 0$.	$a^{-1}ax = a^{-1}b$
Simplify.	$\underbrace{a^{-1}a}_{1}x = a^{-1}b$ $x = a^{-1}b$

Recall that a^{-1}, or $\frac{1}{a}$, is the *multiplicative inverse* of a because $a^{-1}a = 1$. And we call 1 the *multiplicative identity*, because any number multiplied by 1 is itself. Before we solve matrix equations, we need to define the *multiplicative identity matrix* and the *multiplicative inverse matrix*.

A square matrix of order $n \times n$ with 1s along the **main diagonal** (a_{ii}) and 0s for all other elements is called the **multiplicative identity matrix I_n**.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since a real number multiplied by 1 is itself ($a \cdot 1 = a$), we expect that a matrix multiplied by the appropriate identity matrix should result in itself. Remember, the order in which matrices are multiplied makes a difference. Notice the appropriate identity matrix may differ, depending on the order of multiplication, but the identity matrix will always be square.

$$A_{m\times n}I_n = A_{m\times n} \quad \text{and} \quad I_m A_{m\times n} = A_{m\times n}$$

EXAMPLE 10 **Multiplying a Matrix by the Multiplicative Identity Matrix I_n**

For $A = \begin{bmatrix} -2 & 4 & 1 \\ 3 & 7 & -1 \end{bmatrix}$, find I_2A.

Solution:

Write the two matrices. $A = \begin{bmatrix} -2 & 4 & 1 \\ 3 & 7 & -1 \end{bmatrix} \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Find the product I_2A.

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 4 & 1 \\ 3 & 7 & -1 \end{bmatrix}$$

$$I_2A = \begin{bmatrix} (1)(-2) + (0)(3) & (1)(4) + (0)(7) & (1)(1) + (0)(-1) \\ (0)(-2) + (1)(3) & (0)(4) + (1)(7) & (0)(1) + (1)(-1) \end{bmatrix}$$

$$I_2A = \boxed{\begin{bmatrix} -2 & 4 & 1 \\ 3 & 7 & -1 \end{bmatrix}} = A$$

YOUR TURN For A in Example 10, find AI_3.

Answer:
$AI_3 = \begin{bmatrix} -2 & 4 & 1 \\ 3 & 7 & -1 \end{bmatrix} = A$

The identity matrix I_n will assist us in developing the concept of an *inverse of a square matrix.*

Study Tip

- Only a *square* matrix can have an inverse.
- Not all square matrices have inverses.

DEFINITION **Inverse of a Square Matrix**

Let A be a square $n \times n$ matrix. If there exists a square $n \times n$ matrix A^{-1} such that

$$AA^{-1} = I_n \quad \text{and} \quad A^{-1}A = I_n$$

then A^{-1}, stated as "A inverse," is the **inverse** of A.

It is important to note that only a square matrix can have an inverse. Even then, not all square matrices have inverses.

EXAMPLE 11 Multiplying a Matrix by Its Inverse

Verify that the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$.

Solution:

Show that $AA^{-1} = I_2$ and $A^{-1}A = I_2$.

Find the product AA^{-1}.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} (1)(-5) + (3)(2) & (1)(3) + (3)(-1) \\ (2)(-5) + (5)(2) & (2)(3) + (5)(-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

Find the product $A^{-1}A$.

$$\begin{aligned} A^{-1}A &= \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} (-5)(1) + (3)(2) & (-5)(3) + (3)(5) \\ (2)(1) + (-1)(2) & (2)(3) + (-1)(5) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

■ **YOUR TURN** Verify that the inverse of $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$.

■ **Answer:** $AA^{-1} = A^{-1}A = I_2$

Technology Tip

Enter the matrices as A, and A^{-1} as B.

```
[A]
          [[1 3]
           [2 5]]
[B]
        [[-5 3 ]
          [2  -1]]
```

Enter AA^{-1} as AB, and A^{-1} as BA.

```
[A][B]
          [[1 0]
           [0 1]]
[B][A]
          [[1 0]
           [0 1]]
```

Now that we can show that two matrices are inverses of one another, let us describe the process for finding an inverse, if it exists. If an inverse A^{-1} exists, then the matrix A is said to be **nonsingular**. If the inverse does not exist, then the matrix A is said to be **singular**.

Let $A = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$ and the inverse be $A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, where w, x, y, and z are variables to be determined. A matrix and its inverse must satisfy the identity $AA^{-1} = I_2$.

Words	**Math**
The product of a matrix and its inverse is the identity matrix.	$\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Multiply the two matrices on the left.	$\begin{bmatrix} w - y & x - z \\ 2w - 3y & 2x - 3z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Equate corresponding matrix elements.	$\begin{aligned} w - y &= 1 \\ 2w - 3y &= 0 \end{aligned}$ and $\begin{aligned} x - z &= 0 \\ 2x - 3z &= 1 \end{aligned}$

Notice that there are two systems of equations, both of which can be solved by several methods (elimination, substitution, or augmented matrices). We will find that $w = 3$, $x = -1$, $y = 2$, and $z = -1$. Therefore, we know the inverse is $A^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$. But, instead, let us use augmented matrices in order to develop the general procedure.

Write the two systems of equations as two augmented matrices:

$$\begin{array}{cc} w & y \end{array} \qquad \begin{array}{cc} x & z \end{array}$$
$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & -3 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -3 & 1 \end{array}\right]$$

Since the left side is the same for each augmented matrix, we can combine these two matrices into one matrix, thereby simultaneously solving both systems of equations.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

Notice that the right side of the vertical line is the identity matrix I_2.

Using Gauss–Jordan elimination, transform the matrix on the left to the identity matrix.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

$$R_2 - 2R_1 \rightarrow R_2 \quad \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$-R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$R_1 + R_2 \rightarrow R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

The matrix on the right of the vertical line is the inverse $A^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$.

FINDING THE INVERSE OF A SQUARE MATRIX

To find the inverse of an $n \times n$ matrix A:

Step 1: Form the matrix $[A \mid I_n]$.

Step 2: Use row operations to transform this entire augmented matrix to $[I_n \mid A^{-1}]$. This is done by applying Gauss–Jordan elimination to reduce A to the identity matrix I_n. If this is not possible, then A is a singular matrix and no inverse exists.

Step 3: Verify the result by showing that $AA^{-1} = I_n$ and $A^{-1}A = I_n$.

EXAMPLE 12 Finding the Inverse of a 2×2 Matrix

Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$.

Solution:

STEP 1 Form the matrix $[A \mid I_2]$.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

STEP 2 Use row operations to transform A into I_2.

$$R_2 - 3R_1 \rightarrow R_2 \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$$

$$-R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$R_1 - 2R_2 \rightarrow R_1 \quad \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

Identify the inverse.

$$\boxed{A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}}$$

Technology Tip

A graphing calculator can be used to find the inverse of A. Enter the matrix A.

```
[A]
          [[1 2]
           [3 5]]
```

To find A^{-1}, press 2nd MATRIX 1:[A] ENTER x^{-1} ENTER.

```
[A]
          [[1 2]
           [3 5]]
[A]-1
        [[-5 2 ]
         [3  -1]]
```

STEP 3 Check.

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A^{-1}A = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

■ **YOUR TURN** Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$.

■ **Answer:** $A^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$

This procedure for finding an inverse of a square matrix is used for all square matrices of order $n \times n$. For the special case of a 2×2 matrix, there is a formula (that will be derived in Exercises 107 and 108) for finding the inverse.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represent any 2×2 matrix; then the inverse matrix is given by

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad ad - bc \neq 0}$$

The denominator $ad - bc$ is called the *determinant* of the matrix A and will be discussed in Section 8.5.

We found the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ in Example 12. Let us now find the inverse using this formula.

WORDS	MATH
Write the formula for A^{-1}.	$A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
Substitute $a = 1$, $b = 2$, $c = 3$, and $d = 5$ into the formula.	$A^{-1} = \dfrac{1}{(1)(5) - (2)(3)}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$
Simplify.	$A^{-1} = (-1)\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$
	$A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

The result is the same as that we found in Example 12.

EXAMPLE 13 Finding That No Inverse Exists: Singular Matrix

Find the inverse of $A = \begin{bmatrix} 1 & -5 \\ -1 & 5 \end{bmatrix}$.

Solution:

STEP 1 Form the matrix $[A \mid I_2]$.

$$\left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ -1 & 5 & 0 & 1 \end{array}\right]$$

STEP 2 Apply row operations to transform A into I_2.

$$R_2 + R_1 \rightarrow R_2 \quad \left[\begin{array}{cc|cc} 1 & -5 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We cannot convert the left-hand side of the augmented matrix to I_2 because of the all-zero row on the left-hand side. Therefore, $\boxed{A \text{ is not invertible}}$; that is, A has no inverse, or A^{-1} does not exist. We say that A is **singular**.

Study Tip

If the determinant of a 2×2 matrix is equal to 0, then its inverse does not exist.

Technology Tip

A graphing calculator can be used to find the inverse of A. Enter the matrix A.

```
[A]
      [[1   2  -1]
       [0   1  -1]
       [-1  0  -2]]
```

To find A^{-1}, press [2nd] [*MATRIX*] [1:[*A*]] [ENTER] [x^{-1}] [ENTER].

```
[A]-1
     [[2   -4  1 ]
      [-1  3   -1]
      [-1  2   -1]]
```

EXAMPLE 14 Finding the Inverse of a 3 × 3 Matrix

Find the inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -2 \end{bmatrix}$.

Solution:

STEP 1 Form the matrix $[A \mid I_3]$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 \end{array}\right]$$

STEP 2 Apply row operations to transform A into I_3.

$$R_3 + R_1 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -3 & 1 & 0 & 1 \end{array}\right]$$

$$R_3 - 2R_2 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$$-R_3 \to R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$\begin{array}{c} R_2 + R_3 \to R_2 \\ R_1 + R_3 \to R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$R_1 - 2R_2 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -4 & 1 \\ 0 & 1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

Identify the inverse.

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -4 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{bmatrix}}$$

STEP 3 Check. $AA^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & -4 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$$A^{-1}A = \begin{bmatrix} 2 & -4 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

■ **YOUR TURN** Find the inverse of $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$.

■ **Answer:** $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

Solving Systems of Linear Equations Using Matrix Algebra and Inverses of Square Matrices

We can solve systems of linear equations using matrix algebra. We will use a system of three equations and three variables to demonstrate the procedure. However, it can be extended to any square system.

Linear System of Equations	**Matrix Form of the System**
$a_1x + b_1y + c_1z = d_1$ $a_2x + b_2y + c_2z = d_2$ $a_3x + b_3y + c_3z = d_3$	$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{B}$

Recall that a system of linear equations has a unique solution, no solution, or infinitely many solutions. If a system of n equations in n variables has a unique solution, it can be found using the following procedure:

Words	**Math**
Write the system of linear equations as a matrix equation.	$A_{n\times n}X_{n\times 1} = B_{n\times 1}$
Multiply both sides of the equation by A^{-1}.	$A^{-1}AX = A^{-1}B$
A matrix times its inverse is the identity matrix.	$I_nX = A^{-1}B$
A matrix times the identity matrix is equal to itself.	$X = A^{-1}B$

Notice the order in which the right side is multiplied, $X_{n\times 1} = A^{-1}_{n\times n}B_{n\times 1}$, and remember that matrix multiplication is not commutative. Therefore, you multiply both sides of the matrix equation in the same order.

Solving a system of linear equations using matrix algebra: Unique solution

If a system of linear equations is represented by the matrix equation $AX = B$, where A is a nonsingular square matrix, then the system has a unique solution given by

$$X = A^{-1}B$$

EXAMPLE 15 Solving a System of Linear Equations Using Matrix Algebra

Solve the system of equations using matrix algebra.

$$\begin{aligned} x + y + z &= 2 \\ x \quad\; + z &= 1 \\ x - y - z &= -4 \end{aligned}$$

Solution:

Write the system in matrix form.

$$AX = B$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

Find the inverse of A.

Form the matrix $[A \mid I_3]$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$-R_2 \to R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$R_3 + 2R_2 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array}\right]$$

$$-\tfrac{1}{2}R_3 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right]$$

$$R_1 - R_3 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right]$$

$$R_1 - R_2 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right]$$

Identify the inverse.

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}$$

The solution to the system is $X = A^{-1}B$.

$$X = A^{-1}B = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & -1 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}$$

Technology Tip

Enter the matrix A.

```
[A]
      [[1  1   1 ]
       [1  0   1 ]
       [1  -1  -1]]
```

Now use the graphing calculator to find the inverse of A, A^{-1}.
[2nd] [*MATRIX*] [1:[*A*]] [ENTER] [x^{-1}] [ENTER].

```
      [[1  1   1 ]
       [1  0   1 ]
       [1  -1  -1]]
[A]⁻¹
  [[.5   0   .5 ]
   [1    -1  0  ]
   [-.5  1   -.5]]
```

To show elements using fractions, press [2nd] [*MATRIX*] [1:[*A*]] [ENTER] [x^{-1}] [MATH] [1: *Frac*] [ENTER] [ENTER].

```
      [[1  1   1 ]
       [1  0   1 ]
       [1  -1  -1]]
[A]⁻¹▸Frac
[[1/2   0   1/2 ]
 [1     -1  0   ]
 [-1/2  1   -1/2]]
```

Now enter the matrix B:

```
[B]
            [[2 ]
             [1 ]
             [-4]]
```

To find X enter $A^{-1}B$, press [2nd] [*ANS*] [2nd] [*MATRIX*] [2:*B*] [ENTER] [ENTER].

```
Ans[B]
            [[-1]
             [1 ]
             [2 ]]
```

The solution to the system is $x = -1$, $y = 1$, and $z = 2$.

Simplify.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\boxed{x = -1,\ y = 1,\ z = 2}$$

■ **YOUR TURN** Solve the system of equations using matrix algebra.

$$\begin{aligned} x + \ \ y - z &= 3 \\ y + z &= 1 \\ 2x + 3y + z &= 5 \end{aligned}$$

■ **Answer:** $x = 0,\ y = 2,\ z = -1$

Cryptography Applications

Cryptography is the practice of hiding information, or secret communication. Let's assume you want to send your ATM PIN code over the Internet, but you don't want hackers to be able to retrieve it. You can represent the PIN code in a matrix and then multiply that PIN matrix by a "key" matrix so that it is encrypted. If the person you send it to has the "inverse key" matrix, he can multiply the encrypted matrix he receives by the inverse key matrix and the result will be the original PIN matrix. Although PIN numbers are typically four digits, we will assume two digits to illustrate the process.

Words	**Math**
Suppose the two-digit ATM PIN is 13.	$P = [1 \quad 3]$
Apply any 2×2 nonsingular matrix as the "key" (encryption) matrix.	$K = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$
Multiply the PIN and encryption matrices.	$PK = [1 \quad 3]\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ $= [1(2) + 3(5) \quad 1(3) + 3(8)]$ $= [17 \quad 27]$
The receiver of the encrypted matrix sees only $[17 \quad 27]$.	
The decoding "key" is the inverse matrix K^{-1}.	$K^{-1} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$

Study Tip

$$K = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

$$K^{-1} = \frac{1}{(2)(8) - (3)(5)}\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$$

Any receiver who has the decoding key can multiply the received encrypted matrix by the decoding "key" matrix. The result is the original transmitted PIN number.

$$[17 \quad 27]\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = [17(8) + 27(-5) \qquad 17(-3) + 27(2)] = [1 \quad 3]$$

SECTION 8.4 SUMMARY

Matrices can be used to represent data. Operations such as equality, addition, subtraction, and scalar multiplication are performed entry by entry. Two matrices can be added or subtracted only if they have the same order. Matrix multiplication, however, requires that the number of columns in the first matrix is equal to the number of rows in the second matrix and is performed using a row-by-column procedure.

Matrix Multiplication Is Not Commutative: $AB \neq BA$

Operation	Order Requirement
Equality	Same: $A_{m\times n} = B_{m\times n}$
Addition	Same: $A_{m\times n} + B_{m\times n}$
Subtraction	Same: $A_{m\times n} - B_{m\times n}$
Scalar multiplication	None: $kA_{m\times n}$
Matrix multiplication	$A_{m\times n}\, B_{n\times p} = (AB)_{m\times p}$

Systems of linear equations can be solved using matrix equations.

System of Linear Equations	A	X	B	Matrix Equation: $AX = B$
$\begin{aligned} x - y + z &= 2 \\ 2x + 2y - 3z &= -3 \\ x + y + z &= 6 \end{aligned}$	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$

If this system of linear equations has a unique solution, then the solution is represented by

$$X = A^{-1}B$$

A^{-1} is the inverse of A, that is, $AA^{-1} = A^{-1}A = I$, and is found by

$$[A_{n\times n} | I_n] \rightarrow [I_n | A^{-1}_{n\times n}]$$

SECTION 8.4 EXERCISES

SKILLS

In Exercises 1–8, state the order of each matrix.

1. $\begin{bmatrix} -1 & 2 & 4 \\ 7 & -3 & 9 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 5 \\ 2 & 6 \\ -1 & -4 \end{bmatrix}$

3. $\begin{bmatrix} -4 & 5 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -4 & 5 & 3 & 7 \end{bmatrix}$

5. $\begin{bmatrix} -3 & 4 & 1 \\ 10 & 8 & 0 \\ -2 & 5 & 7 \end{bmatrix}$

6. $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

7. $\begin{bmatrix} -3 & 6 & 0 & 5 \\ 4 & -9 & 2 & 7 \\ 1 & 8 & 3 & 6 \\ 5 & 0 & -4 & 11 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 3 & 6 & 9 \\ 2 & 5 & -7 & 8 \end{bmatrix}$

In Exercises 9–14, solve for the indicated variables.

9. $\begin{bmatrix} 2 & x \\ y & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 1 & 3 \end{bmatrix}$

10. $\begin{bmatrix} -3 & 17 \\ x & y \end{bmatrix} = \begin{bmatrix} -3 & 17 \\ 10 & 12 \end{bmatrix}$

11. $\begin{bmatrix} x+y & 3 \\ x-y & 9 \end{bmatrix} = \begin{bmatrix} -5 & z \\ -1 & 9 \end{bmatrix}$

12. $\begin{bmatrix} x & -4 \\ y & 7 \end{bmatrix} = \begin{bmatrix} 2+y & -4 \\ 5 & 7 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 4 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} x-y & 4 \\ 0 & 2y+x \end{bmatrix}$

14. $\begin{bmatrix} 9 & 2b+1 \\ -5 & 16 \end{bmatrix} = \begin{bmatrix} a^2 & 9 \\ 2a+1 & b^2 \end{bmatrix}$

In Exercises 15–24, perform the indicated operations for each expression, if possible.

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -2 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -2 \end{bmatrix}$$

15. $A + B$ **16.** $C + D$ **17.** $C - D$ **18.** $A - B$ **19.** $B + C$

20. $A + D$ **21.** $D - B$ **22.** $C - A$ **23.** $2A + 3B$ **24.** $2B - 3A$

In Exercises 25–44, perform the indicated operations for each expression, if possible.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 5 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & -3 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 7 & 2 \\ 3 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 1 & -1 \\ 2 & 5 \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 4 \\ -3 & 1 & 5 \end{bmatrix} \quad F = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

25. CD **26.** BF **27.** DC **28.** $(A + E)D$

29. DG **30.** $2A + 3E$ **31.** GD **32.** $ED + C$

33. $-4BD$ **34.** $-3ED$ **35.** $B(A + E)$ **36.** $GC + 5C$

37. $FB + 5A$ **38.** A^2 **39.** $G^2 + 5G$ **40.** $C \cdot (2E)$

41. $(2E) \cdot F$ **42.** $CA + 5C$ **43.** DF **44.** AE

In Exercises 45–50, determine whether B is the multiplicative inverse of A using $AA^{-1} = I$.

45. $A = \begin{bmatrix} 8 & -11 \\ -5 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 11 \\ 5 & 8 \end{bmatrix}$

46. $A = \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix}$

47. $A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{3}{7} \end{bmatrix}$

48. $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$

49. $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

50. $A = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & -2 \\ -1 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

In Exercises 51–62, find A^{-1}, if possible.

51. $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

52. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

53. $A = \begin{bmatrix} \frac{1}{3} & 2 \\ 5 & \frac{3}{4} \end{bmatrix}$

54. $A = \begin{bmatrix} \frac{1}{4} & 2 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$

55. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$

56. $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & -3 \end{bmatrix}$

57. $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

58. $A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & -1 \\ 1 & 0 & -4 \end{bmatrix}$

59. $A = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

60. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix}$

61. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & -1 & -1 \end{bmatrix}$

62. $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -3 \\ 3 & -5 & 1 \end{bmatrix}$

In Exercises 63–74, apply matrix algebra to solve the system of linear equations.

63. $\begin{aligned} 2x - y &= 5 \\ x + y &= 1 \end{aligned}$

64. $\begin{aligned} 2x - 3y &= 12 \\ x + y &= 1 \end{aligned}$

65. $\begin{aligned} 4x - 9y &= -1 \\ 7x - 3y &= \tfrac{5}{2} \end{aligned}$

66. $\begin{aligned} 7x - 3y &= 1 \\ 4x - 5y &= -\tfrac{7}{5} \end{aligned}$

67. $\begin{aligned} x + y + z &= 1 \\ x - y - z &= -1 \\ -x + y - z &= -1 \end{aligned}$

68. $\begin{aligned} x - y + z &= 0 \\ x + y + z &= 2 \\ -x + 2y - 3z &= 1 \end{aligned}$

69. $\begin{aligned} x \quad + z &= 3 \\ y + z &= 1 \\ x - y \quad &= 2 \end{aligned}$

70. $\begin{aligned} x + 2y - 3z &= 1 \\ x - y - z &= 3 \\ x \quad - 4z &= 0 \end{aligned}$

71. $\begin{aligned} 2x + 4y + z &= -5 \\ x + y - z &= 7 \\ x + y \quad &= 0 \end{aligned}$

72. $\begin{aligned} x \quad + z &= 3 \\ x + y - z &= -3 \\ 2x + y - z &= -5 \end{aligned}$

73. $\begin{aligned} x + y - z &= 4 \\ x - y + z &= 2 \\ 2x - y - z &= -3 \end{aligned}$

74. $\begin{aligned} x - y - z &= 0 \\ x + y - 3z &= 2 \\ 3x - 5y + z &= 4 \end{aligned}$

APPLICATIONS

75. Smoking. On January 6 and 10, 2000, the Harris Poll conducted a survey of adult smokers in the United States. When asked, "Have you ever tried to quit smoking?", 70% said yes and 30% said no. Write a 2×1 matrix—call it A—that represents those smokers. When asked what consequences smoking would have on their lives, 89% believed it would increase their chance of getting lung cancer and 84% believed smoking would shorten their lives. Write a 2×1 matrix—call it B—that represents those smokers. If there are 46 million adult smokers in the United States

a. What does $46A$ tell us?
b. What does $46B$ tell us?

76. Women in Science. According to the study of science and engineering indicators by the National Science Foundation (www.nsf.gov), the number of female graduate students in science and engineering disciplines has increased over the last 30 years. In 1981, 24% of mathematics graduate students were female and 23% of graduate students in computer science were female. In 1991, 32% of mathematics graduate students and 21% of computer science graduate students were female. In 2001, 38% of mathematics graduate students and 30% of computer science graduate students were female. Write three 2×1 matrices representing the percentage of female graduate students.

$$A = \begin{bmatrix} \text{\% female–math–1981} \\ \text{\% female–C.S.–1981} \end{bmatrix}$$

$$B = \begin{bmatrix} \text{\% female–math–1991} \\ \text{\% female–C.S.–1991} \end{bmatrix}$$

$$C = \begin{bmatrix} \text{\% female–math–2001} \\ \text{\% female–C.S.–2001} \end{bmatrix}$$

What does $C - B$ tell us? What does $B - A$ tell us? What can you conclude about the number of women pursuing mathematics and computer science graduate degrees?

Note: C.S. = computer science.

77. Registered Voters. According to the U.S. Census Bureau (www.census.gov), in the 2000 national election, 58.9% of men over the age of 18 were registered voters, but only 41.4% voted; and 62.8% of women over 18 were registered voters, but only 43% actually voted. Write a 2×2 matrix with the following data:

$$A = \begin{bmatrix} \text{Percentage of registered male voters} & \text{Percentage of registered female voters} \\ \text{Percent of males who voted} & \text{Percent of females who voted} \end{bmatrix}$$

If we let B be a 2×1 matrix representing the total population of males and females over the age of 18 in the United States, or $B = \begin{bmatrix} 100\text{ M} \\ 110\text{ M} \end{bmatrix}$, what does AB tell us?

78. Job Application. A company has two rubrics for scoring job applicants based on weighting education, experience, and the interview differently.

Matrix *A*

$$\begin{array}{l} \\ \text{Education} \\ \text{Experience} \\ \text{Interview} \end{array} \begin{array}{c} \begin{array}{cc} \text{Rubric 1} & \text{Rubric 2} \end{array} \\ \begin{bmatrix} 0.5 & 0.6 \\ 0.3 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \end{array}$$

Applicants receive a score from 1 to 10 in each category (education, experience, and interview). Two applicants are shown in matrix B.

Matrix *B*

$$\begin{array}{l} \\ \text{Applicant 1} \\ \text{Applicant 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Education} & \text{Experience} & \text{Interview} \end{array} \\ \begin{bmatrix} 8 & 7 & 5 \\ 6 & 8 & 8 \end{bmatrix} \end{array}$$

What is the order of BA? What does each entry in BA tell us?

79. Taxes. The IRS allows an individual to deduct business expenses in the following way: \$0.45 per mile driven, 50% of entertainment costs, and 100% of actual expenses. Represent these deductions in the given order as a row matrix A. In 2006, Jamie had the following business expenses: \$2700 in entertainment, \$15,200 actual expenses, and he drove 7523 miles. Represent Jamie's expenses in the given order as a column matrix B. Multiply these two matrices to find the total amount of business expenses Jamie can claim on his 2006 federal tax form: AB.

80. Tips on Service. Marilyn decides to go to the Safety Harbor Spa for a day of pampering. She is treated to a hot stone massage (\$85), a manicure and pedicure (\$75), and a haircut (\$100). Represent the costs of the individual services as a row matrix A (in the given order). She decides to tip her masseur 25%, her nail tech 20%, and her hair stylist 15%. Represent the tipping percentages as a column matrix B (in the given order). Multiply these matrices to find the total amount in tips AB she needs to add to her final bill.

Use the following tables for Exercises 81 and 82:

The following table gives fuel and electric requirements per mile associated with gasoline and electric automobiles:

	Number of Gallons/Mile	Number of kW-hr/Mile
SUV full size	0.06	0
Hybrid car	0.02	0.1
Electric car	0	0.3

The following table gives an average cost for gasoline and electricity:

Cost per gallon of gasoline	\$3.80
Cost per kW-hr of electricity	\$0.05

81. Environment. Let matrix A represent the gasoline and electricity consumption and matrix B represent the costs of gasoline and electricity. Find AB and describe what the elements of the product matrix represent. *Hint:* A has order 3×2 and B has order 2×1.

82. Environment. Assume you drive 12,000 miles per year. What are the yearly costs associated with driving the three types of cars in Exercise 81?

For Exercises 83 and 84, refer to the following:

The results of a nutritional analysis of one serving of three foods A, B, and C were

$$X = \begin{array}{c} \begin{array}{ccc} \text{Carbohydrates (g)} & \text{Protein (g)} & \text{Fat (g)} \end{array} \\ \begin{bmatrix} 5 & 0 & 2 \\ 5 & 6 & 5 \\ 8 & 4 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ A \\ B \\ C \end{array}$$

It is possible to find the nutritional content of a meal consisting of a combination of the foods A, B, and C by multiplying the matrix X by a second matrix $N = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$, that is, XN, where r is the number of servings of food A, s is the number of servings of food B, and t is the number of servings of food C.

83. Health/Nutrition. Find the matrix N that represents a meal consisting of two servings of food A and one serving of food B. Find the nutritional content of that meal.

84. Health/Nutrition. Find the matrix N that represents a meal consisting of one serving of food A and two servings of food C. Find the nutritional content of that meal.

For Exercises 85 and 86, refer to the following:

Cell phone companies charge users based on the number of minutes talked, the number of text messages sent, and the number of megabytes of data used. The costs for three cell phone providers are given in the following table:

	Minutes	Text Messages	Megabyte of Data
C_1	\$0.04	\$0.05	\$0.15
C_2	\$0.06	\$0.05	\$0.18
C_3	\$0.07	\$0.07	\$0.13

It is possible to find the cost to a cell phone user for each of the three providers by creating a matrix X whose rows are the rows of data in the table and multiplying the matrix X by a second matrix $N = \begin{bmatrix} m \\ t \\ d \end{bmatrix}$, that is, XN, where m is the number of minutes talked, t is the number of text messages sent, and d is the megabytes of data used.

85. Telecommunications/Business. A local business is looking at providing an employee a cell phone for business use. Find the matrix N that represents expected normal cell phone usage of 200 minutes, 25 text messages, and no data usage. Find and interpret XN. Which is the better cell phone provider for this employee?

86. Telecommunications/Business. A local business is looking at providing an employee a cell phone for business use. Find the matrix N that represents expected normal cell phone usage of 125 minutes, 125 text messages, and 320 megabytes of data usage. Find and interpret XN. Which is the better cell phone provider for this employee?

For Exercises 87–92, apply the following decoding scheme:

1	A	10	J	19	S
2	B	11	K	20	T
3	C	12	L	21	U
4	D	13	M	22	V
5	E	14	N	23	W
6	F	15	O	24	X
7	G	16	P	25	Y
8	H	17	Q	26	Z
9	I	18	R		

The encoding matrix is $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$. The encrypted matrices are given below. For each of the following, determine the 3-letter word that is originally transmitted. *Hint:* All six words are parts of the body.

87. Cryptography. $[55 \quad 10 \quad -22]$

88. Cryptography. $[31 \quad 8 \quad -7]$

89. Cryptography. $[21 \quad 12 \quad -2]$

90. Cryptography. $[9 \quad 1 \quad 5]$

91. Cryptography. $[-10 \quad 5 \quad 20]$

92. Cryptography. $[40 \quad 5 \quad -17]$

For Exercises 93 and 94, refer to the following:

The results of a nutritional analysis of one serving of three foods A, B, and C were:

$$\begin{array}{cc} & \begin{array}{ccc} \text{Carbohydrates (g)} & \text{Protein (g)} & \text{Fat (g)} \end{array} \\ Y = & \begin{bmatrix} 8 & 4 & 6 \\ 6 & 10 & 5 \\ 10 & 4 & 8 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

The nutritional content of a meal consisting of a combination of the foods A, B, and C is the product of the matrix Y and a second matrix $N = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$, that is, YN, where r is the number of servings of food A, s is the number of servings of food B, and t is the number of servings of food C.

93. Health/Nutrition. Use the inverse matrix technique to find the number of servings of foods A, B, and C necessary to create a meal of 18 grams of carbohydrates, 21 grams of protein, and 22 grams of fat.

94. Health/Nutrition. Use the inverse matrix technique to find the number of servings of foods A, B, and C necessary to create a meal of 14 grams of carbohydrates, 25 grams of protein, and 16 grams of fat.

For Exercises 95 and 96, refer to the following:

Cell phone companies charge users based on the number of minutes talked, the number of text messages sent, and the number of megabytes of data used. The costs for three cell phone providers are given in the table:

	Minutes	Text Messages	Megabytes of Data
C_1	\$0.03	\$0.06	\$0.15
C_2	\$0.04	\$0.05	\$0.18
C_3	\$0.05	\$0.07	\$0.13

The cost to a cell phone user for each of the three providers is the product of the matrix X whose rows are the rows of data in the table and the matrix $N = \begin{bmatrix} m \\ t \\ d \end{bmatrix}$ where m is the number of minutes talked, t is the number of text messages sent, and d is the megabytes of data used.

95. Telecommunications/Business. A local business is looking at providing an employee a cell phone for business use. The business solicits estimates for their normal monthly usage from three cell phone providers. Company 1 estimates the cost to be \$49.50, Company 2 estimates the cost to be \$52.00, and Company 3 estimates the cost to be \$58.50. Use the inverse matrix technique to find the normal monthly usage for the employee.

96. Telecommunications/Business. A local business is looking at providing an employee a cell phone for business use. The business solicits estimates for their normal monthly usage from three cell phone providers. Company 1 estimates the cost to be \$82.50, Company 2 estimates the cost to be \$85.00, and Company 3 estimates the cost to be \$92.50. Use the inverse matrix technique to find the normal monthly usage for the employee.

CATCH THE MISTAKE

In Exercises 97–100, explain the mistake that is made.

97. Multiply $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}$.

Solution:

Multiply corresponding elements.

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} (3)(-1) & (2)(3) \\ (1)(-2) & (4)(5) \end{bmatrix}$$

Simplify. $$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ -2 & 20 \end{bmatrix}$$

This is incorrect. What mistake was made?

98. Multiply $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix}$.

Solution:

Multiply using column-by-row method.

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} (3)(-1) + (1)(3) & (2)(-1) + (4)(3) \\ (3)(-2) + (1)(5) & (2)(-2) + (4)(5) \end{bmatrix}$$

Simplify. $$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ -1 & 16 \end{bmatrix}$$

This is incorrect. What mistake was made?

99. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 0 \end{bmatrix}$.

Solution:

Write the matrix $[A \mid I_3]$. $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

Use Gaussian elimination to reduce A.

$$\begin{array}{r} R_2 + R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\tfrac{1}{2}R_2 \to R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}$ is incorrect because $AA^{-1} \neq I_3$.

What mistake was made?

100. Find the inverse of A given that $A = \begin{bmatrix} 2 & 5 \\ 3 & 10 \end{bmatrix}$.

Solution: $$A^{-1} = \frac{1}{A} \qquad A^{-1} = \frac{1}{\begin{bmatrix} 2 & 5 \\ 3 & 10 \end{bmatrix}}$$

Simplify. $$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{10} \end{bmatrix}$$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 101–106, determine whether the statements are true or false.

101. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then $AB = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{22}b_{22} \end{bmatrix}$.

102. If AB is defined, then $AB = BA$.

103. AB is defined only if the number of columns in A equals the number of rows in B.

104. $A + B$ is defined only if A and B have the same order.

105. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & \frac{1}{a_{12}} \\ \frac{1}{a_{21}} & \frac{1}{a_{22}} \end{bmatrix}$.

106. All square matrices have inverses.

107. For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, find A^2.

108. In order for $A^2_{m\times n}$ to be defined, what condition (with respect to m and n) must be met?

109. For what values of x does the inverse of A not exist, given $A = \begin{bmatrix} x & 6 \\ 3 & 2 \end{bmatrix}$?

110. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Find A^{-1}. Assume $abc \neq 0$.

CHALLENGE

111. For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ find $A, A^2, A^3, \ldots$. What is A^n?

112. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ find $A, A^2, A^3, \ldots$. What is A^n?

113. If $A_{m\times n}B_{n\times p}$ is defined, explain why $(A_{m\times n}\,B_{n\times p})^2$ is not defined for $m \neq p$.

114. Given $C_{n\times m}$ and $A_{m\times n} = B_{m\times n}$, explain why $AC \neq CB$, if $m \neq n$.

115. Verify that $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, provided $ad - bc \neq 0$.

116. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and form the matrix $[A \mid I_2]$. Apply row operations to transform into $[I_2 \mid A^{-1}]$. Show $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ such that $ad - bc \neq 0$.

117. Why does the square matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ not have an inverse?

118. Why does the square matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ not have an inverse?

TECHNOLOGY

In Exercises 119–124, apply a graphing utility to perform the indicated matrix operations, if possible.

$$A = \begin{bmatrix} 1 & 7 & 9 & 2 \\ -3 & -6 & 15 & 11 \\ 0 & 3 & 2 & 5 \\ 9 & 8 & -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 9 \\ 8 & 6 \\ -4 & -2 \\ 3 & 1 \end{bmatrix}$$

119. AB **120.** BA **121.** BB **122.** AA

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -3 & 0 & 2 \\ 4 & -6 & 0 \end{bmatrix}$$

123. A^2 **124.** A^5

In Exercises 125 and 126, apply a graphing utility to perform the indicated matrix operations.

$$A = \begin{bmatrix} 1 & 7 & 9 & 2 \\ -3 & -6 & 15 & 11 \\ 0 & 3 & 2 & 5 \\ 9 & 8 & -4 & 1 \end{bmatrix}$$

125. Find A^{-1}. **126.** Find AA^{-1}.

PREVIEW TO CALCULUS

In calculus, when finding the inverse of a vector function, it is fundamental that the matrix of partial derivatives is not singular. In Exercises 127–130, find the inverse of each matrix.

127. $\begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$ **128.** $\begin{bmatrix} 1 & 1 \\ uy & ux \end{bmatrix}$ **129.** $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **130.** $\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

SECTION 8.5 THE DETERMINANT OF A SQUARE MATRIX AND CRAMER'S RULE

SKILLS OBJECTIVES

- Find the determinant of a 2×2 matrix.
- Find the determinant of an $n \times n$ matrix.
- Use Cramer's rule to solve a square system of linear equations.

CONCEPTUAL OBJECTIVES

- Derive Cramer's rule.
- Understand that if a determinant of a matrix is equal to zero, then that matrix does not have an inverse.
- Understand that Cramer's rule can be used to find only a unique solution.

In Section 8.3, we discussed Gauss–Jordan elimination as a way to solve systems of linear equations using augmented matrices. Then in Section 8.4, we employed matrix algebra and inverses to solve systems of linear equations that are square (same number of equations as variables). In this section, we will describe another method, called Cramer's rule, for solving systems of linear equations. Cramer's rule is applicable only to square systems. *Determinants* of square matrices play a vital role in Cramer's rule and indicate whether a matrix has an inverse.

Determinant of a 2 × 2 Matrix

Every square matrix A has a number associated with it called its *determinant*, denoted $\det(A)$ or $|A|$.

DEFINITION **Determinant of a 2 × 2 Matrix**

The **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Although the symbol for determinant, $|\,|$, looks like absolute value bars, the determinant can be any real number (positive, negative, or zero). The determinant of a 2×2 matrix is found by finding the product of the main diagonal entries (top left to bottom right) and subtracting the product of the entries along the other diagonal (bottom left to top right).

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Study Tip

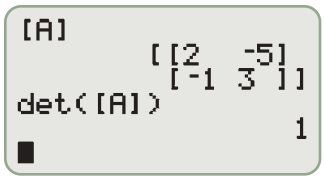
The determinant of a 2×2 matrix is found by finding the product of the main diagonal entries and subtracting the product of the other diagonal entries.

Technology Tip

A graphing calculator can be used to find the determinant of each matrix.

Enter the matrix $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ as A.

Press [2nd] [*MATRIX*]. Use [▶] to highlight [*MATH*] [1:*det*(] [ENTER] [2nd] [*MATRIX*] [ENTER] [)] [ENTER].



■ **Answer:** -1

EXAMPLE 1 Finding the Determinant of a 2 × 2 Matrix

Find the determinant of each matrix.

a. $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 0.5 & 0.2 \\ -3.0 & -4.2 \end{bmatrix}$ **c.** $\begin{bmatrix} \frac{2}{3} & 1 \\ 2 & 3 \end{bmatrix}$

Solution:

a. $\begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix} = (2)(3) - (-1)(-5) = 6 - 5 = \boxed{1}$

b. $\begin{vmatrix} 0.5 & 0.2 \\ -3 & -4.2 \end{vmatrix} = (0.5)(-4.2) - (-3)(0.2) = -2.1 + 0.6 = \boxed{-1.5}$

c. $\begin{vmatrix} \frac{2}{3} & 1 \\ 2 & 3 \end{vmatrix} = \left(\frac{2}{3}\right)(3) - (2)(1) = 2 - 2 = \boxed{0}$

In Example 1, we see that determinants are real numbers that can be positive, negative, or zero. Although evaluating determinants of 2×2 matrices is a simple process, one **common mistake** is reversing the difference: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq bc - ad$.

■ **YOUR TURN** Evaluate the determinant $\begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix}$.

Determinant of an $n \times n$ Matrix

In order to define the *determinant* of a 3×3 or a general $n \times n$ (where $n \geq 3$) matrix, we first define *minors* and *cofactors* of a square matrix.

DEFINITION **Minor and Cofactor**

Let A be a square matrix of order $n \times n$. Then:

- The **minor** M_{ij} of the entry a_{ij} is the determinant of the $(n - 1) \times (n - 1)$ matrix obtained when the ith row and jth column of A are deleted.
- The **cofactor** C_{ij} of the entry a_{ij} is given by $C_{ij} = (-1)^{i+j}M_{ij}$.

The following table illustrates entries, minors, and cofactors of the matrix:

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 0 \\ 5 & -2 & 3 \end{bmatrix}$$

ENTRY a_{ij}	MINOR M_{ij}	COFACTOR C_{ij}
$a_{11} = 1$	For M_{11}, delete the first row and first column: $\begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 0 \\ 5 & -2 & 3 \end{bmatrix}$ $M_{11} = \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} = -3 - 0 = -3$	$C_{11} = (-1)^{1+1}M_{11} = (1)(-3) = -3$
$a_{32} = -2$	For M_{32}, delete the third row and second column: $\begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 0 \\ 5 & -2 & 3 \end{bmatrix}$ $M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} = 0 - 8 = -8$	$C_{32} = (-1)^{3+2}M_{32} = (-1)(-8) = 8$

Notice that the cofactor is simply the minor multiplied by either 1 or -1, depending on whether $i + j$ is even or odd. Therefore, we can make the following sign pattern for 3×3 and 4×4 matrices and obtain the cofactor by multiplying the minor with the appropriate sign ($+1$ or -1):

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

DEFINITION **Determinant of an $n \times n$ Matrix**

Let A be an $n \times n$ matrix. Then the **determinant** of A is found by summing the entries in *any* row of A (or column of A) multiplied by each entries' respective cofactor.

If A is a 3×3 matrix, the determinant can be given by

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

this is called **expanding the determinant by the first row**. It is important to note that any row or column can be used. Typically, the row or column with the most zeros is selected because it makes the arithmetic simpler.

Combining the definitions of minors, cofactors, and determinants, we now give a general definition for the determinant of a 3×3 matrix.

Row 1 expansion: $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Column 1 expansion: $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Whichever row or column is expanded, an alternating sign scheme is used (see sign arrays above). Notice that in either of the expansions above, each 2×2 determinant obtained is found by crossing out the row and column containing the entry that is multiplying the determinant.

EXAMPLE 2 Finding the Determinant of a 3 × 3 Matrix

For the given matrix, expand the determinant by the *first row*.

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 5 & -2 \\ -3 & 7 & 4 \end{bmatrix}$$

Solution:

Expand the determinant by the **first** row. Remember the alternating **sign**.

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 5 & -2 \\ -3 & 7 & 4 \end{vmatrix} = +2\begin{vmatrix} 5 & -2 \\ 7 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & -2 \\ -3 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 5 \\ -3 & 7 \end{vmatrix}$$

Evaluate the resulting 2 × 2 determinants.

$$= 2[(5)(4) - (7)(-2)] - 1[(-1)(4) - (-3)(-2)] + 3[(-1)(7) - (-3)(5)]$$

$$= 2[20 + 14] - [-4 - 6] + 3[-7 + 15]$$

Simplify. $= 2(34) - (-10) + 3(8)$

$$= 68 + 10 + 24$$

$$= \boxed{102}$$

YOUR TURN For the given matrix, expand the determinant by the first row.

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & 4 \\ 7 & -1 & 6 \end{bmatrix}$$

Study Tip

The determinant by the third column is also 102. It does not matter on which row or column the expansion occurs.

■ **Answer:** 156

Determinants can be expanded by any row *or* column. Typically, the row or column with the most zeros is selected to simplify the arithmetic.

EXAMPLE 3 Finding the Determinant of a 3 × 3 Matrix

Find the determinant of the matrix $\begin{vmatrix} -1 & 2 & 0 \\ 4 & 7 & 1 \\ 5 & 3 & 0 \end{vmatrix}$.

Solution:

Since there are two 0s in the third column, expand the determinant by the third column. Recall the sign array.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{vmatrix} -1 & 2 & 0 \\ 4 & 7 & 1 \\ 5 & 3 & 0 \end{vmatrix} = +0\begin{vmatrix} 4 & 7 \\ 5 & 3 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 5 & 3 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 \\ 4 & 7 \end{vmatrix}$$

There is no need to calculate the two determinants that are multiplied by 0s, since 0 times any real number is zero.

$$\begin{vmatrix} -1 & 2 & 0 \\ 4 & 7 & 1 \\ 5 & 3 & 0 \end{vmatrix} = 0 - 1\underbrace{\begin{vmatrix} -1 & 2 \\ 5 & 3 \end{vmatrix}}_{-3-10} + 0$$

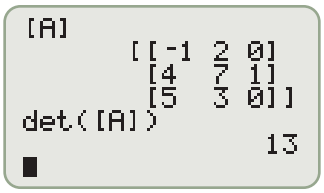
Simplify. $= -1(-13) = \boxed{13}$

YOUR TURN Evaluate the determinant $\begin{vmatrix} 1 & -2 & 1 \\ -1 & 0 & 3 \\ -4 & 0 & 2 \end{vmatrix}$.

Technology Tip

Enter the matrix as *A* and find the determinant.

Press [2nd] [*MATRIX*]. Use [▶] to highlight [*MATH*] [1:*det*(] [2nd] [*MATRIX*] [ENTER] [)] [ENTER].



■ **Answer:** 20

EXAMPLE 4 Finding the Determinant of a 4×4 Matrix

Find the determinant of the matrix $\begin{vmatrix} 1 & -2 & 3 & 4 \\ -4 & 0 & -1 & 0 \\ -3 & 9 & 6 & 5 \\ -5 & 7 & 2 & 1 \end{vmatrix}$.

Solution:

Since there are two 0s in the second row, expand the determinant by the second row. Recall the sign array for a 4×4 matrix.

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ \mathbf{-4} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ -3 & 9 & 6 & 5 \\ -5 & 7 & 2 & 1 \end{bmatrix} = \mathbf{-(-4)}\begin{vmatrix} -2 & 3 & 4 \\ 9 & 6 & 5 \\ 7 & 2 & 1 \end{vmatrix} \mathbf{+\ 0\ -(-1)}\begin{vmatrix} 1 & -2 & 4 \\ -3 & 9 & 5 \\ -5 & 7 & 1 \end{vmatrix} \mathbf{+\ 0}$$

Evaluate the two 3×3 determinants.

$$\begin{aligned} \begin{vmatrix} -2 & 3 & 4 \\ 9 & 6 & 5 \\ 7 & 2 & 1 \end{vmatrix} &= -2\begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 9 & 5 \\ 7 & 1 \end{vmatrix} + 4\begin{vmatrix} 9 & 6 \\ 7 & 2 \end{vmatrix} \\ &= -2(6 - 10) - 3(9 - 35) + 4(18 - 42) \\ &= -2(-4) - 3(-26) + 4(-24) \\ &= 8 + 78 - 96 \\ &= -10 \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} 1 & -2 & 4 \\ -3 & 9 & 5 \\ -5 & 7 & 1 \end{vmatrix} &= 1\begin{vmatrix} 9 & 5 \\ 7 & 1 \end{vmatrix} - (-2)\begin{vmatrix} -3 & 5 \\ -5 & 1 \end{vmatrix} + 4\begin{vmatrix} -3 & 9 \\ -5 & 7 \end{vmatrix} \\ &= 1(9 - 35) + 2(-3 + 25) + 4(-21 + 45) \\ &= -26 + 2(22) + 4(24) \\ &= -26 + 44 + 96 \\ &= 114 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ -4 & 0 & -1 & 0 \\ -3 & 9 & 6 & 5 \\ -5 & 7 & 2 & 1 \end{bmatrix} = 4\underbrace{\begin{vmatrix} -2 & 3 & 4 \\ 9 & 6 & 5 \\ 7 & 2 & 1 \end{vmatrix}}_{-10} + \underbrace{\begin{vmatrix} 1 & -2 & 4 \\ -3 & 9 & 5 \\ -5 & 7 & 1 \end{vmatrix}}_{114} = 4(-10) + 114 = \boxed{74}$$

Cramer's Rule: Systems of Linear Equations in Two Variables

Let's now apply determinants of 2×2 matrices to solve systems of linear equations in two variables. We begin by solving the general system of two linear equations in two variables:

$$\begin{aligned} &(1) \quad a_1x + b_1y = c_1 \\ &(2) \quad a_2x + b_2y = c_2 \end{aligned}$$

Solve for x using elimination (eliminate y).

Multiply (1) by b_2. $\quad b_2a_1x + b_2b_1y = b_2c_1$

Multiply (2) by $-b_1$. $\quad -b_1a_2x - b_1b_2y = -b_1c_2$

Add the two new equations to eliminate y. $\quad (a_1b_2 - a_2b_1)x = (b_2c_1 - b_1c_2)$

Divide both sides by $(a_1b_2 - a_2b_1)$.

$$x = \frac{(b_2c_1 - b_1c_2)}{(a_1b_2 - a_2b_1)}$$

Write both the numerator and the denominator as determinants.

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Solve for y using elimination (eliminate x).

Multiply (1) by $-a_2$. $\quad -a_2a_1x - a_2b_1y = -a_2c_1$

Multiply (2) by a_1. $\quad a_1a_2x + a_1b_2y = a_1c_2$

Add the two new equations to eliminate x. $\quad (a_1b_2 - a_2b_1)y = (a_1c_2 - a_2c_1)$

Divide both sides by $(a_1b_2 - a_2b_1)$.

$$y = \frac{(a_1c_2 - a_2c_1)}{(a_1b_2 - a_2b_1)}$$

Write both the numerator and the denominator as determinants.

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Notice that the solutions for x and y involve three determinants. If we let

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

$$\text{then} \quad x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D}.$$

Notice that the real number D is the determinant of the coefficient matrix of the system and cannot equal zero ($D \neq 0$) or there will be no unique solution. These formulas for solving a system of two linear equations in two variables are known as *Cramer's rule*.

Cramer's rule for solving systems of two linear equations in two variables

For the system of linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

let

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

If $D \neq 0$, then the solution to the system of linear equations is

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D}$$

If $D = 0$, then the system of linear equations has either no solution or infinitely many solutions.

Notice that the determinants D_x and D_y are similar to the determinant D. A three-step procedure is outlined for setting up the three determinants for a system of two linear equations in two variables:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Step 1: Set up D.

Apply the coefficients of x and y.

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Step 2: Set up D_x.

Start with D and replace the coefficients of x (column 1) with the constants on the right side of the equal sign.

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

Step 3: Set up D_y.

Start with D and replace the coefficients of y (column 2) with the constants on the right side of the equal sign.

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Study Tip

Cramer's rule is only applicable to square systems of linear equations.

EXAMPLE 5 Using Cramer's Rule to Solve a System of Two Linear Equations

Apply Cramer's rule to solve the system.

$$x + 3y = 1$$
$$2x + y = -3$$

Technology Tip

A graphing calculator can be used to solve the system using Cramer's rule. Enter the matrix A for the determinant D_x, B for D_y, C for D.

```
[A]
        [[1  3]
         [-3 1]]
[B]
        [[1  1 ]
         [2  -3]]
```

```
[C]
        [[1 3]
         [2 1]]
```

To solve for x and y, enter D_x/D as A/C for x and D_y/D as B/C for y.

```
det([A])/det([C]
)
                -2
det([B])/det([C]
)
                 1
```

■ **Answer:** $x = 5$, $y = -6$

Solution:

Set up the three determinants.

$$D = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$D_x = \begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix}$$

$$D_y = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$

Evaluate the determinants.

$$D = 1 - 6 = -5$$
$$D_x = 1 - (-9) = 10$$
$$D_y = -3 - 2 = -5$$

Solve for x and y.

$$x = \frac{D_x}{D} = \frac{10}{-5} = -2$$

$$y = \frac{D_y}{D} = \frac{-5}{-5} = 1$$

$$\boxed{x = -2,\ y = 1}$$

■ **YOUR TURN** Apply Cramer's rule to solve the system.

$$\begin{aligned} 5x + 4y &= 1 \\ -3x - 2y &= -3 \end{aligned}$$

Recall from Section 8.1 that systems of two linear equations in two variables led to one of three possible outcomes: a unique solution, no solution, and infinitely many solutions. When $D = 0$, Cramer's rule does not apply and the system is either inconsistent (has no solution) or contains dependent equations (has infinitely many solutions).

Cramer's Rule: Systems of Linear Equations in Three Variables

Cramer's rule can also be used to solve higher order systems of linear equations. The following box summarizes Cramer's rule for solving a system of three equations in three variables:

Cramer's rule: Solution for systems of three equations in three variables

The system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

has the solution

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D} \qquad D \neq 0$$

where the determinants are given as follows:

Display the coefficients of x, y, and z.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Replace the coefficients of x (column 1) in D with the constants on the right side of the equal sign.

$$D_x = \begin{vmatrix} \mathbf{d_1} & b_1 & c_1 \\ \mathbf{d_2} & b_2 & c_2 \\ \mathbf{d_3} & b_3 & c_3 \end{vmatrix}$$

Replace the coefficients of y (column 2) in D with the constants on the right side of the equal sign.

$$D_y = \begin{vmatrix} a_1 & \mathbf{d_1} & c_1 \\ a_2 & \mathbf{d_2} & c_2 \\ a_3 & \mathbf{d_3} & c_3 \end{vmatrix}$$

Replace the coefficients of z (column 3) in D with the constants on the right side of the equal sign.

$$D_z = \begin{vmatrix} a_1 & b_1 & \mathbf{d_1} \\ a_2 & b_2 & \mathbf{d_2} \\ a_3 & b_3 & \mathbf{d_3} \end{vmatrix}$$

EXAMPLE 6 Using Cramer's Rule to Solve a System of Three Linear Equations

Use Cramer's rule to solve the system.

$$\begin{aligned} 3x - 2y + 3z &= -3 \\ 5x + 3y + 8z &= -2 \\ x + y + 3z &= 1 \end{aligned}$$

Solution:

Set up the four determinants.

D contains the coefficients of x, y, and z.

$$D = \begin{vmatrix} 3 & -2 & 3 \\ 5 & 3 & 8 \\ 1 & 1 & 3 \end{vmatrix}$$

Replace a column with constants on the right side of the equation.

$$D_x = \begin{vmatrix} \mathbf{3} & -2 & 3 \\ \mathbf{-2} & 3 & 8 \\ \mathbf{1} & 1 & 3 \end{vmatrix} \qquad D_y = \begin{vmatrix} 3 & \mathbf{-3} & 3 \\ 5 & \mathbf{-2} & 8 \\ 1 & \mathbf{1} & 3 \end{vmatrix} \qquad D_z = \begin{vmatrix} 3 & -2 & \mathbf{-3} \\ 5 & 3 & \mathbf{-2} \\ 1 & 1 & \mathbf{1} \end{vmatrix}$$

Evaluate the determinants.

$$\begin{aligned} D &= 3(9 - 8) - (-2)(15 - 8) + 3(5 - 3) = 23 \\ D_x &= -3(9 - 8) - (-2)(-6 - 8) + 3(-2 - 3) = -46 \\ D_y &= 3(-6 - 8) - (-3)(15 - 8) + 3(5 + 2) = 0 \\ D_z &= 3(3 + 2) - (-2)(5 + 2) - 3(5 - 3) = 23 \end{aligned}$$

Solve for x, y, and z.

$$x = \frac{D_x}{D} = \frac{-46}{23} = -2 \qquad y = \frac{D_y}{D} = \frac{0}{23} = 0 \qquad z = \frac{D_z}{D} = \frac{23}{23} = 1$$

$$\boxed{x = -2,\ y = 0,\ z = 1}$$

■ **YOUR TURN** Use Cramer's rule to solve the system.

$$\begin{aligned} 2x + 3y + z &= -1 \\ x - y - z &= 0 \\ -3x - 2y + 3z &= 10 \end{aligned}$$

■ **Answer:** $x = 1$, $y = -2$, $z = 3$

As was the case in two equations, when $D = 0$, Cramer's rule does not apply and the system of three equations is either inconsistent (no solution) or contains dependent equations (infinitely many solutions).

SECTION 8.5 SUMMARY

In this section, **determinants** were discussed for square matrices.

ORDER	DETERMINANT	ARRAY
2×2	$\det(A) = \lvert A \rvert = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$	
3×3	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ Expansion by first row (any row or column can be used)	$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Cramer's rule was developed for 2×2 and 3×3 matrices, but it can be extended to general $n \times n$ matrices. When the coefficient determinant is equal to zero ($D = 0$), then the system is either inconsistent (and has no solution) or represents dependent equations (and has infinitely many solutions), and Cramer's rule does not apply.

SYSTEM	ORDER	SOLUTION	DETERMINANTS
$a_1x + b_1y = c_1$ $a_2x + b_2y = c_2$	2×2	$x = \dfrac{D_x}{D} \qquad y = \dfrac{D_y}{D}$	$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$ $D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ $D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$
$a_1x + b_1y + c_1z = d_1$ $a_2x + b_2y + c_2z = d_2$ $a_3x + b_3y + c_3z = d_3$	3×3	$x = \dfrac{D_x}{D} \qquad y = \dfrac{D_y}{D} \qquad z = \dfrac{D_z}{D}$	$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ $D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

SECTION 8.5 EXERCISES

SKILLS

In Exercises 1–10, evaluate each 2 × 2 determinant.

1. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ **2.** $\begin{vmatrix} 1 & -2 \\ -3 & -4 \end{vmatrix}$ **3.** $\begin{vmatrix} 7 & 9 \\ -5 & -2 \end{vmatrix}$ **4.** $\begin{vmatrix} -3 & -11 \\ 7 & 15 \end{vmatrix}$ **5.** $\begin{vmatrix} 0 & 7 \\ 4 & -1 \end{vmatrix}$

6. $\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$ **7.** $\begin{vmatrix} -1.2 & 2.4 \\ -0.5 & 1.5 \end{vmatrix}$ **8.** $\begin{vmatrix} -1.0 & 1.4 \\ 1.5 & -2.8 \end{vmatrix}$ **9.** $\begin{vmatrix} \frac{3}{4} & \frac{1}{3} \\ 2 & \frac{8}{9} \end{vmatrix}$ **10.** $\begin{vmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{2}{3} & -\frac{8}{9} \end{vmatrix}$

In Exercises 11–30, use Cramer's rule to solve each system of equations, if possible.

11. $\begin{aligned} x + y &= -1 \\ x - y &= 11 \end{aligned}$ **12.** $\begin{aligned} x + y &= -1 \\ x - y &= -9 \end{aligned}$ **13.** $\begin{aligned} 3x + 2y &= -4 \\ -2x + y &= 5 \end{aligned}$ **14.** $\begin{aligned} 5x + 3y &= 1 \\ 4x - 7y &= -18 \end{aligned}$

15. $\begin{aligned} 3x - 2y &= -1 \\ 5x + 4y &= -31 \end{aligned}$ **16.** $\begin{aligned} x - 4y &= -7 \\ 3x + 8y &= 19 \end{aligned}$ **17.** $\begin{aligned} 7x - 3y &= -29 \\ 5x + 2y &= 0 \end{aligned}$ **18.** $\begin{aligned} 6x - 2y &= 24 \\ 4x + 7y &= 41 \end{aligned}$

19. $\begin{aligned} 3x + 5y &= 16 \\ y - x &= 0 \end{aligned}$ **20.** $\begin{aligned} -2x - 3y &= 15 \\ 7y + 4x &= -33 \end{aligned}$ **21.** $\begin{aligned} 3x - 5y &= 7 \\ -6x + 10y &= -21 \end{aligned}$ **22.** $\begin{aligned} 3x - 5y &= 7 \\ 6x - 10y &= 14 \end{aligned}$

23. $\begin{aligned} 2x - 3y &= 4 \\ -10x + 15y &= -20 \end{aligned}$ **24.** $\begin{aligned} 2x - 3y &= 2 \\ 10x - 15y &= 20 \end{aligned}$ **25.** $\begin{aligned} 3x + \tfrac{1}{2}y &= 1 \\ 4x + \tfrac{1}{3}y &= \tfrac{5}{3} \end{aligned}$ **26.** $\begin{aligned} \tfrac{3}{2}x + \tfrac{9}{4}y &= \tfrac{9}{8} \\ \tfrac{1}{3}x + \tfrac{1}{4}y &= \tfrac{1}{12} \end{aligned}$

27. $\begin{aligned} 0.3x - 0.5y &= -0.6 \\ 0.2x + y &= 2.4 \end{aligned}$ **28.** $\begin{aligned} 0.5x - 0.4y &= -3.6 \\ 10x + 3.6y &= -14 \end{aligned}$ **29.** $\begin{aligned} y &= 17x + 7 \\ y &= -15x + 7 \end{aligned}$ **30.** $\begin{aligned} 9x &= -45 - 2y \\ 4x &= -3y - 20 \end{aligned}$

In Exercises 31–42, evaluate each 3 × 3 determinant.

31. $\begin{vmatrix} 3 & 1 & 0 \\ 2 & 0 & -1 \\ -4 & 1 & 0 \end{vmatrix}$ **32.** $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & -3 & 5 \end{vmatrix}$ **33.** $\begin{vmatrix} 2 & 1 & -5 \\ 3 & 0 & -1 \\ 4 & 0 & 7 \end{vmatrix}$ **34.** $\begin{vmatrix} 2 & 1 & -5 \\ 3 & -7 & 0 \\ 4 & -6 & 0 \end{vmatrix}$

35. $\begin{vmatrix} 1 & 1 & -5 \\ 3 & -7 & -4 \\ 4 & -6 & 9 \end{vmatrix}$ **36.** $\begin{vmatrix} -3 & 2 & -5 \\ 1 & 8 & 2 \\ 4 & -6 & 9 \end{vmatrix}$ **37.** $\begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & -2 & 1 \end{vmatrix}$ **38.** $\begin{vmatrix} -7 & 2 & 5 \\ \frac{7}{8} & 3 & 4 \\ -1 & 4 & 6 \end{vmatrix}$

39. $\begin{vmatrix} -3 & 1 & 5 \\ 2 & 0 & 6 \\ 4 & 7 & -9 \end{vmatrix}$ **40.** $\begin{vmatrix} 1 & -1 & 5 \\ 3 & -3 & 6 \\ 4 & 9 & 0 \end{vmatrix}$ **41.** $\begin{vmatrix} -2 & 1 & -7 \\ 4 & -2 & 14 \\ 0 & 1 & 8 \end{vmatrix}$ **42.** $\begin{vmatrix} 5 & -2 & -1 \\ 4 & -9 & -3 \\ 2 & 8 & -6 \end{vmatrix}$

In Exercises 43–58, apply Cramer's rule to solve each system of equations, if possible.

43. $\begin{aligned} x + y - z &= 0 \\ x - y + z &= 4 \\ x + y + z &= 10 \end{aligned}$ **44.** $\begin{aligned} -x + y + z &= -4 \\ x + y - z &= 0 \\ x + y + z &= 2 \end{aligned}$ **45.** $\begin{aligned} 3x + 8y + 2z &= 28 \\ -2x + 5y + 3z &= 34 \\ 4x + 9y + 2z &= 29 \end{aligned}$ **46.** $\begin{aligned} 7x + 2y - z &= -1 \\ 6x + 5y + z &= 16 \\ -5x - 4y + 3z &= -5 \end{aligned}$

47. $\begin{aligned} 3x \phantom{{}+4y} + 5z &= 11 \\ 4y + 3z &= -9 \\ 2x - y \phantom{{}+3z} &= 7 \end{aligned}$ **48.** $\begin{aligned} 3x \phantom{{}-2y} - 2z &= 7 \\ 4x \phantom{{}-2y} + z &= 24 \\ 6x - 2y \phantom{{}+z} &= 10 \end{aligned}$ **49.** $\begin{aligned} x + y - z &= 5 \\ x - y + z &= -1 \\ -2x - 2y + 2z &= -10 \end{aligned}$ **50.** $\begin{aligned} x + y - z &= 3 \\ x - y + z &= -2 \\ -2x - 2y + 2z &= -6 \end{aligned}$

51. $$\begin{aligned} x + y + z &= 9 \\ x - y + z &= 3 \\ -x + y - z &= 5 \end{aligned}$$

52. $$\begin{aligned} x + y + z &= 6 \\ x - y - z &= 0 \\ -x + y + z &= 7 \end{aligned}$$

53. $$\begin{aligned} x + 2y + 3z &= 11 \\ -2x + 3y + 5z &= 29 \\ 4x - y + 8z &= 19 \end{aligned}$$

54. $$\begin{aligned} 8x - 2y + 5z &= 36 \\ 3x + y - z &= 17 \\ 2x - 6y + 4z &= -2 \end{aligned}$$

55. $$\begin{aligned} x - 4y + 7z &= 49 \\ -3x + 2y - z &= -17 \\ 5x + 8y - 2z &= -24 \end{aligned}$$

56. $$\begin{aligned} \tfrac{1}{2}x - 2y + 7z &= 25 \\ x + \tfrac{1}{4}y - 4z &= -2 \\ -4x + 5y &= -56 \end{aligned}$$

57. $$\begin{aligned} 2x + 7y - 4z &= -5.5 \\ -x - 4y - 5z &= -19 \\ 4x - 2y - 9z &= -38 \end{aligned}$$

58. $$\begin{aligned} 4x - 2y + z &= -15 \\ 3x + y - 2z &= -20 \\ -6x + y + 5z &= 51 \end{aligned}$$

APPLICATIONS

In Exercises 59 and 60, three points, (x_1, y_1), (x_2, y_2), and (x_3, y_3), are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

59. Geometry. Apply determinants to determine whether the points, $(-2, -1)$, $(1, 5)$, and $(3, 9)$, are collinear.

60. Geometry. Apply determinants to determine whether the points, $(2, -6)$, $(-7, 30)$, and $(5, -18)$, are collinear.

For Exercises 61–64, the area of a triangle with vertices, (x_1, y_1), (x_2, y_2), and (x_3, y_3), is given by

$$\text{Area} = \pm\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

where the sign is chosen so that the area is positive.

61. Geometry. Apply determinants to find the area of a triangle with vertices, $(3, 2)$, $(5, 2)$, and $(3, -4)$. Check your answer by plotting these vertices in a Cartesian plane and using the formula for area of a right triangle.

62. Geometry. Apply determinants to find the area of a triangle with vertices, $(2, 3)$, $(7, 3)$, and $(7, 7)$. Check your answer by plotting these vertices in a Cartesian plane and using the formula for area of a right triangle.

63. Geometry. Apply determinants to find the area of a triangle with vertices, $(1, 2)$, $(3, 4)$, and $(-2, 5)$.

64. Geometry. Apply determinants to find the area of a triangle with vertices, $(-1, -2)$, $(3, 4)$, and $(2, 1)$.

65. Geometry. An equation of a line that passes through two points (x_1, y_1) and (x_2, y_2) can be expressed as a determinant equation as follows:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Apply the determinant to write an equation of the line passing through the points $(1, 2)$ and $(2, 4)$. Expand the determinant and express the equation of the line in slope–intercept form.

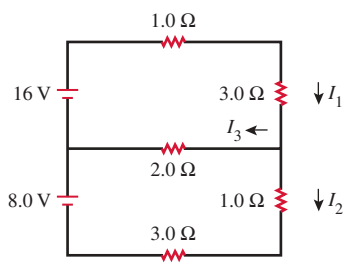
66. Geometry. If three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line), then the following determinant equation must be satisfied:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Determine whether $(0, 5)$, $(2, 0)$, and $(1, 2)$ are collinear.

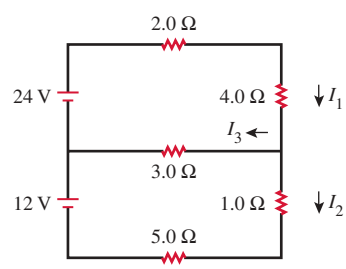
67. Electricity: Circuit Theory. The following equations come from circuit theory. Find the currents I_1, I_2, and I_3.

$$\begin{aligned} I_1 &= I_2 + I_3 \\ 16 &= 4I_1 + 2I_3 \\ 24 &= 4I_1 + 4I_2 \end{aligned}$$



68. Electricity: Circuit Theory. The following equations come from circuit theory. Find the currents I_1, I_2, and I_3.

$$\begin{aligned} I_1 &= I_2 + I_3 \\ 24 &= 6I_1 + 3I_3 \\ 36 &= 6I_1 + 6I_2 \end{aligned}$$



CATCH THE MISTAKE

In Exercises 69–72, explain the mistake that is made.

69. Evaluate the determinant $\begin{vmatrix} 2 & 1 & 3 \\ -3 & 0 & 2 \\ 1 & 4 & -1 \end{vmatrix}$.

Solution:

Expand the 3×3 determinant in terms of the 2×2 determinants.

$$\begin{vmatrix} 2 & 1 & 3 \\ -3 & 0 & 2 \\ 1 & 4 & -1 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ 4 & -1 \end{vmatrix} + 1\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} -3 & 0 \\ 1 & 4 \end{vmatrix}$$

Expand the 2×2 determinants. $= 2(0 - 8) + 1(3 - 2) + 3(-12 - 0)$

Simplify. $= -16 + 1 - 36 = -51$

This is incorrect. What mistake was made?

70. Evaluate the determinant $\begin{vmatrix} 2 & 1 & 3 \\ -3 & 0 & 2 \\ 1 & 4 & -1 \end{vmatrix}$.

Solution:

Expand the 3×3 determinant in terms of the 2×2 determinants.

$$\begin{vmatrix} 2 & 1 & 3 \\ -3 & 0 & 2 \\ 1 & 4 & -1 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ 4 & -1 \end{vmatrix} - 1\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}$$

Expand the 2×2 determinants. $= 2(0 - 8) - 1(3 - 2) + 3(3 - 2)$

Simplify. $= -16 - 1 + 3 = -14$

This is incorrect. What mistake was made?

71. Solve the system of linear equations.

$$\begin{aligned} 2x + 3y &= 6 \\ -x - y &= -3 \end{aligned}$$

Solution:

Set up the determinants.

$$D = \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix}, D_x = \begin{vmatrix} 2 & 6 \\ -1 & -3 \end{vmatrix}, \text{ and } D_y = \begin{vmatrix} 6 & 3 \\ -3 & -1 \end{vmatrix}$$

Evaluate the determinants. $D = 1$, $D_x = 0$, and $D_y = 3$

Solve for x and y. $x = \dfrac{D_x}{D} = \dfrac{0}{1} = 0$ and $y = \dfrac{D_y}{D} = \dfrac{3}{1} = 3$

$x = 0$, $y = 3$ is incorrect. What mistake was made?

72. Solve the system of linear equations.

$$\begin{aligned} 4x - 6y &= 0 \\ 4x + 6y &= 4 \end{aligned}$$

Solution:

Set up the determinants.

$$D = \begin{vmatrix} 4 & -6 \\ 4 & 6 \end{vmatrix}, D_x = \begin{vmatrix} 0 & -6 \\ 4 & 6 \end{vmatrix}, \text{ and } D_y = \begin{vmatrix} 4 & 0 \\ 4 & 4 \end{vmatrix}$$

Evaluate the determinants. $D = 48$, $D_x = 24$, and $D_y = 16$

Solve for x and y. $x = \dfrac{D}{D_x} = \dfrac{48}{24} = 2$ and $y = \dfrac{D_y}{D} = \dfrac{48}{16} = 3$

$x = 2$, $y = 3$ is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 73–76, determine whether each statement is true or false.

73. The value of a determinant changes sign if any two rows are interchanged.

74. If all the entries in any column are equal to zero, the value of the determinant is 0.

75. $\begin{vmatrix} 2 & 6 & 4 \\ 0 & 2 & 8 \\ 4 & 0 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 2 & 0 & 5 \end{vmatrix}$

76. $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 2 & 8 \\ 3 & 1 & 2 \end{vmatrix} = 0$

77. Calculate the determinant $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$.

78. Calculate the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix}$.

CHALLENGE

79. Evaluate the determinant:

$$\begin{vmatrix} 1 & -2 & -1 & 3 \\ 4 & 0 & 1 & 2 \\ 0 & 3 & 2 & 4 \\ 1 & -3 & 5 & -4 \end{vmatrix}$$

80. For the system of equations

$$\begin{aligned} 3x + 2y &= 5 \\ ax - 4y &= 1 \end{aligned}$$

find a that guarantees no unique solution.

81. Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - b_1a_2c_3$$

by expanding down the second column.

82. Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3b_1a_2$$

by expanding across the third row.

83. Show that

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (a - b)(a - c)(b - c)$$

84. For the system of equations

$$\begin{aligned} x + 3y + 2z &= 0 \\ x + ay + 4z &= 0 \\ 2y + az &= 0 \end{aligned}$$

find the value(s) of a that guarantees no unique solution.

TECHNOLOGY

In Exercises 85–88, apply a graphing utility to evaluate the determinants.

85. $\begin{vmatrix} 1 & 1 & -5 \\ 3 & -7 & -4 \\ 4 & -6 & 9 \end{vmatrix}$ Compare with your answer to Exercise 35.

86. $\begin{vmatrix} -3 & 2 & -5 \\ 1 & 8 & 2 \\ 4 & -6 & 9 \end{vmatrix}$ Compare with your answer to Exercise 36.

87. $\begin{vmatrix} -3 & 2 & -1 & 3 \\ 4 & 1 & 5 & 2 \\ 17 & 2 & 2 & 8 \\ 13 & -4 & 10 & -11 \end{vmatrix}$

88. $\begin{vmatrix} -3 & 21 & 19 & 3 \\ 4 & 1 & 16 & 2 \\ 17 & 31 & 2 & 5 \\ 13 & -4 & 10 & 2 \end{vmatrix}$

In Exercises 89 and 90, apply Cramer's rule to solve each system of equations and a graphing utility to evaluate the determinants.

89.
$$\begin{aligned} 3.1x + 1.6y - 4.8z &= -33.76 \\ 5.2x - 3.4y + 0.5z &= -36.68 \\ 0.5x - 6.4y + 11.4z &= 25.96 \end{aligned}$$

90.
$$\begin{aligned} -9.2x + 2.7y + 5.1z &= -89.20 \\ 4.3x - 6.9y - 7.6z &= 38.89 \\ 2.8x - 3.9y - 3.5z &= 34.08 \end{aligned}$$

PREVIEW TO CALCULUS

In calculus, determinants are used when evaluating double and triple integrals through a change of variables. In these cases, the elements of the determinant are functions.

In Exercises 91–94, find each determinant.

91. $\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

92. $\begin{vmatrix} 2x & 2y \\ 2x & 2y - 2 \end{vmatrix}$

93. $\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$

94. $\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$

SECTION 8.6

PARTIAL FRACTIONS

SKILLS OBJECTIVES

- Decompose rational expressions into sums of partial fractions when the denominators contain:
  - Distinct linear factors
  - Repeated linear factors
  - Distinct irreducible quadratic factors
  - Repeated irreducible quadratic factors

CONCEPTUAL OBJECTIVE

- Understand the connection between partial-fraction decomposition and systems of linear equations.

Performing Partial Fraction Decomposition

In Chapter 2, we studied polynomial functions, and in Section 2.6, we discussed ratios of polynomial functions, called rational functions. Rational expressions are of the form

$$\frac{n(x)}{d(x)} \qquad d(x) \neq 0$$

where the numerator $n(x)$ and the denominator $d(x)$ are polynomials. Examples of rational expressions are

$$\frac{4x - 1}{2x + 3} \qquad \frac{2x + 5}{x^2 - 1} \qquad \frac{3x^4 - 2x + 5}{x^2 + 2x + 4}$$

Suppose we are asked to add two rational expressions: $\dfrac{2}{x + 1} + \dfrac{5}{x - 3}$.

We already possess the skills to accomplish this. We first identify the least common denominator $(x + 1)(x - 3)$ and combine the fractions into a single expression.

$$\frac{2}{x + 1} + \frac{5}{x - 3} = \frac{2(x - 3) + 5(x + 1)}{(x + 1)(x - 3)} = \frac{2x - 6 + 5x + 5}{(x + 1)(x - 3)} = \frac{7x - 1}{x^2 - 2x - 3}$$

How do we do this in reverse? For example, how do we start with $\dfrac{7x - 1}{x^2 - 2x - 3}$ and write this expression as a sum of two simpler expressions?

$$\frac{7x - 1}{x^2 - 2x - 3} = \overbrace{\underbrace{\frac{2}{x + 1}}_{\text{Partial Fraction}} + \underbrace{\frac{5}{x - 3}}_{\text{Partial Fraction}}}^{\text{Partial-Fraction Decomposition}}$$

Each of the two expressions on the right is called a **partial fraction**. The sum of these fractions is called the **partial-fraction decomposition** of $\dfrac{7x - 1}{x^2 - 2x - 3}$.

Partial-fraction decomposition is an important tool in calculus. Calculus operations such as differentiation and integration are often made simpler if you apply partial fractions. The reason partial fractions were not discussed until now is because partial-fraction decomposition *requires the ability to solve systems of linear equations*. Since partial-fraction decomposition is made possible by the techniques of solving systems of linear equations, we consider partial fractions an important application of systems of linear equations.

As mentioned earlier, a rational expression is the ratio of two polynomial expressions $n(x)/d(x)$ and we assume that $n(x)$ and $d(x)$ are polynomials with no common factors other than 1. If the degree of $n(x)$ is less than the degree of $d(x)$, then the rational expression

$n(x)/d(x)$ is said to be **proper**. If the degree of $n(x)$ is greater than or equal to the degree of $d(x)$, the rational expression is said to be **improper**. If the rational expression is improper, it should first be divided using long division.

$$\frac{n(x)}{d(x)} = Q(x) + \frac{r(x)}{d(x)}$$

The result is the sum of a quotient $Q(x)$ and a rational expression, which is the ratio of the remainder $r(x)$ and the divisor $d(x)$. The rational expression $r(x)/d(x)$ is proper, and the techniques outlined in this section can be applied to its partial-fraction decomposition.

Partial-fraction decomposition of proper rational expressions always begins with factoring the denominator $d(x)$. The goal is to write $d(x)$ as a product of distinct linear factors, but that may not always be possible. Sometimes $d(x)$ can be factored into a product of linear factors, where one or more are repeated. And, sometimes the factored form of $d(x)$ contains irreducible quadratic factors, such as $x^2 + 1$. There are times when the irreducible quadratic factors are repeated, such as $(x^2 + 1)^2$. A procedure is now outlined for partial-fraction decomposition.

PARTIAL-FRACTION DECOMPOSITION

To write a rational expression $\dfrac{n(x)}{d(x)}$ as a sum of partial fractions:

Step 1: Determine whether the rational expression is proper or improper.

- Proper: degree of $n(x) <$ degree of $d(x)$
- Improper: degree of $n(x) \geq$ degree of $d(x)$

Step 2: If proper, proceed to Step 3.

If improper, divide $\dfrac{n(x)}{d(x)}$ using polynomial (long) division and write the result as $\dfrac{n(x)}{d(x)} = Q(x) + \dfrac{r(x)}{d(x)}$ and proceed to Step 3 with $\dfrac{r(x)}{d(x)}$.

Step 3: Factor $d(x)$. One of four possible cases will arise:

Case 1 Distinct (nonrepeated) *linear* factors: $(ax + b)$

Example: $d(x) = (3x - 1)(x + 2)$

Case 2 One or more repeated linear factors: $(ax + b)^m \qquad m \geq 2$

Example: $d(x) = (x + 5)^2(x - 3)$

Case 3 One or more distinct irreducible $(ax^2 + bx + c = 0$ has no real roots$)$ quadratic factors: $(ax^2 + bx + c)$

Example: $d(x) = (x^2 + 4)(x + 1)(x - 2)$

Case 4 One or more repeated irreducible quadratic factors: $(ax^2 + bx + c)^m$

Example: $d(x) = (x^2 + x + 1)^2(x + 1)(x - 2)$

Step 4: Decompose the rational expression into a sum of partial fractions according to the procedure outlined in each case in this section.

Step 4 depends on which cases, or types of factors, arise. It is important to note that these four cases are not exclusive and combinations of different types of factors will appear.

Distinct Linear Factors

CASE 1: $d(x)$ HAS ONLY DISTINCT (NONREPEATED) LINEAR FACTORS

If $d(x)$ is a polynomial of degree p, and it can be factored into p linear factors

$$d(x) = \underbrace{(ax + b)(cx + d)\ldots}_{p \text{ linear factors}}$$

where no two factors are the same, then the partial-fraction decomposition of $\dfrac{n(x)}{d(x)}$ can be written as

$$\frac{n(x)}{d(x)} = \frac{A}{(ax + b)} + \frac{B}{(cx + d)} + \cdots$$

where the numerators, A, B, and so on are constants to be determined.

The goal is to write a proper rational expression as the sum of proper rational expressions. Therefore, if the denominator is a linear factor (degree 1), then the numerator is a constant (degree 0).

EXAMPLE 1 Partial-Fraction Decomposition with Distinct Linear Factors

Find the partial-fraction decomposition of $\dfrac{5x + 13}{x^2 + 4x - 5}$.

Solution:

Factor the denominator. $\dfrac{5x + 13}{(x - 1)(x + 5)}$

Express as a sum of two partial fractions. $\dfrac{5x + 13}{(x - 1)(x + 5)} = \dfrac{A}{(x - 1)} + \dfrac{B}{(x + 5)}$

Multiply the two sides of the equation by the LCD $(x - 1)(x + 5)$. $5x + 13 = A(x + 5) + B(x - 1)$

Eliminate the parentheses. $5x + 13 = Ax + 5A + Bx - B$

Group the x's and constants on the right. $5x + 13 = (A + B)x + (5A - B)$

Identify like terms. $5x + 13 = (A + B)x + (5A - B)$

Equate the **coefficients of x**. $5 = A + B$

Equate the **constant** terms. $13 = 5A - B$

Solve the system of two linear equations using any method to solve for A and B. $A = 3, B = 2$

Substitute $A = 3$, $B = 2$ into the partial-fraction decomposition.

$$\boxed{\frac{5x + 13}{(x - 1)(x + 5)} = \frac{3}{(x - 1)} + \frac{2}{(x + 5)}}$$

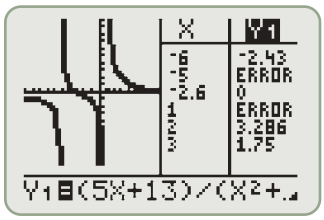
Check by adding the partial fractions.

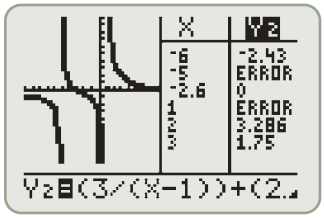
$$\frac{3}{(x - 1)} + \frac{2}{(x + 5)} = \frac{3(x + 5) + 2(x - 1)}{(x - 1)(x + 5)} = \frac{5x + 13}{x^2 + 4x - 5}$$

■ **YOUR TURN** Find the partial-fraction decomposition of $\dfrac{4x - 13}{x^2 - 3x - 10}$.

Technology Tip

Use a TI to check the graph of $Y_1 = \dfrac{5x + 13}{x^2 + 4x - 5}$ and its partial-fraction decomposition $Y_2 = \dfrac{3}{x - 1} + \dfrac{2}{x + 5}$. The graphs and tables of values are shown.





■ **Answer:** $\dfrac{4x - 13}{x^2 - 3x - 10} = \dfrac{3}{x + 2} + \dfrac{1}{x - 5}$

In Example 1, we started with a rational expression that had a numerator of degree 1 and a denominator of degree 2. Partial-fraction decomposition enabled us to write that rational expression as a sum of two rational expressions with degree 0 numerators and degree 1 denominators.

Repeated Linear Factors

CASE 2: $d(x)$ HAS AT LEAST ONE REPEATED LINEAR FACTOR

If $d(x)$ can be factored into a product of linear factors, then the partial-fraction decomposition will proceed as in Case 1, with the exception of a repeated factor $(ax + b)^m$, $m \geq 2$. Any linear factor repeated m times will result in the sum of m partial fractions

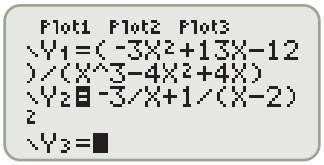
$$\frac{A}{(ax + b)} + \frac{B}{(ax + b)^2} + \frac{C}{(ax + b)^3} + \cdots + \frac{M}{(ax + b)^m}$$

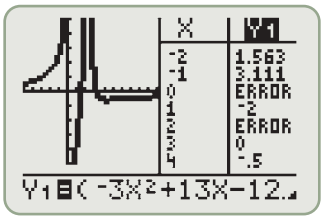
where the numerators, $A, B, C, \ldots, M$ are constants to be determined.

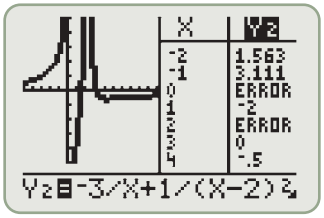
Note that if $d(x)$ is of degree p, the general form of the decomposition will have p partial fractions. If some numerator constants turn out to be zero, then the final decomposition may have fewer than p partial fractions.

Technology Tip

Use a TI to check the graph of $Y_1 = \dfrac{-3x^2 + 13x - 12}{x^3 - 4x^2 + 4x}$ and its partial-fraction decomposition $Y_2 = \dfrac{-3}{x} + \dfrac{1}{(x - 2)^2}$. The graphs and tables of values are shown.







EXAMPLE 2 Partial-Fraction Decomposition with a Repeated Linear Factor

Find the partial-fraction decomposition of $\dfrac{-3x^2 + 13x - 12}{x^3 - 4x^2 + 4x}$.

Solution:

Factor the denominator. $\dfrac{-3x^2 + 13x - 12}{x(x - 2)^2}$

Express as a sum of three partial fractions. $\dfrac{-3x^2 + 13x - 12}{x(x - 2)^2} = \dfrac{A}{x} + \dfrac{B}{(x - 2)} + \dfrac{C}{(x - 2)^2}$

Multiply both sides by the LCD $x(x - 2)^2$. $-3x^2 + 13x - 12 = A(x - 2)^2 + Bx(x - 2) + Cx$

Eliminate the parentheses. $-3x^2 + 13x - 12 = Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx$

Group like terms on the right. $-3x^2 + 13x - 12 = (A + B)x^2 + (-4A - 2B + C)x + 4A$

Identify like terms on both sides. $-3x^2 + 13x - 12 = (A + B)x^2 + (-4A - 2B + C)x + 4A$

Equate the **coefficients of x^2**. $-3 = A + B$ (1)

Equate the **coefficients of x**. $13 = -4A - 2B + C$ (2)

Equate the **constant** terms. $-12 = 4A$ (3)

Solve the system of three equations for A, B, and C.

Solve (3) for A.	$A = -3$
Substitute $A = -3$ into (1).	$B = 0$
Substitute $A = -3$ and $B = 0$ into (2).	$C = 1$

Substitute $A = -3$, $B = 0$, $C = 1$ into the partial-fraction decomposition.

$$\frac{-3x^2 + 13x - 12}{x(x-2)^2} = \frac{-3}{x} + \frac{0}{(x-2)} + \frac{1}{(x-2)^2}$$

$$\boxed{\frac{-3x^2 + 13x - 12}{x^3 - 4x^2 + 4x} = \frac{-3}{x} + \frac{1}{(x-2)^2}}$$

Check by adding the partial fractions.

$$\frac{-3}{x} + \frac{1}{(x-2)^2} = \frac{-3(x-2)^2 + 1(x)}{x(x-2)^2} = \frac{-3x^2 + 13x - 12}{x^3 - 4x^2 + 4x}$$

■ **YOUR TURN** Find the partial-fraction decomposition of $\dfrac{x^2 + 1}{x^3 + 2x^2 + x}$.

■ **Answer:**
$$\frac{x^2 + 1}{x^3 + 2x^2 + x} = \frac{1}{x} - \frac{2}{(x+1)^2}$$

EXAMPLE 3 Partial-Fraction Decomposition with Multiple Repeated Linear Factors

Find the partial-fraction decomposition of $\dfrac{2x^3 + 6x^2 + 6x + 9}{x^4 + 6x^3 + 9x^2}$.

Solution:

Factor the denominator.	$\dfrac{2x^3 + 6x^2 + 6x + 9}{x^2(x+3)^2}$
Express as a sum of four partial fractions.	$\dfrac{2x^3 + 6x^2 + 6x + 9}{x^2(x+3)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x+3)} + \dfrac{D}{(x+3)^2}$
Multiply both sides by the LCD $x^2(x+3)^2$.	$2x^3 + 6x^2 + 6x + 9 = Ax(x+3)^2 + B(x+3)^2 + Cx^2(x+3) + Dx^2$
Eliminate the parentheses.	$2x^3 + 6x^2 + 6x + 9 = Ax^3 + 6Ax^2 + 9Ax + Bx^2 + 6Bx + 9B + Cx^3 + 3Cx^2 + Dx^2$
Group like terms on the right.	$2x^3 + 6x^2 + 6x + 9 = (A + C)x^3 + (6A + B + 3C + D)x^2 + (9A + 6B)x + 9B$
Identify like terms on both sides.	$2x^3 + 6x^2 + 6x + 9 = (A + C)x^3 + (6A + B + 3C + D)x^2 + (9A + 6B)x + 9B$
Equate the **coefficients of x^3**.	$2 = A + C$ (1)
Equate the **coefficients of x^2**.	$6 = 6A + B + 3C + D$ (2)
Equate the **coefficients of x**.	$6 = 9A + 6B$ (3)
Equate the **constant** terms.	$9 = 9B$ (4)

Solve the system of four equations for A, B, C, and D.

Solve Equation (4) for B.	$B = 1$
Substitute $B = 1$ into Equation (3) and solve for A.	$A = 0$
Substitute $A = 0$ into Equation (1) and solve for C.	$C = 2$
Substitute $A = 0$, $B = 1$, and $C = 2$ into Equation (2) and solve for D.	$D = -1$

Substitute $A = 0$, $B = 1$, $C = 2$, $D = -1$ into the partial-fraction decomposition.

$$\frac{2x^3 + 6x^2 + 6x + 9}{x^2(x+3)^2} = \frac{0}{x} + \frac{1}{x^2} + \frac{2}{(x+3)} + \frac{-1}{(x+3)^2}$$

$$\boxed{\frac{2x^3 + 6x^2 + 6x + 9}{x^2(x+3)^2} = \frac{1}{x^2} + \frac{2}{(x+3)} - \frac{1}{(x+3)^2}}$$

Check by adding the partial fractions.

$$\frac{1}{x^2} + \frac{2}{(x+3)} - \frac{1}{(x+3)^2} = \frac{(x+3)^2 + 2x^2(x+3) - 1(x^2)}{x^2(x+3)^2}$$

$$= \frac{2x^3 + 6x^2 + 6x + 9}{x^4 + 6x^3 + 9x^2}$$

■ **YOUR TURN** Find the partial-fraction decomposition of $\dfrac{2x^3 + 2x + 1}{x^4 + 2x^3 + x^2}$.

■ **Answer:** $\dfrac{2x^3 + 2x + 1}{x^4 + 2x^3 + x^2} = \dfrac{1}{x^2} + \dfrac{2}{(x+1)} - \dfrac{3}{(x+1)^2}$

Distinct Irreducible Quadratic Factors

There will be times when a polynomial cannot be factored into a product of linear factors with real coefficients. For example, $x^2 + 4$, $x^2 + x + 1$, and $9x^2 + 3x + 2$ are all examples of *irreducible quadratic* expressions. The general form of an **irreducible quadratic factor** is given by

$$ax^2 + bx + c \quad \text{where } ax^2 + bx + c = 0 \text{ has no real roots}$$

CASE 3: $d(x)$ HAS A DISTINCT IRREDUCIBLE QUADRATIC FACTOR

If the factored form of $d(x)$ contains an irreducible quadratic factor $ax^2 + bx + c$, then the partial-fraction decomposition will contain a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined.

Study Tip

In a partial-fraction decomposition, the degree of the numerator is always 1 less than the degree of the denominator.

Recall that for a proper rational expression, the degree of the numerator is less than the degree of the denominator. For irreducible quadratic (degree 2) denominators we assume a linear (degree 1) numerator. For example,

$$\frac{7x^2 + 2}{(2x+1)(x^2+1)} = \underbrace{\frac{A}{(2x+1)}}_{\frac{\text{Constant numerator}}{\text{Linear factor}}} + \underbrace{\frac{Bx + C}{(x^2+1)}}_{\frac{\text{Linear numerator}}{\text{Quadratic factor}}}$$

A constant is used in the numerator when the denominator consists of a linear expression and a linear expression is used in the numerator when the denominator consists of a quadratic expression.

EXAMPLE 4 Partial-Fraction Decomposition with an Irreducible Quadratic Factor

Find the partial-fraction decomposition of $\dfrac{7x^2 + 2}{(2x + 1)(x^2 + 1)}$.

Solution:

The denominator is already in factored form. $\dfrac{7x^2 + 2}{(2x + 1)(x^2 + 1)}$

Express as a sum of two partial fractions. $\dfrac{7x^2 + 2}{(2x + 1)(x^2 + 1)} = \dfrac{A}{(2x + 1)} + \dfrac{Bx + C}{(x^2 + 1)}$

Multiply both sides by the LCD $(2x + 1)(x^2 + 1)$. $7x^2 + 2 = A(x^2 + 1) + (Bx + C)(2x + 1)$

Eliminate the parentheses. $7x^2 + 2 = Ax^2 + A + 2Bx^2 + Bx + 2Cx + C$

Group like terms on the right. $7x^2 + 2 = (A + 2B)x^2 + (B + 2C)x + (A + C)$

Identify like terms on both sides. $7x^2 + 0x + 2 = (A + 2B)x^2 + (B + 2C)x + (A + C)$

Equate the **coefficients of x^2**. $7 = A + 2B$

Equate the **coefficients of x**. $0 = B + 2C$

Equate the **constant** terms. $2 = A + C$

Solve the system of three equations for A, B, and C. $A = 3, B = 2, C = -1$

Substitute $A = 3, B = 2, C = -1$ into the partial-fraction decomposition.

$$\boxed{\frac{7x^2 + 2}{(2x + 1)(x^2 + 1)} = \frac{3}{(2x + 1)} + \frac{2x - 1}{(x^2 + 1)}}$$

Check by adding the partial fractions.

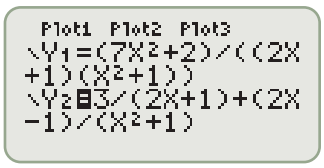
$$\frac{3}{(2x + 1)} + \frac{2x - 1}{(x^2 + 1)} = \frac{3(x^2 + 1) + (2x - 1)(2x + 1)}{(2x + 1)(x^2 + 1)} = \frac{7x^2 + 2}{(2x + 1)(x^2 + 1)}$$

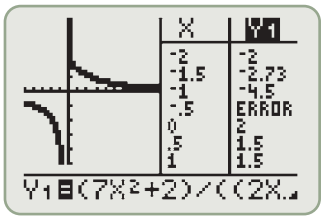
■ **YOUR TURN** Find the partial-fraction decomposition of $\dfrac{-2x^2 + x + 6}{(x - 1)(x^2 + 4)}$.

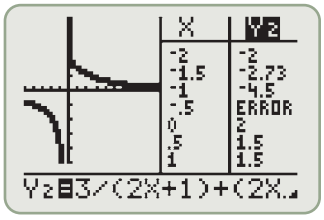
■ **Answer:** $\dfrac{-2x^2 + x + 6}{(x - 1)(x^2 + 4)} = \dfrac{1}{x - 1} - \dfrac{3x + 2}{x^2 + 4}$

Technology Tip

Use a TI to check the graph of $Y_1 = \dfrac{7x^2 + 2}{(2x + 1)(x^2 + 1)}$ and its partial-fraction decomposition $Y_2 = \dfrac{3}{2x + 1} + \dfrac{2x - 1}{x^2 + 1}$. The graphs and tables of values are shown.







Repeated Irreducible Quadratic Factors

CASE 4: $d(x)$ HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

If the factored form of $d(x)$ contains an irreducible quadratic factor $(ax^2 + bx + c)^m$, where $b^2 - 4ac < 0$, then the partial-fraction decomposition will contain a series of terms of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where A_i and B_i with $i = 1, 2, \ldots, m$, are constants to be determined.

EXAMPLE 5 Partial-Fraction Decomposition with a Repeated Irreducible Quadratic Factor

Find the partial-fraction decomposition of $\dfrac{x^3 - x^2 + 3x + 2}{(x^2 + 1)^2}$.

Solution:

The denominator is already in factored form. $\dfrac{x^3 - x^2 + 3x + 2}{(x^2 + 1)^2}$

Express as a sum of two partial fractions. $\dfrac{x^3 - x^2 + 3x + 2}{(x^2 + 1)^2} = \dfrac{Ax + B}{x^2 + 1} + \dfrac{Cx + D}{(x^2 + 1)^2}$

Multiply both sides by the LCD $(x^2 + 1)^2$. $x^3 - x^2 + 3x + 2 = (Ax + B)(x^2 + 1) + Cx + D$

Eliminate the parentheses. $x^3 - x^2 + 3x + 2 = Ax^3 + Bx^2 + Ax + B + Cx + D$

Group like terms on the right. $x^3 - x^2 + 3x + 2 = Ax^3 + Bx^2 + (A + C)x + (B + D)$

Identify like terms on both sides. $x^3 - x^2 + 3x + 2 = Ax^3 + Bx^2 + (A + C)x + (B + D)$

Equate the **coefficients of x^3**. $1 = A$ (1)

Equate the **coefficients of x^2**. $-1 = B$ (2)

Equate the **coefficients of x**. $3 = A + C$ (3)

Equate the **constant** terms. $2 = B + D$ (4)

Substitute $A = 1$ into Equation (3) and solve for C. $C = 2$

Substitute $B = -1$ into Equation (4) and solve for D. $D = 3$

Substitute $A = 1$, $B = -1$, $C = 2$, $D = 3$ into the partial-fraction decomposition.

$$\boxed{\frac{x^3 - x^2 + 3x + 2}{(x^2 + 1)^2} = \frac{x - 1}{x^2 + 1} + \frac{2x + 3}{(x^2 + 1)^2}}$$

Check by adding the partial fractions.

$$\frac{x - 1}{x^2 + 1} + \frac{2x + 3}{(x^2 + 1)^2} = \frac{(x - 1)(x^2 + 1) + (2x + 3)}{(x^2 + 1)^2} = \frac{x^3 - x^2 + 3x + 2}{(x^2 + 1)^2}$$

■ **YOUR TURN** Find the partial-fraction decomposition of $\dfrac{3x^3 + x^2 + 4x - 1}{(x^2 + 4)^2}$.

■ **Answer:** $\dfrac{3x^3 + x^2 + 4x - 1}{(x^2 + 4)^2} = \dfrac{3x + 1}{x^2 + 4} - \dfrac{8x + 5}{(x^2 + 4)^2}$

Combinations of All Four Cases

As you probably can imagine, there are rational expressions that have combinations of all four cases, which can lead to a system of several equations when solving for the unknown constants in the numerators of the partial fractions.

EXAMPLE 6 Partial-Fraction Decomposition

Find the partial-fraction decomposition of $\dfrac{x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8}{x^2(x^2 + 2)^2}$.

Solution:

The denominator is already in factored form.

$$\frac{x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8}{x^2(x^2 + 2)^2}$$

Express as a sum of partial fractions.

There are repeated linear and irreducible quadratic factors.

$$\frac{x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8}{x^2(x^2 + 2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{(x^2 + 2)} + \frac{Ex + F}{(x^2 + 2)^2}$$

Multiply both sides by the LCD $x^2(x^2 + 2)^2$.

$$\begin{aligned} &x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8 \\ &\quad = Ax(x^2 + 2)^2 + B(x^2 + 2)^2 + (Cx + D)x^2(x^2 + 2) + (Ex + F)x^2 \end{aligned}$$

Eliminate the parentheses.

$$\begin{aligned} &x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8 \\ &\quad = Ax^5 + 4Ax^3 + 4Ax + Bx^4 + 4Bx^2 + 4B + Cx^5 + 2Cx^3 + Dx^4 + 2Dx^2 + Ex^3 + Fx^2 \end{aligned}$$

Group like terms on the right.

$$\begin{aligned} &x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8 \\ &\quad = (A + C)x^5 + (B + D)x^4 + (4A + 2C + E)x^3 + (4B + 2D + F)x^2 + 4Ax + 4B \end{aligned}$$

Equating the coefficients of like terms leads to six equations.

$$\begin{aligned} A + C &= 1 \\ B + D &= 1 \\ 4A + 2C + E &= 4 \\ 4B + 2D + F &= -3 \\ 4A &= 4 \\ 4B &= -8 \end{aligned}$$

Solve this system of equations.

$$A = 1, \qquad B = -2, \qquad C = 0, \qquad D = 3, \qquad E = 0, \qquad F = -1$$

Substitute $A = 1$, $B = -2$, $C = 0$, $D = 3$, $E = 0$, $F = -1$ into the partial-fraction decomposition.

$$\frac{x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8}{x^2(x^2 + 2)^2} = \frac{1}{x} + \frac{-2}{x^2} + \frac{0x + 3}{(x^2 + 2)} + \frac{0x + -1}{(x^2 + 2)^2}$$

$$\boxed{\frac{x^5 + x^4 + 4x^3 - 3x^2 + 4x - 8}{x^2(x^2 + 2)^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{3}{(x^2 + 2)} - \frac{1}{(x^2 + 2)^2}}$$

Check by adding the partial fractions.

SECTION 8.6 SUMMARY

A rational expression $\frac{n(x)}{d(x)}$ is

- **Proper:** If the degree of the numerator is less than the degree of the denominator.
- **Improper:** If the degree of the numerator is equal to or greater than the degree of the denominator.

Partial-Fraction Decomposition of Proper Rational Expressions

1. Distinct (nonrepeated) linear factors

$$\text{Example: } \frac{3x - 10}{(x - 5)(x + 4)} = \frac{A}{x - 5} + \frac{B}{x + 4}$$

2. Repeated linear factors

$$\text{Example: } \frac{2x + 5}{(x - 3)^2(x + 1)} = \frac{A}{x - 3} + \frac{B}{(x - 3)^2} + \frac{C}{x + 1}$$

3. Distinct irreducible quadratic factors

$$\text{Example: } \frac{1 - x}{(x^2 + 1)(x^2 + 8)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 8}$$

4. Repeated irreducible quadratic factors

$$\text{Example: } \frac{4x^2 - 3x + 2}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

SECTION 8.6 EXERCISES

SKILLS

In Exercises 1–6, match the rational expression (1–6) with the form of the partial-fraction decomposition (a–f).

1. $\frac{3x + 2}{x(x^2 - 25)}$ **2.** $\frac{3x + 2}{x(x^2 + 25)}$ **3.** $\frac{3x + 2}{x^2(x^2 + 25)}$ **4.** $\frac{3x + 2}{x^2(x^2 - 25)}$ **5.** $\frac{3x + 2}{x(x^2 + 25)^2}$ **6.** $\frac{3x + 2}{x^2(x^2 + 25)^2}$

a. $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 25}$ **b.** $\frac{A}{x} + \frac{Bx + C}{x^2 + 25} + \frac{Dx + E}{(x^2 + 25)^2}$ **c.** $\frac{A}{x} + \frac{Bx + C}{x^2 + 25}$

d. $\frac{A}{x} + \frac{B}{x + 5} + \frac{C}{x - 5}$ **e.** $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 25} + \frac{Ex + F}{(x^2 + 25)^2}$ **f.** $\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 5} + \frac{D}{x - 5}$

In Exercises 7–14, write the form of the partial-fraction decomposition. Do not solve for the constants.

7. $\frac{9}{x^2 - x - 20}$ **8.** $\frac{8}{x^2 - 3x - 10}$ **9.** $\frac{2x + 5}{x^3 - 4x^2}$ **10.** $\frac{x^2 + 2x - 1}{x^4 - 9x^2}$

11. $\frac{2x^3 - 4x^2 + 7x + 3}{(x^2 + x + 5)}$ **12.** $\frac{2x^3 + 5x^2 + 6}{(x^2 - 3x + 7)}$ **13.** $\frac{3x^3 - x + 9}{(x^2 + 10)^2}$ **14.** $\frac{5x^3 + 2x^2 + 4}{(x^2 + 13)^2}$

In Exercises 15–40, find the partial-fraction decomposition for each rational function.

15. $\frac{1}{x(x + 1)}$ **16.** $\frac{1}{x(x - 1)}$ **17.** $\frac{x}{x(x - 1)}$ **18.** $\frac{x}{x(x + 1)}$

19. $\frac{9x - 11}{(x - 3)(x + 5)}$ **20.** $\frac{8x - 13}{(x - 2)(x + 1)}$ **21.** $\frac{3x + 1}{(x - 1)^2}$ **22.** $\frac{9y - 2}{(y - 1)^2}$

23. $\frac{4x - 3}{x^2 + 6x + 9}$ **24.** $\frac{3x + 1}{x^2 + 4x + 4}$ **25.** $\frac{4x^2 - 32x + 72}{(x + 1)(x - 5)^2}$ **26.** $\frac{4x^2 - 7x - 3}{(x + 2)(x - 1)^2}$

27. $\dfrac{5x^2 + 28x - 6}{(x + 4)(x^2 + 3)}$

28. $\dfrac{x^2 + 5x + 4}{(x - 2)(x^2 + 2)}$

29. $\dfrac{-2x^2 - 17x + 11}{(x - 7)(3x^2 - 7x + 5)}$

30. $\dfrac{14x^2 + 8x + 40}{(x + 5)(2x^2 - 3x + 5)}$

31. $\dfrac{x^3}{(x^2 + 9)^2}$

32. $\dfrac{x^2}{(x^2 + 9)^2}$

33. $\dfrac{2x^3 - 3x^2 + 7x - 2}{(x^2 + 1)^2}$

34. $\dfrac{-x^3 + 2x^2 - 3x + 15}{(x^2 + 8)^2}$

35. $\dfrac{3x + 1}{x^4 - 1}$

36. $\dfrac{2 - x}{x^4 - 81}$

37. $\dfrac{5x^2 + 9x - 8}{(x - 1)(x^2 + 2x - 1)}$

38. $\dfrac{10x^2 - 5x + 29}{(x - 3)(x^2 + 4x + 5)}$

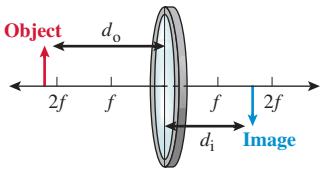
39. $\dfrac{3x}{x^3 - 1}$

40. $\dfrac{5x + 2}{x^3 - 8}$

APPLICATIONS

41. Optics. The relationship between the distance of an object to a lens d_o, the distance to the image d_i, and the focal length f of the lens is given by

$$\frac{f(d_i + d_o)}{d_i d_o} = 1$$



Use partial-fraction decomposition to write the lens law in terms of sums of fractions. What does each term represent?

42. Sums. Find the partial-fraction decomposition of $\dfrac{1}{n(n + 1)}$, and apply it to find the sum of

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{999 \cdot 1000}$$

In Exercises 43 and 44, refer to the following:

Laplace transforms are used to solve differential equations. The Laplace transform of $f(t)$ is denoted by $L\{f(t)\}$; thus, $L\{e^{3t}\}$ is the Laplace transform of $f(t) = e^{3t}$. It is known that $L\{e^{kt}\} = \dfrac{1}{s - k}$ and $L\{e^{-kt}\} = \dfrac{1}{s + k}$. Then the inverse Laplace transform of $g(s) = \dfrac{1}{s - k}$ is $L^{-1}\left\{\dfrac{1}{s - k}\right\} = e^{kt}$. Inverse Laplace transforms are linear:

$$L^{-1}\{f(t) + g(t)\} = L^{-1}\{f(t)\} + L^{-1}\{g(t)\}$$

43. Laplace Transform. Use partial fractions to find the inverse Laplace transform of $\dfrac{9 + s}{4 - s^2}$.

44. Laplace Transform. Use partial fractions to find the inverse Laplace transform of $\dfrac{2s^2 + 3s - 2}{s(s + 1)(s - 2)}$.

CATCH THE MISTAKE

In Exercises 45 and 46, explain the mistake that is made.

45. Find the partial-fraction decomposition of $\dfrac{3x^2 + 3x + 1}{x(x^2 + 1)}$.

Solution:

Write the partial-fraction decomposition form. $\dfrac{3x^2 + 3x + 1}{x(x^2 + 1)} = \dfrac{A}{x} + \dfrac{B}{x^2 + 1}$

Multiply both sides by the LCD $x(x^2 + 1)$. $3x^2 + 3x + 1 = A(x^2 + 1) + Bx$

Eliminate the parentheses. $3x^2 + 3x + 1 = Ax^2 + Bx + A$

Matching like terms eads to three equations. $A = 3$, $B = 3$, and $A = 1$

This is incorrect. What mistake was made?

46. Find the partial-fraction decomposition of $\dfrac{3x^4 - x - 1}{x(x - 1)}$.

Solution:

Write the partial-fraction decomposition form. $\dfrac{3x^4 - x - 1}{x(x - 1)} = \dfrac{A}{x} + \dfrac{B}{x - 1}$

Multiply both sides by the LCD $x(x - 1)$. $3x^4 - x - 1 = A(x - 1) + Bx$

Eliminate the parentheses and group like terms. $3x^4 - x - 1 = (A + B)x - A$

Compare like coefficients. $A = 1$, $B = -2$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 47–52, determine whether each statement is true or false.

47. Partial-fraction decomposition can be employed only when the degree of the numerator is greater than the degree of the denominator.

48. The degree of the denominator of a proper rational expression is equal to the number of partial fractions in its decomposition.

49. Partial-fraction decomposition depends on the factors of the denominator.

50. A rational function can always be decomposed into partial fractions with linear or irreducible quadratic factors in each denominator.

51. The partial-fraction decomposition of a rational function $\dfrac{f(x)}{(x-a)^n}$ has the form $\dfrac{A_1}{(x-a)} + \dfrac{A_2}{(x-a)^2} + \cdots + \dfrac{A_n}{(x-a)^n}$, where all the numbers A_i are nonzero.

52. The rational function $\dfrac{1}{x^3+1}$ cannot be decomposed into partial fractions.

CHALLENGE

For Exercises 53–58, find the partial-fraction decomposition.

53. $\dfrac{x^2+4x-8}{x^3-x^2-4x+4}$

54. $\dfrac{ax+b}{x^2-c^2}$ a, b, c are real numbers.

55. $\dfrac{2x^3+x^2-x-1}{x^4+x^3}$

56. $\dfrac{-x^3+2x-2}{x^5-x^4}$

57. $\dfrac{x^5+2}{(x^2+1)^3}$

58. $\dfrac{x^2-4}{(x^2+1)^3}$

TECHNOLOGY

59. Apply a graphing utility to graph $y_1 = \dfrac{5x+4}{x^2+x-2}$ and $y_2 = \dfrac{3}{x-1} + \dfrac{2}{x+2}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

60. Apply a graphing utility to graph $y_1 = \dfrac{2x^2+2x-5}{x^3+5x}$ and $y_2 = \dfrac{3x+2}{x^2+5} - \dfrac{1}{x}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

61. Apply a graphing utility to graph $y_1 = \dfrac{x^9+8x-1}{x^5(x^2+1)^3}$ and $y_2 = \dfrac{4}{x} - \dfrac{1}{x^5} + \dfrac{2}{x^2+1} - \dfrac{3x+2}{(x^2+1)^2}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

62. Apply a graphing utility to graph $y_1 = \dfrac{x^3+2x+6}{(x+3)(x^2-4)^3}$ and $y_2 = \dfrac{2}{x+3} + \dfrac{x+3}{(x^2-4)^3}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

63. Apply a graphing utility to graph $y_1 = \dfrac{2x^3-8x+16}{(x-2)^2(x^2+4)}$ and $y_2 = \dfrac{1}{x-2} + \dfrac{2}{(x-2)^2} + \dfrac{x+4}{x^2+4}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

64. Apply a graphing utility to graph $y_1 = \dfrac{3x^3+14x^2+6x+51}{(x^2+3x-4)(x^2+2x+5)}$ and $y_2 = \dfrac{2}{x-1} - \dfrac{1}{x+4} + \dfrac{2x-3}{x^2+2x+5}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

PREVIEW TO CALCULUS

In calculus, partial fractions are used to calculate the sums of infinite series. In Exercises 65–68, find the partial-fraction decomposition of the summand.

65. $\displaystyle\sum_{k=1}^{\infty} \frac{9}{k(k+3)}$

66. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

67. $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2}$

68. $\displaystyle\sum_{k=1}^{\infty} \frac{4}{k(k+1)(k+2)}$

SECTION 8.7

SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES

SKILLS OBJECTIVES

- Graph a linear inequality in two variables.
- Graph a system of linear inequalities in two variables.
- Solve an optimization problem using linear programming.

CONCEPTUAL OBJECTIVES

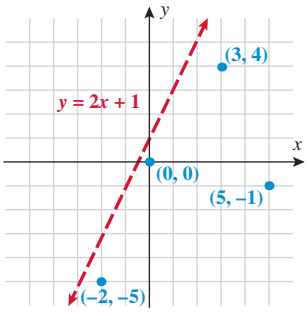
- Interpret the difference between solid and dashed lines.
- Interpret an overlapped shaded region as a solution.

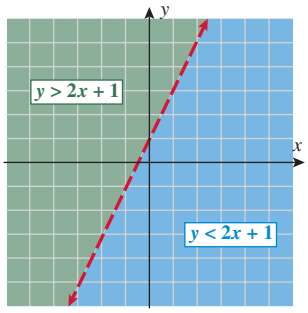
Linear Inequalities in Two Variables

Recall in Section 0.6 that $y = 2x + 1$ is an *equation in two variables* whose graph is a line in the xy-plane. We now turn our attention to **linear inequalities in two variables**. For example, if we change the $=$ in $y = 2x + 1$ to $<$, we get $y < 2x + 1$. The solution to this inequality in two variables is the set of all points (x, y) that make this inequality true. Some solutions to this inequality are $(-2, -5)$, $(0, 0)$, $(3, 4)$, $(5, -1)$, ...

In fact, the entire region ***below*** the line $y = 2x + 1$ satisfies the inequality $y < 2x + 1$. If we reverse the sign of the inequality to get $y > 2x + 1$, then the entire region ***above*** the line $y = 2x + 1$ represents the solution to the inequality.

Any line divides the xy-plane into two **half-planes**. For example, the line $y = 2x + 1$ divides the xy-plane into two half-planes represented as $y > 2x + 1$ and $y < 2x + 1$. Recall that with inequalities in one variable we used the notation of parentheses and brackets to denote the type of inequality (strict or nonstrict). We use a similar notation with linear inequalities in two variables. If the inequality is a strict inequality, $<$ or $>$, then the line is *dashed*, and, if the inequality includes the equal sign, $\leq$ or $\geq$, then a *solid* line is used. The following box summarizes the procedure for graphing a linear inequality in two variables.





GRAPHING A LINEAR INEQUALITY IN TWO VARIABLES

Step 1: **Change** the inequality sign, $<$, $\leq$, $\geq$, or $>$, to an equal sign, $=$.

Step 2: **Draw** the line that corresponds to the resulting equation in Step 1.
- If the inequality is strict, $<$ or $>$, use a **dashed** line.
- If the inequality is not strict, $\leq$ or $\geq$, use a **solid** line.

Step 3: **Test** a point.
- Select a point in one half-plane and test to see whether it satisfies the inequality. If it does, then so do all the points in that region (half-plane). If not, then none of the points in that half-plane satisfy the inequality.
- Repeat this step for the other half-plane.

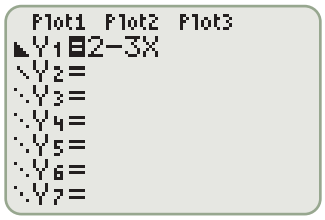
Step 4: **Shade** the half-plane that satisfies the inequality.

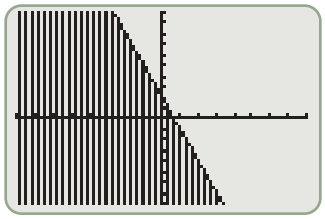
Study Tip

A dashed line means that the points that lie on the line are not included in the solution of the linear inequality.

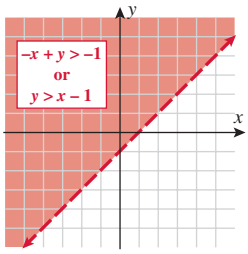
Technology Tip

The graphing calculator can be used to help in shading the linear inequality $3x + y < 2$. However, it will not show whether the line is solid or dashed. First solve for y, $y < -3x + 2$. Then, enter $y_1 = -3x + 2$. Since $y_1 < -3x + 2$, the region below the dashed line is shaded.





■ **Answer:**



EXAMPLE 1 Graphing a Strict Linear Inequality in Two Variables

Graph the inequality $3x + y < 2$.

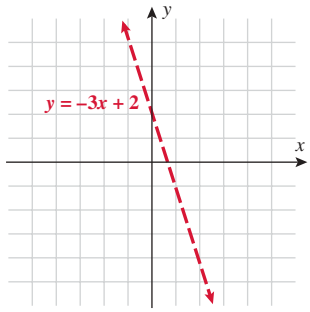
Solution:

STEP 1 Change the inequality sign to an equal sign. $3x + y = 2$

STEP 2 Draw the line.

Convert from standard form to slope–intercept form. $y = -3x + 2$

Since the inequality $<$ is a strict inequality, use a **dashed** line.



STEP 3 Test points in each half-plane.

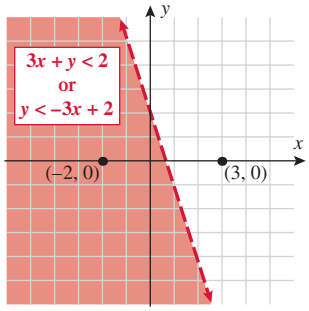
Substitute (3, 0) into $3x + y < 2$. $3(3) + 0 < 2$

The point (3, 0) does not satisfy the inequality. $9 < 2$

Substitute $(-2, 0)$ into $3x + y < 2$. $3(-2) + 0 < 2$

The point $(-2, 0)$ does satisfy the inequality. $-6 < 2$

STEP 4 Shade the region containing the point $(-2, 0)$.



■ **YOUR TURN** Graph the inequality $-x + y > -1$.

EXAMPLE 2 Graphing a Nonstrict Linear Inequality in Two Variables

Graph the inequality $2x - 3y \geq 6$.

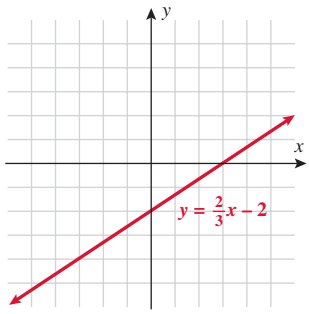
Solution:

STEP 1 Change the inequality sign to an equal sign. $2x - 3y = 6$

STEP 2 Draw the line.

Convert from standard form to slope–intercept form. $y = \frac{2}{3}x - 2$

Since the inequality $\geq$ is not a strict inequality, use a **solid** line.



STEP 3 Test points in each half-plane.

Substitute (5, 0) into $2x - 3y \geq 6$. $2(5) - 3(0) \geq 6$

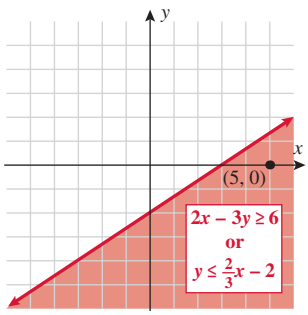
The point (5, 0) satisfies the inequality. $10 \geq 6$

Substitute (0, 0) into $2x - 3y \geq 6$. $2(0) - 3(0) \geq 6$

$0 \geq 6$

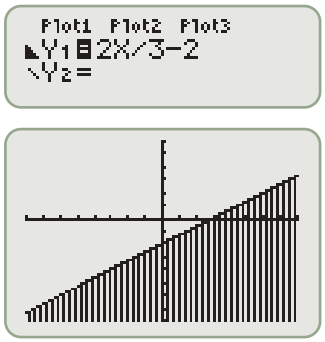
The point (0, 0) does not satisfy the inequality.

STEP 4 Shade the region containing the point (5, 0).

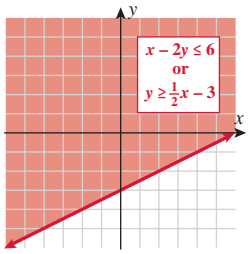


■ **YOUR TURN** Graph the inequality $x - 2y \leq 6$.

Technology Tip



■ **Answer:**



Systems of Linear Inequalities in Two Variables

Systems of linear inequalities are similar to *systems of linear equations.* In systems of linear equations we sought the points that satisfied *all* of the equations. The **solution set of a system of inequalities** contains the points that satisfy *all* of the inequalities. The graph of a system of inequalities can be obtained by simultaneously graphing each individual inequality and finding where the shaded regions intersect (or overlap), if at all.

EXAMPLE 3 Solving a System of Two Linear Inequalities

Graph the system of inequalities:

$$x + y \geq -2$$
$$x + y \leq 2$$

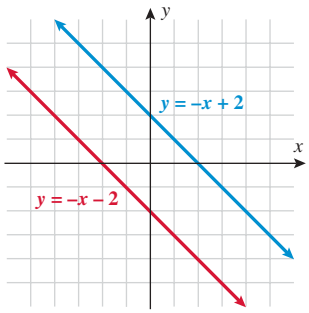
Solution:

STEP 1 Change the inequality signs to equal signs.

$$x + y = -2$$
$$x + y = 2$$

STEP 2 Draw the two lines.

Because the inequality signs are not strict, use solid lines.



STEP 3 Test points for each inequality.

$x + y \geq -2$

Substitute $(-4, 0)$ into $x + y \geq -2$. $\quad -4 \geq -2$

The point $(-4, 0)$ does not satisfy the inequality.

Substitute $(0, 0)$ into $x + y \geq -2$. $\quad 0 \geq -2$

The point $(0, 0)$ does satisfy the inequality.

$x + y \leq 2$

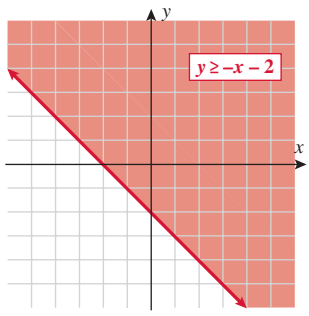
Substitute $(0, 0)$ into $x + y \leq 2$. $\quad 0 \leq 2$

The point $(0, 0)$ does satisfy the inequality.

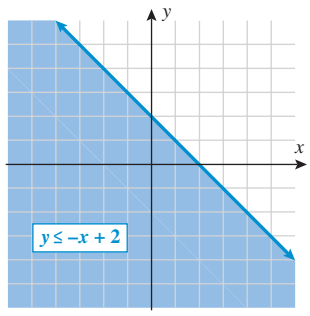
Substitute $(4, 0)$ into $x + y \leq 2$. $\quad 4 \leq 2$

The point $(4, 0)$ does not satisfy the inequality.

STEP 4 For $x + y \geq -2$, shade the region *above* that includes $(0, 0)$.

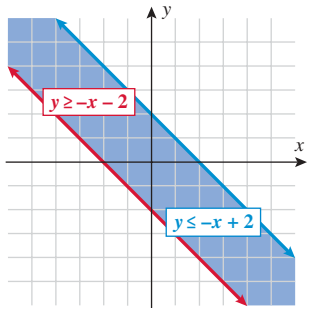


For $x + y \leq 2$, shade the region *below* that includes $(0, 0)$.



STEP 5 All of the points in the overlapping region and on the lines constitute the solution.

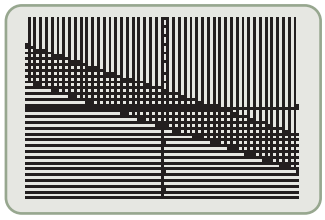
Notice that three sample points $(0, 0)$, $(-1, 1)$, and $(1, -1)$ all lie in the shaded region and all three satisfy both inequalities.



Technology Tip

Solve for y in each inequality first. Enter $y_1 \geq -x - 2$ and $y_2 \leq -x + 2$.

The overlapping region is the solution.



EXAMPLE 4 Solving a System of Two Linear Inequalities with No Solution

Graph the system of inequalities:

$$x + y \le -2$$
$$x + y \ge 2$$

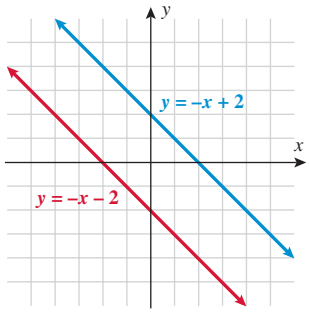
Solution:

STEP 1 Change the inequality signs to equal signs.

$$x + y = -2$$
$$x + y = 2$$

STEP 2 Draw the two lines.

Because the inequality signs are not strict, use solid lines.



STEP 3 Test points for each inequality.

$x + y \le -2$

Substitute $(-4, 0)$ into $x + y \le -2$. $\quad -4 \le -2$

The point $(-4, 0)$ does satisfy the inequality.

Substitute $(0, 0)$ into $x + y \le -2$. $\quad 0 \le -2$

The point $(0, 0)$ does not satisfy the inequality.

$x + y \ge 2$

Substitute $(0, 0)$ into $x + y \ge 2$. $\quad 0 \ge 2$

The point $(0, 0)$ does not satisfy the inequality.

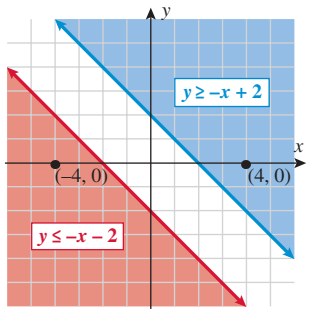
Substitute $(4, 0)$ into $x + y \ge 2$. $\quad 4 \ge 2$

The point $(4, 0)$ does satisfy the inequality.

STEP 4 For $x + y \le -2$, shade the region *below* that includes $(-4, 0)$.

For $x + y \le 2$, shade the region *above* that includes $(4, 0)$.

STEP 5 There is no overlapping region. Therefore, no points satisfy both inequalities. We say there is $\boxed{\text{no solution}}$.



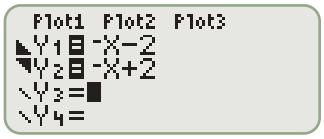
YOUR TURN Graph the solution to the system of inequalities.

a. $y > x + 1$
$y < x - 1$

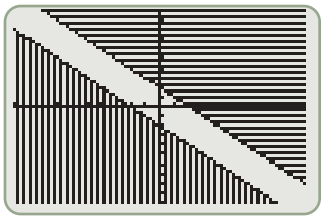
b. $y < x + 1$
$y > x - 1$

Technology Tip

Solve for y in each inequality first. Enter $y_1 \le -x - 2$ and $y_2 \ge -x + 2$.



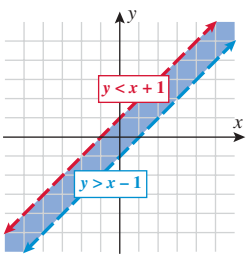
There is no overlapping region. Therefore, there is no solution to the system of inequalities.



Answer:

a. no solution

b.



Thus far we have addressed only systems of two linear inequalities. Systems with more than two inequalities are treated in a similar manner. The solution is the set of all points that satisfy *all* of the inequalities. When there are more than two linear inequalities, the solution may be a **bounded** region. We can algebraically determine where the lines intersect by setting the y-values equal to each other.

EXAMPLE 5 Solving a System of Multiple Linear Inequalities

Solve the system of inequalities:

$$\begin{aligned} y &\le x \\ y &\ge -x \\ y &< 3 \end{aligned}$$

Solution:

STEP 1 Change the inequalities to equal signs.

$$\begin{aligned} y &= x \\ y &= -x \\ y &= 3 \end{aligned}$$

STEP 2 Draw the three lines.

To determine the points of intersection, set the y-values equal.

Point where $y = x$ and $y = -x$ intersect:

$$\begin{aligned} x &= -x \\ x &= 0 \end{aligned}$$

Substitute $x = 0$ into $y = x$. $(0, 0)$

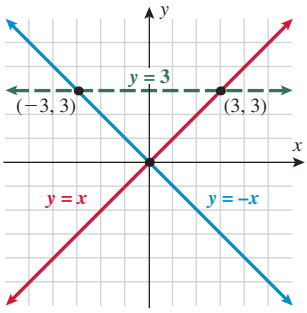
Point where $y = -x$ and $y = 3$ intersect:

$$\begin{aligned} -x &= 3 \\ x &= -3 \end{aligned}$$

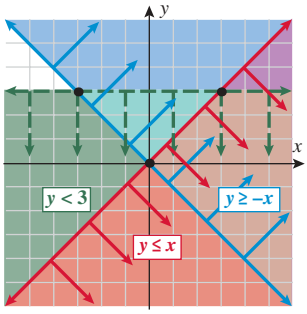
$(-3, 3)$

Point where $y = 3$ and $y = x$ intersect:

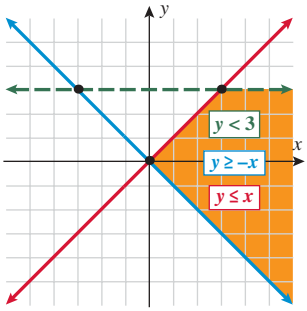
$$\begin{aligned} x &= 3 \\ (3, 3) \end{aligned}$$



STEP 3 Test points to determine the shaded half-planes corresponding to $y \le x$, $y \ge -x$, and $y < 3$.



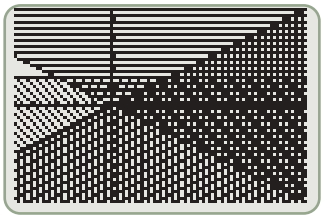
STEP 4 All of the points in the overlapping region (orange) and along the boundaries of the region corresponding to the lines $y = -x$ and $y = x$ constitute the solution.



Technology Tip

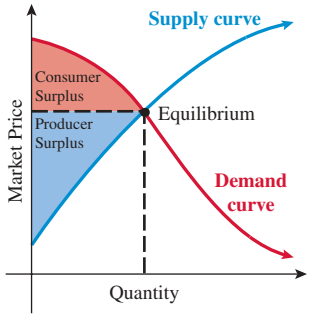
Solve for y in each inequality first. Enter $y_1 \le x$, $y_2 \ge -x$, and $y_3 < 3$.

The overlapping region is the solution to the system of inequalities.



Applications

In economics, the point where the supply and demand curves intersect is called the **equilibrium point**. **Consumer surplus** is a measure of the amount that consumers benefit by being able to purchase a product for a price less than the maximum they would be willing to pay. **Producer surplus** is a measure of the amount that producers benefit by selling at a market price that is higher than the least they would be willing to sell for.



EXAMPLE 6 Consumer Surplus and Producer Surplus

The Tesla Motors Roadster is the first electric car that is able to travel 245 miles on a single charge. The price of a 2013 model is approximately $90,000 (including tax and incentives).

Courtesy Tesla Motors

Suppose the supply and demand equations for this electric car are given by

$$P = 90{,}000 - 0.1x \qquad \text{(Demand)}$$
$$P = 10{,}000 + 0.3x \qquad \text{(Supply)}$$

where P is the price in dollars and x is the number of cars produced. Calculate the consumer surplus and the producer surplus for these two equations.

Solution:

Find the equilibrium point.

$$90{,}000 - 0.1x = 10{,}000 + 0.3x$$
$$0.4x = 80{,}000$$
$$x = 200{,}000$$

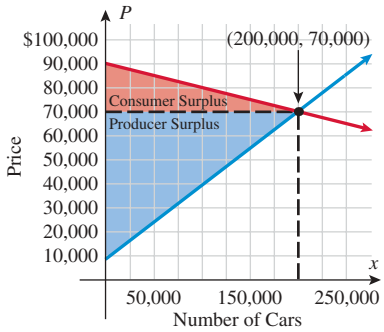
Let $x = 200{,}000$ in either the supply or demand equation.

$$P = 90{,}000 - 0.1(200{,}000) = 70{,}000$$
$$P = 10{,}000 + 0.3(200{,}000) = 70{,}000$$

According to these models, if the price of a Tesla Motors Roadster is $70,000, then 200,000 cars will be sold and there will be no surplus.

Write the systems of linear inequalities that correspond to consumer surplus and producer surplus.

Consumer Surplus	Producer Surplus
$P \le 90{,}000 - 0.1x$	$P \ge 10{,}000 + 0.3x$
$P \ge 70{,}000$	$P \le 70{,}000$
$x \ge 0$	$x \ge 0$



The consumer surplus is the area of the red triangle.

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(200{,}000)(20{,}000)$$
$$= 2{,}000{,}000{,}000$$

The consumer surplus is \$2B.

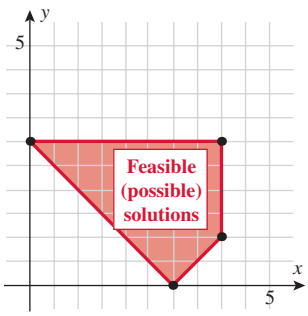
The producer surplus is the area of the blue triangle.

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(200{,}000)(60{,}000)$$
$$= 6{,}000{,}000{,}000$$

The producer surplus is \$6B.

The graph of the systems of linear inequalities in Example 6 are said to be **bounded**, whereas the graphs of the systems of linear inequalities in Examples 3–5 are said to be **unbounded**. Any points that correspond to boundary lines intersecting are called **corner points** or **vertices**. In Example 6, the vertices corresponding to the consumer surplus are the points (0, 90,000), (0, 70,000), and (200,000, 70,000), and the vertices corresponding to the producer surplus are the points (0, 70,000), (0, 10,000), and (200,000, 70,000).

The Linear Programming Model



Often we seek to maximize or minimize a function subject to constraints. This process is called **optimization**. When the function we seek to minimize or maximize is linear and the constraints are given in terms of linear inequalities, a graphing approach to such problems is called **linear programming**. In linear programming, we start with a linear equation, called the **objective function**, that represents the quantity that is to be maximized or minimized.

The goal is to minimize or maximize the objective function $z = Ax + By$ subject to *constraints*. In other words, find the points (x, y) that make the value of z the largest (or smallest). The **constraints** are a system of linear inequalities, and the common shaded region represents the **feasible** (possible) **solutions**.

If the constraints form a bounded region, the maximum or minimum value of the objective function will occur using the coordinates of one of the vertices. If the region is not bounded, then if an optimal solution exists, it will occur at a vertex. A procedure for solving linear programming problems is outlined below:

Solving an Optimization Problem Using Linear Programming

Step 1: Write the objective function. This expression represents the quantity that is to be minimized or maximized.

Step 2: Write the constraints. This is a system of linear inequalities.

Step 3: Graph the constraints. Graph the system of linear inequalities and shade the common region, which contains the feasible solutions.

Step 4: Identify the vertices. The corner points (vertices) of the shaded region represent possible solutions for maximizing or minimizing the objective function.

Step 5: Evaluate the objective function for each vertex. For each corner point of the shaded region, substitute the coordinates into the objective function and list the value of the objective function.

Step 6: Identify the optimal solution. The largest (maximum) or smallest (minimum) value of the objective function in Step 5 is the optimal solution.

EXAMPLE 7 Maximizing an Objective Function

Find the maximum value of $z = 2x + y$ subject to the constraints:

$$x \geq 1 \qquad x \leq 4 \qquad x + y \leq 5 \qquad y \geq 0$$

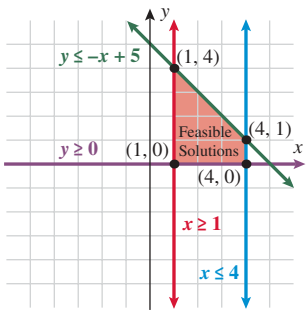
Solution:

STEP 1 Write the objective function. $z = 2x + y$

STEP 2 Write the constraints.

$x \geq 1$
$x \leq 4$
$y \leq -x + 5$
$y \geq 0$

STEP 3 Graph the constraints.



STEP 4 Identify the vertices. (1, 4), (4, 1), (1, 0), (4, 0)

STEP 5 Evaluate the objective function for each vertex.

VERTEX	x	y	OBJECTIVE FUNCTION: $z = 2x + y$
(1, 4)	1	4	$2(1) + 4 = \mathbf{6}$
(4, 1)	4	1	$2(4) + 1 = \mathbf{9}$
(1, 0)	1	0	$2(1) + 0 = \mathbf{2}$
(4, 0)	4	0	$2(4) + 0 = \mathbf{8}$

STEP 6 The maximum value of z is **9**, subject to the given constraints when $x = 4$ and $y = 1$.

YOUR TURN Find the maximum value of $z = x + 3y$ subject to the constraints:

$$x \geq 1 \qquad x \leq 3 \qquad y \leq -x + 3 \qquad y \geq 0$$

Study Tip

The bounded region is the region that satisfies *all* of the constraints. Only vertices of the bounded region correspond to possible solutions. Even though $y = -x + 5$ and $y = 0$ intersect at $x = 5$, that point of intersection is outside the shaded region and therefore is *not* one of the vertices.

Answer: The maximum value of z is **7**, which occurs when $x = 1$ and $y = 2$.

EXAMPLE 8 Minimizing an Objective Function

Find the minimum value of $z = 4x + 5y$ subject to the constraints:

$$x \geq 0 \qquad 2x + y \leq 6 \qquad x + y \leq 5 \qquad y \geq 0$$

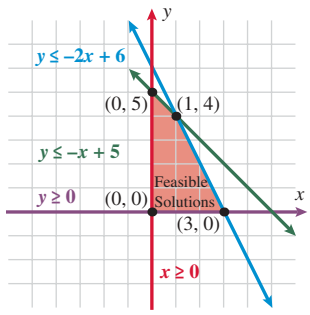
Solution:

STEP 1 Write the objective function. $z = 4x + 5y$

STEP 2 Write the constraints.

$x \geq 0$
$y \leq -2x + 6$
$y \leq -x + 5$
$y \geq 0$

STEP 3 Graph the constraints.



STEP 4 Identify the vertices. (0, 0), (0, 5), (1, 4), (3, 0)

STEP 5 Evaluate the objective function for each vertex.

VERTEX	x	y	OBJECTIVE FUNCTION: $z = 4x + 5y$
(0, 0)	0	0	$4(0) + 5(0) = \mathbf{0}$
(0, 5)	0	5	$4(0) + 5(5) = \mathbf{25}$
(1, 4)	1	4	$4(1) + 5(4) = \mathbf{24}$
(3, 0)	3	0	$4(3) + 5(0) = \mathbf{12}$

STEP 6 The minimum value of z is **0**, which occurs when $x = 0$ and $y = 0$.

YOUR TURN Find the minimum value of $z = 2x + 3y$ subject to the constraints:

$$x \geq 1 \qquad 2x + y \leq 8 \qquad x + y \geq 4$$

Answer: The minimum value of z is **8**, which occurs when $x = 4$ and $y = 0$.

Study Tip

Maxima or minima of objective functions only occur at the vertices of the shaded region corresponding to the constraints.

EXAMPLE 9 Solving an Optimization Problem Using Linear Programming: Unbounded Region

Find the maximum value and minimum value of $z = 7x + 3y$ subject to the constraints:

$$y \geq 0 \qquad -2x + y \leq 0 \qquad -x + y \geq -4$$

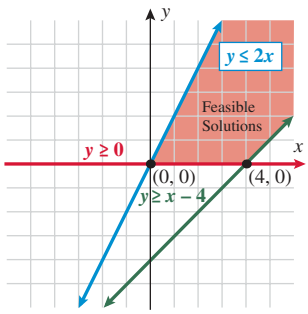
Solution:

STEP 1 Write the objective function. $z = 7x + 3y$

STEP 2 Write the constraints.

$y \geq 0$
$y \leq 2x$
$y \geq x - 4$

STEP 3 Graph the constraints.



STEP 4 Identify the vertices. (0, 0), (4, 0)

STEP 5 Evaluate the objective function for each vertex.

VERTEX	x	y	OBJECTIVE FUNCTION: $z = 7x + 3y$
(0, 0)	0	0	$7(0) + 3(0) = \mathbf{0}$
(4, 0)	4	0	$7(4) + 3(0) = \mathbf{28}$

STEP 6 The minimum value of z is **0**, which occurs when $x = 0$ and $y = 0$.

There is no maximum value, because if we select a point in the shaded region, say, (3, 3), the objective function at (3, 3) is equal to 30, which is greater than 28.

When the feasible solutions are contained in a bounded region, then a maximum and a minimum exist and are each located at one of the vertices. If the feasible solutions are contained in an unbounded region, then if a maximum or minimum exists, it is located at one of the vertices.

SECTION 8.7 SUMMARY

Graphing a Linear Inequality

1. Change the inequality sign to an equal sign.
2. Draw the line $y = mx + b$. (Dashed for strict inequalities and solid for nonstrict inequalities.)
3. Test a point. (Select a point in one-half plane and test the inequality. Repeat this step for the other half-plane.)
4. Shade the half-plane of the overlapping region that satisfies the linear inequality.

Graphing a System of Linear Inequalities

- Draw the individual linear inequalities.
- The overlapped shaded region, if it exists, is the solution.

Linear Programming Model

1. Write the objective function.
2. Write the constraints.
3. Graph the constraints.
4. Identify the vertices.
5. Evaluate the objective function for each vertex.
6. Identify the optimal solution.

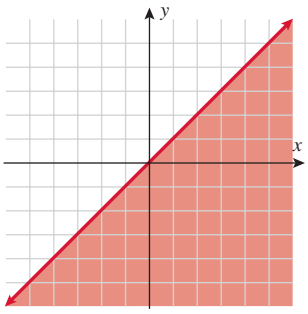
SECTION 8.7 EXERCISES

SKILLS

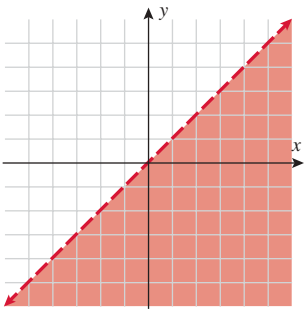
In Exercises 1–4, match the linear inequality with the correct graph.

1. $y > x$ **2.** $y \geq x$ **3.** $y < x$ **4.** $y \leq x$

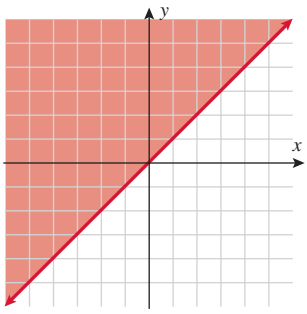
a.



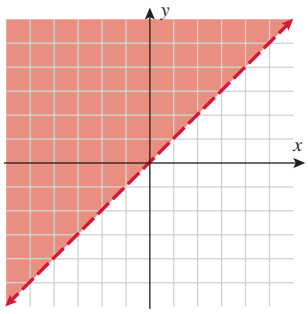
b.



c.



d.



In Exercises 5–20, graph each linear inequality.

5. $y > x - 1$ **6.** $y \geq -x + 1$ **7.** $y \leq -x$ **8.** $y > -x$

9. $y \leq -3x + 2$ **10.** $y < 2x + 3$ **11.** $y \leq -2x + 1$ **12.** $y > 3x - 2$

13. $3x + 4y < 2$ **14.** $2x + 3y > -6$ **15.** $5x + 3y < 15$ **16.** $4x - 5y \leq 20$

17. $4x - 2y \geq 6$ **18.** $6x - 3y \geq 9$ **19.** $6x + 4y \leq 12$ **20.** $5x - 2y \geq 10$

In Exercises 21–50, graph each system of inequalities or indicate that the system has no solution.

21. $y \geq x - 1$, $y \leq x + 1$ **22.** $y > x + 1$, $y < x - 1$ **23.** $y > 2x + 1$, $y < 2x - 1$ **24.** $y \leq 2x - 1$, $y \geq 2x + 1$ **25.** $y \geq 2x$, $y \leq 2x$

26. $y > 2x$, $y < 2x$ **27.** $x > -2$, $x < 4$ **28.** $y < 3$, $y > 0$ **29.** $x \geq 2$, $y \leq x$ **30.** $y \leq 3$, $y \geq x$

31. $y > x$, $x < 0$, $y < 4$

32. $y \le x$, $x \ge 0$, $y \le 1$

33. $x + y > 2$, $y < 1$, $x > 0$

34. $x + y < 4$, $x > 0$, $y \ge 1$

35. $-x + y > 1$, $y < 3$, $x > 0$

36. $x - y > 2$, $y < 4$, $x \ge 0$

37. $x + 3y > 6$, $y < 1$, $x \ge 1$

38. $x + 2y > 4$, $y < 1$, $x \ge 0$

39. $y \ge x - 1$, $y \le -x + 3$, $y < x + 2$

40. $y < 4 - x$, $y > x - 4$, $y > -x - 4$

41. $x + y > -4$, $-x + y < 2$, $y \ge -1$, $y \le 1$

42. $y < x + 2$, $y > x - 2$, $y < -x + 2$, $y > -x - 2$

43. $y < x + 3$, $x + y \ge 1$, $y \ge 1$, $y \le 3$

44. $y \le -x + 2$, $y - x \ge -3$, $y \ge -2$, $y \le 1$

45. $y + x < 2$, $y + x \ge 4$, $y \ge -2$, $y \le 1$

46. $y - x < 3$, $y + x > 3$, $y \le -2$, $y \ge -4$

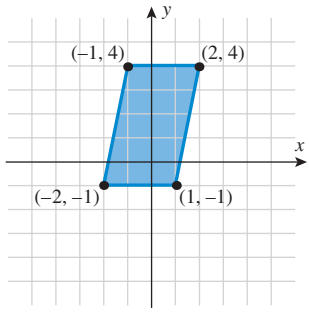
47. $2x - y < 2$, $2x + y > 2$, $y < 2$

48. $3x - y > 3$, $3x + y < 3$, $y < -2$

49. $x + 4y > 5$, $x - 4y < 5$, $x > 6$

50. $2x - 3y < 6$, $2x + 3y > 6$, $x < 4$

In Exercises 51–54, find the value of the objective function at each of the vertices. What is the maximum value of the objective function? What is the minimum value of the objective function?



51. Objective function: $z = 2x + 3y$

52. Objective function: $z = 3x + 2y$

53. Objective function: $z = 1.5x + 4.5y$

54. Objective function: $z = \frac{2}{3}x + \frac{3}{5}y$

In Exercises 55–62, minimize or maximize each objective function subject to the constraints.

55. Minimize $z = 7x + 4y$ subject to

$x \ge 0 \quad y \ge 0 \quad -x + y \le 4$

56. Maximize $z = 3x + 5y$ subject to

$x \ge 0 \quad y \ge 0 \quad -x + y \ge 4$

57. Maximize $z = 4x + 3y$ subject to

$x \ge 0 \quad y \le -x + 4 \quad y \ge -x$

58. Minimize $z = 4x + 3y$ subject to

$x \ge 0 \quad y \ge 0$
$x + y \le 10 \quad x + y \ge 0$

59. Minimize $z = 2.5x + 3.1y$ subject to

$x \ge 0 \quad y \ge 0 \quad x \le 4$
$-x + y \le 2 \quad x + y \le 6$

60. Maximize $z = 2.5x - 3.1y$ subject to

$x \ge 1 \quad y \le 7 \quad x \le 3$
$-x + y \ge 2 \quad x + y \ge 6$

61. Maximize $z = \frac{1}{4}x + \frac{2}{5}y$ subject to

$x + y \ge 5 \quad x + y \le 7$
$-x + y \le 5 \quad -x + y \ge 3$

62. Minimize $z = \frac{1}{3}x - \frac{2}{5}y$ subject to

$x + y \ge 6 \quad x + y \le 8$
$-x + y \le 6 \quad -x + y \ge 4$

APPLICATIONS

For Exercises 63–66, employ the following supply and demand equations:

Demand: $P = 80 - 0.01x$
Supply: $P = 20 + 0.02x$

where P is the price in dollars when x units are produced.

63. Consumer Surplus. Write a system of linear inequalities corresponding to the consumer surplus.

64. Producer Surplus. Write a system of linear inequalities corresponding to the producer surplus.

65. Consumer Surplus. Calculate the consumer surplus given the supply and demand equations.

66. Producer Surplus. Calculate the producer surplus given the supply and demand equations.

67. Hurricanes. After back-to-back-to-back-to-back hurricanes (Charley, Frances, Ivan, and Jeanne) in Florida in the summer of 2004, FEMA sent disaster relief trucks to Florida. Floridians mainly needed drinking water and generators. Each truck could carry no more than 6000 pounds of cargo or 2400 cubic feet of cargo. Each case of bottled water takes up 1 cubic foot of space and weighs 25 pounds. Each generator takes up 20 cubic feet and weighs 150 pounds. Let x represent the number of cases of water and y represent the number of generators, and write a system of linear inequalities that describes the number of generators and cases of water each truck can haul to Florida.

68. Hurricanes. Repeat Exercise 67 with a smaller truck and different supplies. Suppose the smaller trucks that can haul 2000 pounds and 1500 cubic feet of cargo are used to haul plywood and tarps. A case of plywood is 60 cubic feet and weighs 500 pounds. A case of tarps is 10 cubic feet and weighs 50 pounds. Letting x represent the number of cases of plywood and y represent the number of cases of tarps, write a system of linear inequalities that describes the number of cases of tarps and plywood each truck can haul to Florida. Graph the system of linear inequalities.

69. Hurricanes. After the 2004 hurricanes in Florida, a student at Valencia Community College decided to create two T-shirts to sell. One T-shirt said, "I survived Charley on Friday the Thirteenth," and the second said, "I survived Charley, Frances, Ivan, and Jeanne." The Charley T-shirt costs him \$7 to make and he sold it for \$13. The other T-shirt cost him \$5 to make and he sold it for \$10. He did not want to invest more than \$1000. He estimated that the total demand would not exceed 180 T-shirts. Find the number of each type of T-shirt he needed to make to yield maximum profit.

70. Hurricanes. After Hurricane Charley devastated central Florida unexpectedly, Orlando residents prepared for Hurricane Frances by boarding up windows and filling up their cars with gas. It took 5 hours of standing in line to get plywood, and lines for gas were just as time-consuming. A student at Seminole Community College decided to do a spoof of the "Got Milk" ads and created two T-shirts: "Got Plywood" showing a line of people in a home improvement store, and "Got Gas" showing a street lined with cars waiting to pump gasoline. The "Got Plywood" shirts cost \$8 to make, and she sold them for \$13. The "Got Gas" shirts cost \$6 to make, and she sold them for \$10. She decided to limit her costs to \$1400. She estimated that demand for these T-shirts would not exceed 200 T-shirts. Find the number of each type of T-shirt she should have made to yield maximum profit.

71. Health. A diet must be designed to provide at least 275 units of calcium, 125 units of iron, and 200 units of Vitamin B. Each ounce of food A contains 10 units of calcium, 15 units of iron, and 20 units of vitamin B. Each ounce of food B contains 20 units of calcium, 10 units of iron, and 15 units of vitamin B.

a. Find a system of inequalities to describe the different quantities of food that may be used (let x = the number of ounces of food A and y = the number of ounces of food B).

b. Graph the system of inequalities.

c. Using the graph found in part (b), find two possible solutions (there are infinitely many).

72. Health. A diet must be designed to provide at least 350 units of calcium, 175 units of iron, and 225 units of Vitamin B. Each ounce of food A contains 15 units of calcium, 25 units of iron, and 20 units of vitamin B. Each ounce of food B contains 25 units of calcium, 10 units of iron, and 10 units of vitamin B.

a. Find a system of inequalities to describe the different quantities of food that may be used (let x = the number of ounces of food A and y = the number of ounces of food B).

b. Graph the system of inequalities.

c. Using the graph found in part (b), find two possible solutions (there are infinitely many).

73. Business. A manufacturer produces two types of computer mouse: USB wireless mouse and a Bluetooth mouse. Past sales indicate that it is necessary to produce at least twice as many USB wireless mice than Bluetooth mice. To meet demand, the manufacturer must produce at least 1000 computer mice per hour.

a. Find a system of inequalities describing the production levels of computer mice. Let x be the production level for USB wireless mouse and y be the production level for Bluetooth mouse.

b. Graph the system of inequalities describing the production levels of computer mice.

c. Use your graph in part (b) to find two possible solutions.

74. Business. A manufacturer produces two types of mechanical pencil lead: 0.5 millimeter and 0.7 millimeter. Past sales indicate that it is necessary to produce at least 50% more 0.5 millimeter lead than 0.7 millimeter lead. To meet demand, the manufacturer must produce at least 10,000 pieces of pencil lead per hour.

a. Find a system of inequalities describing the production levels of pencil lead. Let x be the production level for 0.5 millimeter pencil lead and y be the production level for 0.7 millimeter pencil lead.

b. Graph the system of inequalities describing the production levels of pencil lead.

c. Use your graph in part (b) to find two possible solutions.

75. Computer Business. A computer science major and a business major decide to start a small business that builds and sells desktop computers and laptop computers. They buy the parts, assemble them, load the operating system, and sell the computers to other students. The costs for parts, time to assemble each computer, and profit are summarized in the following table:

	DESKTOP	LAPTOP
Cost of parts	\$700	\$400
Time to assemble (hours)	5	3
Profit	\$500	\$300

They were able to get a small business loan in the amount of \$10,000 to cover costs. They plan on making these computers over the summer and selling them the first day of class. They can dedicate at most only 90 hours to assembling these computers. They estimate that the demand for laptops will be at least three times as great as the demand for desktops. How many of each type of computer should they make to maximize profit?

76. Computer Business. Repeat Exercise 75 if the two students are able to get a loan for \$30,000 to cover costs and they can dedicate at most 120 hours to assembling the computers.

77. Passenger Ratio. The Eurostar is a high-speed train that travels between London, Brussels, and Paris. There are 30 cars on each departure. Each train car is designated first-class or second-class. Based on demand for each type of fare, there should always be at least two but no more than four first-class train cars. The management wants to claim that the ratio of first-class to second-class cars never exceeds 1:8. If the profit on each first-class train car is twice as much as the profit on each second-class train car, find the number of each class of train car that will generate a maximum profit.

78. Passenger Ratio. Repeat Exercise 77. This time, assume that there has to be at least one first-class train car and that the profit from each first-class train car is 1.2 times as much as the profit from each second-class train car. The ratio of first-class to second-class cannot exceed 1:10.

79. Production. A manufacturer of skis produces two models: a regular ski and a slalom ski. A set of regular skis produces a \$25 profit and a set of slalom skis produces a profit of \$50. The manufacturer expects a customer demand of at least 200 pairs of regular skis and at least 80 pair of slalom skis. The maximum number of pairs of skis that can be produced by this company is 400. How many of each model of skis should be produced to maximize profits?

80. Donut Inventory. A well-known donut store makes two popular types of donuts: crème-filled and jelly-filled. The manager knows from past statistics that the number of dozens of donuts sold is at least 10, but no more than 30. To prepare the donuts for frying, the baker needs (on the average) 3 minutes for a dozen crème-filled and 2 minutes for jelly-filled. The baker has at most two hours available per day to prepare the donuts. How many dozens of each type should be prepared to maximize the daily profit if there is a \$1.20 profit for each dozen crème-filled and \$1.80 profit for each dozen jelly-filled donuts?

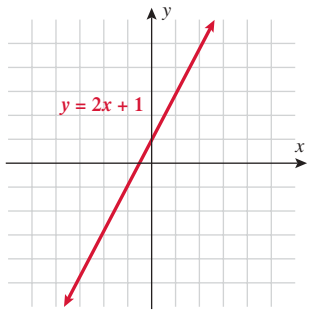
CATCH THE MISTAKE

In Exercises 81 and 82, explain the mistake that is made.

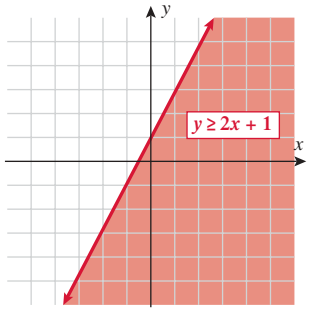
81. Graph the inequality $y \geq 2x + 1$.

Solution:

Graph the line $y = 2x + 1$ with a solid line.



Since the inequality is $\geq$, shade to the *right*.

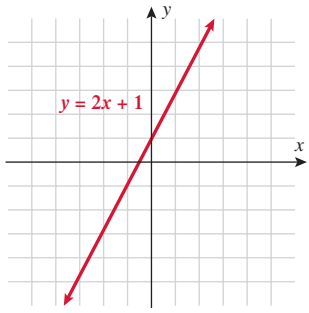


This is incorrect. What mistake was made?

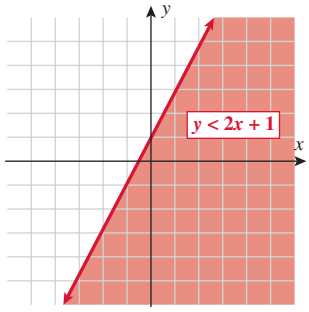
82. Graph the inequality $y < 2x + 1$.

Solution:

Graph the line $y = 2x + 1$ with a solid line.



Since the inequality is $<$, shade *below*.



This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 83–88, determine whether each statement is true or false.

83. A linear inequality always has a solution that is a half-plane.

84. A dashed curve is used for strict inequalities.

85. A solid curve is used for strict inequalities.

86. A system of linear inequalities always has a solution.

87. An objective function always has a maximum or minimum.

88. An objective function subject to constraints that correspond to a bounded region always has a maximum and a minimum.

CHALLENGE

In Exercises 89 and 90, for the system of linear inequalities, assume *a*, *b*, *c*, and *d* are real numbers.

$$\begin{aligned} x &\geq a \\ x &< b \\ y &> c \\ y &\leq d \end{aligned}$$

89. Describe the solution when $a < b$ and $c < d$.

90. What will the solution be if $a > b$ and $c > d$?

For Exercises 91 and 92, use the following system of linear inequalities:

$$\begin{aligned} y &\leq ax + b \\ y &\geq -ax + b \end{aligned}$$

91. If a and b are positive real numbers, graph the solution.

92. If a and b are negative real numbers, graph the solution.

93. Maximize the objective function $z = 2x + y$ subject to the conditions, where $a > 2$.

$$\begin{aligned} ax + y &\geq -a \\ -ax + y &\leq a \\ ax + y &\leq a \\ -ax + y &\geq -a \end{aligned}$$

94. Maximize the objective function $z = x + 2y$ subject to the conditions, where $a > b > 0$.

$$\begin{aligned} x + y &\geq a \\ -x + y &\leq a \\ x + y &\leq a + b \\ -x + y &\geq a - b \end{aligned}$$

TECHNOLOGY

In Exercises 95 and 96, apply a graphing utility to graph the following inequalities.

95. $4x - 2y \geq 6$ (Check with your answer to Exercise 17.)

96. $6x - 3y \geq 9$ (Check with your answer to Exercise 18.)

In Exercises 97 and 98, use a graphing utility to graph each system of inequalities or indicate that the system has no solution.

97. $$\begin{aligned} -0.05x + 0.02y &\geq 0.12 \\ 0.01x + 0.08y &\leq 0.08 \end{aligned}$$

98. $$\begin{aligned} y &\leq 2x + 3 \\ y &> -0.5x + 5 \end{aligned}$$

PREVIEW TO CALCULUS

In calculus, the first steps when solving the problem of finding the area enclosed by a set of curves are similar to those for finding the feasible region in a linear programming problem.

In Exercises 99–102, graph the system of inequalities and identify the vertices, that is, the points of intersection of the given curves.

99. $$\begin{aligned} y &\leq x + 2 \\ y &\geq x^2 \end{aligned}$$

100. $$\begin{aligned} x &\leq 25 \\ x &\geq y^2 \end{aligned}$$

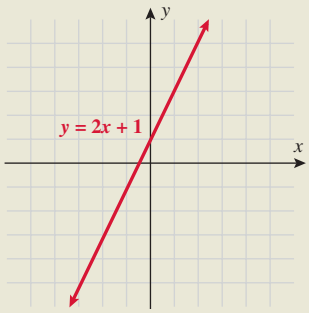
101. $$\begin{aligned} y &\leq x^2 \\ y &\geq x^3 \\ x &\geq 0 \end{aligned}$$

102. $$\begin{aligned} y &\geq x^3 \\ y &\leq -x \\ y &\geq x + 6 \end{aligned}$$

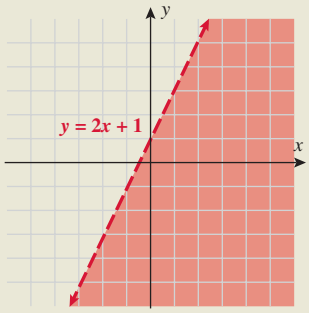
CHAPTER 8 INQUIRY-BASED LEARNING PROJECT

In Section 8.1, you learned how to solve systems of two linear equations in two variables. That is, you determined the set of all points that satisfy both given equations in a system. In Chapter 0 you solved linear inequalities in one variable. Next, you will put these ideas together as you consider systems of linear inequalities in two variables, and their solutions.

1. The following graph shows the line $y = 2x + 1$. Notice that the line divides the Cartesian plane into two half-planes—one below and one above the line.



The set of all the points in the half-plane below the line are in the shaded region.



a. The set of points *on* the line, together with the points in the shaded region, make up the solution set of the inequality: $y \leq 2x + 1$. To get an idea of what this means, choose several points—a few on the line, a few in the shaded region, and a few in the unshaded region—and evaluate the inequality for each. What do you notice?

b. Write an inequality that has as its solution set the points on the line $y = 2x + 1$ together with all the points above the line.

c. Suppose you wanted to graph the solutions to the strict inequality $y < 2x + 1$. How do you think you could alter the graph shown above to do this? Explain.

2. To graph a linear inequality, first graph the associated line (either solid or dashed), then choose the appropriate half-plane to shade. Remember, ALL of the points in one of the half-planes are in the solution set of the given inequality, and NONE of the points in the other half-plane are in the solution set. To determine which half-plane to shade, one need only check one test point.

a. Graph the linear inequality $3x + 6y < 18$.

b. Graph the linear inequality $3x - 6y \leq 18$.

c. Now consider the system of two linear inequalities:

$$3x + 6y > 18$$
$$3x - 6y \leq 18$$

Graph the two associated lines together at the right.

Shade the region(s) that contain(s) the points in the solution set of the system of inequalities. Use some test points, if needed.

d. How is the region you shaded in part (c) related to the regions you shaded in parts (a) and (b)?

MODELING OUR WORLD

In 2005 hybrid vehicles were introduced in the U.S. market. The demand for hybrids, which are typically powered by a combination of gasoline and electric batteries, was based on popular recognition of petroleum as an increasingly scarce nonrenewable resource, as well as consumers' need to combat rising prices at the gas pumps. In addition to achieving greater fuel economy than conventional internal combustion engine vehicles (ICEVs), their use also results in reduced emissions.

An online "Gas Mileage Impact Calculator," created by the American Council for an Energy-Efficient Economy (www.aceee.org), was used to generate the following tables comparing a conventional sedan (four-door) and an SUV versus their respective hybrid counterparts.

Gas Mileage Impact Calculator

	Toyota Camry 2.4L 4, auto $3.75/gallon 15,000 mi/year	Toyota Camry Hybrid 2.4L 4, auto $3.75/gallon 15,000 mi/year
Gas consumption	611 gallons	449 gallons
Gas cost	$2289.75	$1681.99
Fuel economy	25 mpg	33 mpg
Emissions		
Carbon dioxide (greenhouse gas)	11,601 pounds	8522 pounds
Carbon monoxide (poisonous gas)	235 pounds	169 pounds
Nitrogen oxides (lung irritant and smog)	10 pounds	7 pounds
Particulate matter (soot)	255 grams	255 grams
Hydrocarbons (smog)	6 pounds	8 pounds

	Toyota Highlander 3.5L 6, auto stk $3.75/gallon 15,000 mi/year	Toyota Highlander Hybrid 3.3L 6, auto Awd $3.75/gallon 15,000 mi/year
Gas consumption	740 gallons	576 gallons
Gas cost	$2773.44	$2158.33
Fuel economy	20 mpg	26 mpg
Emissions		
Carbon dioxide (greenhouse gas)	14,052 pounds	10,936 pounds
Carbon monoxide (poisonous gas)	229 pounds	187 pounds
Nitrogen oxides (lung irritant and smog)	11 pounds	8 pounds
Particulate matter (soot)	320 grams	399 grams
Hydrocarbons (smog)	7 pounds	16 pounds

MODELING OUR WORLD (*continued*)

The MSRP and mileage comparisons for the 2008 models are given below:

	Camry	Camry Hybrid	Highlander	Highlander Hybrid
MSRP	$19,435	$26,065	$28,035	$34,435
Miles per gallon in city	21	33	18	27
Miles per gallon on highway	31	34	24	25

For the following questions, assume that you drive 15,000 miles per year (all in the city) and the price of gasoline is $3.75 per gallon.

1. Write a linear equation that models the total cost of owning and operating each vehicle y as a function of the number of years of ownership x.

a. Camry **b.** Camry Hybrid

c. Highlander **d.** Highlander Hybrid

2. Write a linear equation that models the total number of pounds of carbon dioxide each vehicle emits y as a function of the number of years of ownership x.

a. Camry **b.** Camry Hybrid

c. Highlander **d.** Highlander Hybrid

3. How many years would you have to own and drive the vehicle for the hybrid to be the better deal?

a. Camry Hybrid versus Camry **b.** Highlander Hybrid versus Highlander

4. How many years would you have to own and drive the vehicle for the hybrid to emit 50% less carbon dioxide than its conventional counterpart?

a. Camry Hybrid versus Camry **b.** Highlander Hybrid versus Highlander

Section	Concept	Key Ideas/Formulas
8.1	**Systems of linear equations in two variables**	$A_1x + B_1y = C_1$ $A_2x + B_2y = C_2$
	Solving systems of linear equations in two variables	**Substitution method** Solve for one variable in terms of the other and substitute that expression into the other equation. **Elimination method** Eliminate a variable by adding multiples of the equations. **Graphing method** Graph the two lines. The solution is the point of intersection. Parallel lines have no solution and identical lines have infinitely many solutions.
	Three methods and three types of solutions	One solution, no solution, infinitely many solutions
8.2	**Systems of linear equations in three variables**	Planes in a three-dimensional coordinate system
	Solving systems of linear equations in three variables	Step 1: Reduce the system to two equations and two unknowns. Step 2: Solve the resulting system from Step 1. Step 3: Substitute solutions found in Step 2 into any of the equations to find the third variable. Step 4: Check.
	Types of solutions	One solution (point), no solution, or infinitely many solutions (line or the same plane)
8.3	**Systems of linear equations and matrices**	
	Matrices	$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
	Augmented matrices	$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \Rightarrow \left[\begin{array}{ccc\|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$
	Row operations on a matrix	1. $R_i \leftrightarrow R_j$ — Interchange row i with row j. 2. $cR_i \rightarrow R_i$ — Multiply row i by the constant c. 3. $cR_i + R_j \rightarrow R_j$ — Multiply row i by the constant c and add to row j, writing the results in row j.
	Row–echelon form of a matrix	A matrix is in **row–echelon form** if it has all three of the following properties: 1. Any rows consisting entirely of 0s are at the bottom of the matrix. 2. For each row that does not consist entirely of 0s, the first (leftmost) nonzero entry is 1 (called the leading 1). 3. For two successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row. If a matrix in row–echelon form has the following additional property, then the matrix is in **reduced row–echelon form**: 4. Every column containing a leading 1 has zeros in every position above and below the leading 1.

Section	Concept	Key Ideas/Formulas
	Gaussian elimination with back-substitution	Step 1: Write the system of equations as an augmented matrix. Step 2: Apply row operations to transform the matrix into row–echelon form. Step 3: Apply back-substitution to identify the solution.
	Gauss–Jordan elimination	Step 1: Write the system of equations as an augmented matrix. Step 2: Apply row operations to transform the matrix into *reduced* row–echelon form. Step 3: Identify the solution.
	Inconsistent and dependent systems	No solution or infinitely many solutions
8.4	**Matrix algebra**	$\begin{array}{c} \\ \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m \end{array} \begin{array}{c} \begin{array}{cccccc} \text{Column } 1 & \text{Column } 2 & \cdots & \text{Column } j & \cdots & \text{Column } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$
	Equality of matrices	The orders must be the same: $A_{m\times n}$ and $B_{m\times n}$ and corresponding entries are equal.
	Matrix addition and subtraction	The orders must be the same: $A_{m\times n}$ and $B_{m\times n}$. Perform operation entry by entry.
	Scalar and matrix multiplication	The scalar is multiplied by *all* of the entries. For matrix multiplication ■ The orders must satisfy the relationship: $A_{m\times n}$ and $B_{n\times p}$, resulting in $(AB)_{m\times p}$. ■ Perform multiplication row by column. ■ Matrix multiplication is not commutative: $AB \neq BA$.
	Matrix equations	Linear system: $AX = B$. *Front* multiply both sides of the equation by A^{-1} (provided A^{-1} exists), which results in $X = A^{-1}B$.
	Finding the inverse of a square matrix	Only square matrices, $n \times n$, have inverses, $A^{-1}A = I_n$. Step 1: Form the matrix $[A \mid I_n]$. Step 2: Use row operations to transform this matrix to $[I_n \mid A^{-1}]$.
	Solving systems of linear equations using matrix algebra and inverses of square matrices	$AX = B$ Step 1: Find A^{-1}. Step 2: $X = A^{-1}B$.

Section	Concept	Key Ideas/Formulas
8.5	**The determinant of a square matrix and Cramer's rule**	Cramer's rule can only be used to solve a system of linear equations with a unique solution.
	Determinant of a 2×2 matrix	$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
	Determinant of an $n \times n$ matrix	Let A be a square matrix of order $n \times n$. Then: ■ The **minor** M_{ij} of the element a_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained when the ith row and jth column of A are deleted. ■ The **cofactor** C_{ij} of the element a_{ij} is given by $C_{ij} = (-1)^{i+j}M_{ij}$. $\begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 0 \\ 5 & -2 & 3 \end{bmatrix}$ $M_{11} = \begin{vmatrix} -1 & 0 \\ -2 & 3 \end{vmatrix} = -3 - 0 = -3$ $C_{11} = (-1)^{1+1}M_{11} = (1)(-3) = -3$ Sign pattern of cofactors for the determinant of a 3×3 matrix: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ If A is a 3×3 matrix, the determinant can be given by $\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$. This is called **expanding the determinant by the first row**. (Note that any row or column can be used.) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$
	Cramer's rule: Systems of linear equations in two variables	The system $a_1x + b_1y = c_1$ $a_2x + b_2y = c_2$ has the solution $x = \frac{D_x}{D} \quad y = \frac{D_y}{D} \quad \text{if } D \neq 0$ where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

SECTION	CONCEPT	KEY IDEAS/FORMULAS
	Cramer's rule: Systems of linear equations in three variables	The system $$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$ has the solution $$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D} \quad \text{if } D \neq 0$$ where $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$ $$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$
8.6	**Partial fractions**	$\frac{n(x)}{d(x)}$ Factor $d(x)$
	Performing partial fraction decomposition	Write $\frac{n(x)}{d(x)}$ as a sum of partial fractions: *Case 1:* Distinct (nonrepeated) **linear** factors *Case 2:* Repeated **linear** factors *Case 3:* Distinct **irreducible quadratic** factors *Case 4:* Repeated **irreducible quadratic** factors. **Distinct linear factors** $$\frac{n(x)}{d(x)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \cdots$$ **Repeated linear factors** $$\frac{n(x)}{d(x)} = \frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \cdots + \frac{M}{(ax+b)^m}$$ **Distinct irreducible quadratic factors** $$\frac{n(x)}{d(x)} = \frac{Ax+B}{ax^2+bx+c}$$ **Repeated irreducible quadratic factors** $$\frac{n(x)}{d(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$
8.7	**Systems of linear inequalities in two variables**	
	Linear inequalities in two variables	■ $\leq$ or $\geq$ use solid lines. ■ $<$ or $>$ use dashed lines.
	Systems of linear inequalities in two variables	Solutions are determined graphically by finding the common shaded regions.
	The linear programming model	Finding optimal solutions Minimizing or maximizing a function subject to constraints (linear inequalities)

8.1 Systems of Linear Equations in Two Variables

Solve each system of linear equations.

1. $r - s = 3$
$r + s = 3$

2. $3x + 4y = 2$
$x - y = 6$

3. $-4x + 2y = 3$
$4x - y = 5$

4. $0.25x - 0.5y = 0.6$
$0.5x + 0.25y = 0.8$

5. $x + y = 3$
$x - y = 1$

6. $3x + y = 4$
$2x + y = 1$

7. $4c - 4d = 3$
$c + d = 4$

8. $5r + 2s = 1$
$r - s = -3$

9. $y = -\frac{1}{2}x$
$y = \frac{1}{2}x + 2$

10. $2x + 4y = -2$
$4x - 2y = 3$

11. $1.3x - 2.4y = 1.6$
$0.7x - 1.2y = 1.4$

12. $\frac{1}{4}x - \frac{3}{4}y = 12$
$\frac{1}{2}y + \frac{1}{4}x = \frac{1}{2}$

13. $5x - 3y = 21$
$-2x + 7y = -20$

14. $6x - 2y = -2$
$4x + 3y = 16$

15. $10x - 7y = -24$
$7x + 4y = 1$

16. $\frac{1}{3}x - \frac{2}{9}y = \frac{2}{9}$
$\frac{4}{5}x + \frac{3}{4}y = -\frac{3}{4}$

Match each system of equations with its graph.

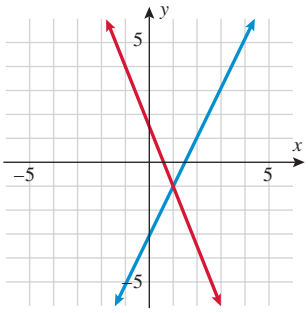
17. $2x - 3y = 4$
$x + 4y = 3$

18. $5x - y = 2$
$5x - y = -2$

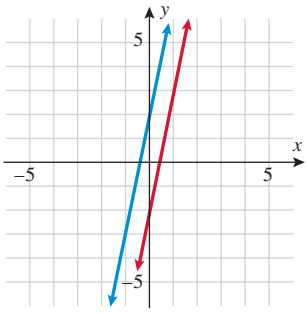
19. $x + 2y = -6$
$2x + 4y = -12$

20. $5x + 2y = 3$
$4x - 2y = 6$

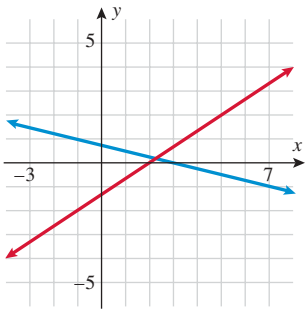
a.



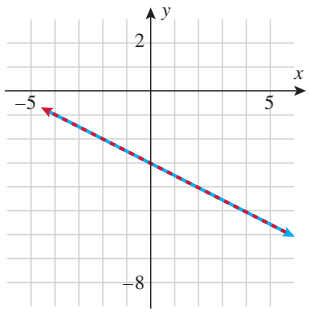
b.



c.



d.



Applications

21. Chemistry. In chemistry lab, Alexandra needs to make a 42-milliliter solution that is 15% NaCl. All that is in the lab is 6% and 18% NaCl. How many milliliters of each solution should she use to obtain the desired mix?

22. Gas Mileage. A Nissan Sentra gets approximately 32 mpg on the highway and 18 mpg in the city. Suppose 265 miles were driven on a full tank (12 gallons) of gasoline. Approximately how many miles were driven in the city and how many on the highway?

8.2 Systems of Linear Equations in Three Variables

Solve each system of linear equations.

23. $x + y + z = 1$
$x - y - z = -3$
$-x + y + z = 3$

24. $x - 2y + z = 3$
$2x - y + z = -4$
$3x - 3y - 5z = 2$

25. $x + y + z = 7$
$x - y - z = 17$
$y + z = 5$

26. $x + z = 3$
$-x + y - z = -1$
$x + y + z = 5$

Applications

27. Fitting a Curve to Data. The average number of flights on a commercial plane that a person takes per year can be modeled by a quadratic function $y = ax^2 + bx + c$, where $a < 0$, and x represents age: $16 \le x \le 65$. The following table gives the average number of flights per year that a person takes on a commercial airline. Determine a quadratic function that models this quantity. *Note:* Coefficients will be approximate.

Age	Number of Flights per Year
16	2
40	6
65	4

28. Investment Portfolio. Danny and Paula decide to invest \$20,000 of their savings. They put some in an IRA account earning 4.5% interest, some in a mutual fund that has been averaging 8% a year, and some in a stock that earned 12% last year. If they put \$4000 more in the IRA than in the mutual fund, and the mutual fund and stock have the same growth in the next year as they did in the previous year, they will earn \$1525 in a year. How much money did they put in each of the three investments?

8.3 Systems of Linear Equations and Matrices

Write the augmented matrix for each system of linear equations.

29. $\begin{aligned} 5x + 7y &= 2 \\ 3x - 4y &= -2 \end{aligned}$

30. $\begin{aligned} 2.3x - 4.5y &= 6.8 \\ -0.4x + 2.1y &= -9.1 \end{aligned}$

31. $\begin{aligned} 2x - z &= 3 \\ y - 3z &= -2 \\ x + 4z &= -3 \end{aligned}$

32. $\begin{aligned} 2y - x + 3z &= 1 \\ 4z - 2y + 3x &= -2 \\ x - y - 4z &= 0 \end{aligned}$

Indicate whether each matrix is in row–echelon form. If it is, state whether it is in *reduced* row–echelon form.

33. $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 2 \end{array}\right]$

34. $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right]$

35. $\left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 2 & 3 \end{array}\right]$

36. $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$

Perform the indicated row operations on each matrix.

37. $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & -2 & 2 \end{array}\right]$ $\quad -\frac{1}{2}R_2 \to R_2$

38. $\left[\begin{array}{cc|c} 1 & 4 & 1 \\ 2 & -2 & 3 \end{array}\right]$ $\quad R_2 - 2R_1 \to R_2$

39. $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 1 & -4 & 8 \end{array}\right]$ $\quad R_2 + R_1 \to R_1$

40. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 6 & 0 \\ 0 & 2 & -2 & 3 & -2 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & -1 & 3 & -3 & 3 \end{array}\right]$ $\quad \begin{aligned} -2R_1 + R_2 &\to R_1 \\ R_4 + R_3 &\to R_4 \end{aligned}$

Apply row operations to transform each matrix to reduced row–echelon form.

41. $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 3 & 4 & 1 \end{array}\right]$

42. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ -2 & 0 & 1 & -2 \end{array}\right]$

43. $\left[\begin{array}{ccc|c} 4 & 1 & -2 & 0 \\ 1 & 0 & -1 & 0 \\ -2 & 1 & 1 & 12 \end{array}\right]$

44. $\left[\begin{array}{ccc|c} 2 & 3 & 2 & 1 \\ 0 & -1 & 1 & -2 \\ 1 & 1 & -1 & 6 \end{array}\right]$

Solve the system of linear equations using augmented matrices.

45. $\begin{aligned} 3x - 2y &= 2 \\ -2x + 4y &= 1 \end{aligned}$

46. $\begin{aligned} 2x - 7y &= 22 \\ x + 5y &= -23 \end{aligned}$

47. $\begin{aligned} 5x - y &= 9 \\ x + 4y &= 6 \end{aligned}$

48. $\begin{aligned} 8x + 7y &= 10 \\ -3x + 5y &= 42 \end{aligned}$

49. $\begin{aligned} x - 2y + z &= 3 \\ 2x - y + z &= -4 \\ 3x - 3y - 5z &= 2 \end{aligned}$

50. $\begin{aligned} 3x - y + 4z &= 18 \\ 5x + 2y - z &= -20 \\ x + 7y - 6z &= -38 \end{aligned}$

51. $\begin{aligned} x - 4y + 10z &= -61 \\ 3x - 5y + 8z &= -52 \\ -5x + y - 2z &= 8 \end{aligned}$

52. $\begin{aligned} 4x - 2y + 5z &= 17 \\ x + 6y - 3z &= -\tfrac{17}{2} \\ -2x + 5y + z &= 2 \end{aligned}$

53. $\begin{aligned} 3x + y + z &= -4 \\ x - 2y + z &= -6 \end{aligned}$

54. $\begin{aligned} 2x - y + 3z &= 6 \\ 3x + 2y - z &= 12 \end{aligned}$

Applications

55. Fitting a Curve to Data. The average number of flights on a commercial plane that a person takes a year can be modeled by a quadratic function $y = ax^2 + bx + c$, where $a < 0$ and x represents age: $16 < x < 65$. The table below gives the average number of flights per year that a person takes on a commercial airline. Determine a quadratic function that models this quantity by solving for a, b, and c using matrices and compare with Exercise 27. *Note:* Coefficients will be approximate.

Age	Number of Flights per Year
16	2
40	6
65	4

56. Investment Portfolio. Danny and Paula decide to invest \$20,000 of their savings in investments. They put some in an IRA account earning 4.5% interest, some in a mutual fund that has been averaging 8% a year, and some in a stock that earned 12% last year. If they put \$3000 more in the mutual fund than in the IRA, and the mutual fund and stock have the same growth in the next year as they did in the previous year, they will earn \$1877.50 in a year. How much money did they put in each of the three investments?

8.4 Matrix Algebra

Calculate the given expression, if possible.

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & -1 \\ 3 & 7 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 0 & 1 \\ 2 & -1 & 4 \\ 0 & 3 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 2 \\ 9 & 7 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}$$

57. $A + C$ **58.** $B + A$ **59.** $B + E$

60. $A + D$ **61.** $2A + D$ **62.** $3E + B$

63. $2D - 3A$ **64.** $3B - 4E$ **65.** $5A - 2D$

66. $5B - 4E$ **67.** AB **68.** BC

69. DA **70.** AD **71.** $BC + E$

72. DB **73.** EC **74.** CE

Determine whether B is the multiplicative inverse of A using $AA^{-1} = I$.

75. $A = \begin{bmatrix} 6 & 4 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -0.5 & 1 \\ 1 & -1.5 \end{bmatrix}$

76. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$

77. $A = \begin{bmatrix} 1 & -2 & 6 \\ 2 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -\frac{1}{7} & \frac{4}{7} & 2 \\ \frac{2}{7} & -\frac{1}{7} & -2 \\ \frac{2}{7} & -\frac{1}{7} & -1 \end{bmatrix}$

78. $A = \begin{bmatrix} 0 & 7 & 6 \\ 1 & 0 & -4 \\ -2 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 2 & 0 & 6 \end{bmatrix}$

Find A^{-1}, if it exists.

79. $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ **80.** $A = \begin{bmatrix} -2 & 7 \\ -4 & 6 \end{bmatrix}$

81. $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ **82.** $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

83. $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & -1 \\ 0 & 1 & -3 \end{bmatrix}$ **84.** $A = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 1 & 2 \\ -3 & -2 & 1 \end{bmatrix}$

85. $A = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ **86.** $A = \begin{bmatrix} -4 & 4 & 3 \\ 1 & 2 & 2 \\ 3 & -1 & 6 \end{bmatrix}$

Solve the system of linear equations using matrix algebra.

87. $\begin{aligned} 3x - y &= 11 \\ 5x + 2y &= 33 \end{aligned}$

88. $\begin{aligned} 6x + 4y &= 15 \\ -3x - 2y &= -1 \end{aligned}$

89. $\begin{aligned} \tfrac{5}{8}x - \tfrac{2}{3}y &= -3 \\ \tfrac{3}{4}x + \tfrac{5}{6}y &= 16 \end{aligned}$

90. $\begin{aligned} x + y - z &= 0 \\ 2x - y + 3z &= 18 \\ 3x - 2y + z &= 17 \end{aligned}$

91. $\begin{aligned} 3x - 2y + 4z &= 11 \\ 6x + 3y - 2z &= 6 \\ x - y + 7z &= 20 \end{aligned}$

92. $\begin{aligned} 2x + 6y - 4z &= 11 \\ -x - 3y + 2z &= -\tfrac{11}{2} \\ 4x + 5y + 6z &= 20 \end{aligned}$

8.5 The Determinant of a Square Matrix and Cramer's Rule

Evaluate each 2 × 2 determinant.

93. $\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix}$ **94.** $\begin{vmatrix} -2 & -4 \\ -3 & 2 \end{vmatrix}$

95. $\begin{vmatrix} 2.4 & -2.3 \\ 3.6 & -1.2 \end{vmatrix}$ **96.** $\begin{vmatrix} -\frac{1}{4} & 4 \\ \frac{3}{4} & -4 \end{vmatrix}$

Employ Cramer's rule to solve each system of equations, if possible.

97. $\begin{aligned} x - y &= 2 \\ x + y &= 4 \end{aligned}$

98. $\begin{aligned} 3x - y &= -17 \\ -x + 5y &= 43 \end{aligned}$

99. $\begin{aligned} 2x + 4y &= 12 \\ x - 2y &= 6 \end{aligned}$

100. $\begin{aligned} -x + y &= 4 \\ 2x - 6y &= -5 \end{aligned}$

101. $\begin{aligned} -3x &= 40 - 2y \\ 2x &= 25 + y \end{aligned}$

102. $\begin{aligned} 3x &= 20 + 4y \\ y - x &= -6 \end{aligned}$

Evaluate each 3 × 3 determinant.

103. $\begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 2 & -1 & 0 \end{vmatrix}$ **104.** $\begin{vmatrix} 0 & -2 & 1 \\ 0 & -3 & 7 \\ 1 & -10 & -3 \end{vmatrix}$

105. $\begin{vmatrix} a & 0 & -b \\ -a & b & c \\ 0 & 0 & -d \end{vmatrix}$ **106.** $\begin{vmatrix} -2 & -4 & 6 \\ 2 & 0 & 3 \\ -1 & 2 & \frac{3}{4} \end{vmatrix}$

Employ Cramer's rule to solve each system of equations, if possible.

107. $\begin{aligned} x + y - 2z &= -2 \\ 2x - y + z &= 3 \\ x + y + z &= 4 \end{aligned}$

108. $\begin{aligned} -x - y + z &= 3 \\ x + 2y - 2z &= 8 \\ 2x + y + 4z &= -4 \end{aligned}$

109. $\begin{aligned} 3x \qquad + 4z &= -1 \\ x + y + 2z &= -3 \\ y - 4z &= -9 \end{aligned}$

110. $\begin{aligned} x + y + z &= 0 \\ -x - 3y + 5z &= -2 \\ 2x + y - 3z &= -4 \end{aligned}$

REVIEW EXERCISES

8.6 Partial Fractions

Write the form of each partial-fraction decomposition. Do not solve for the constants.

111. $\dfrac{4}{(x-1)^2(x+3)(x-5)}$

112. $\dfrac{7}{(x-9)(3x+5)^2(x+4)}$

113. $\dfrac{12}{x(4x+5)(2x+1)^2}$

114. $\dfrac{2}{(x+1)(x-5)(x-9)^2}$

115. $\dfrac{3}{x^2+x-12}$

116. $\dfrac{x^2+3x-2}{x^3+6x^2}$

117. $\dfrac{3x^3+4x^2+56x+62}{(x^2+17)^2}$

118. $\dfrac{x^3+7x^2+10}{(x^2+13)^2}$

Find the partial-fraction decomposition for each rational function.

119. $\dfrac{9x+23}{(x-1)(x+7)}$

120. $\dfrac{12x+1}{(3x+2)(2x-1)}$

121. $\dfrac{13x^2+90x-25}{2x^3-50x}$

122. $\dfrac{5x^2+x+24}{x^3+8x}$

123. $\dfrac{2}{x^2+x}$

124. $\dfrac{x}{x(x+3)}$

125. $\dfrac{5x-17}{x^2+4x+4}$

126. $\dfrac{x^3}{(x^2+64)^2}$

8.7 Systems of Linear Inequalities in Two Variables

Graph each linear inequality.

127. $y \geq -2x+3$

128. $y < x-4$

129. $2x+4y > 5$

130. $5x+2y \leq 4$

131. $y \geq -3x+2$

132. $y < x-2$

133. $3x+8y \leq 16$

134. $4x-9y \leq 18$

Graph each system of inequalities or indicate that the system has no solution.

135. $y \geq x+2$
$y \leq x-2$

136. $y \geq 3x$
$y \leq 3x$

137. $x \leq -2$
$y > x$

138. $x+3y \geq 6$
$2x-y \leq 8$

139. $3x-4y \leq 16$
$5x+3y > 9$

140. $x+y > -4$
$x-y < 3$
$y \geq -2$
$x \leq 8$

Minimize or maximize the objective function subject to the constraints.

141. Minimize $z = 2x+y$ subject to

$$x \geq 0 \qquad y \geq 0 \qquad x+y \leq 3$$

142. Maximize $z = 2x+3y$ subject to

$$x \geq 0 \qquad y \geq 0$$
$$-x+y \leq 0 \qquad x \leq 3$$

143. Minimize $z = 3x-5y$ subject to

$$2x+y > 6 \qquad 2x-y < 6 \qquad x > 0$$

144. Maximize $z = -2x+7y$ subject to

$$3x+y < 7 \qquad x-2y > 1 \qquad x \geq 0$$

Applications

For Exercises 145 and 146, refer to the following:

An art student decides to hand-paint coasters and sell sets at a flea market. She decides to make two types of coaster sets: an ocean watercolor and black-and-white geometric shapes. The cost, profit, and time it takes her to paint each set are summarized in the table below.

	Ocean Watercolor	Geometric Shapes
Cost	\$4	\$2
Profit	\$15	\$8
Hours	3	2

145. **Profit.** If the student's costs cannot exceed \$100 and she can spend only 90 hours total painting the coasters, determine the number of each type she should make to maximize her profit.

146. **Profit.** If the student's costs cannot exceed \$300 and she can spend only 90 hours painting, determine the number of each type she should make to maximize her profit.

Technology Exercises

Section 8.1

147. Apply a graphing utility to graph the two equations $0.4x+0.3y = -0.1$ and $0.5x-0.2y = 1.6$. Find the solution to this system of linear equations.

148. Apply a graphing utility to graph the two equations $\frac{1}{2}x+\frac{3}{10}y = \frac{1}{5}$ and $-\frac{5}{3}x+\frac{1}{2}y = \frac{4}{3}$. Find the solution to this system of linear equations.

Section 8.2

Employ a graphing calculator to solve the system of equations.

149.
$$\begin{aligned} 5x - 3y + 15z &= 21 \\ -2x + 0.8y - 4z &= -8 \\ 2.5x - y + 7.5z &= 12 \end{aligned}$$

150.
$$\begin{aligned} 2x - 1.5y + 3z &= 9.5 \\ 0.5x - 0.375y + 0.75z &= 1.5 \end{aligned}$$

Section 8.3

In Exercises 151 and 152, refer to the following:

You are asked to model a set of three points with a quadratic function $y = ax^2 + bx + c$ and determine the quadratic function.

a. Set up a system of equations; use a graphing utility or graphing calculator to solve the system by entering the coefficients of the augmented matrix.

b. Use the graphing calculator commands $\boxed{\text{STAT}}$ $\boxed{\text{QuadReg}}$ to model the data using a quadratic function. Round your answers to two decimal places.

151. $(-10, 12.5), (3, -2.8), (9, 8.5)$

152. $(-4, 10), (2.5, -9.5), (13.5, 12.6)$

Section 8.4

Apply a graphing utility to perform the indicated matrix operations, if possible.

$$A = \begin{bmatrix} -6 & 0 & 4 \\ 1 & 3 & 5 \\ 2 & -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 1 \\ 0 & 2 \\ -8 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 4 & -3 & 0 \\ 1 & 2 & 5 \end{bmatrix}$$

153. ABC

154. CAB

Apply a graphing utility and matrix algebra to solve the system of linear equations.

155.
$$\begin{aligned} 6.1x - 14.2y &= 75.495 \\ -2.3x + 7.2y &= -36.495 \end{aligned}$$

156.
$$\begin{aligned} 7.2x + 3.2y - 1.7z &= 5.53 \\ -1.3x + 4.1y + 2.8z &= -23.949 \end{aligned}$$

Section 8.5

Apply Cramer's rule to solve each system of equations and a graphing utility to evaluate the determinants.

157.
$$\begin{aligned} 4.5x - 8.7y &= -72.33 \\ -1.4x + 5.3y &= 31.32 \end{aligned}$$

158.
$$\begin{aligned} 1.4x + 3.6y + 7.5z &= 42.08 \\ 2.1x - 5.7y - 4.2z &= 5.37 \\ 1.8x - 2.8y - 6.2z &= -9.86 \end{aligned}$$

Section 8.6

159. Apply a graphing utility to graph $y_1 = \dfrac{x^2 + 4}{x^4 - x^2}$ and $y_2 = -\dfrac{4}{x^2} + \dfrac{5/2}{x - 1} - \dfrac{5/2}{x + 1}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

160. Apply a graphing utility to graph $y_1 = \dfrac{x^3 + 6x^2 + 27x + 38}{(x^2 + 8x + 17)(x^2 + 6x + 13)}$ and $y_2 = \dfrac{2x + 1}{x^2 + 8x + 17} - \dfrac{x - 3}{x^2 + 6x + 13}$ in the same viewing rectangle. Is y_2 the partial-fraction decomposition of y_1?

Section 8.7

In Exercises 161 and 162, use a graphing utility to graph each system of inequalities or indicate that the system has no solution.

161.
$$\begin{aligned} 2x + 5y &\geq -15 \\ y &\leq -\tfrac{2}{3}x - 1 \end{aligned}$$

162.
$$\begin{aligned} y &\leq 0.5x \\ y &> -1.5x + 6 \end{aligned}$$

163. Maximize $z = 6.2x + 1.5y$ subject to

$$\begin{aligned} 4x - 3y &\leq 5.4 \\ 2x + 4.5y &\leq 6.3 \\ 3x - y &\geq -10.7 \end{aligned}$$

164. Minimize $z = 1.6x - 2.8y$ subject to

$$\begin{aligned} y &\geq 3.2x - 4.8 \qquad & x &\geq -2 \\ y &\leq 3.2x + 4.8 \qquad & x &\leq 4 \end{aligned}$$

Solve each system of linear equations using elimination and/or substitution methods.

1. $\begin{aligned} x - 2y &= 1 \\ -x + 3y &= 2 \end{aligned}$

2. $\begin{aligned} 3x + 5y &= -2 \\ 7x + 11y &= -6 \end{aligned}$

3. $\begin{aligned} x - y &= 2 \\ -2x + 2y &= -4 \end{aligned}$

4. $\begin{aligned} 3x - 2y &= 5 \\ 6x - 4y &= 0 \end{aligned}$

5. $\begin{aligned} x + y + z &= -1 \\ 2x + y + z &= 0 \\ -x + y + 2z &= 0 \end{aligned}$

6. $\begin{aligned} 6x + 9y + z &= 5 \\ 2x - 3y + z &= 3 \\ 10x + 12y + 2z &= 9 \end{aligned}$

In Exercises 7 and 8, write the system of linear equations as an augmented matrix.

7. $\begin{aligned} 6x + 9y + z &= 5 \\ 2x - 3y + z &= 3 \\ 10x + 12y + 2z &= 9 \end{aligned}$

8. $\begin{aligned} 3x + 2y - 10z &= 2 \\ x + y - z &= 5 \end{aligned}$

9. Perform the following row operations.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 7 & -1 \\ -3 & -2 & 0 \end{bmatrix} \quad \begin{aligned} R_2 - 2R_1 &\to R_2 \\ R_3 + 3R_1 &\to R_3 \end{aligned}$$

10. Rewrite the following matrix in reduced row–echelon form.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & -2 & 1 \end{array}\right]$$

In Exercises 11 and 12, solve the systems of linear equations using augmented matrices.

11. $\begin{aligned} 6x + 9y + z &= 5 \\ 2x - 3y + z &= 3 \\ 10x + 12y + 2z &= 9 \end{aligned}$

12. $\begin{aligned} 3x + 2y - 10z &= 2 \\ x + y - z &= 5 \end{aligned}$

13. Multiply the matrices, if possible.

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 3 & -5 \\ -1 & 1 \end{bmatrix}$$

14. Add the matrices, if possible.

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 3 & -5 \\ -1 & 1 \end{bmatrix}$$

15. Find the inverse of $\begin{bmatrix} 4 & 3 \\ 5 & -1 \end{bmatrix}$, if it exists.

16. Find the inverse of $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 2 & 0 \\ -1 & 2 & 5 \end{bmatrix}$, if it exists.

17. Solve the system of linear equations with matrix algebra (inverses).

$$\begin{aligned} 3x - y + 4z &= 18 \\ x + 2y + 3z &= 20 \\ -4x + 6y - z &= 11 \end{aligned}$$

Calculate the determinant.

18. $\begin{vmatrix} 7 & -5 \\ 2 & -1 \end{vmatrix}$

19. $\begin{vmatrix} 1 & -2 & -1 \\ 3 & -5 & 2 \\ 4 & -1 & 0 \end{vmatrix}$

In Exercises 20 and 21, solve the system of linear equations using Cramer's rule.

20. $\begin{aligned} x - 2y &= 1 \\ -x + 3y &= 2 \end{aligned}$

21. $\begin{aligned} 3x + 5y - 2z &= -6 \\ 7x + 11y + 3z &= 2 \\ x - y + z &= 4 \end{aligned}$

22. A company has two rubrics for scoring job applicants based on weighting education, experience, and the interview differently.

$$\textbf{Matrix } A\textbf{:} \quad \begin{array}{l} \\ \text{Education} \\ \text{Experience} \\ \text{Interview} \end{array} \begin{array}{c} \begin{array}{cc} \text{Rubric 1} & \text{Rubric 2} \end{array} \\ \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.1 \\ 0.1 & 0.3 \end{bmatrix} \end{array}$$

Applicants receive a score from 1 to 10 in each category (education, experience, and interview). Two applicants are shown in the matrix B.

$$\textbf{Matrix } B\textbf{:} \quad \begin{array}{l} \\ \text{Applicant 1} \\ \text{Applicant 2} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Education} & \text{Experience} & \text{Interview} \end{array} \\ \begin{bmatrix} 4 & 7 & 3 \\ 6 & 5 & 4 \end{bmatrix} \end{array}$$

What is the order of BA? What does each entry in BA tell us?

Write each rational expression as a sum of partial fractions.

23. $\dfrac{2x + 5}{x^2 + x}$

24. $\dfrac{3x - 13}{(x - 5)^2}$

25. $\dfrac{5x - 3}{x(x^2 - 9)}$

26. $\dfrac{1}{2x^2 + 5x - 3}$

Graph the inequalities.

27. $-2x + y < 6$

28. $4x - y \geq 8$

In Exercises 29 and 30, graph the system of inequalities.

29. $\begin{aligned} x + y &\leq 4 \\ -x + y &\geq -2 \end{aligned}$

30. $\begin{aligned} x + 3y &\leq 6 \\ 2x - y &\leq 4 \end{aligned}$

31. Minimize the function $z = 5x + 7y$ subject to the constraints

$$x \geq 0 \qquad y \geq 0 \qquad x + y \leq 3 \qquad -x + y \geq 1$$

32. Find the maximum value of the objective function $z = 3x + 6y$ given the constraints

$$x \geq 0 \qquad y \geq 0$$
$$x + y \leq 6 \qquad -x + 2y \leq 4$$

33. Apply a graphing utility and matrix algebra to solve the system of linear equations.

$$5.6x - 2.7y = 87.28$$
$$-4.2x + 8.4y = -106.26$$

34. You are asked to model a set of three points with a quadratic function $y = ax^2 + bx + c$.

a. Set up a system of equations; use a graphing utility or graphing calculator to solve the system by entering the coefficients of the augmented matrix.

b. Use the graphing calculator commands $\boxed{\text{STAT}}$ $\boxed{\text{QuadReg}}$ to model the data using a quadratic function.

$$(-3, 6), (1, 12), (5, 7)$$

1. Evaluate $g[f(-1]$, with $f(x) = \sqrt{2x + 11}$ and $g(x) = x^3$.

2. Use interval notation to express the domain of the function $G(x) = \dfrac{9}{\sqrt{1 - 5x}}$.

3. Using the function $f(x) = x^2 - 3x + 2$, evaluate the difference quotient $\dfrac{f(x + h) - f(x)}{h}$.

4. Find all the real zeros (and state the multiplicity) of $f(x) = -4x(x - 7)^2(x + 13)^3$.

5. Find the vertex of the parabola $f(x) = -0.04x^2 + 1.2x - 3$.

6. Factor the polynomial $P(x) = x^4 + 8x^2 - 9$ as a product of linear factors.

7. Find the vertical and horizontal asymptotes of the function $f(x) = \dfrac{5x - 7}{3 - x}$.

8. Approximate e^{π} using a calculator. Round your answer to two decimal places.

9. Evaluate $\log_5 0.2$ exactly.

10. Solve $5^{2x-1} = 11$ for x. Round the answer to three decimal places.

11. Evaluate $\log_2 6$ using the change-of-base formula. Round your answer to three decimal places.

12. Solve $\ln(5x - 6) = 2$. Round your answer to three decimal places.

13. Give the exact value of $\cos 30°$.

14. How much money should be put in a savings account now that earns 4.7% a year compounded weekly, if you want to have $65,000 in 17 years?

15. The terminal side of angle θ in standard position passes through the point $(-5, 2)$. Calculate the exact values of the six trigonometric function for angle θ.

16. Find all values of θ, where $0° \le \theta \le 360°$, when $\cos\theta = -\dfrac{\sqrt{3}}{2}$.

17. Graph the function $y = \tan\left(\frac{1}{4}x\right)$ over the interval $-2\pi \le x \le 2\pi$.

18. Verify the identity $\cos(3x) = \cos x(1 - 4\sin^2 x)$.

19. State the domain and range of the function $y = 5\tan\left(x - \dfrac{\pi}{2}\right)$.

20. Simplify the trigonometric expression $\dfrac{\sec^4 x - 1}{\sec^2 x + 1}$.

21. Use the half-angle identities to find the exact value of $\tan\left(-\dfrac{3\pi}{8}\right)$.

22. Write the product $7\sin(-2x)\sin(5x)$ as a sum or difference of sines and/or cosines.

23. Solve the triangle $\beta = 106.3°$, $\gamma = 37.4°$, $a = 76.1$ m.

24. Find the angle (rounded to the nearest degree) between the vectors $\langle 2, 3\rangle$ and $\langle -4, -5\rangle$.

25. Find all complex solutions to $x^3 - 27 = 0$.

26. Graph $\theta = -\dfrac{\pi}{4}$.

27. Given

$$A = \begin{bmatrix} 3 & 4 & -7 \\ 0 & 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 8 & -2 & 6 \\ 9 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix}$$

find CB.

28. Solve the system using Gauss–Jordan elimination.

$$\begin{aligned} x - 2y + 3z &= 11 \\ 4x + 5y - z &= -8 \\ 3x + y - 2z &= 1 \end{aligned}$$

29. Use Cramer's rule to solve the system of equations.

$$\begin{aligned} 7x + 5y &= 1 \\ -x + 4y &= -1 \end{aligned}$$

30. Write the matrix equation, find the inverse of the coefficient matrix, and solve the system using matrix algebra.

$$\begin{aligned} 2x + 5y &= -1 \\ -x + 4y &= 7 \end{aligned}$$

31. Graph the system of linear inequalities.

$$\begin{aligned} y &> -x \\ y &\ge -3 \\ x &\le 3 \end{aligned}$$

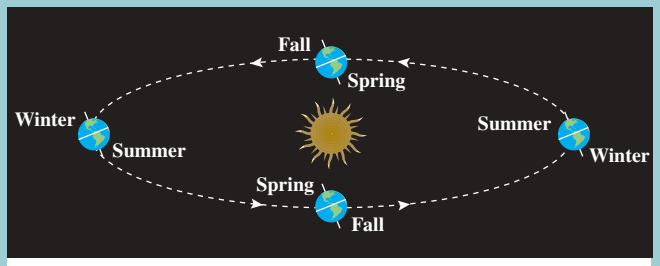
9

Conics, Systems of Nonlinear Equations and Inequalities, and Parametric Equations

© Photodisc/Age Fotostock America, Inc.

Health Korvola/Getty Images, Inc.

We will now study three types of conic sections or conics: the parabola, the ellipse, and the hyperbola. The trajectory of a basketball is a *parabola*, the Earth's orbit around the Sun is an *ellipse*, and the shape of a cooling tower is a *hyperbola*.



Earth's orbit around the Sun is an *ellipse*.

IN THIS CHAPTER we define the three conic sections: the parabola, the ellipse, and the hyperbola. Algebraic equations and the graphs of these conics are discussed. We solve systems of nonlinear equations and inequalities involving parabolas, ellipses, and hyperbolas. We then will determine how rotating the axes changes the equation of a conic, and with our results we will be able to identify the graph of a general second-degree equation as one of the three conics. We will discuss the equations of the conics first in rectangular coordinates and then in polar coordinates. Finally, we will look at parametric equations, which give orientation along a plane curve.

CONICS, SYSTEMS OF NONLINEAR EQUATIONS AND INEQUALITIES, AND PARAMETRIC EQUATIONS

9.1 Conic Basics
- Three Types of Conics

9.2 The Parabola
- Parabola with a Vertex at the Origin
- Parabola with a Vertex at the Point (h, k)

9.3 The Ellipse
- Ellipse Centered at the Origin
- Ellipse Centered at the Point (h, k)

9.4 The Hyperbola
- Hyperbola Centered at the Origin
- Hyperbola Centered at the Point (h, k)

9.5 Systems of Nonlinear Equations
- Solving a System of Nonlinear Equations

9.6 Systems of Nonlinear Inequalities
- Nonlinear Inequalities in Two Variables
- Systems of Nonlinear Inequalities

9.7 Rotation of Axes
- Rotation of Axes Formulas
- The Angle of Rotation Necessary to Transform a General Second-Degree Equation into a Familiar Equation of a Conic

9.8 Polar Equations of Conics
- Equations of Conics in Polar Coordinates

9.9 Parametric Equations and Graphs
- Parametric Equations of a Curve
- Applications of Parametric Equations

LEARNING OBJECTIVES

- Identify if a second-degree equation in two variables corresponds to a parabola, an ellipse, or a hyperbola.
- Graph parabolas whose vertex is at the point (h, k).
- Graph an ellipse whose center is at the point (h, k).
- Graph a hyperbola whose center is at the point (h, k).
- Solve systems of nonlinear equations.
- Graph systems of nonlinear inequalities.
- Transform general second-degree equations into recognizable equations of conics by analyzing the rotation of axes.
- Express equations of conics in polar coordinates.
- Express projectile motion using parametric equations.

SECTION 9.1 CONIC BASICS

SKILLS OBJECTIVES

- Learn the name of each conic section.
- Define conics.
- Recognize the algebraic equation associated with each conic.

CONCEPTUAL OBJECTIVES

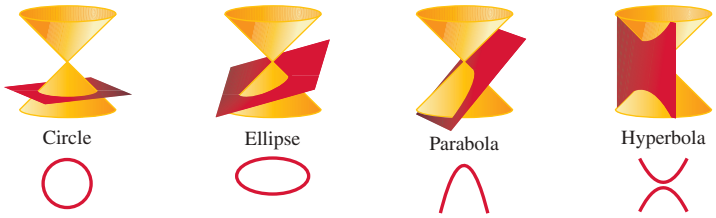
- Understand each conic as an intersection of a plane and a cone.
- Understand how the three equations of the conic sections are related to the general form of a second-degree equation in two variables.

Three Types of Conics

Names of Conics

The word *conic* is derived from the word *cone*. Let's start with a (right circular) **double cone** (see the figure on the left).

Conic sections are curves that result from the intersection of a plane and a double cone. The four conic sections are a **circle**, an **ellipse**, a **parabola**, and a **hyperbola**. **Conics** is an abbreviation for conic sections.



Study Tip

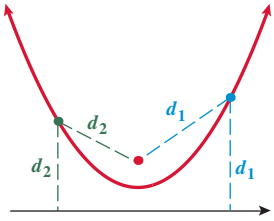
A circle is a special type of ellipse. All circles are ellipses, but not all ellipses are circles.

In Section 0.5, circles were discussed, and we will show that a circle is a particular type of an ellipse. Now we will discuss parabolas, ellipses, and hyperbolas. There are two ways in which we usually describe conics: graphically and algebraically. An entire section will be devoted to each of the three conics, but here we will summarize the definitions of a parabola, an ellipse, and a hyperbola and show how to identify the equations of these conics.

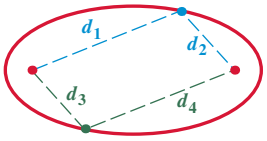
Definitions

You already know that a circle consists of all points equidistant (at a distance equal to the radius) from a point (the center). Ellipses, parabolas, and hyperbolas have similar definitions in that they all have a constant distance (or a sum or difference of distances) to some reference point(s).

A **parabola** is the set of all points that are **equidistant from both a line and a point**. An **ellipse** is the set of all points, the **sum of whose distances to two fixed points is constant**. A **hyperbola** is the set of all points, the **difference of whose distances to two fixed points is a constant**.

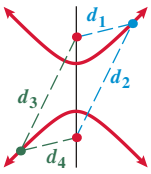


Parabola



$d_1 + d_2 = d_3 + d_4$

Ellipse



$|d_2 - d_1| = |d_4 - d_3|$

Hyperbola

The **general form of a second-degree equation in two variables**, x and y, is given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

If we let $A = 1$, $B = 0$, $C = 1$, $D = 0$, $E = 0$, and $F = -r^2$, this general equation reduces to the equation of a circle centered at the origin: $x^2 + y^2 = r^2$. In fact, all three conics (parabolas, ellipses, and hyperbolas) are special cases of the general second-degree equation.

Recall from Section 0.2 (Quadratic Equations) that the discriminant, $b^2 - 4ac$, determines what types of solutions result from solving a second-degree equation in one variable. If the discriminant is positive, the solutions are two distinct real roots. If the discriminant is zero, the solution is a real repeated root. If the discriminant is negative, the solutions are two complex conjugate roots.

The concept of discriminant is also applicable to second-degree equations in two variables. The discriminant $B^2 - 4AC$ determines the *shape* of the conic section.

Conic	Discriminant
Ellipse	$B^2 - 4AC < 0$
Parabola	$B^2 - 4AC = 0$
Hyperbola	$B^2 - 4AC > 0$

Study Tip

All circles are ellipses since $B^2 - 4AC < 0$.

Using the discriminant to identify the shape of the conic will not work for degenerate cases (when the polynomial factors). For example,

$$2x^2 - xy - y^2 = 0$$

At first glance, one may think this is a hyperbola because $B^2 - 4AC > 0$, but this is a degenerate case.

$$(2x + y)(x - y) = 0$$

$$2x + y = 0 \quad \text{or} \quad x - y = 0$$

$$y = -2x \quad \text{or} \quad y = x$$

The graph is two intersecting lines.

We now identify conics from the general form of a second-degree equation in two variables.

EXAMPLE 1 Determining the Type of Conic

Determine what type of conic corresponds to each of the following equations:

a. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **b.** $y = x^2$ **c.** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Solution:

Write the general form of the second-degree equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

a. Identify A, B, C, D, E, and F. $A = \dfrac{1}{a^2}, B = 0, C = \dfrac{1}{b^2}, D = 0, E = 0, F = -1$

Calculate the discriminant. $B^2 - 4AC = -\dfrac{4}{a^2b^2} < 0$

Since the discriminant is negative, the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is that of an **ellipse**.

Notice that if $a = b = r$, then this equation of an ellipse reduces to the general equation of a circle, $x^2 + y^2 = r^2$, centered at the origin, with radius r.

b. Identify A, B, C, D, E, and F. $A = 1, B = 0, C = 0, D = 0, E = -1, F = 0$

Calculate the discriminant. $B^2 - 4AC = 0$

Since the discriminant is zero, the equation $y = x^2$ is a **parabola**.

c. Identify A, B, C, D, E, and F. $A = \frac{1}{a^2}, B = 0, C = -\frac{1}{b^2}, D = 0, E = 0, F = -1$

Calculate the discriminant. $B^2 - 4AC = \frac{4}{a^2b^2} > 0$

Since the discriminant is positive, the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a **hyperbola**.

Answer: a. ellipse
b. hyperbola
c. parabola

YOUR TURN Determine what type of conic corresponds to each of the following equations:

a. $2x^2 + y^2 = 4$ **b.** $2x^2 = y^2 + 4$ **c.** $2y^2 = x$

In the next three sections, we will discuss the standard forms of equations and the graphs of parabolas, ellipses, and hyperbolas.

SECTION 9.1 SUMMARY

In this section, we defined the three conic sections and determined their general equations with respect to the general form of a second-degree equation in two variables:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The following table summarizes the three conics: ellipse, parabola, and hyperbola.

Conic	Geometric Definition: The Set of All Points	Discriminant
Ellipse	the sum of whose distances to two fixed points is constant	Negative: $B^2 - 4AC < 0$
Parabola	equidistant to both a line and a point	Zero: $B^2 - 4AC = 0$
Hyperbola	the difference of whose distances to two fixed points is a constant	Positive: $B^2 - 4AC > 0$

It is important to note that a circle is a special type of ellipse.

SECTION 9.1 EXERCISES

SKILLS

In Exercises 1–12, identify the conic section as a parabola, ellipse, circle, or hyperbola.

1. $x^2 + xy - y^2 + 2x = -3$ **2.** $x^2 + xy + y^2 + 2x = -3$ **3.** $2x^2 + 2y^2 = 10$ **4.** $x^2 - 4x + y^2 + 2y = 4$

5. $2x^2 - y^2 = 4$ **6.** $2y^2 - x^2 = 16$ **7.** $5x^2 + 20y^2 = 25$ **8.** $4x^2 + 8y^2 = 30$

9. $x^2 - y = 1$ **10.** $y^2 - x = 2$ **11.** $x^2 + y^2 = 10$ **12.** $x^2 + y^2 = 100$

SECTION 9.2 THE PARABOLA

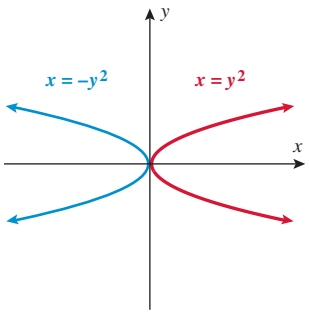
SKILLS OBJECTIVES

- Graph a parabola given the focus, directrix, and vertex.
- Find the equation of a parabola whose vertex is at the origin.
- Find the equation of a parabola whose vertex is at the point (h, k).
- Solve applied problems that involve parabolas.

CONCEPTUAL OBJECTIVES

- Derive the general equation of a parabola.
- Identify, draw, and use the focus, directrix, and axis of symmetry.

Parabola with a Vertex at the Origin



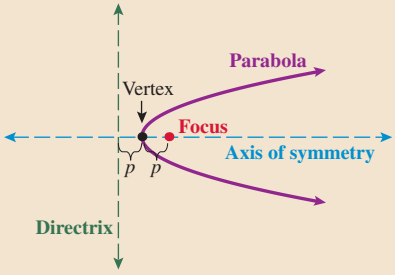
Recall from Section 2.1 that the graphs of quadratic functions such as

$$f(x) = a(x - h)^2 + k \qquad \text{or} \qquad y = ax^2 + bx + c$$

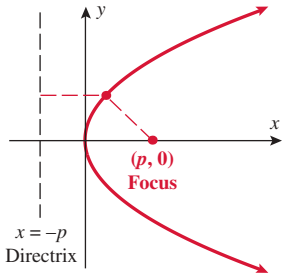
were *parabolas* that opened either upward or downward. We now expand our discussion to *parabolas* that open to the **right** or **left**. We did not discuss these types of parabolas before because they are not functions (they fail the vertical line test).

DEFINITION **Parabola**

A **parabola** is the set of all points in a plane that are equidistant from a fixed line, the **directrix**, and a fixed point not on the line, the **focus**. The line through the focus and perpendicular to the directrix is the **axis of symmetry**. The **vertex** of the parabola is located at the midpoint between the directrix and the focus along the axis of symmetry.



Here p is the distance along the axis of symmetry from the directrix to the vertex and from the vertex to the focus.



Let's consider a parabola with the vertex at the origin and the focus on the positive x-axis. Let the distance from the vertex to the focus be p. Therefore, the focus is located at the point $(p, 0)$. Since the distance from the vertex to the focus is p, the distance from the vertex to the directrix must also be p. Since the axis of symmetry is the x-axis, the directrix must be perpendicular to the x-axis. Therefore, the directrix is given by $x = -p$. Any point, (x, y), must have the same distance to the focus, $(p, 0)$, as it does to the point $(-p, y)$ of the directrix.

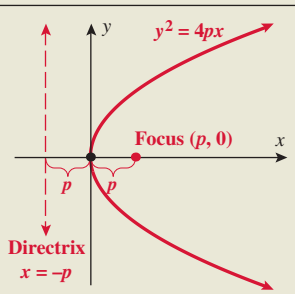
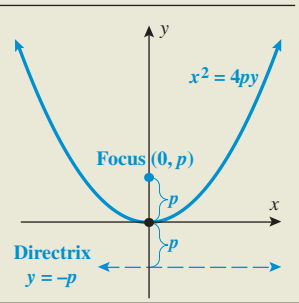
Derivation of the Equation of a Parabola

WORDS	MATH
Calculate the distance from (x, y) to $(p, 0)$ with the distance formula.	$\sqrt{(x - p)^2 + y^2}$
Calculate the distance from (x, y) to $(-p, y)$ with the distance formula.	$\sqrt{(x - (-p))^2 + 0^2}$
Set the two distances equal to one another.	$\sqrt{(x - p)^2 + y^2} = \sqrt{(x + p)^2}$
Recall that $\sqrt{x^2} = \lvert x \rvert$.	$\sqrt{(x - p)^2 + y^2} = \lvert x + p \rvert$
Square both sides of the equation.	$(x - p)^2 + y^2 = (x + p)^2$
Square the binomials inside the parentheses.	$x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2$
Simplify.	$\boxed{y^2 = 4px}$

The equation $y^2 = 4px$ represents a parabola opening right ($p > 0$) with the vertex at the origin. The following box summarizes parabolas that have a vertex at the origin and a focus along either the x-axis or the y-axis:

EQUATION OF A PARABOLA WITH VERTEX AT THE ORIGIN

The standard (conic) form of the equation of a **parabola** with vertex at the origin is given by

EQUATION	$y^2 = 4px$	$x^2 = 4py$
VERTEX	$(0, 0)$	$(0, 0)$
FOCUS	$(p, 0)$	$(0, p)$
DIRECTRIX	$x = -p$	$y = -p$
AXIS OF SYMMETRY	x-axis	y-axis
$p > 0$	opens to the right	opens upward
$p < 0$	opens to the left	opens downward
GRAPH ($p > 0$)		

EXAMPLE 1 **Finding the Focus and Directrix of a Parabola Whose Vertex Is Located at the Origin**

Find the focus and directrix of a parabola whose equation is $y^2 = 8x$.

Solution:

Compare this parabola with the general equation of a parabola. $y^2 = 4px$

$y^2 = 8x$

Let $y^2 = 8x$. $4px = 8x$

Solve for p (assume $x \neq 0$). $4p = 8$

$p = 2$

The focus of a parabola of the form $y^2 = 4px$ is $(p, 0)$. $\boxed{\text{Focus } (2, 0)}$

The directrix of a parabola of the form $y^2 = 4px$ is $x = -p$. $\boxed{\text{Directrix } x = -2}$.

YOUR TURN Find the focus and directrix of a parabola whose equation is $y^2 = 16x$.

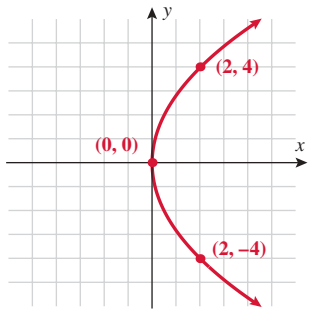
Answer: The focus is (4, 0) and the directrix is $x = -4$.

Graphing a Parabola with a Vertex at the Origin

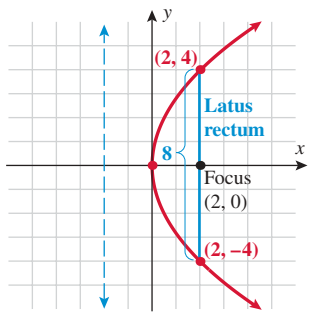
When a seamstress starts with a pattern for a custom-made suit, the pattern is used as a guide. The pattern is not sewn into the suit, but rather removed once it is used to determine the exact shape and size of the fabric to be sewn together. The focus and directrix of a parabola are similar to the pattern used by a seamstress. Although the focus and directrix define a parabola, they do not appear on the graph of a parabola.

We can draw an approximate sketch of a parabola whose vertex is at the origin with three pieces of information. We know that the vertex is located at (0, 0). Additional information that we seek is the direction in which the parabola opens and approximately how wide or narrow to draw the parabolic curve. The direction toward which the parabola opens is found from the equation. An equation of the form $y^2 = 4px$ opens either left or right. It opens right if $p > 0$ and opens left if $p < 0$. An equation of the form $x^2 = 4py$ opens either up or down. It opens up if $p > 0$ and opens down if $p < 0$. How narrow or wide should we draw the parabolic curve? If we select a few points that satisfy the equation, we can use those as graphing aids.

In Example 1, we found that the focus of that parabola is located at (2, 0). If we select the x-coordinate of the focus $x = 2$, and substitute that value into the equation of the parabola $y^2 = 8x$, we find the corresponding y values to be $y = -4$ and $y = 4$. If we plot the three points (0, 0), (2, −4), and (2, 4) and then connect the points with a parabolic curve, we get the graph on the right.



The line segment that passes through the focus (2, 0) is parallel to the directrix $x = -2$, and whose endpoints are on the parabola is called the **latus rectum**. The latus rectum in this case has length 8. The latus rectum is a graphing aid that assists us in determining how wide or how narrow to draw the parabola.



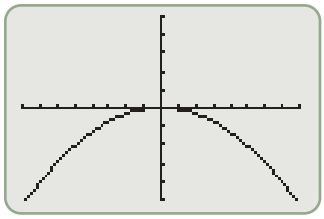
In general, the points on a parabola of the form $y^2 = 4px$ that lie above and below the focus $(p, 0)$ satisfy the equation $y^2 = 4p^2$ and are located at $(p, -2p)$ and $(p, 2p)$. The latus rectum will have length $4|p|$. Similarly, a parabola of the form $x^2 = 4py$ will have a horizontal latus rectum of length $4|p|$.

Study Tip

The focus and directrix define a parabola, but do not appear on its graph.

Technology Tip

To graph $x^2 = -12y$ with a graphing calculator, solve for y first. That is, $y = -\frac{1}{12}x^2$.



EXAMPLE 2 Graphing a Parabola Whose Vertex Is at the Origin, Using the Focus, Directrix, and Latus Rectum as Graphing Aids

Determine the focus, directrix, and length of the latus rectum of the parabola $x^2 = -12y$. Employ these to assist in graphing the parabola.

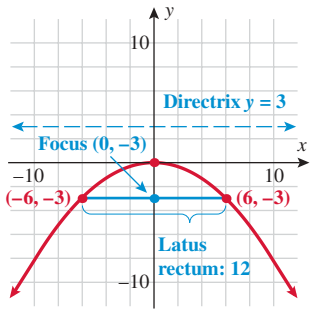
Solution:

Compare this parabola with the general equation of a parabola. $x^2 = 4py \qquad x^2 = -12y$

Solve for p. $4p = -12$

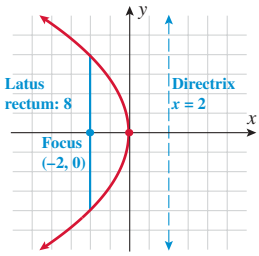
$p = -3$

A parabola of the form $x^2 = 4py$ has focus $(0, p)$, directrix $y = -p$, and a latus rectum of length $4|p|$. For this parabola, $p = -3$; therefore, the focus is $\boxed{(0, -3)}$, the directrix is $\boxed{y = 3}$, and the length of the latus rectum is $\boxed{12}$.



YOUR TURN Find the focus, directrix, and length of the latus rectum of the parabola $y^2 = -8x$, and use these to graph the parabola.

Answer:
The focus is $(-2, 0)$.
The directrix is $x = 2$.
The length of the latus rectum is 8.



Finding the Equation of a Parabola with a Vertex at the Origin

Thus far we have started with the equation of a parabola and then determined its focus and directrix. Let's now reverse the process. For example, if we know the focus and directrix of a parabola, how do we find the equation of the parabola? If we are given the focus and directrix, then we can find the vertex, which is the midpoint between the focus and the directrix. If the vertex is at the origin, then we know the general equation of the parabola that corresponds to the focus.

EXAMPLE 3 Finding the Equation of a Parabola Given the Focus and Directrix When the Vertex Is at the Origin

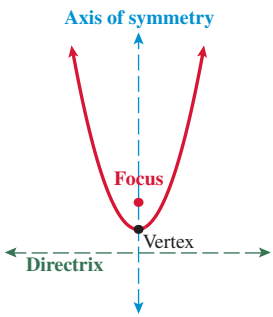
Find the standard form of the equation of a parabola whose focus is at the point $\left(0, \frac{1}{2}\right)$ and whose directrix is $y = -\frac{1}{2}$. Graph the equation.

Solution:

The midpoint of the segment joining the focus and the directrix along the axis of symmetry is the vertex.

Calculate the midpoint between $\left(0, \frac{1}{2}\right)$ and $\left(0, -\frac{1}{2}\right)$.

$$\text{Vertex} = \left(\frac{0+0}{2}, \frac{\frac{1}{2} - \frac{1}{2}}{0}\right) = (0, 0).$$



A parabola with vertex at (0, 0), focus at $(0, p)$, and directrix $y = -p$ corresponds to the equation $x^2 = 4py$.

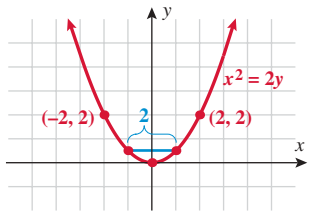
Identify p given that the focus is $(0, p) = \left(0, \frac{1}{2}\right)$. $\quad p = \frac{1}{2}$

Substitute $p = \frac{1}{2}$ into the standard equation of a parabola with vertex at the origin $x^2 = 4py$. $\quad x^2 = 2y$

Now that the equation is known, a few points can be selected, and the parabola can be point-plotted. Alternatively, the length of the latus rectum can be calculated to sketch the approximate width of the parabola.

To graph $x^2 = 2y$, first calculate the latus rectum. $\quad 4|p| = 4\left(\frac{1}{2}\right) = 2$

Label the focus, directrix, and latus rectum, and draw a parabolic curve whose vertex is at the origin that intersects with the latus rectum's endpoints.



■ **YOUR TURN** Find the equation of a parabola whose focus is at the point $(-5, 0)$ and whose directrix is $x = 5$.

■ **Answer:** $y^2 = -20x$

Before we proceed to parabolas with general vertices, let's first make a few observations: The larger the latus rectum, the more rapidly the parabola widens. An alternative approach for graphing the parabola is to plot a few points that satisfy the equation of the parabola, which is the approach in most textbooks.

Parabola with a Vertex at the Point (h, k)

Recall (Section 0.5) that the graph of $x^2 + y^2 = r^2$ is a circle with radius r centered at the origin, whereas the graph of $(x - h)^2 + (y - k)^2 = r^2$ is a circle with radius r centered at the point (h, k). In other words, the center is shifted from the origin to the point (h, k). This same translation (shift) can be used to describe parabolas whose vertex is at the point (h, k).

Study Tip

When $(h, k) = (0, 0)$, the vertex of the parabola is located at the origin.

Equation of a Parabola with Vertex at the Point (h, k)

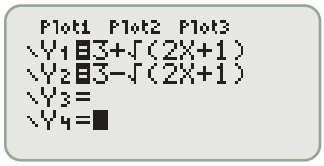
The standard (conic) form of the equation of a parabola with vertex at the point (h, k) is given by

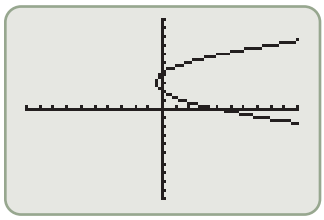
Equation	$(y - k)^2 = 4p(x - h)$	$(x - h)^2 = 4p(y - k)$
Vertex	(h, k)	(h, k)
Focus	$(p + h, k)$	$(h, p + k)$
Directrix	$x = -p + h$	$y = -p + k$
Axis of Symmetry	$y = k$	$x = h$
$p > 0$	opens to the right	opens upward
$p < 0$	opens to the left	opens downward

In order to find the vertex of a parabola given a general second-degree equation, first complete the square (Section 0.2) in order to identify (h, k). Then determine whether the parabola opens up, down, left, or right. Identify points that lie on the graph of the parabola. Intercepts are often the easiest points to find, since they are the points where one of the variables is set equal to zero.

Technology Tip

Use a TI to check the graph of $y^2 - 6y - 2x + 8 = 0$. Use $(y - 3)^2 = 2x + 1$ to solve for y first. That is, $y_1 = 3 + \sqrt{2x + 1}$ or $y_2 = 3 - \sqrt{2x + 1}$.





Study Tip

It is often easier to find the intercepts by converting to the general form of the equation.

EXAMPLE 4 Graphing a Parabola with Vertex (h, k)

Graph the parabola given by the equation $y^2 - 6y - 2x + 8 = 0$.

Solution:

Transform this equation into the form $(y - k)^2 = 4p(x - h)$, since this equation is of degree 2 in y and degree 1 in x. We know this parabola opens either to the left or right.

Complete the square on y: $y^2 - 6y - 2x + 8 = 0$

Isolate the y terms. $y^2 - 6y = 2x - 8$

Add 9 to both sides to complete the square. $y^2 - 6y + 9 = 2x - 8 + 9$

Write the left side as a perfect square. $(y - 3)^2 = 2x + 1$

Factor out a 2 on the right side. $(y - 3)^2 = 2\left(x + \frac{1}{2}\right)$

Compare with $(y - k)^2 = 4p(x - h)$ and identify (h, k) and p.

$$(h, k) = \left(-\frac{1}{2}, 3\right)$$

$$4p = 2 \Rightarrow p = \frac{1}{2}$$

The vertex is at the point $\left(-\frac{1}{2}, 3\right)$, and since $p = \frac{1}{2}$ is positive, the parabola opens to the right. Since the parabola's vertex lies in quadrant II and it opens to the right, we know there are two y-intercepts and one x-intercept. Apply the general equation $y^2 - 6y - 2x + 8 = 0$ to find the intercepts.

Find the y-intercepts (set $x = 0$). $y^2 - 6y + 8 = 0$

Factor. $(y - 2)(y - 4) = 0$

Solve for y. $y = 2$ or $y = 4$

Find the x-intercept (set $y = 0$). $-2x + 8 = 0$

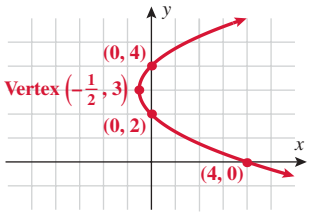
Solve for x. $x = 4$

Label the following points and connect them with a smooth curve:

Vertex: $\left(-\frac{1}{2}, 3\right)$

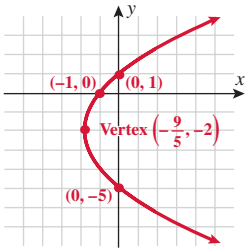
y-intercepts: $(0, 2)$ and $(0, 4)$

x-intercept: $(4, 0)$



YOUR TURN For the equation $y^2 + 4y - 5x - 5 = 0$, identify the vertex and the intercepts, and graph.

Answer:
Vertex: $\left(-\frac{9}{5}, -2\right)$
x-intercept: $x = -1$
y-intercepts: $y = -5$ and $y = 1$



EXAMPLE 5 Graphing a Parabola with Vertex (h, k)

Graph the parabola given by the equation $x^2 - 2x - 8y - 7 = 0$.

Solution:

Transform this equation into the form $(x - h)^2 = 4p(y - k)$, since this equation is degree 2 in x and degree 1 in y. We know this parabola opens either upward or downward.

Complete the square on x: $x^2 - 2x - 8y - 7 = 0$

Isolate the x terms. $x^2 - 2x = 8y + 7$

Add 1 to both sides to complete the square. $x^2 - 2x + 1 = 8y + 7 + 1$

Write the left side as a perfect square. $(x - 1)^2 = 8y + 8$

Factor out the 8 on the right side. $(x - 1)^2 = 8(y + 1)$

Compare with $(x - h)^2 = 4p(y - k)$ and identify (h, k) and p.

$$(h, k) = (1, -1)$$

$$4p = 8 \Rightarrow p = 2$$

The vertex is at the point $(1, -1)$, and since $p = 2$ is positive, the parabola opens upward. Since the parabola's vertex lies in quadrant IV and it opens upward, we know there are two x-intercepts and one y-intercept. Use the general equation $x^2 - 2x - 8y - 7 = 0$ to find the intercepts.

Find the y-intercept (set $x = 0$). $-8y - 7 = 0$

Solve for y. $y = -\dfrac{7}{8}$

Find the x-intercepts (set $y = 0$). $x^2 - 2x - 7 = 0$

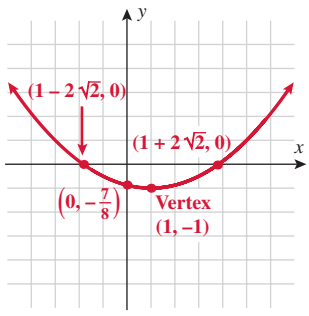
Solve for x. $x = \dfrac{2 \pm \sqrt{4 + 28}}{2} = \dfrac{2 \pm \sqrt{32}}{2} = \dfrac{2 \pm 4\sqrt{2}}{2} = 1 \pm 2\sqrt{2}$

Label the following points and connect with a smooth curve:

Vertex: $(1, -1)$

y-intercept: $\left(0, -\frac{7}{8}\right)$

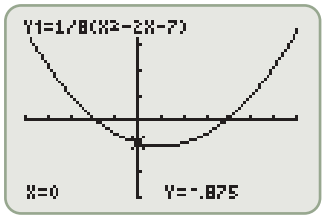
x-intercepts: $\left(1 - 2\sqrt{2}, 0\right)$ and $\left(1 + 2\sqrt{2}, 0\right)$



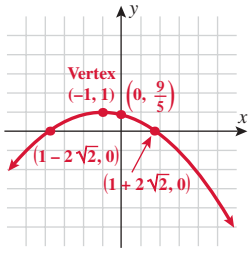
YOUR TURN For the equation $x^2 + 2x + 8y - 7 = 0$, identify the vertex and the intercepts, and graph.

Technology Tip

Use a graphing calculator to check the graph of $x^2 - 2x - 8y - 7 = 0$. Solve for y first: $y = \frac{1}{8}(x^2 - 2x - 7)$

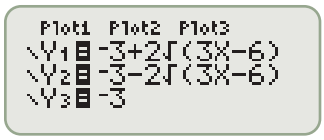


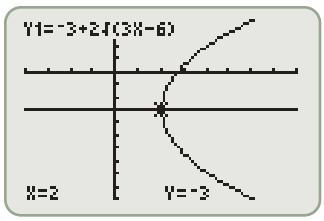
Answer:
Vertex: $(-1, 1)$
x-intercepts: $x = -1 \pm 2\sqrt{2}$
y-intercept: $y = \frac{7}{8}$



Technology Tip

Use a graphing calculator to check the graph of $y^2 + 6y - 12x + 33 = 0$. Use $y^2 + 6y - (12x - 33) = 0$ to solve for y first. That is, $y_1 = -3 + 2\sqrt{3x - 6}$ or $y_2 = -3 - 2\sqrt{3x - 6}$.

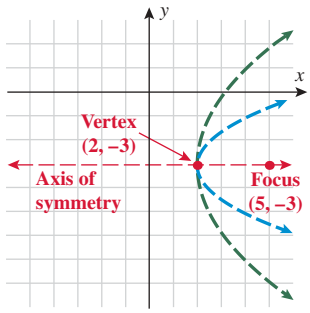




■ **Answer:** $y^2 + 6y + 8x - 7 = 0$

EXAMPLE 6 Finding the Equation of a Parabola with Vertex (h, k)

Find the general form of the equation of a parabola whose vertex is located at the point $(2, -3)$ and whose focus is located at the point $(5, -3)$.



Solution:

Draw a Cartesian plane and label the vertex and focus. The vertex and focus share the same axis of symmetry $y = -3$, and indicate a parabola opening to the right.

Write the standard (conic) equation of a parabola opening to the right. $(y - k)^2 = 4p(x - h) \quad p > 0$

Substitute the vertex $(h, k) = (2, -3)$, into the standard equation. $[y - (-3)]^2 = 4p(x - 2)$

Find p.

The general form of the vertex is (h, k) and the focus is $(h + p, k)$.

For this parabola, the vertex is $(2, -3)$ and the focus is $(5, -3)$.

Find p by taking the difference of the x-coordinates. $p = 3$

Substitute $p = 3$ into $[y - (-3)]^2 = 4p(x - 2)$. $(y + 3)^2 = 4(3)(x - 2)$

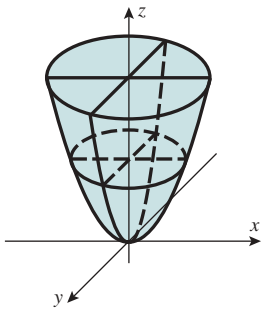
Eliminate the parentheses. $y^2 + 6y + 9 = 12x - 24$

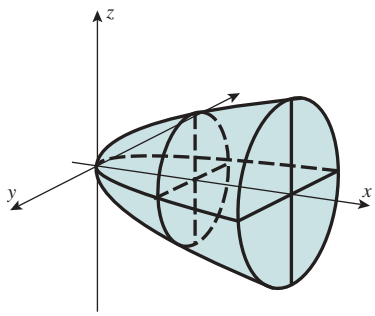
Simplify. $\boxed{y^2 + 6y - 12x + 33 = 0}$

■ **YOUR TURN** Find the equation of the parabola whose vertex is located at $(2, -3)$ and whose focus is located at $(0, -3)$.

Applications

If we start with a parabola in the xy-plane and rotate it around its axis of symmetry, the result will be a three-dimensional paraboloid. Solar cookers illustrate the physical property that the rays of light coming into a parabola should be reflected to the focus. A flashlight reverses this process in that its light source at the focus illuminates a parabolic reflector to direct the beam outward.





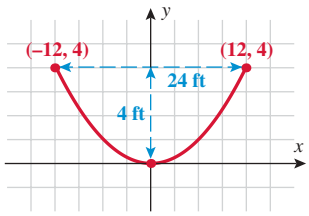
Digital Vision

Satellite dish

A satellite dish is in the shape of a paraboloid. Functioning as an antenna, the parabolic dish collects all of the incoming signals and reflects them to a single point, the focal point, which is where the receiver is located. In Examples 7 and 8, and in the Applications Exercises, the intention is not to find the three-dimensional equation of the paraboloid, but rather the equation of the plane parabola that's rotated to generate the paraboloid.

EXAMPLE 7 Finding the Location of the Receiver in a Satellite Dish

A satellite dish is 24 feet in diameter at its opening and 4 feet deep in its center. Where should the receiver be placed?



Solution:

Draw a parabola with a vertex at the origin representing the center cross section of the satellite dish.

Write the standard equation of a parabola opening upward with vertex at (0, 0).	$x^2 = 4py$
The point (12, 4) lies on the parabola, so substitute (12, 4) into $x^2 = 4py$.	$(12)^2 = 4p(4)$
Simplify.	$144 = 16p$
Solve for p.	$p = 9$
Substitute $p = 9$ into the focus $(0, p)$.	focus: (0, 9)

The receiver should be placed 9 feet from the vertex of the dish.

Parabolic antennas work for sound in addition to light. Have you ever wondered how the sound of the quarterback calling audible plays is heard by the sideline crew? The crew holds a parabolic system with a microphone at the focus. All of the sound in the direction of the parabolic system is reflected toward the focus, where the microphone amplifies and records the sound.

Francis Specker/Icon SMI/NewsCom

EXAMPLE 8 Finding the Equation of a Parabolic Sound Dish

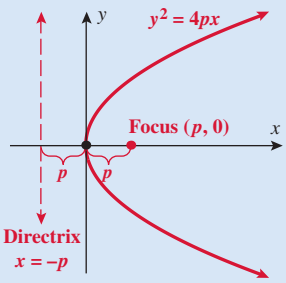
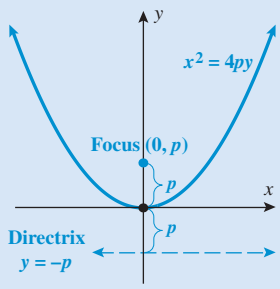
If the parabolic sound dish the sideline crew is holding has a 2-foot diameter at the opening and the microphone is located 6 inches from the vertex, find the equation that governs the center cross section of the parabolic sound dish.

Solution:

Write the standard equation of a parabola opening to the right with the vertex at the origin (0, 0).	$x = 4py^2$
The focus is located 6 inches $\left(\frac{1}{2}\text{ foot}\right)$ from the vertex.	$(p, 0) = \left(\frac{1}{2}, 0\right)$
Solve for p.	$p = \frac{1}{2}$
Let $p = \frac{1}{2}$ in $x = 4py^2$.	$x = 4\left(\frac{1}{2}\right)y^2$
Simplify.	$\boxed{x = 2y^2}$

SECTION 9.2 SUMMARY

In this section, we discussed parabolas whose vertex is at the origin.

EQUATION	$y^2 = 4px$	$x^2 = 4py$
VERTEX	$(0, 0)$	$(0, 0)$
FOCUS	$(p, 0)$	$(0, p)$
DIRECTRIX	$x = -p$	$y = -p$
AXIS OF SYMMETRY	x-axis	y-axis
$p > 0$	opens to the right	opens upward
$p < 0$	opens to the left	opens downward
GRAPH		

For parabolas whose vertex is at the point, (h, k):

EQUATION	$(y - k)^2 = 4p(x - h)$	$(x - h)^2 = 4p(y - k)$
VERTEX	(h, k)	(h, k)
FOCUS	$(p + h, k)$	$(h, p + k)$
DIRECTRIX	$x = -p + h$	$y = -p + k$
AXIS OF SYMMETRY	$y = k$	$x = h$
$p > 0$	opens to the right	opens upward
$p < 0$	opens to the left	opens downward

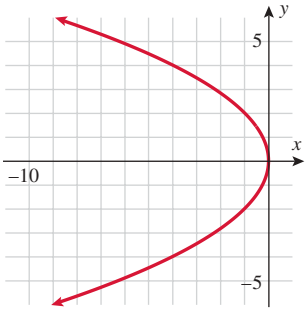
SECTION 9.2 EXERCISES

SKILLS

In Exercises 1–4, match the parabola to the equation.

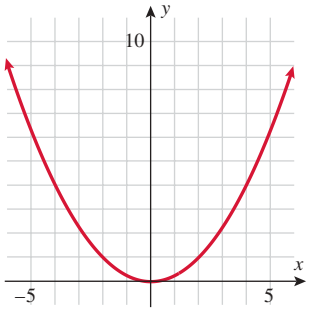
1. $y^2 = 4x$

a.



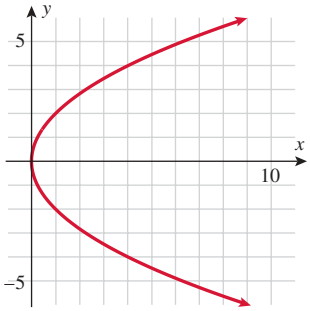
2. $y^2 = -4x$

b.



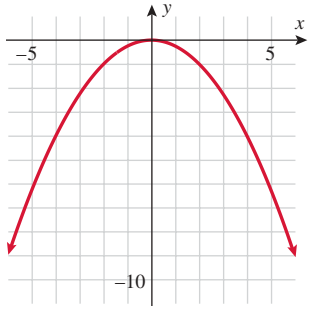
3. $x^2 = -4y$

c.



4. $x^2 = 4y$

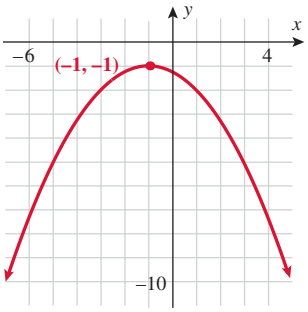
d.



In Exercises 5–8, match the parabola to the equation.

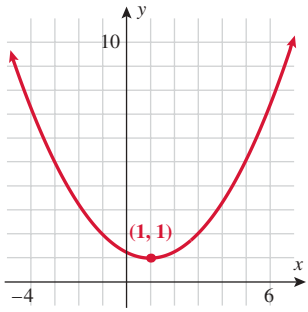
5. $(y - 1)^2 = 4(x - 1)$

a.



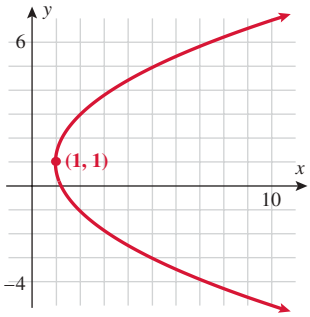
6. $(y + 1)^2 = -4(x - 1)$

b.



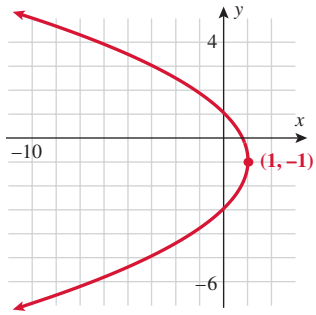
7. $(x + 1)^2 = -4(y + 1)$

c.



8. $(x - 1)^2 = 4(y - 1)$

d.



In Exercises 9–20, find an equation for the parabola described.

9. Vertex at (0, 0); focus at (0, 3)

10. Vertex at (0, 0); focus at (2, 0)

11. Vertex at (0, 0); focus at $(-5, 0)$

12. Vertex at (0, 0); focus at $(0, -4)$

13. Vertex at (3, 5); focus at (3, 7)

14. Vertex at (3, 5); focus at (7, 5)

15. Vertex at (2, 4); focus at (0, 4)

16. Vertex at (2, 4); focus at $(2, -1)$

17. Focus at (2, 4); directrix at $y = -2$

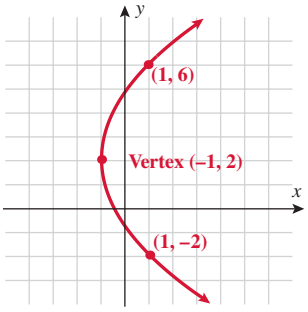
18. Focus at $(2, -2)$; directrix at $y = 4$

19. Focus at $(3, -1)$; directrix at $x = 1$

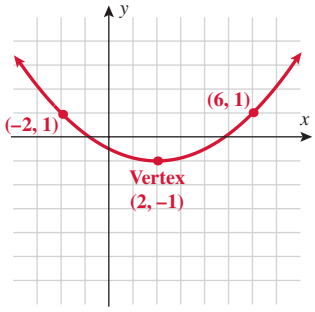
20. Focus at $(-1, 5)$; directrix at $x = 5$

In Exercises 21–24, write an equation for each parabola.

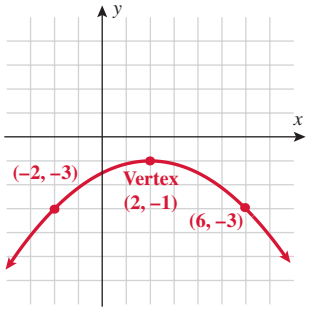
21.



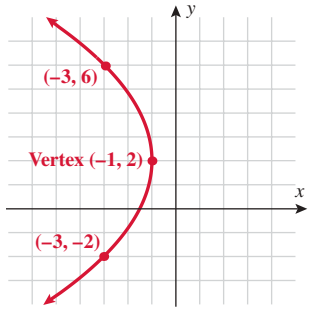
22.



23.



24.



In Exercises 25–32, find the focus, vertex, directrix, and length of latus rectum and graph the parabola.

25. $x^2 = 8y$ **26.** $x^2 = -12y$ **27.** $y^2 = -2x$ **28.** $y^2 = 6x$

29. $x^2 = 16y$ **30.** $x^2 = -8y$ **31.** $y^2 = 4x$ **32.** $y^2 = -16x$

In Exercises 33–44, find the vertex and graph the parabola.

33. $(y - 2)^2 = 4(x + 3)$ **34.** $(y + 2)^2 = -4(x - 1)$ **35.** $(x - 3)^2 = -8(y + 1)$ **36.** $(x + 3)^2 = -8(y - 2)$

37. $(x + 5)^2 = -2y$ **38.** $y^2 = -16(x + 1)$ **39.** $y^2 - 4y - 2x + 4 = 0$ **40.** $x^2 - 6x + 2y + 9 = 0$

41. $y^2 + 2y - 8x - 23 = 0$ **42.** $x^2 - 6x - 4y + 10 = 0$ **43.** $x^2 - x + y - 1 = 0$ **44.** $y^2 + y - x + 1 = 0$

APPLICATIONS

45. Satellite Dish. A satellite dish measures 8 feet across its opening and 2 feet deep at its center. The receiver should be placed at the focus of the parabolic dish. Where is the focus?

46. Satellite Dish. A satellite dish measures 30 feet across its opening and 5 feet deep at its center. The receiver should be placed at the focus of the parabolic dish. Where is the focus?

47. Eyeglass Lens. Eyeglass lenses can be thought of as very wide parabolic curves. If the focus occurs 2 centimeters from the center of the lens and the lens at its opening is 5 centimeters, find an equation that governs the shape of the center cross section of the lens.

48. Optical Lens. A parabolic lens focuses light onto a focal point 3 centimeters from the vertex of the lens. How wide is the lens 0.5 centimeter from the vertex?

Exercises 49 and 50 are examples of solar cookers. Parabolic shapes are often used to generate intense heat by collecting sun rays and focusing all of them at a focal point.

49. Solar Cooker. The parabolic cooker MS-ST10 is delivered as a kit, handily packed in a single carton, with complete assembly instructions and even the necessary tools.

© AP/Wide World Photos

Solar cooker, Ubuntu Village, Johannesburg, South Africa

Thanks to the reflector diameter of 1 meter, it develops an immense power: 1 liter of water boils in significantly less than half an hour. If the rays are focused 40 centimeters from the vertex, find the equation for the parabolic cooker.

50. Le Four Solaire at Font-Romeur "Mirrors of the Solar Furnace." There is a reflector in the Pyrenees Mountains that is eight stories high. It cost $2 million and took 10 years to build. Made of 9000 mirrors arranged in a parabolic formation, it can reach 6000°F just from the Sun hitting it!

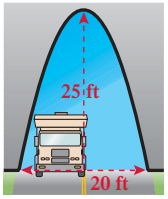
Mark Antman/The Image Works

Solar furnace, Odellio, France

If the diameter of the parabolic mirror is 100 meters and the sunlight is focused 25 meters from the vertex, find the equation for the parabolic dish.

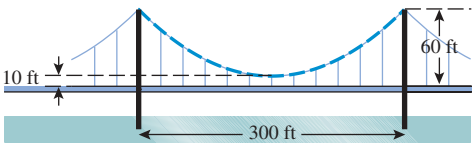
51. Sailing Under a Bridge. A bridge with a parabolic shape has an opening 80 feet wide at the base (where the bridge meets the water), and the height in the center of the bridge is 20 feet. A sailboat whose mast reaches 17 feet above the water is traveling under the bridge 10 feet from the center of the bridge. Will it clear the bridge without scraping its mast? Justify your answer.

52. Driving Under a Bridge. A bridge with a parabolic shape reaches a height of 25 feet in the center of the road, and the width of the bridge opening at ground level is 20 feet combined (both lanes). If an RV is 10 feet tall and 8 feet wide, it won't make it under the bridge if it hugs the center line. Will it clear the bridge if it straddles the center line? Justify your answer.



53. Parabolic Telescope. The Arecibo radio telescope in Puerto Rico has an enormous reflecting surface, or radio mirror. The huge "dish" is 1000 feet in diameter and 167 feet deep and covers an area of about 20 acres. Using these dimensions, determine the focal length of the telescope. Find the equation for the dish portion of the telescope.

54. Suspension Bridge. If one parabolic segment of a suspension bridge is 300 feet and if the cables at the vertex are suspended 10 feet above the bridge, whereas the height of the cables 150 feet from the vertex reaches 60 feet, find the equation of the parabolic path of the suspension cables.



55. Health. In a meditation state, the pulse rate (pulses per minute) can be modeled by $p(t) = 0.18t^2 - 5.4t + 95.5$, where t is in minutes. What is the minimum pulse rate according to this model?

56. Health. In a distress situation, the pulse rate (pulses per minutes) can be modeled by $p(t) = -1.1t^2 + 22t + 80$, where t is the time in seconds. What is the maximum pulse rate according to this model?

57. Business. The profit, in thousands of dollars, for a product is $P(x) = -x^2 + 60x - 500$ where x is the production level in hundreds of units. Find the production level that maximizes the profit. Find the maximum profit.

58. Business. The profit, in thousands of dollars, for a product is $P(x) = -x^2 + 80x - 1200$ where x is the production level in hundreds of units. Find the production level that maximizes the profit. Find the maximum profit.

CATCH THE MISTAKE

In Exercises 59 and 60, explain the mistake that is made.

59. Find an equation for a parabola whose vertex is at the origin and whose focus is at the point (3, 0).

Solution:

Write the general equation for a parabola whose vertex is at the origin. $x^2 = 4py$

The focus of this parabola is $(p, 0) = (3, 0)$. $p = 3$

Substitute $p = 3$ into $x^2 = 4py$. $x^2 = 12y$

This is incorrect. What mistake was made?

60. Find an equation for a parabola whose vertex is at the point (3, 2) and whose focus is located at (5, 2).

Solution:

Write the equation associated with a parabola whose vertex is (3, 2). $(x - h)^2 = 4p(y - k)$

Substitute (3, 2) into $(x - h)^2 = 4p(y - k)$. $(x - 3)^2 = 4p(y - 2)$

The focus is located at (5, 2); therefore, $p = 5$.

Substitute $p = 5$ into $(x - 3)^2 = 4p(y - 2)$. $(x - 3)^2 = 20(y - 2)$

This is incorrect. What mistake(s) was made?

CONCEPTUAL

In Exercises 61–64, determine whether each statement is true or false.

61. The vertex lies on the graph of a parabola.

62. The focus lies on the graph of a parabola.

63. The directrix lies on the graph of a parabola.

64. The endpoints of the latus rectum lie on the graph of a parabola.

In Exercises 65 and 66, use the following equation:

$$\frac{(y - k)^2}{(x - h)} = 4$$

65. Find the directrix of the parabola.

66. Determine whether the parabola opens to the right or to the left.

In Exercises 67 and 68, use the following information about the graph of the parabola:

Axis of symmetry: $x = 6$
Directrix: $y = 4$
Focus: (6, 9)

67. Find the vertex of the parabola.

68. Find the equation of the parabola.

CHALLENGE

69. Derive the standard equation of a parabola with its vertex at the origin, opening upward $x^2 = 4py$. [Calculate the distance d_1 from any point on the parabola (x, y) to the focus $(0, p)$. Calculate the distance d_2 from any point on the parabola (x, y) to the directrix $(-p, y)$. Set $d_1 = d_2$.]

70. Derive the standard equation of a parabola opening right, $y^2 = 4px$. [Calculate the distance d_1 from any point on the parabola (x, y) to the focus $(p, 0)$. Calculate the distance d_2 from any point on the parabola (x, y) to the directrix $(x, -p)$. Set $d_1 = d_2$.]

71. Two parabolas with the same axis of symmetry, $y = 6$, intersect at the point $(4, 2)$. If the directrix of one of these parabolas is the y-axis and the directrix of the other parabola is $x = 8$, find the equations of the parabolas.

72. Two parabolas with the same axis of symmetry, $x = 9$, intersect at the point $(6, -5)$. If the directrix of one of these parabolas is $y = -11$ and the directrix of the other parabola is $y = 1$, find the equations of the parabolas.

73. Find the points of intersection of the parabolas with foci $\left(0, \frac{3}{2}\right)$ and $\left(0, -\frac{3}{4}\right)$, and directresses $y = \frac{1}{2}$ and $y = -\frac{5}{4}$, respectively.

74. Find two parabolas with focus $(1, 2p)$ and vertices $(1, p)$ and $(1, -p)$ that intersect each other.

TECHNOLOGY

75. With a graphing utility, plot the parabola $x^2 - x + y - 1 = 0$. Compare with the sketch you drew for Exercise 43.

76. With a graphing utility, plot the parabola $y^2 + y - x + 1 = 0$. Compare with the sketch you drew for Exercise 44.

77. In your mind, picture the parabola given by $(y + 3.5)^2 = 10(x - 2.5)$. Where is the vertex? Which way does this parabola open? Now plot the parabola with a graphing utility.

78. In your mind, picture the parabola given by $(x + 1.4)^2 = -5(y + 1.7)$. Where is the vertex? Which way does this parabola open? Now plot the parabola with a graphing utility.

79. In your mind, picture the parabola given by $(y - 1.5)^2 = -8(x - 1.8)$. Where is the vertex? Which way does this parabola open? Now plot the parabola with a graphing utility.

80. In your mind, picture the parabola given by $(x + 2.4)^2 = 6(y - 3.2)$. Where is the vertex? Which way does this parabola open? Now plot the parabola with a graphing utility.

PREVIEW TO CALCULUS

In calculus, to find the area between two curves, first we need to find the point of intersection of the two curves. In Exercises 81–84, find the points of intersection of the two parabolas.

81. Parabola I: vertex: $(0, -1)$; directrix: $y = -\frac{5}{4}$
Parabola II: vertex: $(0, 7)$; directrix: $y = \frac{29}{4}$

82. Parabola I: vertex: $(0, 0)$; focus: $(0, 1)$
Parabola II: vertex: $(1, 0)$; focus: $(1, 1)$

83. Parabola I: vertex: $\left(5, \frac{5}{3}\right)$; focus: $\left(5, \frac{29}{12}\right)$
Parabola II: vertex: $\left(\frac{13}{2}, \frac{289}{24}\right)$; focus: $\left(\frac{13}{2}, \frac{253}{24}\right)$

84. Parabola I: focus: $\left(-2, -\frac{35}{4}\right)$; directrix: $y = -\frac{37}{4}$
Parabola II: focus: $\left(2, -\frac{101}{4}\right)$; directrix: $y = -\frac{99}{4}$

SECTION 9.3 THE ELLIPSE

SKILLS OBJECTIVES

- Graph an ellipse given the center, major axis, and minor axis.
- Find the equation of an ellipse centered at the origin.
- Find the equation of an ellipse centered at the point (h, k).
- Solve applied problems that involve ellipses.

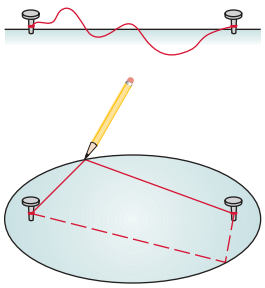
CONCEPTUAL OBJECTIVES

- Derive the general equation of an ellipse.
- Understand the meaning of major and minor axes and foci.
- Understand the properties of an ellipse that result in a circle.
- Interpret eccentricity in terms of the shape of the ellipse.

Ellipse Centered at the Origin

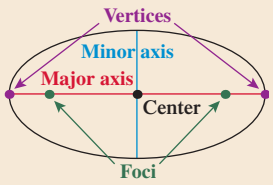
Definition of an Ellipse

If we were to take a piece of string, tie loops at both ends, and tack the ends down so that the string had lots of slack, we would have the picture on the right. If we then took a pencil and pulled the string taut and traced our way around for one full rotation, the result would be an ellipse. See the second figure on the right.



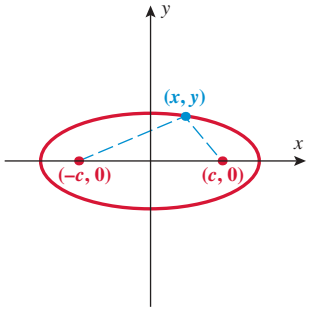
DEFINITION **Ellipse**

An **ellipse** is the set of all points in a plane the sum of whose distances from two fixed points is constant. These two fixed points are called **foci** (plural of focus). A line segment through the foci called the **major axis** intersects the ellipse at the **vertices**. The midpoint of the line segment joining the vertices is called the **center**. The line segment that intersects the center and joins two points on the ellipse and is perpendicular to the major axis is called the **minor axis**.



Let's start with an ellipse whose center is located at the origin. Using graph-shifting techniques, we can later extend the characteristics of an ellipse centered at a point other than the origin. Ellipses can vary from the shape of circles to something quite elongated, either horizontally or vertically, that resembles the shape of a racetrack. We say that the ellipse has either greater (elongated) or lesser (circular) *eccentricity*; as we will see, there is a simple mathematical definition of *eccentricity*. It can be shown that the standard equation of an ellipse with its center at the origin is given by one of two forms, depending on whether the orientation of the major axis of the ellipse is horizontal or vertical. For $a > b > 0$, if the major axis is horizontal, then the equation is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and if the major axis is vertical, then the equation is given by $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Let's consider an ellipse with its center at the origin and the foci on the x-axis. Let the distance from the center to the focus be c. Therefore, the foci are located at the points $(-c, 0)$ and $(c, 0)$. The line segment containing the foci is called the major axis, and it lies along the x-axis. The sum of the two distances from the foci to any point (x, y) must be constant.



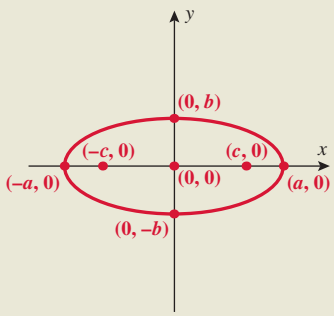
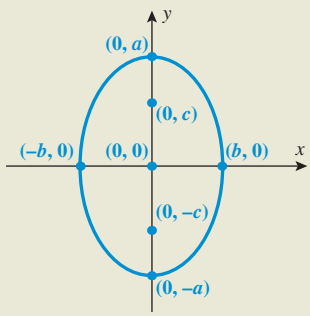
Derivation of the Equation of an Ellipse

Words	Math
Calculate the distance from (x, y) to $(-c, 0)$ by applying the distance formula.	$\sqrt{[x-(-c)]^2+y^2}$
Calculate the distance from (x, y) to $(c, 0)$ by applying the distance formula.	$\sqrt{(x-c)^2+y^2}$
The sum of these two distances is equal to a constant ($2a$ for convenience).	$\sqrt{[x-(-c)]^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$
Isolate one radical.	$\sqrt{[x-(-c)]^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$
Square both sides of the equation.	$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$
Square the binomials inside the parentheses.	$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$
Simplify.	$4cx-4a^2=-4a\sqrt{(x-c)^2+y^2}$
Divide both sides of the equation by –4.	$a^2-cx=a\sqrt{(x-c)^2+y^2}$
Square both sides of the equation.	$(a^2-cx)^2=a^2\left[(x-c)^2+y^2\right]$
Square the binomials inside the parentheses.	$a^4-2a^2cx+c^2x^2=a^2(x^2-2cx+c^2+y^2)$
Distribute the a^2 term.	$a^4-2a^2cx+c^2x^2=a^2x^2-2a^2cx+a^2c^2+a^2y^2$
Group the x and y terms together, respectively, on one side and constants on the other side.	$c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$
Factor out the common factors.	$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2)$
Multiply both sides of the equation by –1.	$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$
We can make the argument that $a>c$ in order for a point to be on the ellipse (and not on the x-axis). Thus, since a and c represent distances and therefore are positive, we know that $a^2>c^2$, or $a^2-c^2>0$. Hence, we can divide both sides of the equation by a^2-c^2, since $a^2-c^2\neq 0$.	$x^2+\dfrac{a^2y^2}{(a^2-c^2)}=a^2$
Let $b^2=a^2-c^2$.	$x^2+\dfrac{a^2y^2}{b^2}=a^2$
Divide both sides of the equation by a^2.	$\boxed{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1}$

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents an ellipse with its center at the origin with the foci along the x-axis, since $a > b$. The following box summarizes ellipses that have their center at the origin and foci along either the x-axis or y-axis:

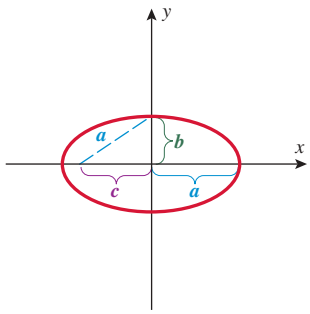
EQUATION OF AN ELLIPSE WITH CENTER AT THE ORIGIN

The **standard form of the equation of an ellipse** with its center at the origin is given by

ORIENTATION OF MAJOR AXIS	Horizontal (along the x-axis)	Vertical (along the y-axis)
EQUATION	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b > 0$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a > b > 0$
FOCI	$(-c, 0)$ and $(c, 0)$ where $c^2 = a^2 - b^2$	$(0, -c)$ and $(0, c)$ where $c^2 = a^2 - b^2$
VERTICES	$(-a, 0)$ and $(a, 0)$	$(0, -a)$ and $(0, a)$
OTHER INTERCEPTS	$(0, b)$ and $(0, -b)$	$(b, 0)$ and $(-b, 0)$
GRAPH		

In both cases, the value of c, the distance along the major axis from the center to the focus, is given by $c^2 = a^2 - b^2$. The length of the major axis is $2a$ and the length of the minor axis is $2b$.

Notice that when $a = b$, the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ simplifies to $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ or $x^2 + y^2 = a^2$, which corresponds to a circle. The vertices correspond to intercepts when an ellipse is centered at the origin. One of the first things we notice about an ellipse is its *eccentricity*. The **eccentricity**, denoted e, is given by $e = \frac{c}{a}$, where $0 < e < 1$. The circle is a limiting form of an ellipse, $c = 0$. In other words, if the eccentricity is close to 0, then the ellipse resembles a circle, whereas if the eccentricity is close to 1, then the ellipse is quite elongated, or eccentric.



Graphing an Ellipse with Center at the Origin

The equation of an ellipse in standard form can be used to graph an ellipse. Although an ellipse is defined in terms of the foci, the foci are not part of the graph. It is important to note that if the divisor of the term with x^2 is larger than the divisor of the term with y^2, then the ellipse is elongated horizontally.

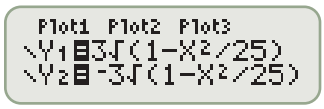
Technology Tip

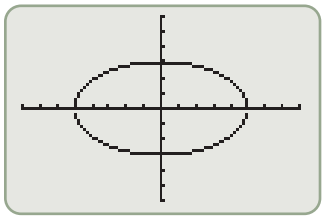
Use a graphing calculator to check the graph of $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Solve for y first. That is,

$y_1 = 3\sqrt{1 - \frac{x^2}{25}}$ and

$y_2 = -3\sqrt{1 - \frac{x^2}{25}}$.





EXAMPLE 1 Graphing an Ellipse with a Horizontal Major Axis

Graph the ellipse given by $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

Solution:

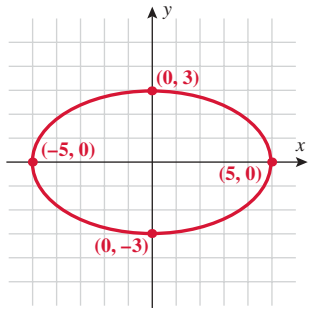
Since $25 > 9$, the major axis is horizontal. $a^2 = 25$ and $b^2 = 9$

Solve for a and b. $a = 5$ and $b = 3$

Identify the vertices: $(-a, 0)$ and $(a, 0)$. $(-5, 0)$ and $(5, 0)$

Identify the endpoints (y-intercepts) on the minor axis: $(0, -b)$ and $(0, b)$. $(0, -3)$ and $(0, 3)$

Graph by labeling the points $(-5, 0)$, $(5, 0)$, $(0, -3)$, and $(0, 3)$ and connecting them with a smooth curve.



If the divisor of x^2 is larger than the divisor of y^2, then the major axis is horizontal along the x-axis, as in Example 1. If the divisor of y^2 is larger than the divisor of x^2, then the major axis is vertical along the y-axis, as you will see in Example 2.

Study Tip

If the divisor of x^2 is larger than the divisor of y^2, then the major axis is horizontal along the x-axis, as in Example 1. If the divisor of y^2 is larger than the divisor of x^2, then the major axis is vertical along the y-axis, as you will see in Example 2.

EXAMPLE 2 Graphing an Ellipse with a Vertical Major Axis

Graph the ellipse given by $16x^2 + y^2 = 16$.

Solution:

Write the equation in standard form by dividing by 16. $\frac{x^2}{1} + \frac{y^2}{16} = 1$

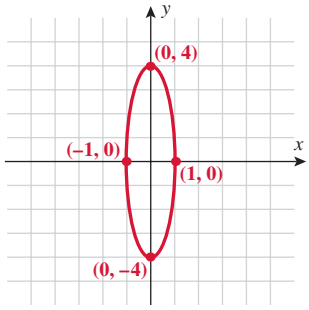
Since $16 > 1$, this ellipse is elongated vertically. $a^2 = 16$ and $b^2 = 1$

Solve for a and b. $a = 4$ and $b = 1$

Identify the vertices: $(0, -a)$ and $(0, a)$. $(0, -4)$ and $(0, 4)$

Identify the x-intercepts on the minor axis: $(-b, 0)$ and $(b, 0)$. $(-1, 0)$ and $(1, 0)$

Graph by labeling the points $(0, -4)$, $(0, 4)$, $(-1, 0)$, and $(1, 0)$ and connecting them with a smooth curve.

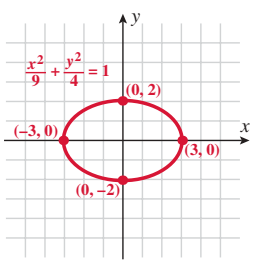


YOUR TURN Graph the ellipses:

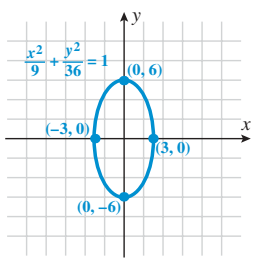
a. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **b.** $\frac{x^2}{9} + \frac{y^2}{36} = 1$

Answer:

a.



b.



Finding the Equation of an Ellipse with Center at the Origin

What if we know the vertices and the foci of an ellipse and want to find the equation to which it corresponds? The axis on which the foci and vertices are located is the major axis. Therefore, we will have the standard equation of an ellipse, and a will be known (from the vertices). Since c is known from the foci, we can use the relation $c^2 = a^2 - b^2$ to determine the unknown b.

EXAMPLE 3 Finding the Equation of an Ellipse Centered at the Origin

Find the standard form of the equation of an ellipse with foci at $(-3, 0)$ and $(3, 0)$ and vertices $(-4, 0)$ and $(4, 0)$.

Solution:

The major axis lies along the x-axis, since it contains the foci and vertices.

Write the corresponding general equation of an ellipse.	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Identify a from the vertices:	
Match vertices $(-4, 0) = (-a, 0)$ and $(4, 0) = (a, 0)$.	$a = 4$
Identify c from the foci:	
Match foci $(-3, 0) = (-c, 0)$ and $(3, 0) = (c, 0)$.	$c = 3$
Substitute $a = 4$ and $c = 3$ into $b^2 = a^2 - c^2$.	$b^2 = 4^2 - 3^2$
Simplify.	$b^2 = 7$
Substitute $a^2 = 16$ and $b^2 = 7$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.	$\frac{x^2}{16} + \frac{y^2}{7} = 1$

The equation of the ellipse is $\boxed{\frac{x^2}{16} + \frac{y^2}{7} = 1}$.

■ **YOUR TURN** Find the standard form of the equation of an ellipse with vertices at $(0, -6)$ and $(0, 6)$ and foci $(0, -5)$ and $(0, 5)$.

■ **Answer:** $\frac{x^2}{11} + \frac{y^2}{36} = 1$

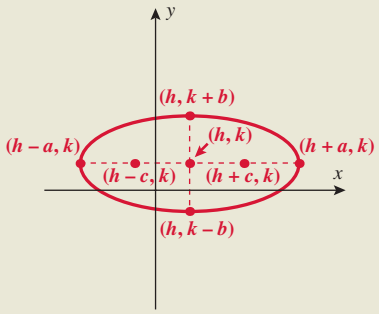
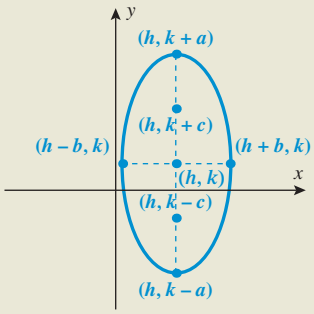
Ellipse Centered at the Point (h, k)

We can use graph-shifting techniques to graph ellipses that are centered at a point other than the origin. For example, to graph $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (assuming h and k are positive constants), start with the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and shift to the right h units and up k units. The center, the vertices, the foci, and the major and minor axes all shift. In other words, the two ellipses are identical in shape and size, except that the ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ is centered at the point (h, k).

The following table summarizes the characteristics of ellipses centered at a point other than the origin:

EQUATION OF AN ELLIPSE WITH CENTER AT THE POINT (h, k)

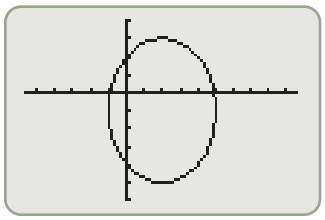
The **standard form of the equation of an ellipse** with its center at the point (h, k) is given by

ORIENTATION OF MAJOR AXIS	Horizontal (parallel to the x-axis)	Vertical (parallel to the y-axis)
EQUATION	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$
GRAPH		
FOCI	$(h - c, k)$ and $(h + c, k)$	$(h, k - c)$ and $(h, k + c)$
VERTICES	$(h - a, k)$ and $(h + a, k)$	$(h, k - a)$ and $(h, k + a)$

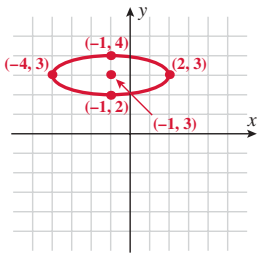
In both cases, $a > b > 0$, $c^2 = a^2 - b^2$, the length of the major axis is $2a$, and the length of the minor axis is $2b$.

Technology Tip

Use a graphing calculator to check the graph of $\dfrac{(x-2)^2}{9} + \dfrac{(y+1)^2}{16} = 1$.



■ **Answer:**



EXAMPLE 4 Graphing an Ellipse with Center (h, k) Given the Equation in Standard Form

Graph the ellipse given by $\dfrac{(x-2)^2}{9} + \dfrac{(y+1)^2}{16} = 1$.

Solution:

Write the equation in the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

$$\frac{(x-2)^2}{3^2} + \frac{[y-(-1)]^2}{4^2} = 1$$

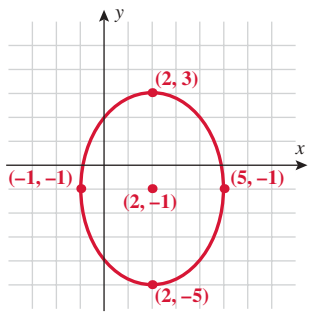
Identify a, b, and the center (h, k).

$a = 4$, $b = 3$, and $(h, k) = (2, -1)$

Draw a graph and label the center: $(2, -1)$.

Since $a = 4$, the vertices are up 4 and down four units from the center: $(2, -5)$ and $(2, 3)$.

Since $b = 3$, the endpoints of the minor axis are to the left and right three units: $(-1, -1)$ and $(5, -1)$.



■ **YOUR TURN** Graph the ellipse given by $\dfrac{(x+1)^2}{9} + \dfrac{(y-3)^2}{1} = 1$.

All active members of the Lambda Chi fraternity are college students, but not all college students are members of the Lambda Chi fraternity. Similarly, all circles are ellipses, but not all ellipses are circles. When $a = b$, the standard equation of an ellipse simplifies to a standard equation of a circle. Recall that when we are given the equation of a circle in general form, we first complete the square in order to express the equation in standard form, which allows the center and radius to be identified. We use that same approach when the equation of an ellipse is given in a general form.

EXAMPLE 5 Graphing an Ellipse with Center (h, k) Given an Equation in General Form

Graph the ellipse given by $4x^2 + 24x + 25y^2 - 50y - 39 = 0$.

Solution:

Transform the general equation into standard form.

Group x terms together and y terms together and add 39 to both sides. $(4x^2 + 24x) + (25y^2 - 50y) = 39$

Factor out the 4 common to the x terms and the 25 common to the y terms. $4(x^2 + 6x) + 25(y^2 - 2y) = 39$

Complete the square on x and y. $4(x^2 + 6x + 9) + 25(y^2 - 2y + 1) = 39 + 4(9) + 25(1)$

Simplify. $4(x + 3)^2 + 25(y - 1)^2 = 100$

Divide by 100. $\dfrac{(x + 3)^2}{25} + \dfrac{(y - 1)^2}{4} = 1$

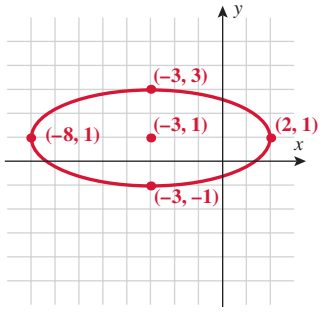
Since $25 > 4$, this is an ellipse with a horizontal major axis.

Now that the equation of the ellipse is in standard form, compare to $\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$ and identify a, b, h, k. $a = 5$, $b = 2$, and $(h, k) = (-3, 1)$

Since $a = 5$, the vertices are five units to left and right of the center. $(-8, 1)$ and $(2, 1)$

Since $b = 2$, the endpoints of the minor axis are up and down two units from the center. $(-3, -1)$ and $(-3, 3)$

Graph.



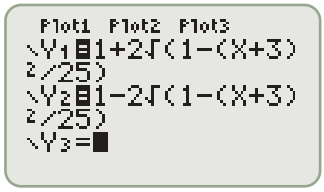
YOUR TURN Write the equation $4x^2 + 32x + y^2 - 2y + 61 = 0$ in standard form. Identify the center, vertices, and endpoints of the minor axis, and graph.

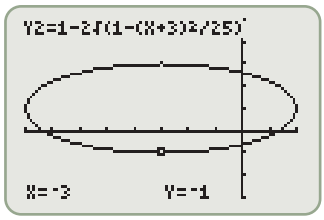
Technology Tip

Use a graphing calculator to check the graph of $4x^2 + 24x + 25y^2 - 50y - 39 = 0$. Use $\dfrac{(x + 3)^2}{25} + \dfrac{(y - 1)^2}{4} = 1$ to solve for y first. That is,

$y_1 = 1 + 2\sqrt{1 - \dfrac{(x + 3)^2}{25}}$ or

$y_2 = 1 - 2\sqrt{1 - \dfrac{(x + 3)^2}{25}}$.





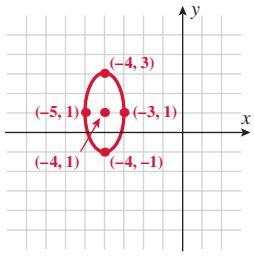
Answer:

$\dfrac{(x + 4)^2}{1} + \dfrac{(y - 1)^2}{4} = 1$

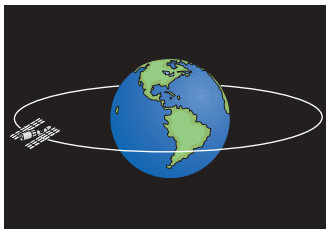
Center: $(-4, 1)$

Vertices: $(-4, -1)$ and $(-4, 3)$

Endpoints of minor axis: $(-5, 1)$ and $(-3, 1)$

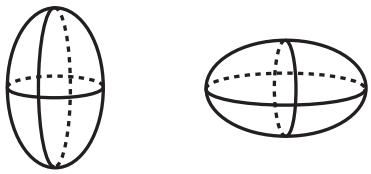


Applications



There are many examples of ellipses all around us. On Earth we have racetracks, and in our solar system, the planets travel in elliptical orbits with the Sun as a focus. Satellites are in elliptical orbits around Earth. Most communications satellites are in a *geosynchronous* (GEO) orbit—they orbit Earth once each day. In order to stay over the same spot on Earth, a *geostationary* satellite has to be directly above the equator; it circles Earth in exactly the time it takes Earth to turn once on its axis, and its orbit has to follow the path of the equator as Earth rotates. Otherwise, from Earth the satellite would appear to move in a north–south line every day.

If we start with an ellipse in the xy-plane and rotate it around its major axis, the result is a three-dimensional ellipsoid.



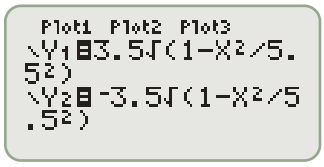
A football and a blimp are two examples of ellipsoids. The ellipsoidal shape allows for a more aerodynamic path.

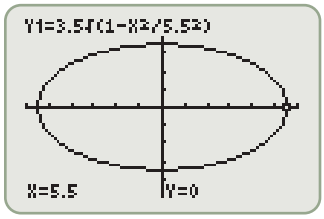
PhotoDisc, Inc.

Peter Phipp/Age Fotostock America, Inc.

Technology Tip

Use a graphing calculator to check the graph of $\frac{x^2}{5.5^2} + \frac{y^2}{3.5^2} = 1$. Solve for y first. That is, $y_1 = 3.5\sqrt{1 - \frac{x^2}{5.5^2}}$ or $y_2 = -3.5\sqrt{1 - \frac{x^2}{5.5^2}}$.





EXAMPLE 6 An Official NFL Football

A longitudinal section (that includes the two vertices and the center) of an official Wilson NFL football is an ellipse. The longitudinal section is approximately 11 inches long and 7 inches wide. Write an equation governing the elliptical longitudinal section.

Solution:

Locate the center of the ellipse at the origin and orient the football horizontally.

Write the general equation of an ellipse centered at the origin. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The length of the major axis is 11 inches. $2a = 11$

Solve for a. $a = 5.5$

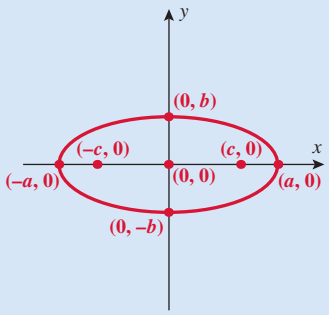
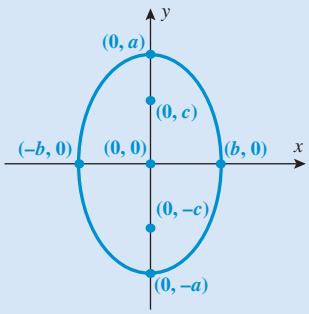
The length of the minor axis is 7 inches. $2b = 7$

Solve for b. $b = 3.5$

Substitute $a = 5.5$ and $b = 3.5$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. $\boxed{\frac{x^2}{5.5^2} + \frac{y^2}{3.5^2} = 1}$

SECTION 9.3 SUMMARY

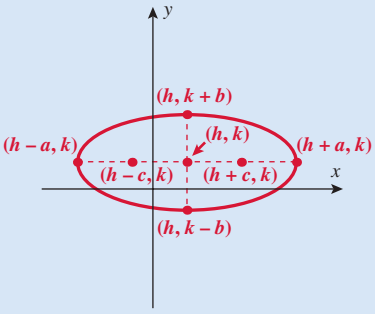
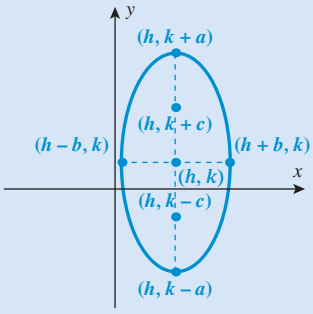
In this section, we first analyzed ellipses that are centered at the origin.

Orientation of Major Axis	Horizontal along the x-axis	Vertical along the y-axis
Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b > 0$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a > b > 0$
Foci*	$(-c, 0)$ and $(c, 0)$	$(0, -c)$ and $(0, c)$
Vertices	$(-a, 0)$ and $(a, 0)$	$(0, -a)$ and $(0, a)$
Other Intercepts	$(0, -b)$ and $(0, b)$	$(-b, 0)$ and $(b, 0)$
Graph		

$*c^2 = a^2 - b^2$

For ellipses centered at the origin, we can graph an ellipse by finding all four intercepts.

For ellipses **centered at the point (h, k),** the major and minor axes and endpoints of the ellipse all shift accordingly. When $a = b$, the ellipse is a circle.

Orientation of Major Axis	Horizontal (parallel to the x-axis)	Vertical (parallel to the y-axis)
Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Graph		
Foci	$(h - c, k)$ and $(h + c, k)$	$(h, k - c)$ and $(h, k + c)$
Vertices	$(h - a, k)$ and $(h + a, k)$	$(h, k - a)$ and $(h, k + a)$

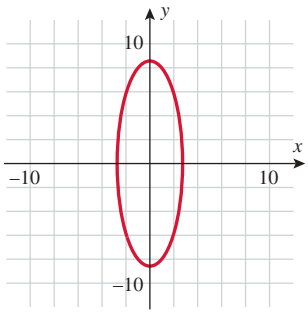
SECTION 9.3 EXERCISES

SKILLS

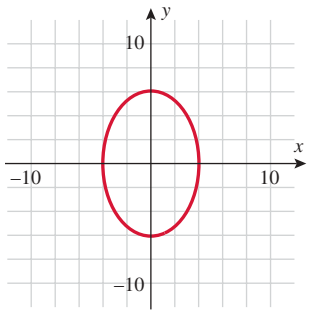
In Exercises 1–4, match the equation to the ellipse.

1. $\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$ **2.** $\dfrac{x^2}{16} + \dfrac{y^2}{36} = 1$ **3.** $\dfrac{x^2}{8} + \dfrac{y^2}{72} = 1$ **4.** $4x^2 + y^2 = 1$

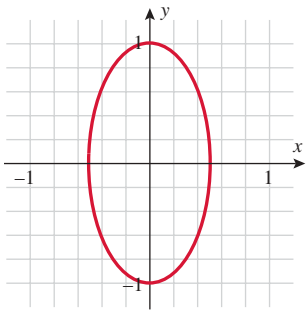
a.



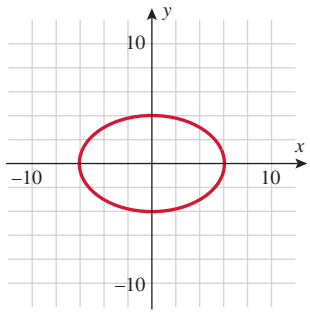
b.



c.



d.



In Exercises 5–16, graph each ellipse. Label the center and vertices.

5. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ **6.** $\dfrac{x^2}{49} + \dfrac{y^2}{9} = 1$ **7.** $\dfrac{x^2}{16} + \dfrac{y^2}{64} = 1$ **8.** $\dfrac{x^2}{25} + \dfrac{y^2}{144} = 1$

9. $\dfrac{x^2}{100} + y^2 = 1$ **10.** $9x^2 + 4y^2 = 36$ **11.** $\dfrac{4}{9}x^2 + 81y^2 = 1$ **12.** $\dfrac{4}{25}x^2 + \dfrac{100}{9}y^2 = 1$

13. $4x^2 + y^2 = 16$ **14.** $x^2 + y^2 = 81$ **15.** $8x^2 + 16y^2 = 32$ **16.** $10x^2 + 25y^2 = 50$

In Exercises 17–24, find the standard form of the equation of an ellipse with the given characteristics.

17. Foci: $(-4, 0)$ and $(4, 0)$ Vertices: $(-6, 0)$ and $(6, 0)$

18. Foci: $(-1, 0)$ and $(1, 0)$ Vertices: $(-3, 0)$ and $(3, 0)$

19. Foci: $(0, -3)$ and $(0, 3)$ Vertices: $(0, -4)$ and $(0, 4)$

20. Foci: $(0, -1)$ and $(0, 1)$ Vertices: $(0, -2)$ and $(0, 2)$

21. Major axis vertical with length of 8, minor axis length of 4, and centered at $(0, 0)$

22. Major axis horizontal with length of 10, minor axis length of 2, and centered at $(0, 0)$

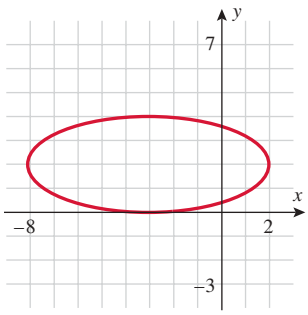
23. Vertices $(0, -7)$ and $(0, 7)$ and endpoints of minor axis $(-3, 0)$ and $(3, 0)$

24. Vertices $(-9, 0)$ and $(9, 0)$ and endpoints of minor axis $(0, -4)$ and $(0, 4)$

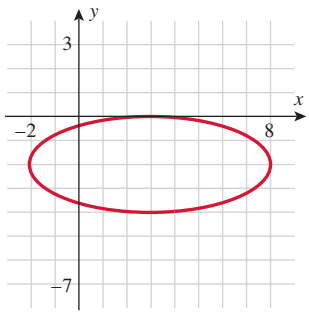
In Exercises 25–28, match each equation with the ellipse.

25. $\dfrac{(x-3)^2}{4} + \dfrac{(y+2)^2}{25} = 1$ **26.** $\dfrac{(x+3)^2}{4} + \dfrac{(y-2)^2}{25} = 1$ **27.** $\dfrac{(x-3)^2}{25} + \dfrac{(y+2)^2}{4} = 1$ **28.** $\dfrac{(x+3)^2}{25} + \dfrac{(y-2)^2}{4} = 1$

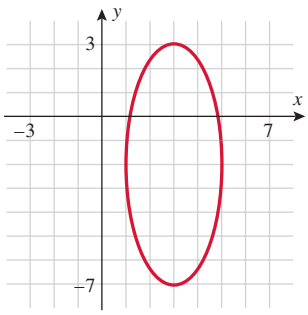
a.



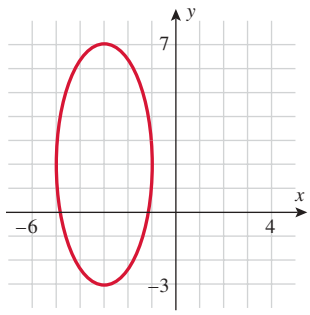
b.



c.



d.



In Exercises 29–38, graph each ellipse. Label the center and vertices.

29. $\dfrac{(x-1)^2}{16} + \dfrac{(y-2)^2}{4} = 1$

30. $\dfrac{(x+1)^2}{36} + \dfrac{(y+2)^2}{9} = 1$

31. $10(x+3)^2 + (y-4)^2 = 80$

32. $3(x+3)^2 + 12(y-4)^2 = 36$

33. $x^2 + 4y^2 - 24y + 32 = 0$

34. $25x^2 + 2y^2 - 4y - 48 = 0$

35. $x^2 - 2x + 2y^2 - 4y - 5 = 0$

36. $9x^2 - 18x + 4y^2 - 27 = 0$

37. $5x^2 + 20x + y^2 + 6y - 21 = 0$

38. $9x^2 + 36x + y^2 + 2y + 36 = 0$

In Exercises 39–46, find the standard form of the equation of an ellipse with the given characteristics.

39. Foci: $(-2, 5)$ and $(6, 5)$ Vertices: $(-3, 5)$ and $(7, 5)$

40. Foci: $(2, -2)$ and $(4, -2)$ Vertices: $(0, -2)$ and $(6, -2)$

41. Foci: $(4, -7)$ and $(4, -1)$ Vertices: $(4, -8)$ and $(4, 0)$

42. Foci: $(2, -6)$ and $(2, -4)$ Vertices:$(2, -7)$ and $(2, -3)$

43. Major axis vertical with length of 8, minor axis length of 4, and centered at $(3, 2)$

44. Major axis horizontal with length of 10, minor axis length of 2, and centered at $(-4, 3)$

45. Vertices $(-1, -9)$ and $(-1, 1)$ and endpoints of minor axis $(-4, -4)$ and $(2, -4)$

46. Vertices $(-2, 3)$ and $(6, 3)$ and endpoints of minor axis $(2, 1)$ and $(2, 5)$

APPLICATIONS

47. Carnival Ride. The Zipper, a favorite carnival ride, maintains an elliptical shape with a major axis of 150 feet and a minor axis of 30 feet. Assuming it is centered at the origin, find an equation for the ellipse.

Courtesy Chance Morgan, Inc.

Zipper

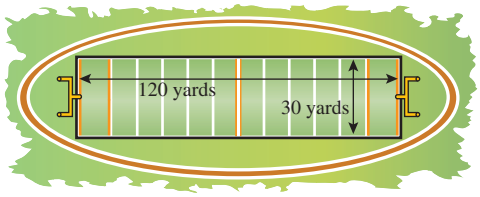
48. Carnival Ride. A Ferris wheel traces an elliptical path with both a major and minor axis of 180 feet. Assuming it is centered at the origin, find an equation for the ellipse (circle).

Tina Buckman/Index Stock

Ferris wheel, Barcelona, Spain

For Exercises 49 and 50, refer to the following information:

A high school wants to build a football field surrounded by an elliptical track. A regulation football field must be 120 yards long and 30 yards wide.



49. Sports Field. Suppose the elliptical track is centered at the origin and has a horizontal major axis of length 150 yards and a minor axis length of 40 yards.

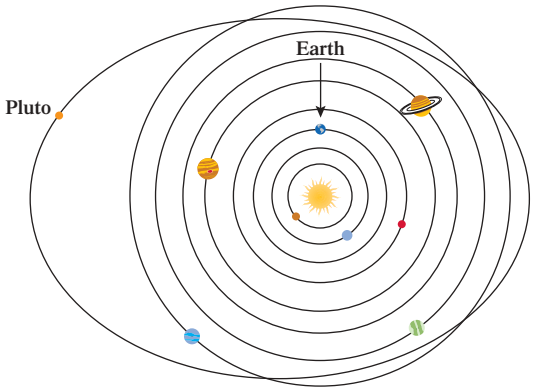
a. Write an equation for the ellipse.

b. Find the width of the track at the end of the field. Will the track completely enclose the football field?

50. Sports Field. Suppose the elliptical track is centered at the origin and has a horizontal major axis of length 150 yards. How long should the minor axis be in order to enclose the field?

For Exercises 51 and 52, refer to orbits in our solar system:

The planets have elliptical orbits with the Sun as one of the foci. Pluto (orange), the planet furthest from the Sun, has a very elongated, or flattened, elliptical orbit, whereas Earth (royal blue) has an almost circular orbit. Because of Pluto's flattened path, it is not always the planet furthest from the Sun.



51. Planetary Orbits. The orbit of the dwarf planet Pluto has approximately the following characteristics (assume the Sun is the focus):

- The length of the major axis $2a$ is approximately 11,827,000,000 kilometers.
- The perihelion distance from the dwarf planet to the Sun is 4,447,000,000 kilometers.

Determine the equation for Pluto's elliptical orbit around the Sun.

52. Planetary Orbits. Earth's orbit has approximately the following characteristics (assume the Sun is the focus):

- The length of the major axis $2a$ is approximately 299,700,000 kilometers.
- The perihelion distance from Earth to the Sun is 147,100,000 kilometers.

Determine the equation for Earth's elliptical orbit around the Sun.

For Exercises 53 and 54, refer to the following information:

Asteroids orbit the Sun in elliptical patterns and often cross paths with Earth's orbit, making life a little tense now and again. A few asteroids have orbits that cross Earth's orbit—called "Apollo asteroids" or "Earth-crossing asteroids." In recent years, asteroids have passed within 100,000 kilometers of Earth!

53. Asteroids. Asteroid 433, or Eros, is the second largest near-Earth asteroid. The semimajor axis is 150 million kilometers and the eccentricity is 0.223, where eccentricity is defined as $e = \sqrt{1 - \frac{b^2}{a^2}}$, where a is the semimajor axis or $2a$ is the major axis, and b is the semiminor axis or $2b$ is the minor axis. Find the equation of Eros's orbit. Round a and b to the nearest million kilometers.

54. Asteroids. The asteroid Toutatis is the largest near-Earth asteroid. The semimajor axis is 350 million kilometers and the eccentricity is 0.634, where eccentricity is defined as $e = \sqrt{1 - \frac{b^2}{a^2}}$, where a is the semimajor axis or $2a$ is the major axis, and b is the semimajor axis or $2b$ is the minor axis. On September 29, 2004, it missed Earth by 961,000 miles. Find the equation of Toutatis's orbit.

55. Halley's Comet. The eccentricity of Halley's Comet is approximately 0.967. If a comet had e almost equal to 1, what would its orbit appear to be from Earth?

56. Halley's Comet. The length of the semimajor axis is 17.8 AU (astronomical units) and the eccentricity is approximately 0.967. Find the equation of Halley's Comet. (Assume 1 AU = 150 million km.)

57. Medicine. The second time that a drug is given to a patient, the relationship between the drug concentration c (in milligrams per cm^3) and the time t (in hours) is given by

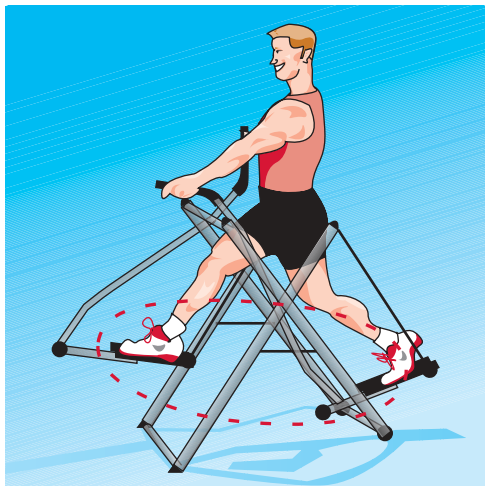
$$t^2 + 9c^2 - 6t - 18c + 9 = 0$$

Determine the highest concentration of drug in the patient's bloodstream.

58. Fuel Transportation. Tanks built to transport fuel and other hazardous materials have an elliptical cross section, which makes them more stable for the transportation of these materials. If the lengths of the major and minor axes of the elliptical cross section are 8 feet and 6 feet, respectively, and the tank is 30 feet long, find the volume of the tank. Round your answer to the nearest integer. *Hint:* The area of an ellipse is $\pi \cdot a \cdot b$.

For Exercises 59 and 60, refer to the following:

An elliptical trainer is an exercise machine that can be used to simulate stair climbing, walking, or running. The stride length is the length of a step on the trainer (forward foot to rear foot). The minimum step-up height is the height of a pedal at its lowest point, while the maximum step-up height is the height of a pedal at its highest point.



59. Health/Exercise. An elliptical trainer has a stride length of 16 inches. The maximum step-up height is 12.5 inches, while the minimum step-up height is 2.5 inches.

a. Find the equation of the ellipse traced by the pedals assuming the origin lies at the pedal axle (center of the ellipse is at the origin).

b. Use the approximation to the perimeter of an ellipse $p = \pi\sqrt{2(a^2 + b^2)}$ to find the distance, to the nearest inch, traveled in one complete step (revolution of a pedal).

c. How many steps, to the nearest step, are necessary to travel a distance of one mile?

60. Health/Exercise. An elliptical trainer has a stride length of 18 inches. The maximum step-up height is 13.5 inches, while the minimum step-up height is 3.5 inches. Find the equation of the ellipse traced by the pedals assuming the origin lies at the pedal axle.

a. Find the equation of the ellipse traced by the pedals assuming the origin lies at the pedal axle (center of the ellipse is at the origin).

b. Use the approximation to the perimeter of an ellipse $p = \pi\sqrt{2(a^2 + b^2)}$ to find the distance, to the nearest inch, traveled in one complete step (revolution of a pedal).

c. How many steps, to the nearest step, are necessary to travel a distance of one mile?

CATCH THE MISTAKE

In Exercises 61 and 62, explain the mistake that is made.

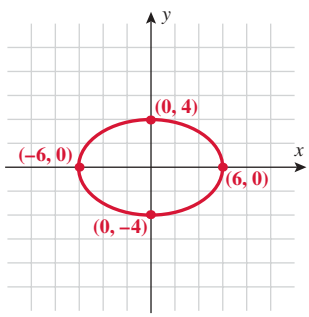
61. Graph the ellipse given by $\frac{x^2}{6} + \frac{y^2}{4} = 1$.

Solution:

Write the standard form of the equation of an ellipse. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Identify a and b. $a = 6, b = 4$

Label the vertices and the endpoints of the minor axis, $(-6, 0)$, $(6, 0)$, $(0, -4)$, $(0, 4)$, and connect with an elliptical curve.



This is incorrect. What mistake was made?

62. Determine the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Solution:

Write the general equation of a horizontal ellipse. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Identify a and b. $a = 4, b = 3$

Substitute $a = 4, b = 3$ into $c^2 = a^2 + b^2$. $c^2 = 4^2 + 3^2$

Solve for c. $c = 5$

Foci are located at $(-5, 0)$ and $(5, 0)$.

The points $(-5, 0)$ and $(5, 0)$ are located outside of the ellipse.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 63–66, determine whether each statement is true or false.

63. If you know the vertices of an ellipse, you can determine the equation for the ellipse.

64. If you know the foci and the endpoints of the minor axis, you can determine the equation for the ellipse.

65. Ellipses centered at the origin have symmetry with respect to the x-axis, y-axis, and the origin.

66. All ellipses are circles, but not all circles are ellipses.

67. How many ellipses, with major and minor axes parallel to the coordinate axes, have focus $(-2, 0)$ and pass through the point $(-2, 2)$?

68. How many ellipses have vertices $(-3, 0)$ and $(3, 0)$?

69. If two ellipses intersect each other, what is the minimum number of intersection points?

70. If two ellipses intersect each other, what is the maximum number of intersection points?

CHALLENGE

71. The eccentricity of an ellipse is defined as $e = \frac{c}{a}$. Compare the eccentricity of the orbit of Pluto to that of Earth (refer to Exercises 51 and 52).

72. The eccentricity of an ellipse is defined as $e = \frac{c}{a}$. Since $a > c > 0$, then $0 < e < 1$. Describe the shape of an ellipse when

a. e is close to zero
b. e is close to one
c. $e = 0.5$

73. Find the equation of an ellipse centered at the origin containing the points (1, 3) and (4, 2).

74. Find the equation of an ellipse centered at the origin containing the points $\left(1, \frac{6\sqrt{5}}{5}\right)$ and $\left(-\frac{5}{3}, 2\right)$.

75. Find the equation of an ellipse centered at $(2, -3)$ that passes through the points $\left(1, -\frac{1}{3}\right)$ and $(5, -3)$.

76. Find the equation of an ellipse centered at $(1, -2)$ that passes through the points $(1, -4)$ and $(2, -2)$.

TECHNOLOGY

77. Graph the following three ellipses: $x^2 + y^2 = 1$, $x^2 + 5y^2 = 1$, and $x^2 + 10y^2 = 1$. What can be said to happen to the ellipse $x^2 + cy^2 = 1$ as c increases?

78. Graph the following three ellipses: $x^2 + y^2 = 1$, $5x^2 + y^2 = 1$, and $10x^2 + y^2 = 1$. What can be said to happen to the ellipse $cx^2 + y^2 = 1$ as c increases?

79. Graph the following three ellipses: $x^2 + y^2 = 1$, $5x^2 + 5y^2 = 1$, and $10x^2 + 10y^2 = 1$. What can be said to happen to the ellipse $cx^2 + cy^2 = 1$ as c increases?

80. Graph the equation $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Notice what a difference the sign makes. Is this an ellipse?

81. Graph the following three ellipses: $x^2 + y^2 = 1$, $0.5x^2 + y^2 = 1$, and $0.05x^2 + y^2 = 1$. What can be said to happen to ellipse $cx^2 + y^2 = 1$ as c decreases?

82. Graph the following three ellipses: $x^2 + y^2 = 1$, $x^2 + 0.5y^2 = 1$, and $x^2 + 0.05y^2 = 1$. What can be said to happen to ellipse $x^2 + cy^2 = 1$ as c decreases?

PREVIEW TO CALCULUS

In calculus, the derivative of a function is used to find its maximum and minimum values. In the case of an ellipse, with major and minor axes parallel to the coordinate axes, the maximum and minimum values correspond to the y-coordinate of the vertices that lie on its vertical axis of symmetry.

In Exercises 83–86, find the maximum and minimum values of each ellipse.

83. $4x^2 + y^2 - 24x + 10y + 57 = 0$

84. $9x^2 + 4y^2 + 72x + 16y + 124 = 0$

85. $81x^2 + 100y^2 - 972x + 1600y + 1216 = 0$

86. $25x^2 + 16y^2 + 200x + 256y - 176 = 0$

SECTION 9.4 THE HYPERBOLA

SKILLS OBJECTIVES

- Find a hyperbola's foci and vertices.
- Find the equation of a hyperbola centered at the origin.
- Graph a hyperbola using asymptotes as graphing aids.
- Find the equation of a hyperbola centered at the point (h, k).
- Solve applied problems that involve hyperbolas.

CONCEPTUAL OBJECTIVES

- Derive the general equation of a hyperbola.
- Identify, apply, and graph the transverse axis, vertices, and foci.
- Use asymptotes to determine the shape of a hyperbola.

Hyperbola Centered at the Origin

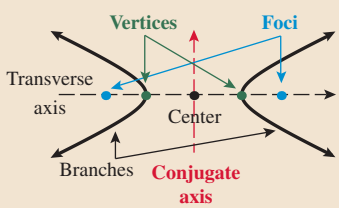
The definition of a hyperbola is similar to the definition of an ellipse. An ellipse is the set of all points, the *sum* of whose distances from two points (the foci) is constant. A *hyperbola* is the set of all points, the *difference* of whose distances from two points (the foci) is constant. What distinguishes their equations is a minus sign.

Ellipse centered at the origin: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

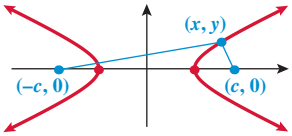
Hyperbola centered at the origin: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

DEFINITION **Hyperbola**

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points is a positive constant. These two fixed points are called **foci**. The hyperbola has two separate curves called **branches**. The two points where the hyperbola intersects the line joining the foci are called **vertices**. The line segment joining the vertices is called the **transverse axis of the hyperbola**. The midpoint of the transverse axis is called the **center**.



Let's consider a hyperbola with the center at the origin and the foci on the x-axis. Let the distance from the center to the focus be c. Therefore, the foci are located at the points $(-c, 0)$ and $(c, 0)$. The difference of the two distances from the foci to any point (x, y) must be constant. We then can follow a similar analysis as done with an ellipse.



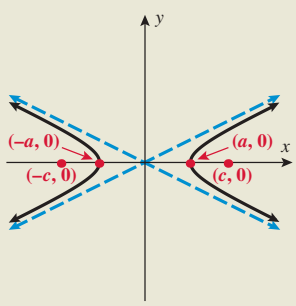
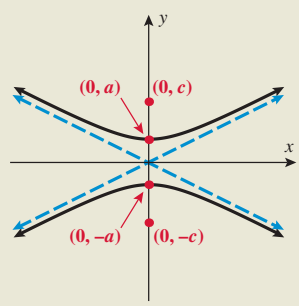
Derivation of the Equation of a Hyperbola

WORDS	MATH
The difference of these two distances is equal to a constant ($2a$ for convenience).	$\sqrt{[x-(-c)]^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$
Following the same procedure that we did with an ellipse leads to:	$(c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2)$
We can make the argument that $c>a$ in order for a point to be on the hyperbola (and not on the x-axis). Therefore, since a and c represent distances and therefore are positive, we know that $c^2>a^2$, or $c^2-a^2>0$. Hence, we can divide both sides of the equation by c^2-a^2, since $c^2-a^2\neq 0$.	$x^2-\dfrac{a^2y^2}{(c^2-a^2)}=a^2$
Let $b^2=c^2-a^2$.	$x^2-\dfrac{a^2y^2}{b^2}=a^2$
Divide both sides of the equation by a^2.	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

The equation $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ represents a hyperbola with its center at the origin, with the foci along the x-axis. The following box summarizes hyperbolas that have their center at the origin and foci along either the x-axis or y-axis:

EQUATION OF A HYPERBOLA WITH CENTER AT THE ORIGIN

The **standard form of the equation of a hyperbola** with its center at the origin is given by

ORIENTATION OF TRANSVERSE AXIS	Horizontal (along the x-axis)	Vertical (along the y-axis)
EQUATION	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$	$\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2}=1$
FOCI	$(-c, 0)$ and $(c, 0)$ where $c^2=a^2+b^2$	$(0, -c)$ and $(0, c)$ where $c^2=a^2+b^2$
ASYMPTOTES	$y=\dfrac{b}{a}x$ and $y=-\dfrac{b}{a}x$	$y=\dfrac{a}{b}x$ and $y=-\dfrac{a}{b}x$
VERTICES	$(-a, 0)$ and $(a, 0)$	$(0, -a)$ and $(0, a)$
TRANSVERSE AXIS	Horizontal length $2a$	Vertical length $2a$
GRAPH		

Note that for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $x = 0$, then $-\frac{y^2}{b^2} = 1$, which yields an imaginary number for y.

However, when $y = 0$, $\frac{x^2}{a^2} = 1$, and therefore $x = \pm a$. The vertices for this hyperbola are $(-a, 0)$ and $(a, 0)$.

EXAMPLE 1 Finding the Foci and Vertices of a Hyperbola Given the Equation

Find the foci and vertices of the hyperbola given by $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

Solution:

Compare to the standard equation of a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. $\quad a^2 = 9, b^2 = 4$

Solve for a and b. $\quad a = 3, b = 2$

Substitute $a = 3$ into the vertices, $(-a, 0)$ and $(a, 0)$. $\quad (-3, 0)$ and $(3, 0)$

Substitute $a = 3, b = 2$ into $c^2 = a^2 + b^2$. $\quad c^2 = 3^2 + 2^2$

Solve for c. $\quad c^2 = 13$

$c = \sqrt{13}$

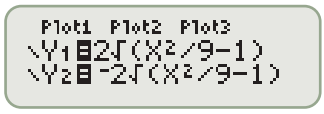
Substitute $c = \sqrt{13}$ into the foci, $(-c, 0)$ and $(c, 0)$. $\quad (-\sqrt{13}, 0)$ and $(\sqrt{13}, 0)$

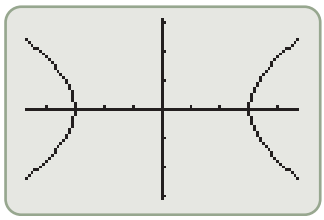
The vertices are $\boxed{(-3, 0)}$ and $\boxed{(3, 0)}$, and the foci are $\boxed{(-\sqrt{13}, 0)}$ and $\boxed{(\sqrt{13}, 0)}$.

■ **YOUR TURN** Find the vertices and foci of the hyperbola $\frac{y^2}{16} - \frac{x^2}{20} = 1$.

Technology Tip

Use a graphing calculator to check the graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Solve for y first. That is, $y_1 = 2\sqrt{\frac{x^2}{9} - 1}$ or $y_2 = -2\sqrt{\frac{x^2}{9} - 1}$.





■ **Answer:** Vertices: $(0, -4)$ and $(0, 4)$
Foci: $(0, -6)$ and $(0, 6)$

EXAMPLE 2 Finding the Equation of a Hyperbola Given Foci and Vertices

Find the standard form of the equation of a hyperbola whose vertices are located at $(0, -4)$ and $(0, 4)$ and whose foci are located at $(0, -5)$ and $(0, 5)$.

Solution:

The center is located at the midpoint of the segment joining the vertices. $\quad \left(\frac{0 + 0}{2}, \frac{-4 + 4}{2}\right) = (0, 0)$

Since the foci and vertices are located on the y-axis, the standard equation is given by: $\quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

The vertices $(0, \pm a)$ and the foci $(0, \pm c)$ can be used to identify a and c. $\quad a = 4, c = 5$

Substitute $a = 4, c = 5$ into $b^2 = c^2 - a^2$. $\quad b^2 = 5^2 - 4^2$

Solve for b. $\quad b^2 = 25 - 16 = 9$

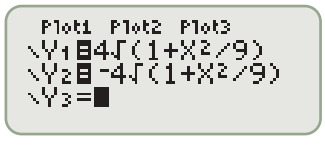
$b = 3$

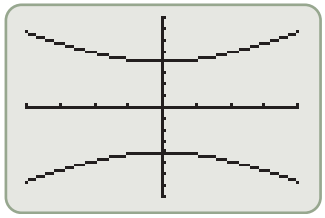
Substitute $a = 4$ and $b = 3$ into $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. $\quad \boxed{\frac{y^2}{16} - \frac{x^2}{9} = 1}$

■ **YOUR TURN** Find the equation of a hyperbola whose vertices are located at $(-2, 0)$ and $(2, 0)$ and whose foci are located at $(-4, 0)$ and $(4, 0)$.

Technology Tip

Use a graphing calculator to check the graph of $\frac{y^2}{16} - \frac{x^2}{9} = 1$. Solve for y first. That is, $y_1 = 4\sqrt{1 + \frac{x^2}{9}}$ or $y_2 = -4\sqrt{1 + \frac{x^2}{9}}$.



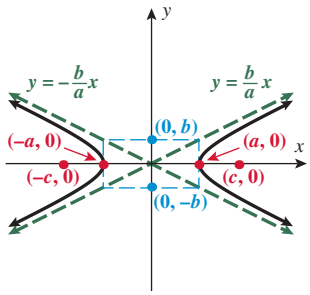


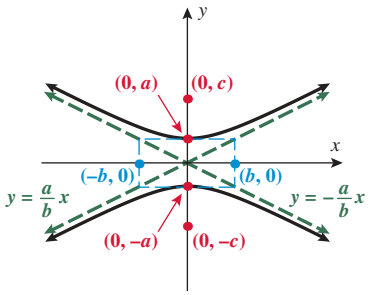
■ **Answer:** $\frac{x^2}{4} - \frac{y^2}{12} = 1$

Graphing a Hyperbola Centered at the Origin

To graph a hyperbola, we use the vertices and asymptotes. The asymptotes are found by the equations $y = \pm\frac{b}{a}x$ or $y = \pm\frac{a}{b}x$, depending on whether the transverse axis is horizontal or vertical. An easy way to draw these graphing aids is to first draw the rectangular box that passes through the vertices and the points $(0, \pm b)$ or $(\pm b, 0)$. The **conjugate axis** is perpendicular to the transverse axis and has length $2b$. The asymptotes pass through the center of the hyperbola and the corners of the rectangular box.

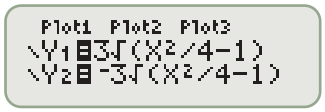
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad\qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

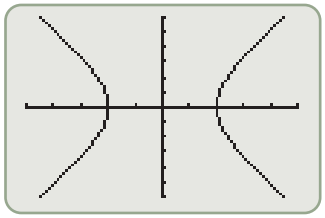




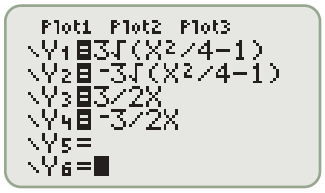
Technology Tip

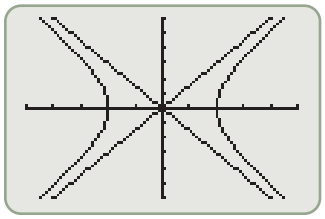
Use a graphing calculator to check the graph of $\frac{x^2}{4} - \frac{y^2}{9} = 1$. Solve for y first. That is, $y_1 = 3\sqrt{\frac{x^2}{4} - 1}$ or $y_2 = -3\sqrt{\frac{x^2}{4} - 1}$.





To add the two asymptotes, enter $y_3 = \frac{3}{2}x$ or $y_4 = -\frac{3}{2}x$.





EXAMPLE 3 Graphing a Hyperbola Centered at the Origin with a Horizontal Transverse Axis

Graph the hyperbola given by $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

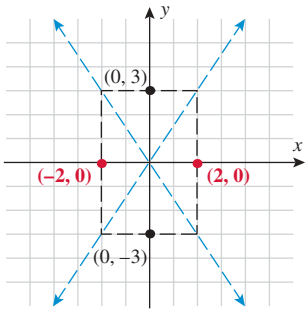
Solution:

Compare $\frac{x^2}{2^2} - \frac{y^2}{3^2} = 1$ to the general equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

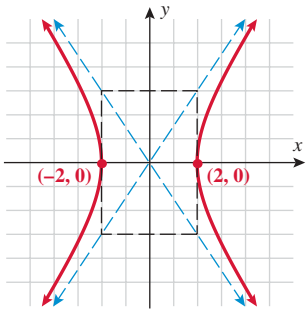
Identify a and b. $\qquad a = 2$ and $b = 3$

The transverse axis of this hyperbola lies on the x-axis.

Label the vertices $(-a, 0) = \mathbf{(-2, 0)}$ and $(a, 0) = \mathbf{(2, 0)}$ and the points $(0, -b) = (0, -3)$ and $(0, b) = (0, 3)$. Draw the rectangular box that passes through those points. Draw the **asymptotes** that pass through the center and the corners of the rectangle.



Draw the two **branches** of the hyperbola, each passing through a vertex and guided by the asymptotes.



In Example 3, if we let $y = 0$, then $\frac{x^2}{4} = 1$ or $x = \pm 2$. Thus the vertices are $(-2, 0)$ and $(2, 0)$, and the transverse axis lies along the x-axis. Note that if $x = 0$, $y = \pm 3i$.

EXAMPLE 4 Graphing a Hyperbola Centered at the Origin with a Vertical Transverse Axis

Graph the hyperbola given by $\frac{y^2}{16} - \frac{x^2}{4} = 1$.

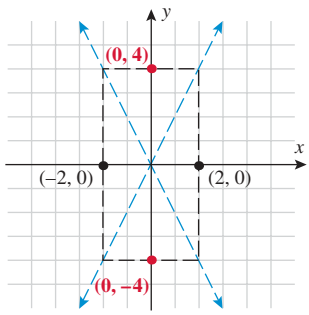
Solution:

Compare $\frac{y^2}{4^2} - \frac{x^2}{2^2} = 1$ to the general equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

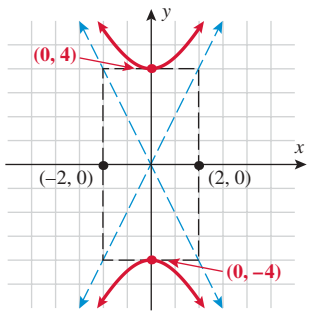
Identify a and b. $a = 4$ and $b = 2$

The transverse axis of this hyperbola lies along the y-axis.

Label the vertices $(0, -a) = \mathbf{(0, -4)}$ and $(0, a) = \mathbf{(0, 4)}$, and the points $(-b, 0) = (-2, 0)$ and $(b, 0) = (2, 0)$. Draw the rectangular box that passes through those points. Draw the **asymptotes** that pass through the center and the corners of the rectangle.



Draw the two **branches** of the hyperbola, each passing through a vertex and guided by the asymptotes.

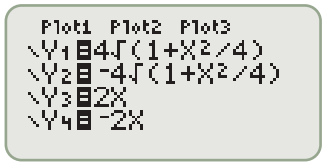


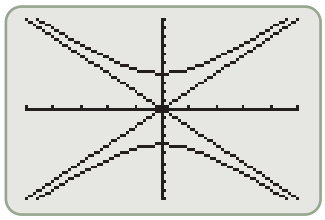
YOUR TURN Graph the hyperbolas:

a. $\frac{y^2}{1} - \frac{x^2}{4} = 1$ **b.** $\frac{x^2}{4} - \frac{y^2}{1} = 1$

Technology Tip

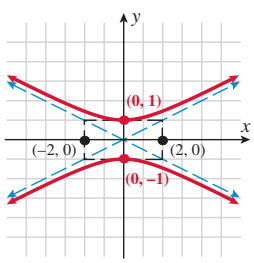
Use a graphing calculator to check the graph of $\frac{y^2}{16} - \frac{x^2}{4} = 1$. Solve for y first. That is, $y_1 = 4\sqrt{1 + \frac{x^2}{4}}$ or $y_2 = -4\sqrt{1 + \frac{x^2}{4}}$. To add the two asymptotes, enter $y_3 = 2x$ or $y_4 = -2x$.



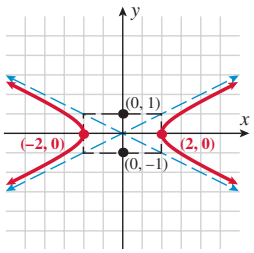


Answer:

a.



b.

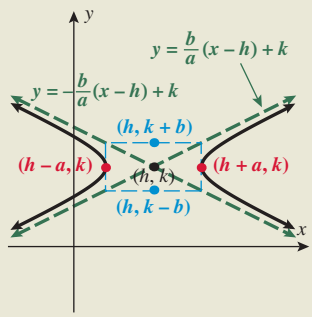
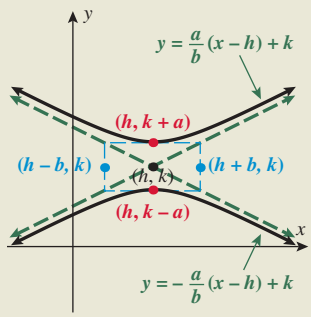


Hyperbola Centered at the Point (h, k)

We can use graph-shifting techniques to graph hyperbolas that are centered at a point other than the origin—say, (h, k). For example, to graph $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, start with the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and shift to the right h units and up k units. The center, the vertices, the foci, the transverse and conjugate axes, and the asymptotes all shift. The following table summarizes the characteristics of hyperbolas centered at a point other than the origin:

Equation of a Hyperbola with Center at the Point (h, k)

The **standard form of the equation of a hyperbola** with its center at the point (h, k) is given by

Orientation of Transverse Axis	Horizontal (parallel to the x-axis)	Vertical (parallel to the y-axis)
Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Vertices	$(h-a, k)$ and $(h+a, k)$	$(h, k-a)$ and $(h, k+a)$
Foci	$(h-c, k)$ and $(h+c, k)$ where $c^2 = a^2 + b^2$	$(h, k-c)$ and $(h, k+c)$ where $c^2 = a^2 + b^2$
Graph		

EXAMPLE 5 Graphing a Hyperbola with Center Not at the Origin

Graph the hyperbola $\dfrac{(y-2)^2}{16} - \dfrac{(x-1)^2}{9} = 1.$

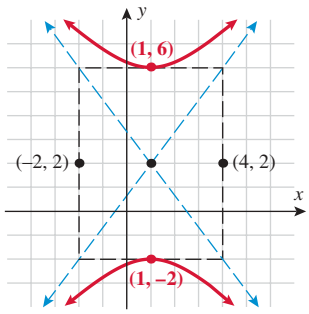
Solution:

Compare $\dfrac{(y-2)^2}{4^2} - \dfrac{(x-1)^2}{3^2} = 1$ to the general equation $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1.$

Identify a, b, and (h, k). $a = 4$, $b = 3$, and $(h, k) = (1, 2)$

The transverse axis of this hyperbola lies along $x = 2$, which is parallel to the y-axis.

Label the vertices $(h, k - a) =$ **(1, −2)** and $(h, k + a) =$ **(1, 6)** and the points $(h - b, k) = (-2, 2)$ and $(h + b, k) = (4, 2)$. Draw the rectangular box that passes through those points. Draw the **asymptotes** that pass through the center $(h, k) = (1, 2)$ and the corners of the rectangle. Draw the two **branches** of the hyperbola, each passing through a vertex and guided by the asymptotes.



Technology Tip

Use a graphing calculator to check the graph of $\dfrac{(y-2)^2}{16} - \dfrac{(x-1)^2}{9} = 1.$

Solve for y first. That is,

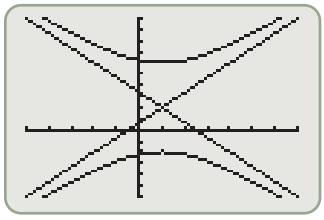
$y_1 = 2 + 4\sqrt{1 + \dfrac{(x-1)^2}{9}}$ or

$y_2 = 2 - 4\sqrt{1 + \dfrac{(x-1)^2}{9}}.$

To add the two asymptotes, enter

$y_3 = \dfrac{4}{3}(x-1) + 2$ or

$y_4 = -\dfrac{4}{3}(x-1) + 2.$



EXAMPLE 6 Transforming an Equation of a Hyperbola to Standard Form

Graph the hyperbola $9x^2 - 16y^2 - 18x + 32y - 151 = 0.$

Solution:

Complete the square on the x terms and y terms, respectively.

$$9(x^2 - 2x) - 16(y^2 - 2y) = 151$$

$$9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 151 + 9 - 16$$

$$9(x-1)^2 - 16(y-1)^2 = 144$$

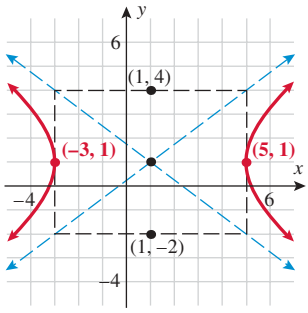
$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

Compare $\dfrac{(x-1)^2}{16} - \dfrac{(y-1)^2}{9} = 1$ to the general form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1.$

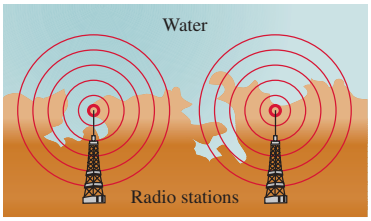
Identify a, b, and (h, k). $a = 4$, $b = 3$, and $(h, k) = (1, 1)$

The transverse axis of this hyperbola lies along $y = 1$.

Label the vertices $(h - a, k) =$ **(−3, 1)** and $(h + a, k) =$ **(5, 1)** and the points $(h, k - b) = (1, -2)$ and $(h, k + b) = (1, 4)$. Draw the rectangular box that passes through these points. Draw the **asymptotes** that pass through the center $(1, 1)$ and the corners of the box. Draw the two **branches** of the hyperbola, each passing through a vertex and guided by the asymptotes.



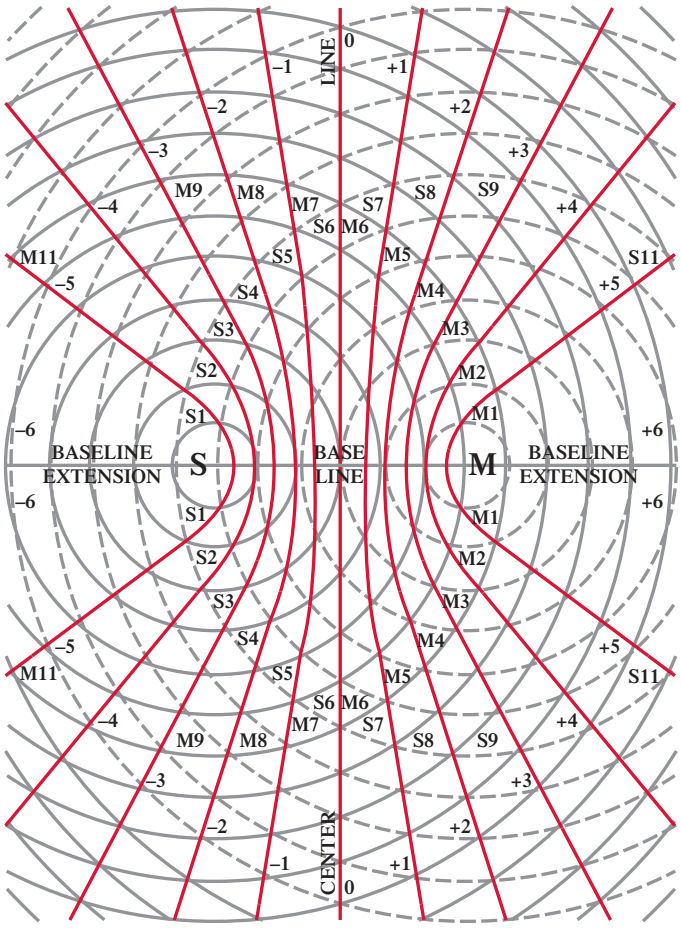
Applications



Nautical navigation is assisted by hyperbolas. For example, suppose that two radio stations on a coast are emitting simultaneous signals. If a boat is at sea, it will be slightly closer to one station than the other station, which results in a small time difference between the received signals from the two stations. Recall that a hyperbola is the set of all points whose differences of the distances from two points (the foci—or the radio stations) are constant. Therefore, if the boat follows the path associated with a constant time difference, that path will be hyperbolic.

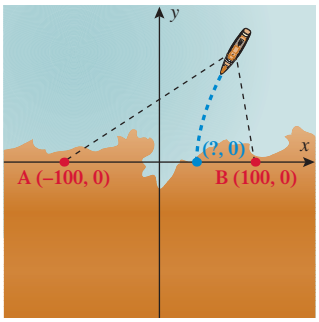
The synchronized signals would intersect one another in associated hyperbolas. Each time difference corresponds to a different path. The radio stations are the foci of the hyperbolas. This principle forms the basis of a hyperbolic radio navigation system known as *LORAN* (**LO**ng-**RA**nge **N**avigation).

There are navigational charts that correspond to different time differences. A ship selects the hyperbolic path that will take it to the desired port, and the loran chart lists the corresponding time difference.



EXAMPLE 7 Nautical Navigation Using Loran

Two LORAN stations are located 200 miles apart along a coast. If a ship records a time difference of 0.00043 second and continues on the hyperbolic path corresponding to that difference, where does it reach shore? Assume that the speed of the radio signal is 186,000 miles per second.



Solution:

Draw the xy-plane and the two stations corresponding to the foci at $(-100, 0)$ and $(100, 0)$. Draw the ship somewhere in quadrant I.

The hyperbola corresponds to a path where the difference of the distances between the ship and the respective stations remains constant. The constant is $2a$, where $(a, 0)$ is a vertex. Find that difference by using $d = rt$.

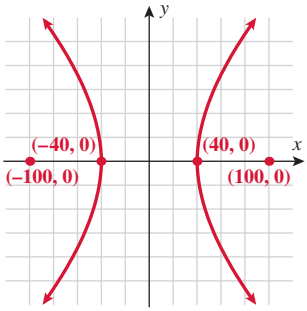
Substitute $r = 186{,}000$ miles/second and $t = 0.00043$ second into $d = rt$.

$$d = (186{,}000 \text{ miles/second})(0.00043 \text{ second}) \approx 80 \text{ miles}$$

Set the constant equal to $2a$. $\quad 2a = 80$

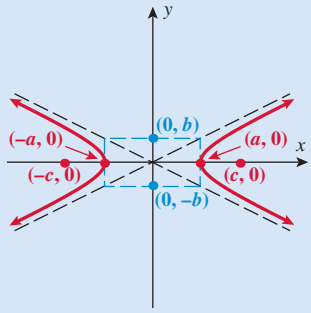
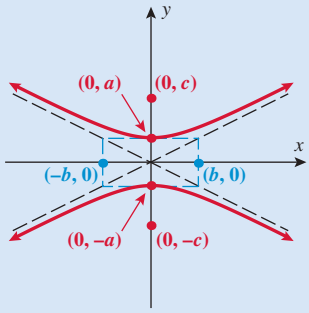
Find a vertex $(a, 0)$. $\quad (40, 0)$

The ship reaches shore between the two stations, 60 miles from station B and 140 miles from station A.

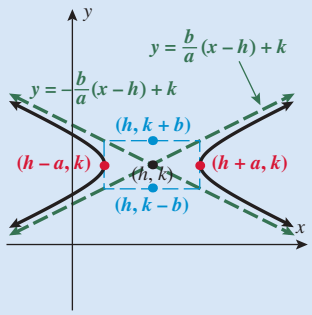
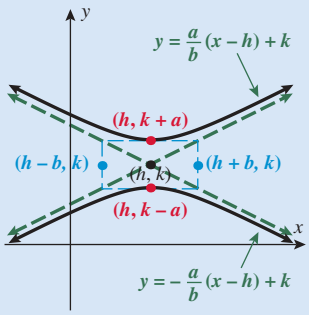


SECTION 9.4 SUMMARY

In this section, we discussed hyperbolas centered at the origin.

EQUATION	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$
TRANSVERSE AXIS	Horizontal (x-axis), length $2a$	Vertical (y-axis), length $2a$
CONJUGATE AXIS	Vertical (y-axis), length $2b$	Horizontal (x-axis), length $2b$
VERTICES	$(-a, 0)$ and $(a, 0)$	$(0, -a)$ and $(0, a)$
FOCI	$(-c, 0)$ and $(c, 0)$ where $c^2 = a^2 + b^2$	$(0, -c)$ and $(0, c)$ where $c^2 = a^2 + b^2$
ASYMPTOTE	$y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$	$y = \dfrac{a}{b}x$ and $y = -\dfrac{a}{b}x$
GRAPH		

For a hyperbola centered at (h, k), the vertices, foci, and asymptotes all shift accordingly.

ORIENTATION OF TRANSVERSE AXIS	Horizontal (parallel to the x-axis)	Vertical (parallel to the y-axis)
EQUATION	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
VERTICES	$(h-a, k)$ and $(h+a, k)$	$(h, k-a)$ and $(h, k+a)$
FOCI	$(h-c, k)$ and $(h+c, k)$ where $c^2 = a^2 + b^2$	$(h, k-c)$ and $(h, k+c)$ where $c^2 = a^2 + b^2$
GRAPH		

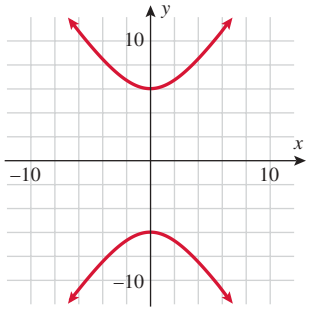
SECTION 9.4 EXERCISES

SKILLS

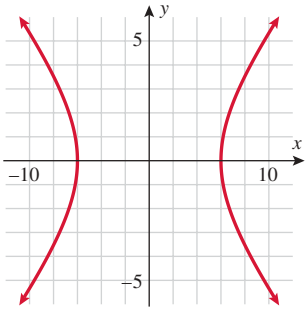
In Exercises 1–4, match each equation with the corresponding hyperbola.

1. $\frac{x^2}{36} - \frac{y^2}{16} = 1$ **2.** $\frac{y^2}{36} - \frac{x^2}{16} = 1$ **3.** $\frac{x^2}{8} - \frac{y^2}{72} = 1$ **4.** $4y^2 - x^2 = 1$

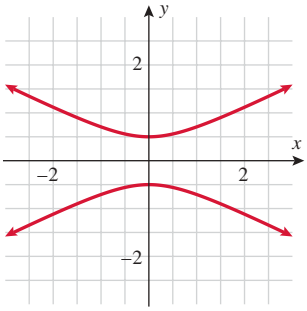
a.



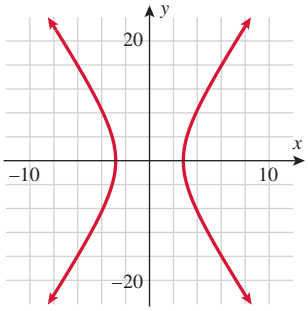
b.



c.



d.



In Exercises 5–16, graph each hyperbola.

5. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ **6.** $\frac{x^2}{49} - \frac{y^2}{9} = 1$ **7.** $\frac{y^2}{16} - \frac{x^2}{64} = 1$ **8.** $\frac{y^2}{144} - \frac{x^2}{25} = 1$

9. $\frac{x^2}{100} - y^2 = 1$ **10.** $9y^2 - 4x^2 = 36$ **11.** $\frac{4y^2}{9} - 81x^2 = 1$ **12.** $\frac{4}{25}x^2 - \frac{100}{9}y^2 = 1$

13. $4x^2 - y^2 = 16$ **14.** $y^2 - x^2 = 81$ **15.** $8y^2 - 16x^2 = 32$ **16.** $10x^2 - 25y^2 = 50$

In Exercises 17–24, find the standard form of an equation of the hyperbola with the given characteristics.

17. Vertices: $(-4, 0)$ and $(4, 0)$ Foci: $(-6, 0)$ and $(6, 0)$

18. Vertices: $(-1, 0)$ and $(1, 0)$ Foci: $(-3, 0)$ and $(3, 0)$

19. Vertices: $(0, -3)$ and $(0, 3)$ Foci: $(0, -4)$ and $(0, 4)$

20. Vertices: $(0, -1)$ and $(0, 1)$ Foci: $(0, -2)$ and $(0, 2)$

21. Center: $(0, 0)$; transverse: x-axis; asymptotes: $y = x$ and $y = -x$

22. Center: $(0, 0)$; transverse: y-axis; asymptotes: $y = x$ and $y = -x$

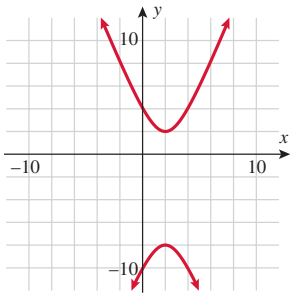
23. Center: $(0, 0)$; transverse axis: y-axis; asymptotes: $y = 2x$ and $y = -2x$

24. Center: $(0, 0)$; transverse axis: x-axis; asymptotes: $y = 2x$ and $y = -2x$

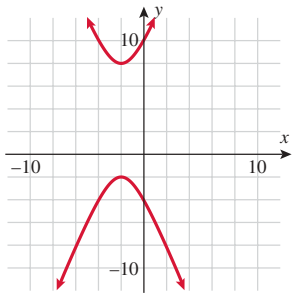
In Exercises 25–28, match each equation with the hyperbola.

25. $\dfrac{(x-3)^2}{4} - \dfrac{(y+2)^2}{25} = 1$ **26.** $\dfrac{(x+3)^2}{4} - \dfrac{(y-2)^2}{25} = 1$ **27.** $\dfrac{(y-3)^2}{25} - \dfrac{(x+2)^2}{4} = 1$ **28.** $\dfrac{(y+3)^2}{25} - \dfrac{(x-2)^2}{4} = 1$

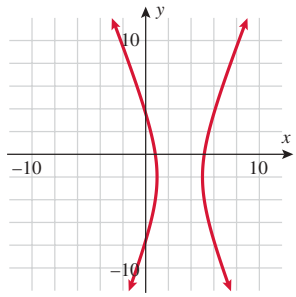
a.



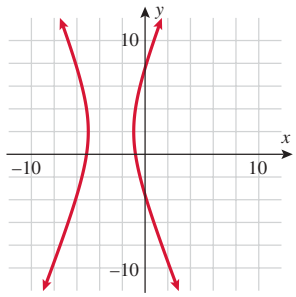
b.



c.



d.



In Exercises 29–38, graph each hyperbola.

29. $\dfrac{(x-1)^2}{16} - \dfrac{(y-2)^2}{4} = 1$ **30.** $\dfrac{(y+1)^2}{36} - \dfrac{(x+2)^2}{9} = 1$ **31.** $10(y+3)^2 - (x-4)^2 = 80$

32. $3(x+3)^2 - 12(y-4)^2 = 36$ **33.** $x^2 - 4x - 4y^2 = 0$ **34.** $-9x^2 + y^2 + 2y - 8 = 0$

35. $-9x^2 - 18x + 4y^2 - 8y - 41 = 0$ **36.** $25x^2 - 50x - 4y^2 - 8y - 79 = 0$

37. $x^2 - 6x - 4y^2 - 16y - 8 = 0$ **38.** $-4x^2 - 16x + y^2 - 2y - 19 = 0$

In Exercises 39–42, find the standard form of the equation of a hyperbola with the given characteristics.

39. Vertices: $(-2, 5)$ and $(6, 5)$ Foci: $(-3, 5)$ and $(7, 5)$

40. Vertices: $(1, -2)$ and $(3, -2)$ Foci: $(0, -2)$ and $(4, -2)$

41. Vertices: $(4, -7)$ and $(4, -1)$ Foci: $(4, -8)$ and $(4, 0)$

42. Vertices: $(2, -6)$ and $(2, -4)$ Foci: $(2, -7)$ and $(2, -3)$

APPLICATIONS

43. Ship Navigation. Two loran stations are located 150 miles apart along a coast. If a ship records a time difference of 0.0005 second and continues on the hyperbolic path corresponding to that difference, where will it reach shore?

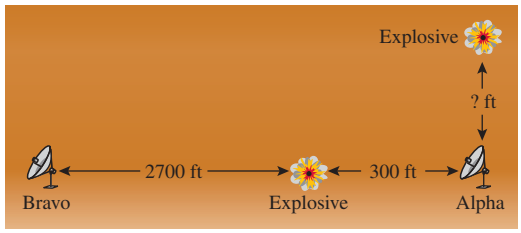
44. Ship Navigation. Two loran stations are located 300 miles apart along a coast. If a ship records a time difference of 0.0007 second and continues on the hyperbolic path corresponding to that difference, where will it reach shore? Round to the nearest mile.

45. Ship Navigation. If the captain of the ship in Exercise 43 wants to reach shore between the stations and 30 miles from one of them, what time difference should he look for?

46. Ship Navigation. If the captain of the ship in Exercise 44 wants to reach shore between the stations and 50 miles from one of them, what time difference should he look for?

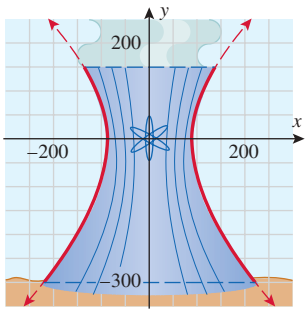
47. Light. If the light from a lamp casts a hyperbolic pattern on the wall due to its lampshade, calculate the equation of the hyperbola if the distance between the vertices is 2 feet and the foci are half a foot from the vertices.

48. Special Ops. A military special ops team is calibrating its recording devices used for passive ascertaining of enemy location. They place two recording stations, alpha and bravo, 3000 feet apart (alpha is due east of bravo). The team detonates small explosives 300 feet west of alpha and records the time it takes each station to register an explosion. The team also sets up a second set of explosives directly north of the alpha station. How many feet north of alpha should the team set off the explosives if it wants to record the same difference in times as on the first explosion?



For Exercises 49 and 50, refer to the following:

Nuclear cooling towers are typically built in the shape of a hyperboloid. The cross section of a cooling tower forms a hyperbola. The cooling tower pictured is 450 feet tall and modeled by the equation $\frac{x^2}{8100} - \frac{y^2}{16{,}900} = 1$



49. Engineering/Design. Find the diameter of the top of the cooling tower to the nearest foot.

50. Engineering/Design. Find the diameter of the base of the tower to the nearest foot.

In Exercises 51–54, refer to the following:

The navigation system loran (long-range navigation) uses the reflection properties of a hyperbola. Two synchronized radio signals are transmitted at a constant speed by two distant radio stations (foci of the hyperbola). Based on the order of arrival and the interval between the signals, the location of the craft along a branch of a hyperbola can be determined. The distance between the radio stations and the craft remains constant. With the help of a third station, the location of the craft can be determined exactly as the intersection of the branches of two hyperbolas.

51. LORAN Navigation System. Two radio stations, located at the same latitude, are separated by 200 kilometers. A vessel navigates following a trajectory parallel to the line connecting A and B, 50 kilometers north of this line. The radio signal transmitted travels at 320 m/μs. The vessel receives the signal from B, 400 μs after receiving the signal from A. Find the location of the vessel.

52. LORAN Navigation System. Two radio stations, located at the same latitude, are separated by 300 kilometers. A vessel navigates following a trajectory parallel to the line connecting A and B, 80 kilometers north of this line. The radio signals transmitted travel at 350 m/μs. The vessel receives the signal from B, 380 μs after receiving the signal from A. Find the location of the vessel.

53. LORAN Navigation System. Two radio stations, located at the same latitude, are separated by 460 kilometers. A vessel navigates following a trajectory parallel to the line connecting A and B, 60 kilometers north of this line. The radio signals transmitted travel at 420 m/μs. The vessel receives the signal from B, 500 μs after receiving the signal from A. Find the location of the vessel.

54. LORAN Navigation System. Two radio stations, located at the same latitude, are separated by 520 kilometers. A vessel navigates following a trajectory parallel to the line connecting A and B, 40 kilometers north of this line. The radio signals transmitted travel at 500 m/μs. The vessel receives the signal from B, 450 μs after receiving the signal from A. Find the location of the vessel.

CATCH THE MISTAKE

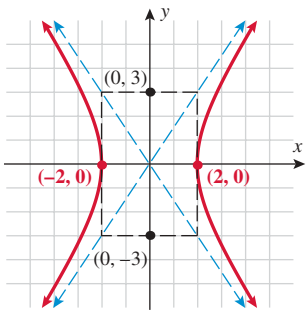
In Exercises 55 and 56, explain the mistake that is made.

55. Graph the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$.

Solution:

Compare the equation to the standard form and solve for a and b.	$a = 2,\ b = 3$
Label the vertices $(-a, 0)$ and $(a, 0)$.	$(-2, 0)$ and $(2, 0)$
Label the points $(0, -b)$ and $(0, b)$.	$(0, -3)$ and $(0, 3)$

Draw the rectangle connecting these four points, and align the asymptotes so that they pass through the center and the corner of the boxes. Then draw the hyperbola using the vertices and asymptotes.



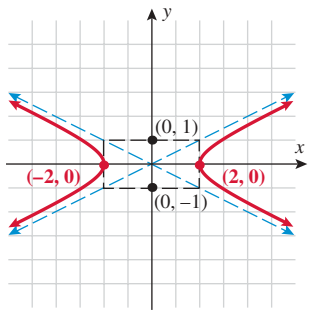
This is incorrect. What mistake was made?

56. Graph the hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$.

Solution:

Compare the equation to the general form and solve for a and b.	$a = 2,\ b = 1$
Label the vertices $(-a, 0)$ and $(a, 0)$.	$(-2, 0)$ and $(2, 0)$
Label the points $(0, -b)$ and $(0, b)$.	$(0, -1)$ and $(0, 1)$

Draw the rectangle connecting these four points, and align the asymptotes so that they pass through the center and the corner of the boxes. Then draw the hyperbola using the vertices and asymptotes.



This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 57–60, determine whether each statement is true or false.

57. If you know the vertices of a hyperbola, you can determine the equation for the hyperbola.

58. If you know the foci and vertices, you can determine the equation for the hyperbola.

59. Hyperbolas centered at the origin have symmetry with respect to the x-axis, y-axis, and the origin.

60. The center and foci are part of the graph of a hyperbola.

61. If the point (p, q) lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, find three other points that lie on the hyperbola.

62. Given the hyperbola $\frac{x^2}{4} - \frac{y^2}{b^2} = 1$, find b such that the asymptotes are perpendicular to each other.

63. A vertical line intersects the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, at the point (p, q), and intersects the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ at the point (p, r). Determine the relationship between q and r. Assume p, r, and q are positive real numbers.

64. Does the line $y = \frac{2b}{a}x$ intersect the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$?

CHALLENGE

65. Find the general equation of a hyperbola whose asymptotes are perpendicular.

66. Find the general equation of a hyperbola whose vertices are $(3, -2)$ and $(-1, -2)$ and whose asymptotes are the lines $y = 2x - 4$ and $y = -2x$.

67. Find the asymptotes of the graph of the hyperbola given by $9y^2 - 16x^2 - 36y - 32x - 124 = 0$.

68. Find the asymptotes of the graph of the hyperbola given by $5x^2 - 4y^2 + 20x + 8y - 4 = 0$.

69. If the line $3x + 5y - 7 = 0$ is perpendicular to one of the asymptotes of the graph of the hyperbola given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$with vertices at $(\pm 3, 0)$, find the foci.

70. If the line $2x - y + 9 = 0$ is perpendicular to one of the asymptotes of the graph of the hyperbola given by $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ with vertices at $(0, \pm 1)$, find the foci.

TECHNOLOGY

71. Graph the following three hyperbolas: $x^2 - y^2 = 1$, $x^2 - 5y^2 = 1$, and $x^2 - 10y^2 = 1$. What can be said to happen to the hyperbola $x^2 - cy^2 = 1$ as c increases?

72. Graph the following three hyperbolas: $x^2 - y^2 = 1$, $5x^2 - y^2 = 1$, and $10x^2 - y^2 = 1$. What can be said to happen to the hyperbola $cx^2 - y^2 = 1$ as c increases?

73. Graph the following three hyperbolas: $x^2 - y^2 = 1$, $0.5x^2 - y^2 = 1$, and $0.05x^2 - y^2 = 1$. What can be said to happen to the hyperbola $cx^2 - y^2 = 1$ as c decreases?

74. Graph the following three hyperbolas: $x^2 - y^2 = 1$, $x^2 - 0.5y^2 = 1$, and $x^2 - 0.05y^2 = 1$. What can be said to happen to the hyperbola $x^2 - cy^2 = 1$ as c decreases?

PREVIEW TO CALCULUS

In Exercises 75 and 76, refer to the following:

In calculus, we study hyperbolic functions. The hyperbolic sine is defined by $\sinh u = \frac{e^u - e^{-u}}{2}$; the hyperbolic cosine is defined by $\cosh u = \frac{e^u + e^{-u}}{2}$.

75. If $x = \cosh u$ and $y = \sinh u$, show that $x^2 - y^2 = 1$.

76. If $x = \frac{e^u + e^{-u}}{e^u - e^{-u}}$ and $y = \frac{2}{e^u - e^{-u}}$, show that $x^2 - y^2 = 1$.

In Exercises 77 and 78, refer to the following:

In calculus, we use the difference quotient $\frac{f(x + h) - f(x)}{h}$ to find the derivative of the function f.

77. Find the derivative of $y = f(x)$, where $y^2 - x^2 = 1$ and $y < 0$.

78. Find the difference quotient of $y = f(x)$, where $4x^2 + y^2 = 1$ and $y > 0$.

SECTION 9.5 SYSTEMS OF NONLINEAR EQUATIONS

SKILLS OBJECTIVES

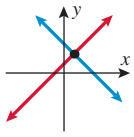
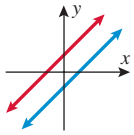
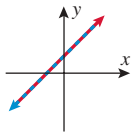
- Solve a system of nonlinear equations with elimination.
- Solve a system of nonlinear equations with substitution.
- Eliminate extraneous solutions.

CONCEPTUAL OBJECTIVES

- Interpret the algebraic solution graphically.
- Understand the types of solutions: distinct number of solutions, no solution, and infinitely many solutions.
- Understand that equations of conic sections are nonlinear equations.

Solving a System of Nonlinear Equations

In Chapter 8, we discussed solving systems of *linear* equations. We applied elimination and substitution to solve systems of linear equations in two variables, and we employed matrices to solve systems of linear equations in three or more variables. Recall that a system of linear equations in two variables has one of three types of solutions:

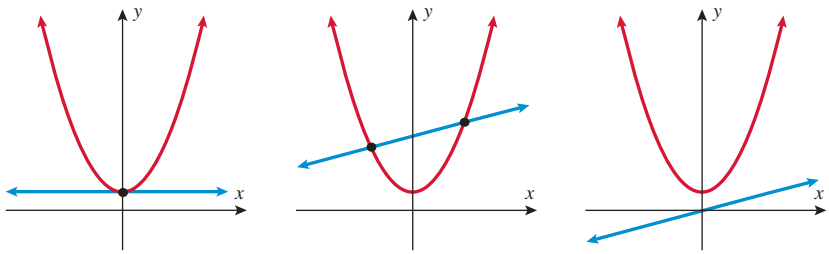
One solution	Two lines that intersect at one point	
No solution	Two parallel lines (never intersect)	
Infinitely many solutions	Two lines that coincide (same line)	

Notice that systems of *linear* equations in two variables always corresponded to *lines*. Now we turn our attention to systems of *nonlinear* equations in two variables. If any of the equations in a system of equations is nonlinear, then the system is a nonlinear system. The following are systems of nonlinear equations:

$$\begin{cases} y = x^2 + 1 \text{ (Parabola)} \\ y = 2x + 2 \text{ (Line)} \end{cases} \quad \begin{cases} x^2 + y^2 = 25 \text{ (Circle)} \\ y = x \text{ (Line)} \end{cases} \quad \begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \text{ (Ellipse)} \\ \dfrac{y^2}{16} - \dfrac{x^2}{25} = 1 \text{ (Hyperbola)} \end{cases}$$

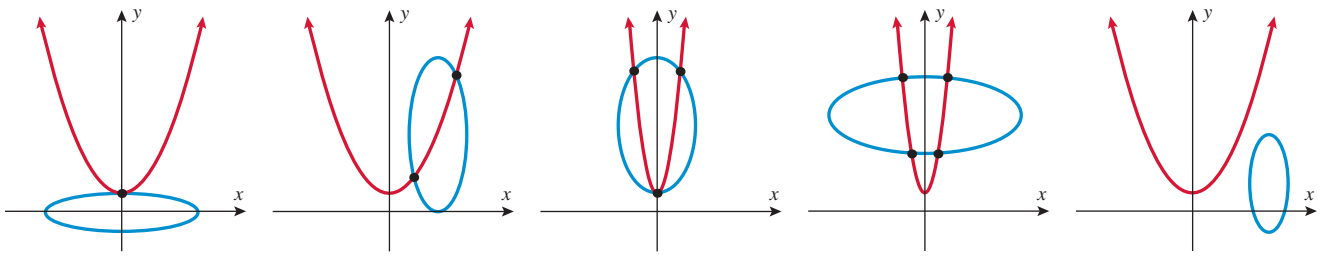
To find the solution to these systems, we ask the question, "At what point(s)—if any—do the graphs of these equations intersect?" Since some nonlinear equations represent conics, this is a convenient time to discuss systems of nonlinear equations.

How many points of intersection do a line and a parabola have? The answer depends on which line and which parabola. As we see in the following graphs, the answer can be one, two, or none.



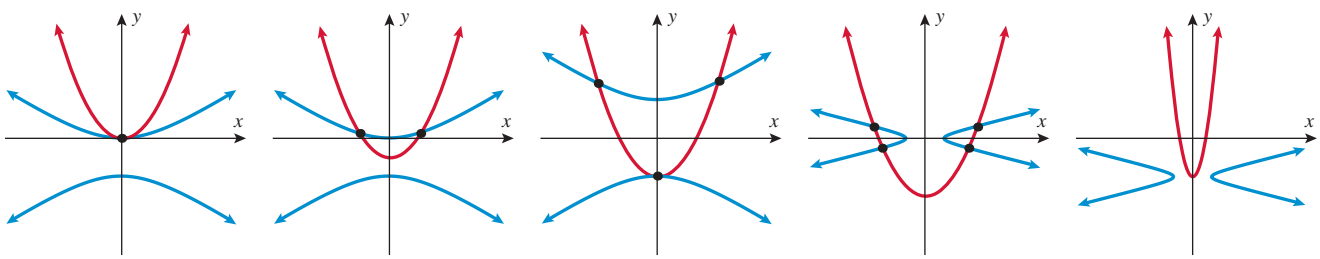
One solution **Two solutions** **No solution**

How many points of intersection do a parabola and an ellipse have? One, two, three, four, or no points of intersection correspond to one solution, two solutions, three solutions, four solutions, or no solution, respectively.



One solution **Two solutions** **Three solutions** **Four solutions** **No solution**

How many points of intersection do a parabola and a hyperbola have? The answer depends on which parabola and which hyperbola. As we see in the following graphs, the answer can be one, two, three, four, or none.



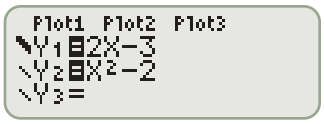
One solution **Two solutions** **Three solutions** **Four solutions** **No solution**

Using Elimination to Solve Systems of Nonlinear Equations

The first three examples in this section use elimination to solve systems of two nonlinear equations. In linear systems, we can eliminate either variable. In nonlinear systems, the variable to eliminate is the one that is raised to the same power in both equations.

Technology Tip

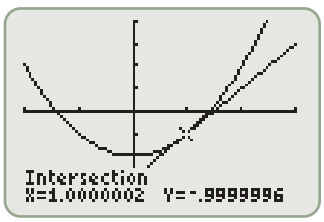
Use a graphing calculator to solve the system of equations. Solve for y in each equation first; that is, $y_1 = 2x - 3$ and $y_2 = x^2 - 2$.



Use the keystrokes:

2nd CALC ▼ 5:Intersect ENTER.

When prompted by the question "First curve?" type ENTER.
When prompted by the question "Second curve?" type ENTER.
When prompted by the question "Guess?" type ENTER.



The graphing calculator supports the solution.

EXAMPLE 1 Solving a System of Two Nonlinear Equations by Elimination: One Solution

Solve the system of equations, and graph the corresponding line and parabola to verify the answer.

Equation (1): $2x - y = 3$

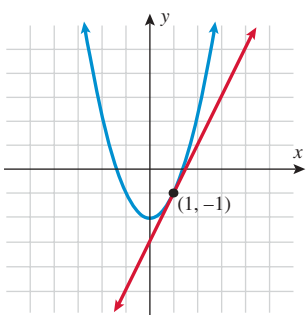
Equation (2): $x^2 - y = 2$

Solution:

Equation (1):	$2x - y = 3$
Multiply both sides of Equation (2) by -1.	$-x^2 + y = -2$
Add.	$2x - x^2 = 1$
Gather all terms to one side.	$x^2 - 2x + 1 = 0$
Factor.	$(x - 1)^2 = 0$
Solve for x.	$x = 1$
Substitute $x = 1$ into original Equation (1).	$2(1) - y = 3$
Solve for y.	$y = -1$

The solution is $x = 1$, $y = -1$, or $(1, -1)$.

Graph the line $y = 2x - 3$ and the parabola $y = x^2 - 2$ and confirm that the point of intersection is $(1, -1)$.



EXAMPLE 2 Solving a System of Two Nonlinear Equations with Elimination: More Than One Solution

Solve the system of equations, and graph the corresponding parabola and circle to verify the answer.

$$\text{Equation (1):} \quad -x^2 + y = -7$$

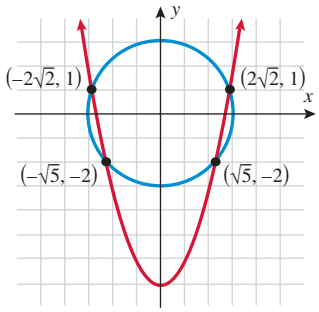
$$\text{Equation (2):} \quad x^2 + y^2 = 9$$

Solution:

Equation (1):	$-x^2 + y = -7$
Equation (2):	$x^2 + y^2 = 9$
Add.	$y^2 + y = 2$
Gather all terms to one side.	$y^2 + y - 2 = 0$
Factor.	$(y + 2)(y - 1) = 0$
Solve for y.	$y = -2$ or $y = 1$
Substitute $y = -2$ into Equation (2).	$x^2 + (-2)^2 = 9$
Solve for x.	$x = \pm\sqrt{5}$
Substitute $y = 1$ into Equation (2).	$x^2 + (1)^2 = 9$
Solve for x.	$x = \pm\sqrt{8} = \pm 2\sqrt{2}$

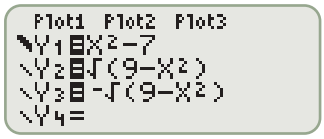
There are four solutions: $\boxed{(-\sqrt{5}, -2), (\sqrt{5}, -2), (-2\sqrt{2}, 1), \text{ and } (2\sqrt{2}, 1)}$

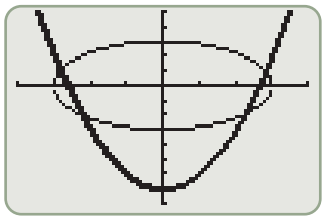
Graph the parabola $y = x^2 - 7$ and the circle $x^2 + y^2 = 9$ and confirm the four points of intersection.



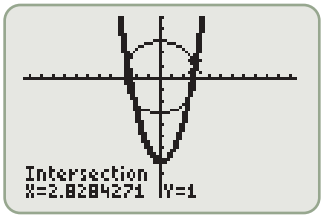
Technology Tip

Use a graphing calculator to solve the system of equations. Solve for y in each equation first; that is, $y_1 = x^2 - 7$, $y_2 = \sqrt{9 - x^2}$, and $y_3 = -\sqrt{9 - x^2}$.



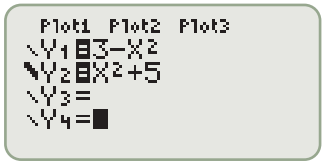


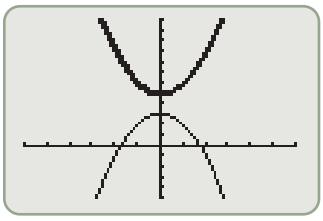
Note: The circle appears elliptical because the x- and y-axes are not of equal scale. The zoom square feature can give the appearance of a circle.



Technology Tip

Use a graphing calculator to solve the system of equations. Solve for y in each equation first; that is, $y_1 = -x^2 + 3$ and $y_2 = x^2 + 5$.





Note: The graphs do not intersect each other. There is no solution to the system.

EXAMPLE 3 Solving a System of Two Nonlinear Equations with Elimination: No Solution

Solve the system of equations, and graph the corresponding parabolas to verify the answer.

Equation (1): $x^2 + y = 3$

Equation (2): $-x^2 + y = 5$

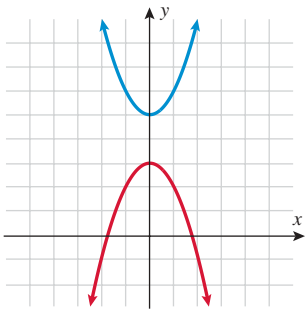
Solution:

Equation (1):	$x^2 + y = 3$
Equation (2):	$\underline{-x^2 + y = 5}$
Add.	$2y = 8$
Solve for y.	$y = 4$
Substitute $y = 4$ into Equation (1).	$x^2 + 4 = 3$
Simplify.	$x^2 = -1$

$x^2 = -1$ has no real solution.

There is $\boxed{\text{no solution}}$ to this system of nonlinear equations.

Graph the parabola $x^2 + y = 3$ and the parabola $y = x^2 + 5$ and confirm there are no points of intersection.



EXAMPLE 4 Solving a System of Nonlinear Equations with Elimination

Solve the system of nonlinear equations with elimination.

$$\text{Equation (1):} \quad \frac{x^2}{4} + y^2 = 1$$

$$\text{Equation (2):} \quad x^2 - y^2 = 1$$

Solution:

Add Equations (1) and (2) to eliminate y^2.

$$\begin{array}{r} \frac{x^2}{4} + y^2 = 1 \\ x^2 - y^2 = 1 \\ \hline \frac{5}{4}x^2 = 2 \end{array}$$

Solve for x.

$$x^2 = \frac{8}{5}$$

$$x = \pm\sqrt{\frac{8}{5}}$$

Let $x = \pm\sqrt{\frac{8}{5}}$ in Equation (2).

$$\left(\pm\sqrt{\frac{8}{5}}\right)^2 - y^2 = 1$$

Solve for y.

$$y^2 = \frac{8}{5} - 1 = \frac{3}{5}$$

$$y = \pm\sqrt{\frac{3}{5}}$$

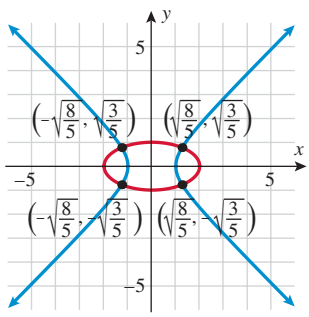
There are four solutions:

$$\boxed{\left(-\sqrt{\frac{8}{5}}, -\sqrt{\frac{3}{5}}\right), \left(-\sqrt{\frac{8}{5}}, \sqrt{\frac{3}{5}}\right), \left(\sqrt{\frac{8}{5}}, -\sqrt{\frac{3}{5}}\right), \text{ and } \left(\sqrt{\frac{8}{5}}, \sqrt{\frac{3}{5}}\right)}$$

A calculator can be used to approximate these solutions:

$$\sqrt{\frac{8}{5}} \approx 1.26$$

$$\sqrt{\frac{3}{5}} \approx 0.77$$

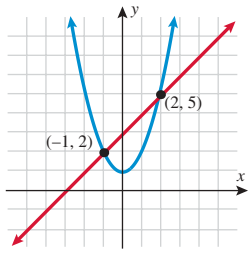


YOUR TURN Solve the following systems of nonlinear equations:

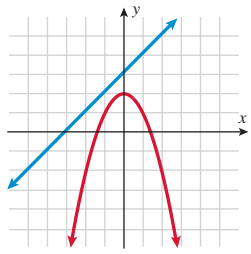
a. $\begin{array}{r} -x + y = 3 \\ x^2 - y = -1 \end{array}$ **b.** $\begin{array}{r} x^2 + y = 2 \\ -x + y = 3 \end{array}$ **c.** $\begin{array}{r} \frac{x^2}{9} + \frac{y^2}{4} = 1 \\ \frac{y^2}{9} - \frac{x^2}{16} = 1 \end{array}$

Answer:

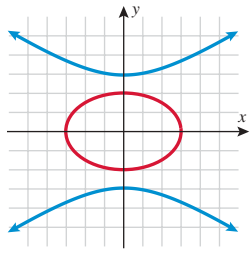
a. $(-1, 2)$ and $(2, 5)$



b. no solution



c. no solution



Using Substitution to Solve Systems of Nonlinear Equations

Study Tip

Extraneous solutions are possible when you have one equation with a linear (or odd) power and one equation with a second-degree (or even) power.

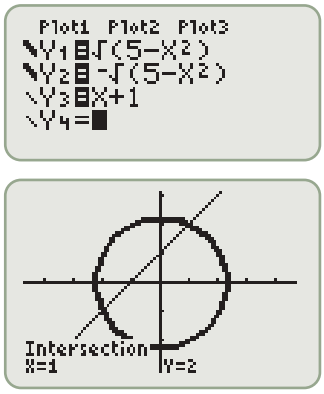
Elimination is based on the idea of eliminating one of the variables and solving the remaining equation in one variable. This is not always possible with nonlinear systems. For example, a system consisting of a circle and a line

$$\begin{aligned} x^2 + y^2 &= 5 \\ -x + y &= 1 \end{aligned}$$

cannot be solved with elimination, because both variables are raised to different powers in each equation. We now turn to the substitution method. It is important to always check solutions, because extraneous solutions are possible.

Technology Tip

Use a graphing calculator to solve the system of equations. Solve for y in each equation first; that is, $y_1 = \sqrt{5 - x^2}$, $y_2 = -\sqrt{5 - x^2}$, and $y_3 = x + 1$.



Note: The graphs support the solutions to the system.

Study Tip

Substitute back into the lowest-degree equation, and always check solutions.

■ **Answer:** (2, 3) and (3, 2)

EXAMPLE 5 Solving a System of Nonlinear Equations with Substitution

Solve the system of equations, and graph the corresponding circle and line to verify the answer.

$$\begin{aligned} &\text{Equation (1):} \quad x^2 + y^2 = 5 \\ &\text{Equation (2):} \quad -x + y = 1 \end{aligned}$$

Solution:

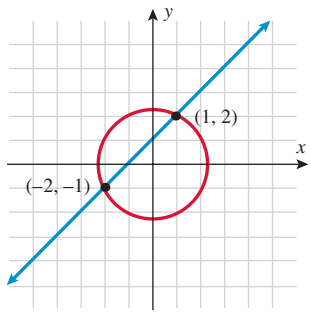
Rewrite Equation (2) with y isolated.

Equation (1):	$x^2 + y^2 = 5$
Equation (2):	$y = x + 1$
Substitute Equation (2), $y = x + 1$, into Equation (1).	$x^2 + (x + 1)^2 = 5$
Eliminate the parentheses.	$x^2 + x^2 + 2x + 1 = 5$
Gather like terms.	$2x^2 + 2x - 4 = 0$
Divide by 2.	$x^2 + x - 2 = 0$
Factor.	$(x + 2)(x - 1) = 0$
Solve for x.	$x = -2$ or $x = 1$
Substitute $x = -2$ into Equation (1).	$(-2)^2 + y^2 = 5$
Solve for y.	$y = -1$ or $y = 1$
Substitute $x = 1$ into Equation (1).	$(1)^2 + y^2 = 5$
Solve for y.	$y = -2$ or $y = 2$

There appear to be four solutions: $(-2, -1)$, $(-2, 1)$, $(1, -2)$, and $(1, 2)$, but a line can intersect a circle in no more than two points. Therefore, at least two solutions are *extraneous*. All four points satisfy Equation (1), but only $(-2, -1)$ and $(1, 2)$ also satisfy Equation (2).

The answer is $\boxed{(-2, -1) \text{ and } (1, 2)}$.

Graph the circle $x^2 + y^2 = 5$ and the line $y = x + 1$ and confirm the two points of intersection.



Note: After solving for x, had we substituted back into the linear Equation (2) instead of Equation (1), extraneous solutions would not have appeared. In general, **substitute back into the lowest-degree equation, and always check solutions**.

■ **YOUR TURN** Solve the system of equations $x^2 + y^2 = 13$ and $x + y = 5$.

In Example 6, the equation $xy = 2$ can also be shown to be a rotated hyperbola (Section 9.7). For now, we can express this equation in terms of a reciprocal function $y = \frac{2}{x}$, a topic we discussed in Section 1.2.

EXAMPLE 6 Solving a System of Nonlinear Equations with Substitution

Solve the system of equations.

$$\text{Equation (1):} \quad x^2 + y^2 = 5$$
$$\text{Equation (2):} \quad xy = 2$$

Solution:

Since Equation (2) tells us that $xy = 2$, we know that neither x nor y can be zero.

Solve Equation (2) for y. $\quad y = \frac{2}{x}$

Substitute $y = \frac{2}{x}$ into Equation (1). $\quad x^2 + \left(\frac{2}{x}\right)^2 = 5$

Eliminate the parentheses. $\quad x^2 + \frac{4}{x^2} = 5$

Multiply by x^2. $\quad x^4 + 4 = 5x^2$

Collect the terms to one side. $\quad x^4 - 5x^2 + 4 = 0$

Factor. $\quad (x^2 - 4)(x^2 - 1) = 0$

Solve for x. $\quad x = \pm 2 \quad \text{or} \quad x = \pm 1$

Substitute $x = -2$ into Equation (2), $xy = 2$, and solve for y. $\quad y = -1$

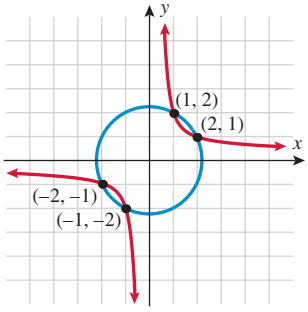
Substitute $x = 2$ into Equation (2), $xy = 2$, and solve for y. $\quad y = 1$

Substitute $x = -1$ into Equation (2), $xy = 2$, and solve for y. $\quad y = -2$

Substitute $x = 1$ into Equation (2), $xy = 2$, and solve for y. $\quad y = 2$

Check to see that there are four solutions: $\boxed{(-2, -1), (-1, -2), (2, 1), \text{ and } (1, 2)}$.

Note: It is important to check the solutions either algebraically or graphically (see the graph on the left).



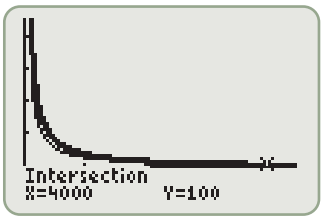
■ **YOUR TURN** Solve the system of equations $x^2 + y^2 = 2$ and $xy = 1$.

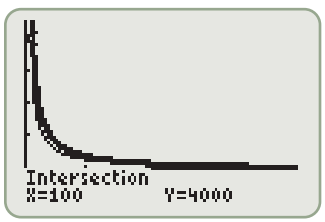
■ **Answer:** $(-1, -1)$ and $(1, 1)$

Applications

Technology Tip

Use a graphing calculator to solve the system of equations. Solve for y in each equation first; that is, $y_1 = \dfrac{400{,}000}{x}$ and $y_2 = 40 + \dfrac{237{,}600}{x - 40}$.





Note: The graphs support the solutions to the system.

EXAMPLE 7 Calculating How Much Fence to Buy

A couple buy a rectangular piece of property advertised as 10 acres (approximately 400,000 square feet). They want two fences to divide the land into an internal grazing area and a surrounding riding path. If they want the riding path to be 20 feet wide, one fence will enclose the property and one internal fence will sit 20 feet inside the outer fence. If the internal grazing field is 237,600 square feet, how many linear feet of fencing should they buy?

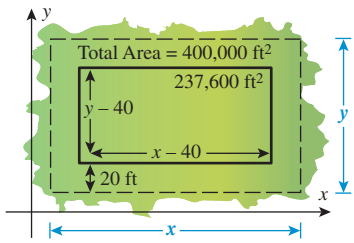
Solution:

Use the five-step procedure for solving word problems from Section 0.1, and use two variables.

STEP 1 Identify the question.

How many linear feet of fence should they buy? Or, what is the sum of the perimeters of the two fences?

STEP 2 Make notes or draw a sketch.



STEP 3 Set up the equations.

$x =$ length of property $\qquad x - 40 =$ length of internal field

$y =$ width of property $\qquad y - 40 =$ width of internal field

Equation (1): $\quad xy = 400{,}000$

Equation (2): $\quad (x - 40)(y - 40) = 237{,}600$

STEP 4 Solve the system of equations.

Substitution Method

Since Equation (1) tells us that $xy = 400{,}000$, we know that neither x nor y can be zero.

Solve Equation (1) for y. $\qquad y = \dfrac{400{,}000}{x}$

Substitute $y = \dfrac{400{,}000}{x}$ into Equation (2). $\qquad (x - 40)\left(\dfrac{400{,}000}{x} - 40\right) = 237{,}600$

Eliminate the parentheses. $\qquad 400{,}000 - 40x - \dfrac{16{,}000{,}000}{x} + 1600 = 237{,}600$

Multiply by the LCD, x. $\qquad 400{,}000x - 40x^2 - 16{,}000{,}000 + 1600x = 237{,}600x$

Collect like terms on one side. $\qquad 40x^2 - 164{,}000x + 16{,}000{,}000 = 0$

Divide by 40. $\qquad x^2 - 4100x + 400{,}000 = 0$

Factor. $\qquad (x - 4000)(x - 100) = 0$

Solve for x. $\qquad x = 4000 \quad \text{or} \quad x = 100$

Substitute $x = 4000$ into the original Equation (1). $\quad 4000y = 400{,}000$

Solve for y. $\quad y = 100$

Substitute $x = 100$ into the original Equation (1). $\quad 100y = 400{,}000$

Solve for y. $\quad y = 4000$

The two solutions yield the same dimensions: 4000×100. The inner field has the dimensions 3960×60. Therefore, the sum of the perimeters of the two fences is

$$2(4000) + 2(100) + 2(3960) + 2(60) = 8000 + 200 + 7920 + 120 = 16{,}240$$

$$\boxed{\text{The couple should buy 16,240 linear feet of fencing.}}$$

STEP 5 Check the solution.

The point (4000, 100) satisfies both Equation (1) and Equation (2).

It is important to note that some nonlinear equations are not conic sections (they could be exponential, logarithmic, or higher-degree polynomial equations). These systems of linear equations are typically solved by the substitution method (see the exercises).

SECTION 9.5 SUMMARY

In this section, systems of two equations were discussed when at least one of the equations is nonlinear (e.g., conics). The substitution method and elimination method can *sometimes* be applied to nonlinear systems. When graphing the two equations, the points of intersection are the solutions of the system. Systems of nonlinear equations can have more than one solution. Also, extraneous solutions can appear, so it is important to always check solutions.

SECTION 9.5 EXERCISES

SKILLS

In Exercises 1–12, solve the system of equations by applying the elimination method.

1. $\begin{aligned} x^2 - y &= -2 \\ -x + y &= 4 \end{aligned}$

2. $\begin{aligned} x^2 + y &= 2 \\ 2x + y &= -1 \end{aligned}$

3. $\begin{aligned} x^2 + y &= 1 \\ 2x + y &= 2 \end{aligned}$

4. $\begin{aligned} x^2 - y &= 2 \\ -2x + y &= -3 \end{aligned}$

5. $\begin{aligned} x^2 + y &= -5 \\ -x + y &= 3 \end{aligned}$

6. $\begin{aligned} x^2 - y &= -7 \\ x + y &= -2 \end{aligned}$

7. $\begin{aligned} x^2 + y^2 &= 1 \\ x^2 - y &= -1 \end{aligned}$

8. $\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + y &= -1 \end{aligned}$

9. $\begin{aligned} x^2 + y^2 &= 3 \\ 4x^2 + y &= 0 \end{aligned}$

10. $\begin{aligned} x^2 + y^2 &= 6 \\ -7x^2 + y &= 0 \end{aligned}$

11. $\begin{aligned} x^2 + y^2 &= -6 \\ -2x^2 + y &= 7 \end{aligned}$

12. $\begin{aligned} x^2 + y^2 &= 5 \\ 3x^2 + y &= 9 \end{aligned}$

In Exercises 13–24, solve the system of equations by applying the substitution method.

13. $\begin{aligned} x + y &= 2 \\ x^2 + y^2 &= 2 \end{aligned}$

14. $\begin{aligned} x - y &= -2 \\ x^2 + y^2 &= 2 \end{aligned}$

15. $\begin{aligned} xy &= 4 \\ x^2 + y^2 &= 10 \end{aligned}$

16. $\begin{aligned} xy &= -3 \\ x^2 + y^2 &= 12 \end{aligned}$

17. $\begin{aligned} y &= x^2 - 3 \\ y &= -4x + 9 \end{aligned}$

18. $\begin{aligned} y &= -x^2 + 5 \\ y &= 3x - 4 \end{aligned}$

19. $\begin{aligned} x^2 + xy - y^2 &= 5 \\ x - y &= -1 \end{aligned}$

20. $\begin{aligned} x^2 + xy + y^2 &= 13 \\ x + y &= -1 \end{aligned}$

21. $\begin{aligned} 2x - y &= 3 \\ x^2 + y^2 - 2x + 6y &= -9 \end{aligned}$

22. $\begin{aligned} x^2 + y^2 - 2x - 4y &= 0 \\ -2x + y &= -3 \end{aligned}$

23. $\begin{aligned} 4x^2 + 12xy + 9y^2 &= 25 \\ -2x + y &= 1 \end{aligned}$

24. $\begin{aligned} -4xy + 4y^2 &= 8 \\ 3x + y &= 2 \end{aligned}$

In Exercises 25–40, solve the system of equations by applying any method.

25. $\begin{aligned} x^3 - y^3 &= 63 \\ x - y &= 3 \end{aligned}$

26. $\begin{aligned} x^3 + y^3 &= -26 \\ x + y &= -2 \end{aligned}$

27. $\begin{aligned} 4x^2 - 3xy &= -5 \\ -x^2 + 3xy &= 8 \end{aligned}$

28. $\begin{aligned} 2x^2 + 5xy &= 2 \\ x^2 - xy &= 1 \end{aligned}$

29. $\begin{aligned} 2x^2 - xy &= 28 \\ 4x^2 - 9xy &= 28 \end{aligned}$

30. $\begin{aligned} -7xy + 2y^2 &= -3 \\ -3xy + y^2 &= 0 \end{aligned}$

31. $\begin{aligned} 4x^2 + 10y^2 &= 26 \\ -2x^2 + 2y^2 &= -6 \end{aligned}$

32. $\begin{aligned} x^3 + y^3 &= 19 \\ x^3 - y^3 &= -35 \end{aligned}$

33. $\begin{aligned} \log_x(2y) &= 3 \\ \log_x(y) &= 2 \end{aligned}$

34. $\begin{aligned} \log_x(y) &= 1 \\ \log_x(2y) &= \tfrac{1}{2} \end{aligned}$

35. $\begin{aligned} \frac{1}{x^3} + \frac{1}{y^2} &= 17 \\ \frac{1}{x^3} - \frac{1}{y^2} &= -1 \end{aligned}$

36. $\begin{aligned} \frac{2}{x^2} + \frac{3}{y^2} &= \frac{5}{6} \\ \frac{4}{x^2} - \frac{9}{y^2} &= 0 \end{aligned}$

37. $\begin{aligned} 2x^2 + 4y^4 &= -2 \\ 6x^2 + 3y^4 &= -1 \end{aligned}$

38. $\begin{aligned} x^2 + y^2 &= -2 \\ x^2 + y^2 &= -1 \end{aligned}$

39. $\begin{aligned} 2x^2 - 5y^2 + 8 &= 0 \\ x^2 - 7y^2 + 4 &= 0 \end{aligned}$

40. $\begin{aligned} x^2 + y^2 &= 4x + 6y - 12 \\ 9x^2 + 4y^2 &= 36x + 24y - 36 \end{aligned}$

In Exercises 41 and 44, graph each equation and find the point(s) of intersection.

41. The parabola $y = x^2 - 6x + 11$ and the line $y = -x + 7$

42. The circle $x^2 + y^2 - 4x - 2y + 5 = 0$ and the line $-x + 3y = 6$

43. The ellipse $9x^2 - 18x + 4y^2 + 8y - 23 = 0$ and the line $-3x + 2y = 1$.

44. The parabola $y = -x^2 + 2x$ and the circle $x^2 + 6x + y^2 - 4y + 12 = 0$.

APPLICATIONS

45. Numbers. The sum of two numbers is 10, and the difference of their squares is 40. Find the numbers.

46. Numbers. The difference of two numbers is 3, and the difference of their squares is 51. Find the numbers.

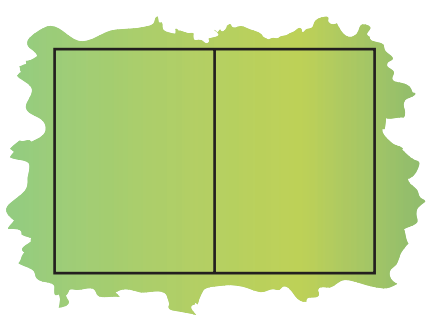
47. Numbers. The product of two numbers is equal to the reciprocal of the difference of their reciprocals. The product of the two numbers is 72. Find the numbers.

48. Numbers. The ratio of the sum of two numbers to the difference of the two numbers is 9. The product of the two numbers is 80. Find the numbers.

49. Geometry. A rectangle has a perimeter of 36 centimeters and an area of 80 square centimeters. Find the dimensions of the rectangle.

50. Geometry. Two concentric circles have perimeters that add up to 16π and areas that add up to 34π. Find the radii of the two circles.

51. Horse Paddock. An equestrian buys a 5-acre rectangular parcel (approximately 200,000 square feet) and is going to fence in the entire property and then divide the parcel into two halves with a fence. If 2200 linear feet of fencing is required, what are the dimensions of the parcel?



52. Dog Run. A family moves into a new home and decides to fence in the yard to give its dog room to roam. If the area that will be fenced in is rectangular and has an area of 11,250 square feet, and the length is twice as much as the width, how many linear feet of fence should the family buy?

53. Footrace. Your college algebra professor and Jeremy Wariner (2004 Olympic Gold Medalist in the men's 400 meter) decided to race. The race was 400 meters and Jeremy gave your professor a 1-minute head start, and still crossed the finish line 1 minute 40 seconds before your professor. If Jeremy ran five times faster than your professor, what was each person's average speed?

54. Footrace. You decided to race Jeremy Wariner for 800 meters. At that distance, Jeremy runs approximately twice as fast as you. He gave you a 1-minute head start and crossed the finish line 20 seconds before you. What were each of your average speeds?

55. Velocity. Two cars start moving simultaneously in the same direction. The first car moves at 50 miles per hour; the speed of the second car is 40 miles per hour. A half-hour later, another car starts moving in the same direction. The third car reaches the first one 1.5 hours after it reached the second car. Find the speed of the third car.

56. Design. Two boxes are constructed to contain the same volume. In the first box, the width is 16 centimeters larger than the depth and the length is five times the depth. In the second box, both the length and width are 4 centimeters shorter and the depth is 25% larger than in the first box. Find the dimensions of the second box.

57. Numbers. Find a number consisting of four digits such that

- the sum of the squares of the thousands and the units is 13.
- the sum of the squares of the hundreds and tens is 85.
- the hundreds is one more than the tens.
- the thousands is one more than the units.
- when 1089 is subtracted from the number, the result has the same digits but in inverse order.

58. Numbers. Find a number consisting of three digits such that

- the sum of the cubes of the hundreds and units is 9.
- the tens is one more than twice the hundreds.
- the hundreds is one more than the units.

CATCH THE MISTAKE

In Exercises 59 and 60, explain the mistake that is made.

59. Solve the system of equations: $\begin{aligned} x^2 + y^2 &= 4 \\ x + y &= 2 \end{aligned}$

Solution:

Multiply the second equation by (-1) and add to the first equation. $x^2 - x = 2$

Subtract 2. $x^2 - x - 2 = 0$

Factor. $(x + 1)(x - 2) = 0$

Solve for x. $x = -1$ and $x = 2$

Substitute $x = -1$ and $x = 2$ into $x + y = 2$. $-1 + y = 2$ and $2 + y = 2$

Solve for y. $y = 3$ and $y = 0$

The answer is $(-1, 3)$ and $(2, 0)$.

This is incorrect. What mistake was made?

60. Solve the system of equations: $\begin{aligned} x^2 + y^2 &= 5 \\ 2x - y &= 0 \end{aligned}$

Solution:

Solve the second equation for y. $y = 2x$

Substitute $y = 2x$ into the first equation. $x^2 + (2x)^2 = 5$

Eliminate the parentheses. $x^2 + 4x^2 = 5$

Gather like terms. $5x^2 = 5$

Solve for x. $x = -1$ and $x = 1$

Substitute $x = -1$ into the first equation. $(-1)^2 + y^2 = 5$

Solve for y. $y = -2$ and $y = 2$

Substitute $x = 1$ into the first equation. $(1)^2 + y^2 = 5$

Solve for y. $y = -2$ and $y = 2$

The answers are $(-1, -2)$, $(-1, 2)$, $(1, -2)$, and $(1, 2)$.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 61–64, determine whether each statement is true or false.

61. A system of equations representing a line and a parabola can intersect in at most three points.

62. A system of equations representing a line and a cubic function can intersect in at most three places.

63. The elimination method can always be used to solve systems of two nonlinear equations.

64. The substitution method always works for solving systems of nonlinear equations.

65. A circle and a line have at most two points of intersection. A circle and a parabola have at most four points of intersection. What is the greatest number of points of intersection that a circle and an nth-degree polynomial can have?

66. A line and a parabola have at most two points of intersection. A line and a cubic function have at most three points of intersection. What is the greatest number of points of intersection that a line and an nth-degree polynomial can have?

CHALLENGE

67. Find a system of equations representing a line and a parabola that has only one real solution.

68. Find a system of equations representing a circle and a parabola that has only one real solution.

In Exercises 69–72, solve each system of equations.

69. $\begin{aligned} x^4 + 2x^2y^2 + y^4 &= 25 \\ x^4 - 2x^2y^2 + y^4 &= 9 \end{aligned}$

70. $\begin{aligned} x^4 + 2x^2y^2 + y^4 &= 169 \\ x^4 - 2x^2y^2 + y^4 &= 25 \end{aligned}$

71. $\begin{aligned} x^4 + 2x^2y^2 + y^4 &= -25 \\ x^4 - 2x^2y^2 + y^4 &= -9 \end{aligned}$

72. $\begin{aligned} x^4 + 2x^2y^2 + y^4 &= -169 \\ x^4 - 2x^2y^2 + y^4 &= -25 \end{aligned}$

TECHNOLOGY

In Exercises 73–78, use a graphing utility to solve the systems of equations.

73. $\begin{aligned} y &= e^x \\ y &= \ln x \end{aligned}$

74. $\begin{aligned} y &= 10^x \\ y &= \log x \end{aligned}$

75. $\begin{aligned} 2x^3 + 4y^2 &= 3 \\ xy^3 &= 7 \end{aligned}$

76. $\begin{aligned} 3x^4 - 2xy + 5y^2 &= 19 \\ x^4y &= 5 \end{aligned}$

77. $\begin{aligned} 5x^3 + 2y^2 &= 40 \\ x^3y &= 5 \end{aligned}$

78. $\begin{aligned} 4x^4 + 2xy + 3y^2 &= 60 \\ x^4y &= 8 - 3x^4 \end{aligned}$

PREVIEW TO CALCULUS

In calculus, when finding the derivative of equations in two variables, we typically use implicit differentiation. A more direct approach is used when an equation can be solved for one variable in terms of the other variable.

In Exercises 79–82, solve each equation for y in terms of x.

79. $x^2 + 4y^2 = 8, y < 0$

80. $y^2 + 2xy + 4 = 0, y > 0$

81. $x^3y^3 = 9y, y > 0$

82. $3xy = -x^3y^2, y < 0$

SECTION 9.6

SYSTEMS OF NONLINEAR INEQUALITIES

SKILLS OBJECTIVES

- Graph a nonlinear inequality in two variables.
- Graph a system of nonlinear inequalities in two variables.

CONCEPTUAL OBJECTIVES

- Understand that a nonlinear inequality in two variables may be represented by either a bounded or an unbounded region.
- Interpret an overlapping shaded region as a solution.

Nonlinear Inequalities in Two Variables

Linear inequalities are expressed in the form $Ax + By \leq C$. Specific expressions can involve either of the strict or either of the nonstrict inequalities. Examples of **nonlinear inequalities in two variables** are

$$9x^2 + 16y^2 \geq 1 \qquad x^2 + y^2 > 1 \qquad y \leq -x^2 + 3 \qquad \text{and} \qquad \frac{x^2}{20} - \frac{y^2}{81} < 1$$

We follow the same procedure as we did with linear inequalities. We change the inequality to an equal sign, graph the resulting nonlinear equation, test points from the two regions, and shade the region that makes the inequality true. For strict inequalities, $<$ or $>$, we use dashed curves, and for nonstrict inequalities, $\leq$ or $\geq$, we use solid curves.

EXAMPLE 1 Graphing a Strict Nonlinear Inequality in Two Variables

Graph the inequality $x^2 + y^2 > 1$.

Solution:

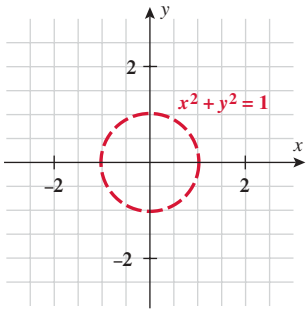
STEP 1 Change the inequality sign to an equal sign. $x^2 + y^2 = 1$

The equation is the equation of a circle.

STEP 2 Draw the graph of the circle.

The center is (0, 0) and the radius is 1.

Since the inequality $>$ is a strict inequality, draw the circle as a **dashed** curve.



STEP 3 Test points in each region (outside the circle and inside the circle).

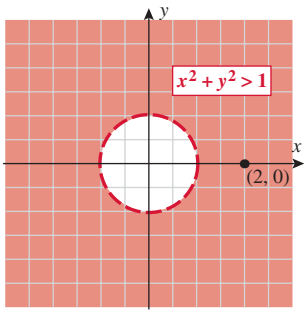
Substitute (2, 0) into $x^2 + y^2 > 1$. $4 \geq 1$

The point (2, 0) satisfies the inequality.

Substitute (0, 0) into $x^2 + y^2 > 1$. $0 \geq 1$

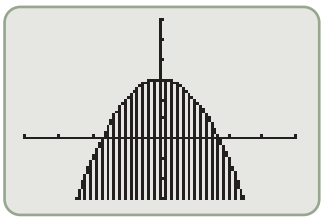
The point (0, 0) does not satisfy the inequality.

STEP 4 Shade the region containing the point (2, 0).



Technology Tip

Use a graphing calculator to graph the inequality $y \le -x^2 + 3$. Enter $y_1 = -x^2 + 3$. For $\le$, use the arrow key to move the cursor to the left of Y_1 and type ENTER until you see ◣.



Note: The parabola should be drawn solid.

EXAMPLE 2 Graphing a Nonstrict Nonlinear Inequality in Two Variables

Graph the inequality $y \le -x^2 + 3$.

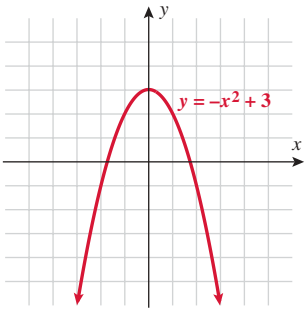
Solution:

STEP 1 Change the inequality sign to an equal sign. $y = -x^2 + 3$

The equation is that of a parabola.

STEP 2 Graph the parabola.

Reflect the base function, $f(x) = x^2$, about the x-axis and shift up three units. Since the inequality $\le$ is a nonstrict inequality, draw the parabola as a **solid** curve.



STEP 3 Test points in each region (inside the parabola and outside the parabola).

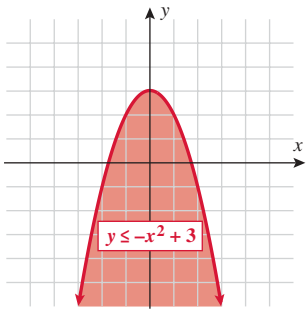
Substitute (3, 0) into $y \le -x^2 + 3$. $0 \le -6$

The point (3, 0) does not satisfy the inequality.

Substitute (0, 0) into $y \le -x^2 + 3$. $0 \le 3$

The point (0, 0) does satisfy the inequality.

STEP 4 Shade the region containing the point (0, 0).

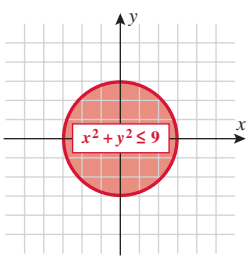


YOUR TURN Graph the following inequalities:

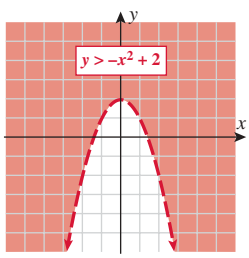
a. $x^2 + y^2 \le 9$ **b.** $y > -x^2 + 2$

■ **Answer:**

a.



b.



Systems of Nonlinear Inequalities

To solve a system of inequalities, first graph the inequalities and shade the region containing the points that satisfy each inequality. The overlap of all the shaded regions is the solution.

EXAMPLE 3 Graphing a System of Inequalities

Graph the solution to the system of inequalities: $y \geq x^2 - 1$
$y < x + 1$

Solution:

STEP 1 Change the inequality signs to equal signs.

$$y = x^2 - 1$$
$$y = x + 1$$

STEP 2 The resulting equations represent a parabola (to be drawn solid) and a line (to be drawn dashed). Graph the two equations.

To determine the points of intersection, set the y-values equal to each other. $x^2 - 1 = x + 1$

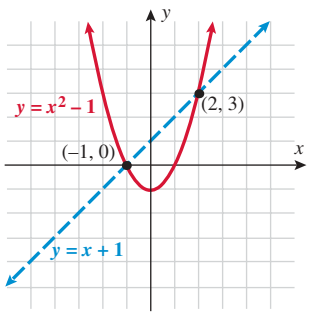
Write the quadratic equation in standard form. $x^2 - x - 2 = 0$

Factor. $(x - 2)(x + 1) = 0$

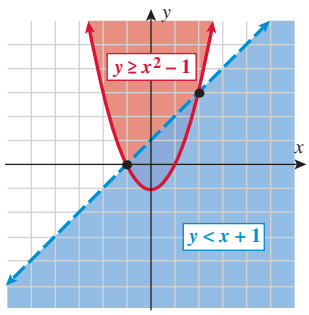
Solve for x. $x = 2$ or $x = -1$

Substitute $x = 2$ into $y = x + 1$. $(2, 3)$

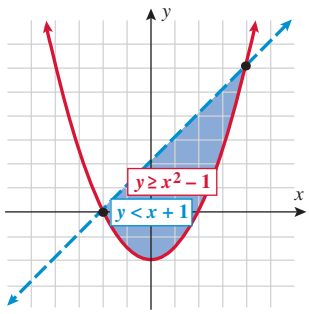
Substitute $x = -1$ into $y = x + 1$. $(-1, 0)$



STEP 3 Test points and shade the regions.



STEP 4 Shade the common region.



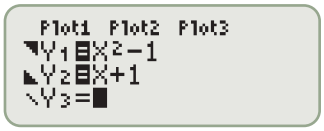
■ **YOUR TURN** Graph the solution to the system of inequalities: $x^2 + y^2 < 9$
$y > 0$

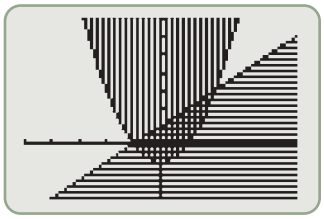
Technology Tip

Use a graphing calculator to graph the solution to the system of inequalities $y \geq x^2 - 1$ and $y < x + 1$.

First enter $y_2 = x^2 - 1$. For $\geq$, use the arrow key to move the cursor to the left of Y_1 and type ENTER until you see ◥. Next enter $y_2 = x + 1$.

For $<$, use the arrow key to move the cursor to the left of Y_1 and type ENTER until you see ◣.



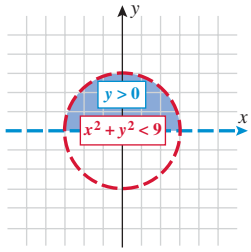


Note: The parabola should be drawn solid, and the line should be drawn dashed.

Study Tip

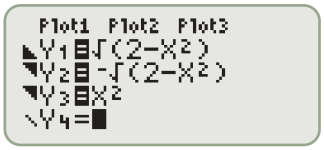
The points of intersection correspond to the vertices of the bounded region.

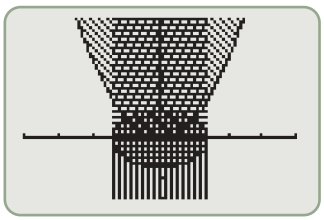
■ **Answer:**



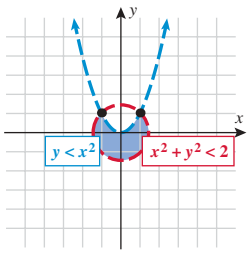
Technology Tip

Use a graphing calculator to graph the solution to the system of inequalities $x^2 + y^2 < 2$ and $y \geq x^2$. First enter $y_1 < \sqrt{2 - x^2}$ and $y_2 > -\sqrt{2 - x^2}$. Use the arrow key to move the cursor to the left of Y_1 and Y_2 and type ENTER until you see ◣ for < and ◥ for >. Next enter $y_3 = x^2$. For ≥, use the arrow key to move the cursor to the left of Y_3 and type ENTER until you see ◥.





■ **Answer:**



EXAMPLE 4 Solving a System of Nonlinear Inequalities

Solve the system of inequalities: $x^2 + y^2 < 2$
$y \geq x^2$

Solution:

STEP 1 Change the inequality signs to equal signs.

$$x^2 + y^2 = 2$$
$$y = x^2$$

STEP 2 The resulting equations correspond to a circle (to be drawn dashed) and a parabola (to be drawn solid). Graph the two inequalities.

To determine the points of intersection, solve the system of equations by substitution.

$$x^2 + (\underbrace{x^2}_{y})^2 = 2$$

$$x^4 + x^2 - 2 = 0$$

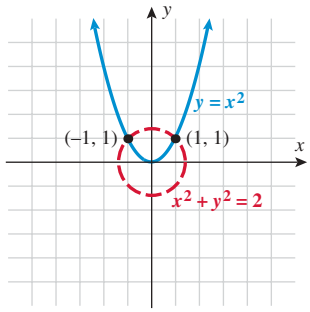
Factor.

$$(x^2 + 2)(x^2 - 1) = 0$$

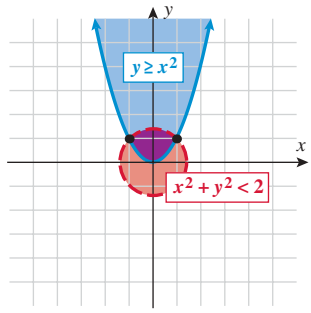
Solve for x.

$$\underbrace{x^2 = -2}_{\text{no solution}} \quad \text{or} \quad \underbrace{x^2 = 1}_{x = \pm 1}$$

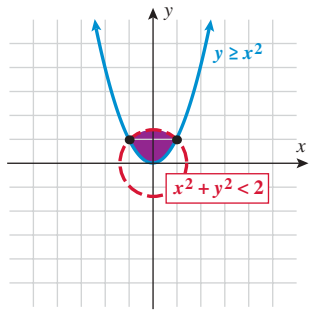
The points of intersection are $(-1, 1)$ and $(1, 1)$.



STEP 3 Test points and shade the region.



STEP 4 Identify the common region as the solution.



■ **YOUR TURN** Solve the system of inequalities: $x^2 + y^2 < 2$
$y < x^2$

It is important to note that any inequality based on an equation whose graph is not a line is considered a nonlinear inequality.

EXAMPLE 5 Solving a System of Nonlinear Inequalities

Solve the system of inequalities: $(x - 1)^2 + \frac{y^2}{4} < 1$

$$y \geq \sqrt{x}$$

Solution:

STEP 1 Change the inequality signs to equal signs.

$$(x - 1)^2 + \frac{y^2}{4} = 1$$
$$y = \sqrt{x}$$

STEP 2 The resulting equations correspond to an ellipse (to be drawn dashed) and the square-root function (to be drawn solid). Graph the two inequalities.

To determine the points of intersection, solve the system of equations by substitution.

$$(x - 1)^2 + \frac{(\overbrace{\sqrt{x}}^{y})^2}{4} = 1$$

Multiply by 4. $\quad 4(x - 1)^2 + x = 4$

Expand the binomial squared. $\quad 4(x^2 - 2x + 1) + x = 4$

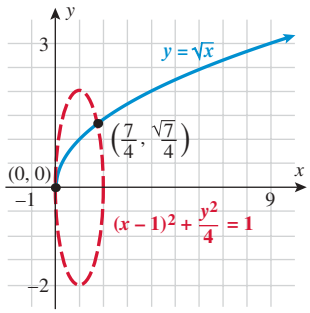
Distribute. $\quad 4x^2 - 8x + 4 + x = 4$

Combine like terms and gather terms to one side. $\quad 4x^2 - 7x = 0$

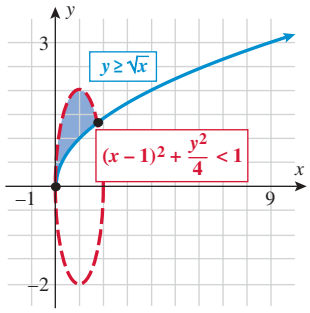
Factor. $\quad x(4x - 7) = 0$

Solve for x. $\quad x = 0 \quad \text{and} \quad x = \frac{7}{4}$

The points of intersection are (0, 0) and $\left(\frac{7}{4}, \sqrt{\frac{7}{4}}\right)$.

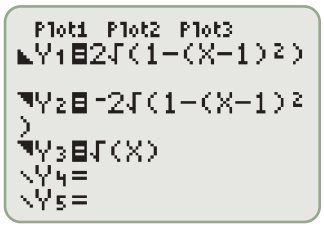


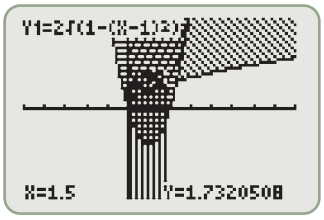
STEP 3 Shade the solution.



Technology Tip

Use a graphing calculator to graph the solution to the system of inequalities $(x - 1)^2 + \frac{y^2}{4} < 1$ and $y \geq \sqrt{x}$. First enter $y_1 < 2\sqrt{1 - (x - 1)^2}$ and $y_2 > -2\sqrt{1 - (x - 1)^2}$. Use the arrow key to move the cursor to the left of Y_1 and Y_2 and type ENTER until you see ◣ for < and ◥ for >. Next enter $y_3 \geq \sqrt{x}$. For ≥, use the arrow key to move the cursor to the left of Y_3 and type ENTER until you see ◥.





Note: The ellipse should be drawn dashed, and the square-root function should be drawn solid.

SECTION 9.6 SUMMARY

In this section, we discussed nonlinear inequalities in two variables. Sometimes these result in bounded regions (e.g., $x^2 + y^2 \le 1$), and sometimes these result in unbounded regions (e.g., $x^2 + y^2 > 1$).

When solving systems of inequalities, we first graph each of the inequalities separately and then look for the intersection (overlap) of all shaded regions.

SECTION 9.6 EXERCISES

SKILLS

In Exercises 1–12, match the nonlinear inequality with the correct graph.

1. $x^2 + y^2 < 25$

2. $x^2 + y^2 \le 9$

3. $\frac{x^2}{9} + \frac{y^2}{16} \ge 1$

4. $\frac{x^2}{4} + \frac{y^2}{9} > 1$

5. $y \ge x^2 - 3$

6. $x^2 \ge 16y$

7. $x \ge y^2 - 4$

8. $\frac{x^2}{9} + \frac{y^2}{25} \ge 1$

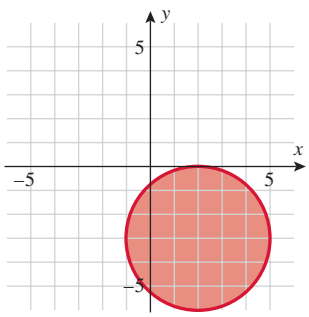
9. $9x^2 + 9y^2 < 36$

10. $(x - 2)^2 + (y + 3)^2 \le 9$

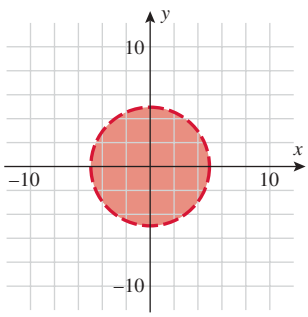
11. $\frac{x^2}{4} - \frac{y^2}{9} \ge 1$

12. $\frac{y^2}{16} - \frac{x^2}{9} < 1$

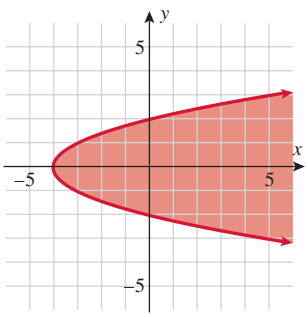
a.



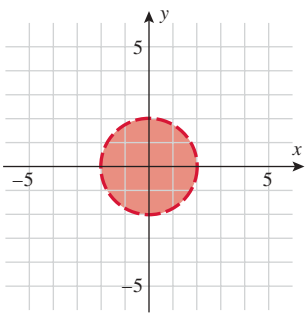
b.



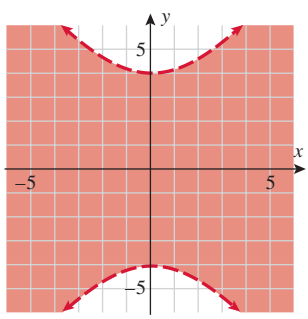
c.



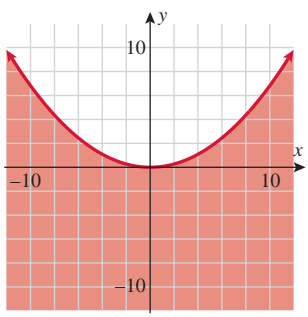
d.



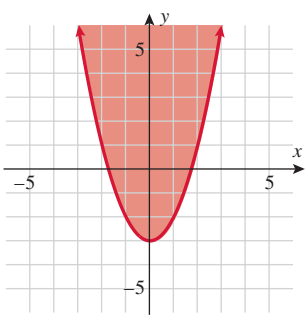
e.



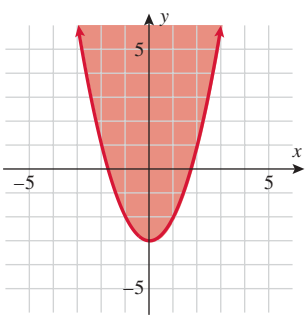
f.



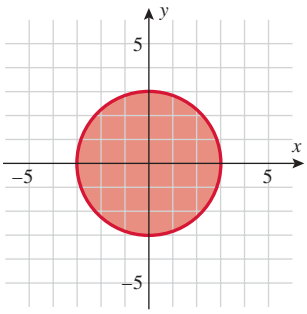
g.



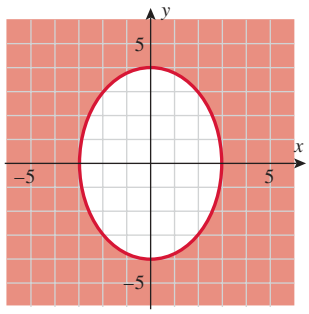
h.



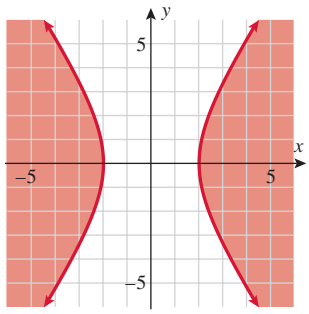
i.



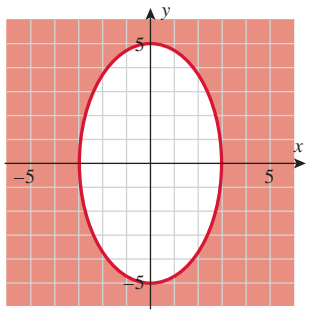
j.



k.



l.



In Exercises 13–30, graph the nonlinear inequality.

13. $y \le x^2 - 2$

14. $y \ge -x^2 + 3$

15. $x^2 + y^2 > 4$

16. $x^2 + y^2 < 16$

17. $x^2 + y^2 - 2x + 4y + 4 \ge 0$

18. $x^2 + y^2 + 2x - 2y - 2 \le 0$

19. $3x^2 + 4y^2 \le 12$

20. $\dfrac{(x-2)^2}{9} + \dfrac{(y+1)^2}{25} > 1$

21. $9x^2 + 16y^2 - 18x + 96y + 9 > 0$

22. $\dfrac{(x-2)^2}{4} - \dfrac{(y+3)^2}{1} \ge 1$

23. $9x^2 - 4y^2 \ge 26$

24. $\dfrac{(y+1)^2}{9} - \dfrac{(x+2)^2}{16} < 1$

25. $36x^2 - 9y^2 \ge 324$

26. $25x^2 - 36y^2 + 200x + 144y - 644 \ge 0$

27. $y \ge e^x$

28. $y \le \ln x$

29. $y < -x^3$

30. $y > -x^4$

In Exercises 31–50, graph each system of inequalities or indicate that the system has no solution.

31. $y < x + 1$
$y \le x^2$

32. $y < x^2 + 4x$
$y \le 3 - x$

33. $y \ge 2 + x$
$y \le 4 - x^2$

34. $y \ge (x-2)^2$
$y \le 4 - x$

35. $y \le -(x+2)^2$
$y > -5 + x$

36. $y \ge (x-1)^2 + 2$
$y \le 10 - x$

37. $-x^2 + y > -1$
$x^2 + y < 1$

38. $x < -y^2 + 1$
$x > y^2 - 1$

39. $y \ge x^2$
$x \ge y^2$

40. $y < x^2$
$x > y^2$

41. $x^2 + y^2 < 36$
$2x + y > 3$

42. $x^2 + y^2 < 36$
$y > 6$

43. $x^2 + y^2 < 25$
$y \ge 6 + x$

44. $(x-1)^2 + (y+2)^2 \le 36$
$y \ge x - 3$

45. $x^2 + y^2 \le 9$
$y \ge 1 + x^2$

46. $x^2 + y^2 \ge 16$
$x^2 + (y-3)^2 \le 9$

47. $x^2 - y^2 < 4$
$y > 1 - x^2$

48. $\dfrac{x^2}{4} - \dfrac{y^2}{9} \le 1$
$y \ge x - 5$

49. $y < e^x$
$y > \ln x \quad x > 0$

50. $y < 10^x$
$y > \log x \quad x > 0$

APPLICATIONS

51. Find the area enclosed by the system of inequalities.

$$x^2 + y^2 < 9$$
$$x > 0$$

52. Find the area enclosed by the system of inequalities.

$$x^2 + y^2 \le 5$$
$$x \le 0$$
$$y \ge 0$$

In Exercises 53 and 54, refer to the following:

The area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by $ab\pi$.

53. Find the area enclosed by the system of inequalities.

$$4x^2 + y^2 \le 16$$
$$x \le 0$$
$$y \ge 0$$

54. Find the area enclosed by the system of inequalities.

$$9x^2 + 4y^2 \ge 36$$
$$x^2 + y^2 \le 9$$

In Exercises 55 and 56, refer to the following:

The area below $y = x^2$, above $y = 0$, and between $x = 0$ and $x = a$ is $\frac{a^3}{3}$.

55. Find the area enclosed by the system of inequalities.

$$\begin{aligned} y &\le x^2 \\ x &\ge 0 \\ x &\le 6 \\ y &\ge x - 6 \end{aligned}$$

56. Find the area enclosed by the system of inequalities.

$$\begin{aligned} y &\le x^2 + 4 \\ y &\ge x \\ x &\ge -3 \\ x &\le 3 \end{aligned}$$

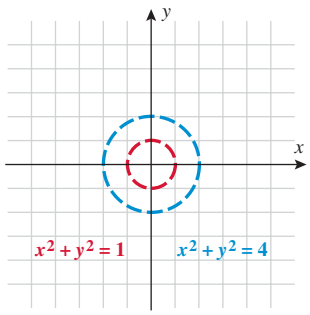
CATCH THE MISTAKE

In Exercises 57 and 58, explain the mistake that is made.

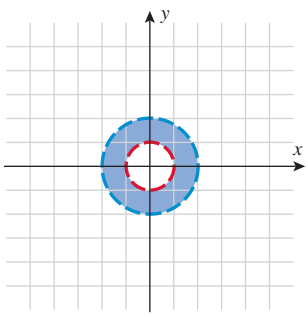
57. Graph the system of inequalities: $\begin{aligned} x^2 + y^2 &< 1 \\ x^2 + y^2 &> 4 \end{aligned}$

Solution:

Draw the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



Shade outside $x^2 + y^2 = 1$ and inside $x^2 + y^2 = 4$.



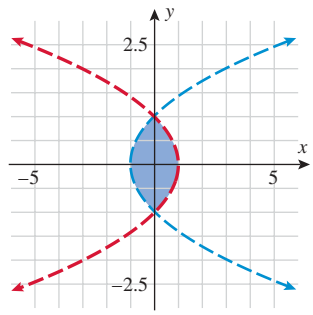
This is incorrect. What mistake was made?

58. Graph the system of inequalities: $\begin{aligned} x &> -y^2 + 1 \\ x &< y^2 - 1 \end{aligned}$

Solution:

Draw the parabolas $x = -y^2 + 1$ and $x = y^2 - 1$.

Shade the region between the curves.



This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 59–66, determine whether each statement is true or false.

59. A nonlinear inequality always represents a bounded region.

60. A system of inequalities always has a solution.

61. The solution to the following system of equations is symmetric with respect to the y-axis:

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &\le 1 \\ y &\ge x^2 - 2a \end{aligned}$$

62. The solution to the following system of equations is symmetric with respect to the origin:

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &\le 1 \\ y &\ge \sqrt[3]{x} \end{aligned}$$

63. The solution to the following system of equations is symmetric with respect to the origin:

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &\ge 1 \\ \frac{x^2}{4a^2} + \frac{y^2}{a^2} &\le 1 \end{aligned}$$

64. The solution to the following system of equations is bounded:

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &\le 1 \\ x &\ge -2a \\ x &\le 2a \end{aligned}$$

65. The following systems of inequalities have the same solution:

$$\begin{aligned} 16x^2 - 25y^2 &\geq 400 \\ x &\geq -6 \\ x &\leq 6 \end{aligned} \qquad \begin{aligned} \frac{y^2}{25} - \frac{x^2}{16} &\geq 1 \\ y &\geq -6 \\ y &\leq 6 \end{aligned}$$

66. The solution to the following system of inequalities is unbounded:

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &\leq 1 \\ \frac{y^2}{a^2} - \frac{x^2}{b^2} &\leq 1 \end{aligned}$$

CHALLENGE

67. For the system of nonlinear inequalities $\begin{aligned} x^2 + y^2 &\geq a^2 \\ x^2 + y^2 &\leq b^2 \end{aligned}$, what restriction must be placed on the values of a and b for this system to have a solution? Assume that a and b are real numbers.

68. Can $x^2 + y^2 < -1$ ever have a real solution? What types of numbers would x and/or y have to be to satisfy this inequality?

69. Find a positive real number a such that the area enclosed by the curves is the same.

$$x^2 + y^2 = 144 \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{4^2} = 1$$

70. If the area of the regions enclosed by $x^2 + y^2 = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are equal, what can you say about a and b?

71. If the solution to

$$\begin{aligned} 4x^2 + 9y^2 &\leq 36 \\ (x - h)^2 &\leq 4y \end{aligned}$$

is symmetric with respect to the y-axis, what can you say about h?

72. The solution to

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &\leq 1 \\ y &\geq x \end{aligned}$$

is located in quadrants I, II, and III. If the sections in quadrants I and III have the same area, what can you say about a and b?

TECHNOLOGY

In Exercises 73–80, use a graphing utility to graph the inequalities.

73. $x^2 + y^2 - 2x + 4y + 4 \geq 0$

74. $x^2 + y^2 + 2x - 2y - 2 \leq 0$

75. $y \geq e^x$

76. $y \leq \ln x$

77. $\begin{aligned} y &< e^x \\ y &> \ln x \qquad x > 0 \end{aligned}$

78. $\begin{aligned} y &< 10^x \\ y &> \log x \qquad x > 0 \end{aligned}$

79. $x^2 - 4y^2 + 5x - 6y + 18 \geq 0$

80. $x^2 - 2xy + 4y^2 + 10x - 25 \leq 0$

PREVIEW TO CALCULUS

In calculus, the problem of finding the area enclosed by a set of curves can be seen as the problem of finding the area enclosed by a system of inequalities.

In Exercises 81–84, graph the system of inequalities.

81. $\begin{aligned} y &\leq x^3 - x \\ y &\geq x^2 - 1 \end{aligned}$

82. $\begin{aligned} y &\leq x^3 \\ y &\geq 2x - x^2 \end{aligned}$

83. $\begin{aligned} y &\geq x^3 - 2x^2 \\ y &\leq x^2 \\ y &\leq 5 \end{aligned}$

84. $\begin{aligned} y &\leq \sqrt{1 - x^2} \\ y &\geq x^2 \\ y &\leq 2x \end{aligned}$

SECTION 9.7 ROTATION OF AXES

SKILLS OBJECTIVES

- Transform general second-degree equations into recognizable equations of conics by analyzing rotation of axes.
- Determine the angle of rotation that will transform a general second-degree equation into a familiar equation of a conic section.
- Graph a rotated conic.

CONCEPTUAL OBJECTIVE

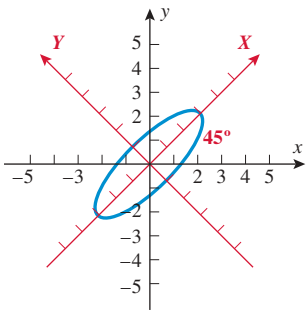
- Understand how the equation of a conic section is altered by rotation of axes.

Rotation of Axes Formulas

In Sections 9.1 through 9.4, we learned to recognize equations of parabolas, ellipses, and hyperbolas that were centered at any point in the Cartesian plane and whose vertices and foci were aligned either along or parallel to either the x-axis or the y-axis. We learned, for example, that the equation of an ellipse centered at the origin takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where the major and minor axes are, respectively, either the x- or the y-axis depending on whether a is greater than or less than b. Now let us look at an equation of a conic section whose graph is *not* aligned with the x- or y-axis: the equation $5x^2 - 8xy + 5y^2 - 9 = 0$.



This graph can be thought of as an ellipse that started with the major axis along the x-axis and the minor axis along the y-axis and then was rotated counterclockwise 45°. A new XY-coordinate system can be introduced that has the same origin but is rotated by a certain amount from the standard xy-coordinate system. In this example, the major axis of the ellipse lies along the new X-axis and the minor axis lies along the new Y-axis. We will see that we can write the equation of this ellipse as

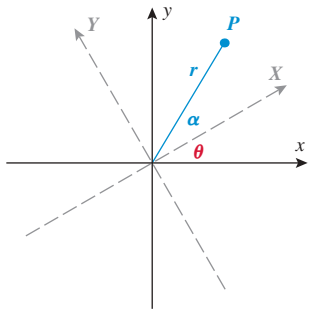
$$\frac{X^2}{9} + \frac{Y^2}{1} = 1$$

We will now develop the *rotation of axes formulas*, which allow us to transform the generalized second-degree equation in xy, that is, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, into an equation in XY of a conic that is familiar to us.

WORDS

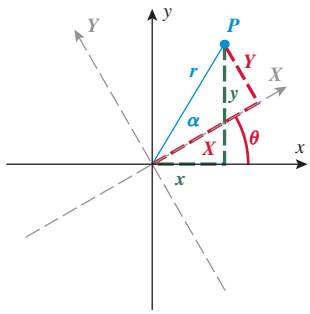
Let the new XY-coordinate system be displaced from the xy-coordinate system by rotation through an angle θ. Let P represent some point a distance r from the origin.

MATH



We can represent the point P as either the point (x, y) or the point (X, Y).

We define the angle α as the angle r makes with the X-axis and $\alpha + \theta$ as the angle r makes with the x-axis.



We can represent the point P in polar coordinates using the following relationships:

$$x = r\cos(\alpha + \theta)$$
$$y = r\sin(\alpha + \theta)$$
$$X = r\cos\alpha$$
$$Y = r\sin\alpha$$

Let us now derive the relationships between the two coordinate systems.

WORDS	**MATH**
Start with the x-term and write the cosine identity for a sum.	$x = r\cos(\alpha + \theta)$ $= r(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$
Eliminate the parentheses and group r with the α-terms.	$x = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$
Substitute according to the relationships $X = r\cos\alpha$ and $Y = r\sin\alpha$.	$\boxed{x = X\cos\theta - Y\sin\theta}$
Start with the y-term and write the sine identity for a sum.	$y = r\sin(\alpha + \theta)$ $= r(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$
Eliminate the parentheses and group r with the α-terms.	$y = (r\sin\alpha)\cos\theta + (r\cos\alpha)\sin\theta$
Substitute according to the relationships $X = r\cos\alpha$ and $Y = r\sin\alpha$.	$\boxed{y = Y\cos\theta + X\sin\theta}$

By treating the highlighted equations for x and y as a system of linear equations in X and Y, we can then solve for X and Y in terms of x and y. The results are summarized in the following box:

ROTATION OF AXES FORMULAS

Suppose that the x- and y-axes in the rectangular coordinate plane are rotated through an acute angle θ to produce the X- and Y-axes. Then, the coordinates (x, y) and (X, Y) are related according to the following equations:

$$\begin{aligned} x &= X\cos\theta - Y\sin\theta \\ y &= X\sin\theta + Y\cos\theta \end{aligned} \quad \text{or} \quad \begin{aligned} X &= x\cos\theta + y\sin\theta \\ Y &= -x\sin\theta + y\cos\theta \end{aligned}$$

EXAMPLE 1 Rotating the Axes

If the xy-coordinate axes are rotated 60°, find the XY-coordinates of the point $(x, y) = (-3, 4)$.

Solution:

Start with the rotation formulas.

$$X = x\cos\theta + y\sin\theta$$
$$Y = -x\sin\theta + y\cos\theta$$

Let $x = -3$, $y = 4$, and $\theta = 60°$.

$$X = -3\cos 60° + 4\sin 60°$$
$$Y = -(-3)\sin 60° + 4\cos 60°$$

Simplify.

$$X = -\underbrace{3\cos 60°}_{\frac{1}{2}} + \underbrace{4\sin 60°}_{\frac{\sqrt{3}}{2}}$$
$$Y = \underbrace{3\sin 60°}_{\frac{\sqrt{3}}{2}} + \underbrace{4\cos 60°}_{\frac{1}{2}}$$
$$X = -\frac{3}{2} + 2\sqrt{3}$$
$$Y = \frac{3\sqrt{3}}{2} + 2$$

The XY-coordinates are $\boxed{\left(-\frac{3}{2} + 2\sqrt{3}, \frac{3\sqrt{3}}{2} + 2\right)}$.

■ **YOUR TURN** If the xy-coordinate axes are rotated 30°, find the XY-coordinates of the point $(x, y) = (3, -4)$.

■ **Answer:** $\left(\frac{3\sqrt{3}}{2} - 2, -\frac{3}{2} - 2\sqrt{3}\right)$

EXAMPLE 2 Rotating an Ellipse

Show that the graph of the equation $5x^2 - 8xy + 5y^2 - 9 = 0$ is an ellipse aligning with coordinate axes that are rotated by 45°.

Solution:

Start with the rotation formulas.

$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

Let $\theta = 45°$.

$$x = \underbrace{X\cos 45°}_{\frac{\sqrt{2}}{2}} - \underbrace{Y\sin 45°}_{\frac{\sqrt{2}}{2}}$$
$$y = \underbrace{X\sin 45°}_{\frac{\sqrt{2}}{2}} + \underbrace{Y\cos 45°}_{\frac{\sqrt{2}}{2}}$$

Simplify.

$$x = \frac{\sqrt{2}}{2}(X - Y)$$
$$y = \frac{\sqrt{2}}{2}(X + Y)$$

Substitute $x = \frac{\sqrt{2}}{2}(X - Y)$ and $y = \frac{\sqrt{2}}{2}(X + Y)$ into $5x^2 - 8xy + 5y^2 - 9 = 0$.

$$5\left[\frac{\sqrt{2}}{2}(X - Y)\right]^2 - 8\left[\frac{\sqrt{2}}{2}(X - Y)\right]\left[\frac{\sqrt{2}}{2}(X + Y)\right] + 5\left[\frac{\sqrt{2}}{2}(X + Y)\right]^2 - 9 = 0$$

Simplify.

$$\frac{5}{2}\left(X^2 - 2XY + Y^2\right) - 4\left(X^2 - Y^2\right) + \frac{5}{2}\left(X^2 + 2XY + Y^2\right) - 9 = 0$$

$$\frac{5}{2}X^2 - 5XY + \frac{5}{2}Y^2 - 4X^2 + 4Y^2 + \frac{5}{2}X^2 + 5XY + \frac{5}{2}Y^2 = 9$$

Combine like terms.

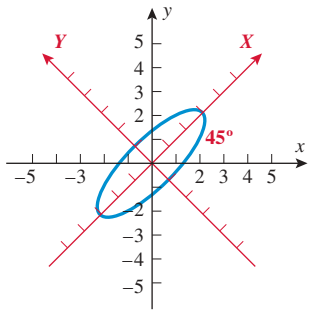
$$X^2 + 9Y^2 = 9$$

Divide by 9.

$$\boxed{\frac{X^2}{9} + \frac{Y^2}{1} = 1}$$

This (as discussed earlier) is an ellipse whose major axis is along the X-axis.

The vertices are at the points $(X, Y) = (\pm 3, 0)$.



Technology Tip

To graph the equation $5x^2 - 8xy + 5y^2 - 9 = 0$ with a graphing calculator, you need to solve for y using the quadratic formula.

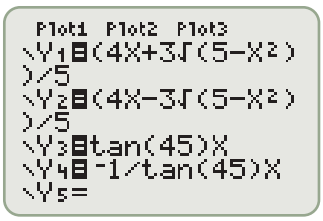
$5y^2 - 8xy + 5x^2 - 9 = 0$

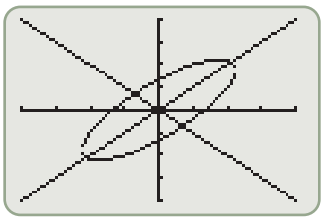
$a = 5,\ b = -8x,\ c = 5x^2 - 9$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{8 \pm 6\sqrt{5 - x^2}}{10}$$

Now enter $y_1 = \frac{4x + 3\sqrt{5 - x^2}}{5}$ and $y_2 = \frac{4x - 3\sqrt{5 - x^2}}{5}$. To graph the X- and Y-axes, enter $y_3 = \tan(45)x$ and $y_4 = -\frac{1}{\tan(45)}x$.





The Angle of Rotation Necessary to Transform a General Second-Degree Equation into a Familiar Equation of a Conic

In Section 9.1, we stated that the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

corresponds to a graph of a conic. Which type of conic it is depends on the value of the discriminant, $B^2 - 4AC$. In Sections 9.2–9.4, we discussed graphs of parabolas, ellipses, and hyperbolas with vertices along either the axes or lines parallel (or perpendicular) to the axes. In all cases the value of B was taken to be zero. When the value of B is nonzero, the result is a conic with vertices along the new XY-axes (or, respectively, parallel and perpendicular to them), which are the original xy-axes rotated through an angle θ. If given θ, we can determine the rotation equations as illustrated in Example 2, but how do we find the angle θ that represents the *angle of rotation*?

To find the angle of rotation, let us start with a general second-degree polynomial equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

We want to transform this equation into an equation in X and Y that does not contain an XY-term. Suppose we rotate our coordinates by an angle θ and use the rotation equations

$$x = X\cos\theta - Y\sin\theta \qquad y = X\sin\theta + Y\cos\theta$$

in the general second-degree polynomial equation; then the result is

$$\begin{aligned} &A(X\cos\theta - Y\sin\theta)^2 + B(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) \\ &+ C(X\cos\theta + Y\sin\theta)^2 + D(X\cos\theta - Y\sin\theta) + E(X\sin\theta + Y\cos\theta) + F = 0 \end{aligned}$$

If we expand these expressions and collect like terms, the result is an equation of the form

$$aX^2 + bXY + cY^2 + dX + eY + f = 0$$

where

$$\begin{aligned} a &= A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta \\ b &= B\left(\cos^2\theta - \sin^2\theta\right) + 2(C - A)\sin\theta\cos\theta \\ c &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\ d &= D\cos\theta + E\sin\theta \\ e &= -D\sin\theta + E\cos\theta \\ f &= F \end{aligned}$$

WORDS	MATH
We do not want this new equation to have an XY-term, so we set $b = 0$.	$B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta = 0$
We can use the double-angle formulas to simplify.	$B\underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos(2\theta)} + (C - A)\underbrace{2\sin\theta\cos\theta}_{\sin(2\theta)} = 0$
Subtract the $\sin(2\theta)$ term from both sides of the equation.	$B\cos(2\theta) = (A - C)\sin(2\theta)$
Divide by $B\sin(2\theta)$.	$\dfrac{B\cos(2\theta)}{B\sin(2\theta)} = \dfrac{(A - C)\sin(2\theta)}{B\sin(2\theta)}$
Simplify.	$\boxed{\cot(2\theta) = \dfrac{A - C}{B}}$

ANGLE OF ROTATION FORMULA

To transform the equation of a conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

into an equation in X and Y without an XY-term, rotate the xy-axes by an acute angle θ that satisfies the equation

$$\cot(2\theta) = \frac{A - C}{B} \quad \text{or} \quad \tan(2\theta) = \frac{B}{A - C}$$

Notice that the trigonometric equation $\cot(2\theta) = \dfrac{A - C}{B}$ or $\tan(2\theta) = \dfrac{B}{A - C}$ can be solved exactly for some values of θ (Example 3) and will have to be approximated with a calculator for other values of θ (Example 4).

EXAMPLE 3 Determining the Angle of Rotation I: The Value of the Cotangent Function Is That of a Known (Special) Angle

Determine the angle of rotation necessary to transform the following equation into an equation in X and Y with no XY-term:

$$3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$$

Solution:

Identify the A, B, and C parameters in the equation.	$\underbrace{3}_{A}x^2 + \underbrace{2\sqrt{3}}_{B}xy + \underbrace{1}_{C}y^2 + 2x - 2\sqrt{3}y = 0$
Write the rotation formula.	$\cot(2\theta) = \dfrac{A - C}{B}$
Let $A = 3$, $B = 2\sqrt{3}$, and $C = 1$.	$\cot(2\theta) = \dfrac{3 - 1}{2\sqrt{3}}$
Simplify.	$\cot(2\theta) = \dfrac{1}{\sqrt{3}}$
Apply the reciprocal identity.	$\tan(2\theta) = \sqrt{3}$

From our knowledge of trigonometric exact values, we know that $2\theta = 60°$ or $\boxed{\theta = 30°}$.

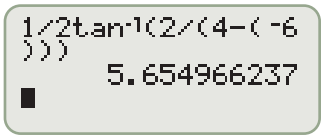
Technology Tip

To find the angle of rotation, use $\theta = \dfrac{1}{2}\tan^{-1}\left(\dfrac{B}{A - C}\right)$ and set a graphing calculator to degree mode. Substitute $A = 3$, $B = 2\sqrt{3}$, $C = 1$.

```
1/2tan⁻¹(2√(3)/(3
-1))
                30
```

Technology Tip

To find the angle of rotation, use $\theta = \frac{1}{2}\tan^{-1}\left(\frac{B}{A - C}\right)$ and set a graphing calculator to degree mode. Substitute $A = 4$, $B = 2$, and $C = -6$.



EXAMPLE 4 Determining the Angle of Rotation II: The Argument of the Cotangent Function Needs to Be Approximated with a Calculator

Determine the angle of rotation necessary to transform the following equation into an equation in X and Y with no XY-term. Round to the nearest tenth of a degree.

$$4x^2 + 2xy - 6y^2 - 5x + y - 2 = 0$$

Solution:

Identify the A, B, and C parameters in the equation.	$\underbrace{4}_{A}x^2 + \underbrace{2}_{B}xy - \underbrace{6}_{C}y^2 - 5x + y - 2 = 0$
Write the rotation formula.	$\cot(2\theta) = \frac{A - C}{B}$
Let $A = 4$, $B = 2$, and $C = -6$.	$\cot(2\theta) = \frac{4 - (-6)}{2}$
Simplify.	$\cot(2\theta) = 5$
Apply the reciprocal identity.	$\tan(2\theta) = \frac{1}{5} = 0.2$
Write the result as an inverse tangent function.	$2\theta = \tan^{-1}(0.2)$
With a calculator evaluate the right side of the equation.	$2\theta \approx 11.31°$
Solve for θ and round to the nearest tenth of a degree.	$\boxed{\theta = 5.7°}$

Special attention must be given when evaluating the inverse tangent function on a calculator, as the result is always in quadrant I or IV. If 2θ turns out to be negative, then 180° must be added so that 2θ is in quadrant II (as opposed to quadrant IV). Then θ will be an acute angle lying in quadrant I.

Recall that we stated (without proof) in Section 9.1 that we can identify a general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

as that of a particular conic depending on the discriminant.

Parabola	$B^2 - 4AC = 0$
Ellipse	$B^2 - 4AC < 0$
Hyperbola	$B^2 - 4AC > 0$

EXAMPLE 5 Graphing a Rotated Conic

For the equation $x^2 + 2xy + y^2 - \sqrt{2}x - 3\sqrt{2}y + 6 = 0$:

a. Determine which conic the equation represents.
b. Find the rotation angle required to eliminate the XY-term in the new coordinate system.
c. Transform the equation in x and y into an equation in X and Y.
d. Graph the resulting conic.

Solution (a):

Identify A, B, and C.

$$\underbrace{1}_{A}x^2 + \underbrace{2}_{B}xy + \underbrace{1}_{C}y^2 - \sqrt{2}x - 3\sqrt{2}y + 6 = 0$$

$$A = 1, B = 2, C = 1$$

Compute the discriminant. $B^2 - 4AC = 2^2 - 4(1)(1) = 0$

Since the discriminant equals zero, the equation represents a **parabola**.

Solution (b):

Write the rotation formula. $\cot(2\theta) = \dfrac{A - C}{B}$

Let $A = 1, B = 2$, and $C = 1$. $\cot(2\theta) = \dfrac{1 - 1}{2}$

Simplify. $\cot(2\theta) = 0$

Write the cotangent function in terms of the sine and cosine functions. $\dfrac{\cos(2\theta)}{\sin(2\theta)} = 0$

The numerator must equal zero. $\cos(2\theta) = 0$

From our knowledge of trigonometric exact values, we know that $2\theta = 90°$ or $\boxed{\theta = 45°}$.

Technology Tip

To find the angle of rotation, use $\theta = \frac{1}{2}\tan^{-1}\left(\frac{B}{A - C}\right)$ and set the calculator to degree mode. Substitute $A = 1$, $B = 2$, and $C = 1$.

```
1/2tan⁻¹(2/(1-1))
```

```
ERR:DIVIDE BY 0
1:Quit
2:Goto
```

The calculator displays an error message, which means that $\cos(2\theta) = 0$. Therefore, $2\theta = 90°$ or $\theta = 45°$.

To graph the equation $x^2 + 2xy + y^2 - \sqrt{2}x - 3\sqrt{2}y + 6 = 0$ with a graphing calculator, you need to solve for y using the quadratic formula.

$y^2 + y(2x - 3\sqrt{2}) + (x^2 - \sqrt{2}x + 6) = 0$

$a = 1$, $b = 2x - 3\sqrt{2}$,

$c = x^2 - \sqrt{2}x + 6$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-2x + 3\sqrt{2} \pm \sqrt{-8\sqrt{2}x - 6}}{2}$$

Now enter

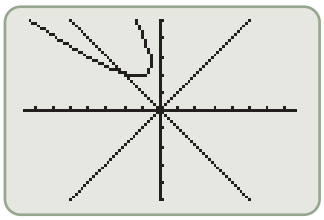
$$y_1 = \frac{-2x + 3\sqrt{2} + \sqrt{-8\sqrt{2}x - 6}}{2}$$

and

$$y_2 = \frac{-2x + 3\sqrt{2} - \sqrt{-8\sqrt{2}x - 6}}{2}.$$

To graph the X- and Y-axes, enter $y_3 = \tan(45)x$ and

$$y_4 = -\frac{1}{\tan(45)}x.$$



Solution (c):

Start with the equation $x^2 + 2xy + y^2 - \sqrt{2}x - 3\sqrt{2}y + 6 = 0$, and use the rotation formulas with $\theta = 45°$.

$$x = X\cos 45° - Y\sin 45° = \frac{\sqrt{2}}{2}(X - Y)$$

$$y = X\sin 45° + Y\cos 45° = \frac{\sqrt{2}}{2}(X + Y)$$

Find x^2, xy, and y^2.

$$x^2 = \left[\frac{\sqrt{2}}{2}(X - Y)\right]^2 = \frac{1}{2}\left(X^2 - 2XY + Y^2\right)$$

$$xy = \left[\frac{\sqrt{2}}{2}(X - Y)\right]\left[\frac{\sqrt{2}}{2}(X + Y)\right] = \frac{1}{2}\left(X^2 - Y^2\right)$$

$$y^2 = \left[\frac{\sqrt{2}}{2}(X + Y)\right]^2 = \frac{1}{2}\left(X^2 + 2XY + Y^2\right)$$

Substitute the values for x, y, x^2, xy, and y^2 into the original equation.

$$x^2 + 2xy + y^2 - \sqrt{2}x - 3\sqrt{2}y + 6 = 0$$

$$\frac{1}{2}\left(X^2 - 2XY + Y^2\right) + 2\frac{1}{2}\left(X^2 - Y^2\right) + \frac{1}{2}\left(X^2 + 2XY + Y^2\right) - \sqrt{2}\left[\frac{\sqrt{2}}{2}(X - Y)\right] - 3\sqrt{2}\left[\frac{\sqrt{2}}{2}(X + Y)\right] + 6 = 0$$

Eliminate the parentheses and combine like terms.

$$2X^2 - 4X - 2Y + 6 = 0$$

Divide by 2.

$$X^2 - 2X - Y + 3 = 0$$

Add Y.

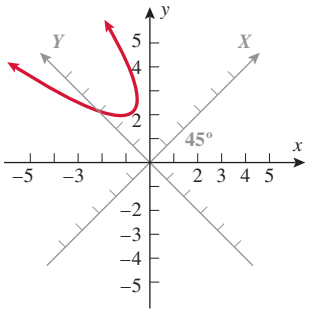
$$Y = \left(X^2 - 2X\right) + 3$$

Complete the square on X.

$$Y = (X - 1)^2 + 2$$

Solution (d):

This is a parabola opening upward in the XY-coordinate system shifted to the right one unit and up two units.



SECTION 9.7 SUMMARY

In this section, we found that the graph of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can represent conics in a system of rotated axes.

The following are the rotation formulas relating the xy-coordinate system to a rotated coordinate system with axes X and Y

$$x = X\cos\theta - Y\sin\theta$$

$$y = X\sin\theta + Y\cos\theta$$

where the rotation angle θ is found from the equation

$$\cot(2\theta) = \frac{A - C}{B} \quad \text{or} \quad \tan(2\theta) = \frac{B}{A - C}$$

SECTION 9.7 EXERCISES

SKILLS

In Exercises 1–8, the coordinates of a point in the *xy*-coordinate system are given. Assuming that the *XY*-axes are found by rotating the *xy*-axes by the given angle θ, find the corresponding coordinates for the point in the *XY*-system.

1. $(2, 4)$, $\theta = 45°$

2. $(5, 1)$, $\theta = 60°$

3. $(-3, 2)$, $\theta = 30°$

4. $(-4, 6)$, $\theta = 45°$

5. $(-1, -3)$, $\theta = 60°$

6. $(4, -4)$, $\theta = 45°$

7. $(0, 3)$, $\theta = 60°$

8. $(-2, 0)$, $\theta = 30°$

In Exercises 9–24, (a) identify the type of conic from the discriminant, (b) transform the equation in *x* and *y* into an equation in *X* and *Y* (without an *XY*-term) by rotating the *x*- and *y*-axes by the indicated angle θ to arrive at the new *X*- and *Y*-axes, and (c) graph the resulting equation (showing both sets of axes).

9. $xy - 1 = 0$, $\theta = 45°$

10. $xy - 4 = 0$, $\theta = 45°$

11. $x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y - 1 = 0$, $\theta = 45°$

12. $2x^2 - 4xy + 2y^2 - \sqrt{2}x + 1 = 0$, $\theta = 45°$

13. $y^2 - \sqrt{3}xy + 3 = 0$, $\theta = 30°$

14. $x^2 - \sqrt{3}xy - 3 = 0$, $\theta = 60°$

15. $7x^2 - 2\sqrt{3}xy + 5y^2 - 8 = 0$, $\theta = 60°$

16. $4x^2 + \sqrt{3}xy + 3y^2 - 45 = 0$, $\theta = 30°$

17. $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y - 2 = 0$, $\theta = 30°$

18. $x^2 + 2\sqrt{3}xy + 3y^2 - 2\sqrt{3}x + 2y - 4 = 0$, $\theta = 60°$

19. $7x^2 + 4\sqrt{3}xy + 3y^2 - 9 = 0$, $\theta = \dfrac{\pi}{6}$

20. $37x^2 + 42\sqrt{3}xy + 79y^2 - 400 = 0$, $\theta = \dfrac{\pi}{3}$

21. $7x^2 - 10\sqrt{3}xy - 3y^2 + 24 = 0$, $\theta = \dfrac{\pi}{3}$

22. $9x^2 + 14\sqrt{3}xy - 5y^2 + 48 = 0$, $\theta = \dfrac{\pi}{6}$

23. $x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y - 8 = 0$, $\theta = \dfrac{\pi}{4}$

24. $x^2 + 2xy + y^2 + 3\sqrt{2}x + \sqrt{2}y = 0$, $\theta = \dfrac{\pi}{4}$

In Exercises 25–38, determine the angle of rotation necessary to transform the equation in *x* and *y* into an equation in *X* and *Y* with no *XY*-term.

25. $x^2 + 4xy + y^2 - 4 = 0$

26. $3x^2 + 5xy + 3y^2 - 2 = 0$

27. $2x^2 + \sqrt{3}xy + 3y^2 - 1 = 0$

28. $4x^2 + \sqrt{3}xy + 3y^2 - 1 = 0$

29. $2x^2 + \sqrt{3}xy + y^2 - 5 = 0$

30. $2\sqrt{3}x^2 + xy + 3\sqrt{3}y^2 + 1 = 0$

31. $\sqrt{2}x^2 + xy + \sqrt{2}y^2 - 1 = 0$

32. $x^2 + 10xy + y^2 + 2 = 0$

33. $12\sqrt{3}x^2 + 4xy + 8\sqrt{3}y^2 - 1 = 0$

34. $4x^2 + 2xy + 2y^2 - 7 = 0$

35. $5x^2 + 6xy + 4y^2 - 1 = 0$

36. $x^2 + 2xy + 12y^2 + 3 = 0$

37. $3x^2 + 10xy + 5y^2 - 1 = 0$

38. $10x^2 + 3xy + 2y^2 + 3 = 0$

In Exercises 39–48, graph the second-degree equation. (*Hint:* Transform the equation into an equation that contains no *xy*-term.)

39. $21x^2 + 10\sqrt{3}xy + 31y^2 - 144 = 0$

40. $5x^2 + 6xy + 5y^2 - 8 = 0$

41. $8x^2 - 20xy + 8y^2 + 18 = 0$

42. $3y^2 - 26\sqrt{3}xy - 23x^2 - 144 = 0$

43. $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y - 12 = 0$

44. $3x^2 - 2\sqrt{3}xy + y^2 - 2x - 2\sqrt{3}y - 4 = 0$

45. $37x^2 - 42\sqrt{3}xy + 79y^2 - 400 = 0$

46. $71x^2 - 58\sqrt{3}xy + 13y^2 + 400 = 0$

47. $x^2 + 2xy + y^2 + 5\sqrt{2}x + 3\sqrt{2}y = 0$

48. $7x^2 - 4\sqrt{3}xy + 3y^2 - 9 = 0$

CONCEPTUAL

In Exercises 49–52, determine whether each statement is true or false.

49. The graph of the equation $x^2 + kxy + 9y^2 = 5$, where k is any positive constant less than 6, is an ellipse.

50. The graph of the equation $x^2 + kxy + 9y^2 = 5$, where k is any constant greater than 6, is a parabola.

51. The reciprocal function is a rotated hyperbola.

52. The equation $\sqrt{x} + \sqrt{y} = 3$ can be transformed into the equation $X^2 + Y^2 = 9$.

CHALLENGE

53. Determine the equation in X and Y that corresponds to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when the axes are rotated through

a. 90° **b.** 180°

54. Determine the equation in X and Y that corresponds to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ when the axes are rotated through

a. 90° **b.** 180°

55. Identify the conic section with equation $y^2 + ax^2 = x$ for $a < 0, a > 0, a = 0$, and $a = 1$.

56. Identify the conic section with equation $x^2 - ay^2 = y$ for $a < 0, a > 0, a = 0$, and $a = 1$.

TECHNOLOGY

For Exercises 57–62, refer to the following:

To use a function-driven software or graphing utility to graph a general second-degree equation, you need to solve for y. Let us consider a general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Group y^2 terms together, y terms together, and the remaining terms together.

$$Ax^2 + \underline{\underline{Bxy}} + \underline{Cy^2} + Dx + \underline{\underline{Ey}} + F = 0$$

$$Cy^2 + (Bxy + Ey) + \left(Ax^2 + Dx + F\right) = 0$$

Factor out the common y in the first set of parentheses.

$$Cy^2 + y(Bx + E) + \left(Ax^2 + Dx + F\right) = 0$$

Now this is a quadratic equation in y: $ay^2 + by + c = 0$.

Use the quadratic formula to solve for y.

$$Cy^2 + y(Bx + E) + \left(Ax^2 + Dx + F\right) = 0$$

$$a = C, b = Bx + E, c = Ax^2 + Dx + F$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad y = \frac{-(Bx + E) \pm \sqrt{(Bx + E)^2 - 4(C)\left(Ax^2 + Dx + F\right)}}{2(C)}$$

$$y = \frac{-(Bx + E) \pm \sqrt{B^2x^2 + 2BEx + E^2 - 4ACx^2 - 4CDx - 4CF}}{2C}$$

$$y = \frac{-(Bx + E) \pm \sqrt{\left(B^2 - 4AC\right)x^2 + (2BE - 4CD)x + \left(E^2 - 4CF\right)}}{2C}$$

Case I: $B^2 - 4AC = 0 \rightarrow$ The second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is a parabola.

$$y = \frac{-(Bx + E) \pm \sqrt{(2BE - 4CD)x + \left(E^2 - 4CF\right)}}{2C}$$

Case II: $B^2 - 4AC < 0 \rightarrow$ The second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is an ellipse.

$$y = \frac{-(Bx + E) \pm \sqrt{\left(B^2 - 4AC\right)x^2 + (2BE - 4CD)x + \left(E^2 - 4CF\right)}}{2C}$$

Case III: $B^2 - 4AC > 0 \rightarrow$ The second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is a hyperbola.

$$y = \frac{-(Bx + E) \pm \sqrt{\left(B^2 - 4AC\right)x^2 + (2BE - 4CD)x + \left(E^2 - 4CF\right)}}{2C}$$

57. Use a graphing utility to explore the second-degree equation $3x^2 + 2\sqrt{3}xy + y^2 + Dx + Ey + F = 0$ for the following values of D, E, and F:

a. $D = 1, E = 3, F = 2$

b. $D = -1, E = -3, F = 2$

Show the angle of rotation to the nearest degree. Explain the differences.

58. Use a graphing utility to explore the second-degree equation $x^2 + 3xy + 3y^2 + Dx + Ey + F = 0$ for the following values of D, E, and F:

a. $D = 2, E = 6, F = -1$

b. $D = 6, E = 2, F = -1$

Show the angle of rotation to the nearest degree. Explain the differences.

59. Use a graphing utility to explore the second-degree equation $2x^2 + 3xy + y^2 + Dx + Ey + F = 0$ for the following values of D, E, and F:

a. $D = 2, E = 1, F = -2$

b. $D = 2, E = 1, F = 2$

Show the angle of rotation to the nearest degree. Explain the differences.

60. Use a graphing utility to explore the second-degree equation $2\sqrt{3}x^2 + xy + \sqrt{3}y^2 + Dx + Ey + F = 0$ for the following values of D, E, and F:

a. $D = 2, E = 1, F = -1$

b. $D = 2, E = 6, F = -1$

Show the angle of rotation to the nearest degree. Explain the differences.

61. Use a graphing utility to explore the second-degree equation $Ax^2 + Bxy + Cy^2 + 2x + y - 1 = 0$ for the following values of A, B, and C:

a. $A = 4, B = -4, C = 1$

b. $A = 4, B = 4, C = -1$

c. $A = 1, B = -4, C = 4$

Show the angle of rotation to the nearest degree. Explain the differences.

62. Use a graphing utility to explore the second-degree equation $Ax^2 + Bxy + Cy^2 + 3x + 5y - 2 = 0$ for the following values of A, B, and C:

a. $A = 1, B = -4, C = 4$

b. $A = 1, B = 4, D = -4$

Show the angle of rotation to the nearest degree. Explain the differences.

PREVIEW TO CALCULUS

In calculus, when finding the area between two curves, we need to find the points of intersection of the curves. In Exercises 63–66, find the points of intersection of the rotated conic sections.

63. $x^2 + 2xy = 10$
$3x^2 - xy = 2$

64. $x^2 - 3xy + 2y^2 = 0$
$x^2 + xy = 6$

65. $2x^2 - 7xy + 2y^2 = -1$
$x^2 - 3xy + y^2 = 1$

66. $4x^2 + xy + 4y^2 = 22$
$-3x^2 + 2xy - 3y^2 = -11$

SECTION 9.8 POLAR EQUATIONS OF CONICS

SKILLS OBJECTIVES

- Define conics in terms of eccentricity.
- Express equations of conics in polar form.
- Graph the polar equations of conics.

CONCEPTUAL OBJECTIVE

- Define all conics in terms of a focus and a directrix.

Equations of Conics in Polar Coordinates

In Section 9.1, we discussed parabolas, ellipses, and hyperbolas in terms of geometric definitions. Then in Sections 9.2–9.4, we examined the rectangular equations of these conics. The equations for ellipses and hyperbolas when their centers are at the origin were simpler than when they were not (when the conics were shifted). In Section 7.5, we discussed polar coordinates and graphing of polar equations. In this section, we develop a more unified definition of the three conics in terms of a single focus and a directrix. You will see in this section that if the *focus* is located at the origin, then equations of conics are simpler when written in polar coordinates.

Alternative Definition of Conics

Recall that when we work with rectangular coordinates, we define a parabola (Sections 9.1 and 9.2) in terms of a fixed point (focus) and a line (directrix), whereas we define an ellipse and hyperbola (Sections 9.1, 9.3, and 9.4) in terms of two fixed points (the foci). However, it is possible to define all three conics in terms of a single focus and a directrix.

The following alternative representation of conics depends on a parameter called *eccentricity*.

ALTERNATIVE DESCRIPTION OF CONICS

Let D be a fixed line (the **directrix**), F be a fixed point (a **focus**) not on D, and e be a fixed positive number (**eccentricity**). The set of all points P such that the ratio of the distance from P to F to the distance from P to D equals the constant e defines a conic section.

$$\frac{d(P, F)}{d(P, D)} = e$$

- If $e = 1$, the conic is a **parabola**.
- If $e < 1$, the conic is an **ellipse**.
- If $e > 1$, the conic is a **hyperbola**.

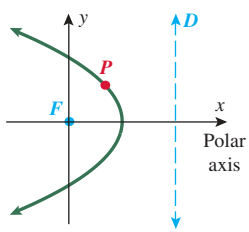
When $e = 1$, the result is a parabola, described by the same definition we used previously in Section 9.1. When $e \neq 1$, the result is either an ellipse or a hyperbola. The major axis of an ellipse passes through the focus and is perpendicular to the directrix. The transverse axis of a hyperbola also passes through the focus and is perpendicular to the directrix. If we let c represent the distance from the focus to the center and a represent the distance from the vertex to the center, then eccentricity is given by

$$e = \frac{c}{a}$$

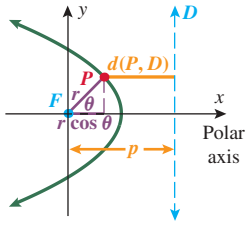
In polar coordinates, if we locate the focus of a conic at the pole and the directrix is either perpendicular or parallel to the polar axis, then we have four possible scenarios:

- The directrix is *perpendicular* to the polar axis and p units to the *right* of the pole.
- The directrix is *perpendicular* to the polar axis and p units to the *left* of the pole.
- The directrix is *parallel* to the polar axis and p units *above* the pole.
- The directrix is *parallel* to the polar axis and p units *below* the pole.

Let us take the case in which the directrix is perpendicular to the polar axis and p units to the right of the pole.



In polar coordinates (r, θ), we see that the distance from the focus to a point P is equal to r, that is, $d(P, F) = r$, and the distance from P to the closest point on the directrix is $d(P, D) = p - r\cos\theta$.



Words	Math
Substitute $d(P, F) = r$ and $d(P, D) = p - r\cos\theta$ into the formula for eccentricity, $\dfrac{d(P, F)}{d(P, D)} = e$.	$\dfrac{r}{p - r\cos\theta} = e$
Multiply the result by $p - r\cos\theta$.	$r = e(p - r\cos\theta)$
Eliminate the parentheses.	$r = ep - er\cos\theta$
Add $er\cos\theta$ to both sides of the equation.	$r + er\cos\theta = ep$
Factor out the common r.	$r + (1 + e\cos\theta) = ep$
Divide both sides by $1 + e\cos\theta$.	$\boxed{r = \dfrac{ep}{1 + e\cos\theta}}$

We need not derive the other three cases here, but note that if the directrix is perpendicular to the polar axis and p units to the *left* of the pole, the resulting polar equation is

$$r = \frac{ep}{1 - e\cos\theta}$$

If the directrix is parallel to the polar axis, the directrix is either above $(y = p)$ or below $(y = -p)$ the polar axis and we get the sine function instead of the cosine function, as summarized in the following box:

Polar Equations of Conics

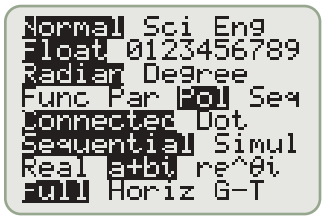
The following polar equations represent conics with one focus at the origin and with eccentricity e. It is assumed that the positive x-axis represents the polar axis.

Equation	Description
$r = \dfrac{ep}{1 + e\cos\theta}$	The directrix is *vertical* and p units to the *right* of the pole.
$r = \dfrac{ep}{1 - e\cos\theta}$	The directrix is *vertical* and p units to the *left* of the pole.
$r = \dfrac{ep}{1 + e\sin\theta}$	The directrix is *horizontal* and p units *above* the pole.
$r = \dfrac{ep}{1 - e\sin\theta}$	The directrix is *horizontal* and p units *below* the pole.

Eccentricity	The Conic Is a _____	The ___ Is Perpendicular to the Directrix
$e = 1$	Parabola	Axis of symmetry
$e < 1$	Ellipse	Major axis
$e > 1$	Hyperbola	Transverse axis

Technology Tip

Be sure to set the calculator to radian and polar modes.



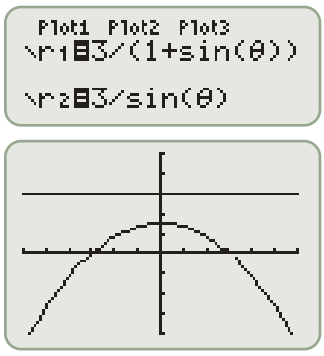
Use [Y =] to enter the polar equation $r = \dfrac{3}{1 + \sin\theta}$.

[r1 =] [3] [÷] [(] [1] [+] [sin] [X, T, θ, n] [)] [)]

To enter the equation of the directrix $y = 3$, use its polar form.

$$y = 3 \quad r\sin\theta = 3 \quad r = \frac{3}{\sin\theta}$$

Now enter [r2 =] [3] [÷] [sin] [X, T, θ, n] [)].



EXAMPLE 1 Finding the Polar Equation of a Conic

Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = 3$.

Solution:

The directrix is horizontal and above the pole. $$r = \frac{ep}{1 + e\sin\theta}$$

A parabola has eccentricity $e = 1$, and we know that $p = 3$. $$\boxed{r = \frac{3}{1 + \sin\theta}}$$

■ **Answer:** $r = \dfrac{3}{1 - \cos\theta}$

■ **YOUR TURN** Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $x = -3$.

Technology Tip

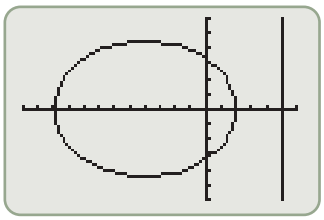
Use $\boxed{\text{Y =}}$ to enter the polar equation $r = \dfrac{10}{3 + 2\cos\theta}$.

$\boxed{\text{r1 =}}$ $\boxed{10}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{3}$ $\boxed{+}$ $\boxed{2}$ $\boxed{\cos}$ $\boxed{\text{X, T, }\theta\text{, n}}$ $\boxed{)}$ $\boxed{)}$

To enter the equation of the directrix $x = 5$, use its polar form.

$$x = 5 \quad r\cos\theta = 5 \quad r = \frac{5}{\cos\theta}$$

$\boxed{\text{r2 =}}$ $\boxed{5}$ $\boxed{\div}$ $\boxed{\cos}$ $\boxed{\text{X, T, }\theta\text{, n}}$ $\boxed{)}$



■ **Answer:** hyperbola, $e = 5$, with transverse axis along the y-axis

EXAMPLE 2 Identifying a Conic from Its Equation

Identify the type of conic represented by the equation $r = \dfrac{10}{3 + 2\cos\theta}$.

Solution:

To identify the type of conic, we need to rewrite the equation in the form: $$r = \frac{ep}{1 \pm e\cos\theta}$$

Divide the numerator and denominator by 3. $$r = \frac{\frac{10}{3}}{\left(1 + \frac{2}{3}\cos\theta\right)}$$

Identify e in the denominator. $$= \frac{\frac{10}{3}}{\left(1 + \underbrace{\frac{2}{3}}_{e}\cos\theta\right)}$$

The numerator is equal to ep. $$= \frac{\overbrace{5}^{p} \cdot \overbrace{\frac{2}{3}}^{e}}{\left(1 + \underbrace{\frac{2}{3}}_{e}\cos\theta\right)}$$

Since $e = \frac{2}{3} < 1$, the conic is an $\boxed{\text{ellipse}}$. The directrix is $x = 5$, so the major axis is along the x-axis (perpendicular to the directrix).

■ **YOUR TURN** Identify the type of conic represented by the equation

$$r = \frac{10}{2 - 10\sin\theta}$$

In Example 2 we found that the polar equation $r = \dfrac{10}{3 + 2\cos\theta}$ is an ellipse with its major axis along the x-axis. We will graph this ellipse in Example 3.

EXAMPLE 3 Graphing a Conic from Its Equation

The graph of the polar equation $r = \dfrac{10}{3 + 2\cos\theta}$ is an ellipse.

a. Find the vertices.
b. Find the center of the ellipse.
c. Find the lengths of the major and minor axes.
d. Graph the ellipse.

Solution (a):

From Example 2 we see that $e = \frac{2}{3}$, which corresponds to an ellipse, and $x = 5$ is the directrix.

The major axis is perpendicular to the directrix. Therefore, the major axis lies along the polar axis. To find the vertices (which lie along the major axis), let $\theta = 0$ and $\theta = \pi$.

$\theta = 0$: $\quad r = \dfrac{10}{3 + 2\cos\theta} = \dfrac{10}{5} = 2$

$\theta = \pi$: $\quad r = \dfrac{10}{3 + 2\cos\pi} = \dfrac{10}{1} = 10$

The vertices are the points $\boxed{V_1 = (2, 0)}$ and $\boxed{V_2 = (10, \pi)}$.

Solution (b):

The vertices in rectangular coordinates are $V_1 = (2, 0)$ and $V_2 = (-10, 0)$.

The midpoint (in rectangular coordinates) between the two vertices is the point $(-4, 0)$, which corresponds to the point $\boxed{(4, \pi)}$ in polar coordinates.

Solution (c):

The length of the major axis, $2a$, is the distance between the vertices. $\quad \boxed{2a = 12}$

The length $a = 6$ corresponds to the distance from the center to a vertex.

Apply the formula $e = \dfrac{c}{a}$ with $a = 6$ and $e = \frac{2}{3}$ to find c. $\quad c = ae = 6\left(\dfrac{2}{3}\right) = 4$

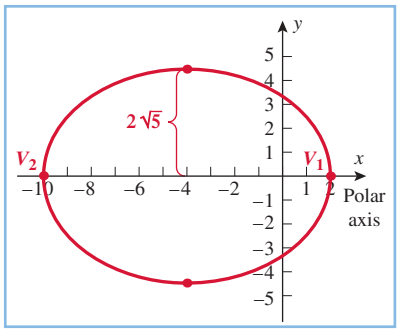
Let $a = 6$ and $c = 4$ in $b^2 = a^2 - c^2$. $\quad b^2 = 6^2 - 4^2 = 20$

Solve for b. $\quad b = 2\sqrt{5}$

The length of the minor axis is $\boxed{2b = 4\sqrt{5}}$.

Solution (d):

Graph the ellipse.

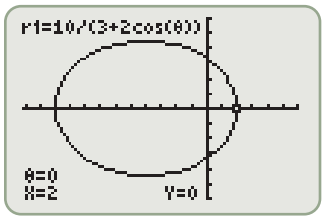


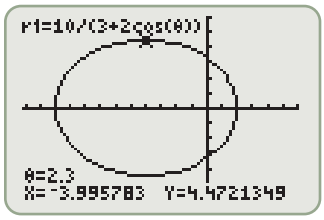
Technology Tip

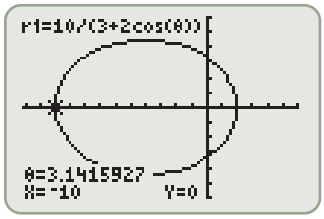
Use $\boxed{\text{Y =}}$ to enter the polar equation $r = \dfrac{10}{3 + 2\cos\theta}$.

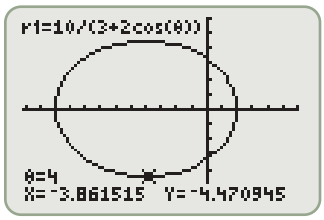
$\boxed{\text{r1 =}}$ $\boxed{10}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{3}$ $\boxed{+}$ $\boxed{2}$ $\boxed{\cos}$ $\boxed{\text{X, T, }\theta\text{, n}}$ $\boxed{)}$ $\boxed{)}$

Use the $\boxed{\text{TRACE}}$ key to trace the vertices of the ellipse.





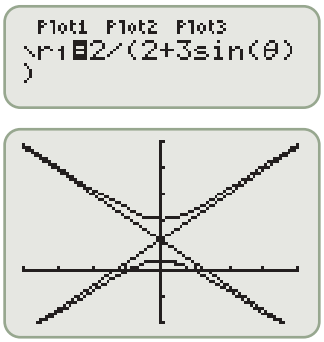




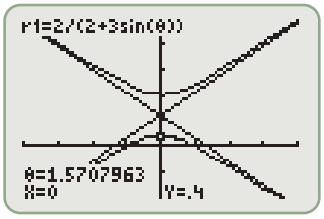
Technology Tip

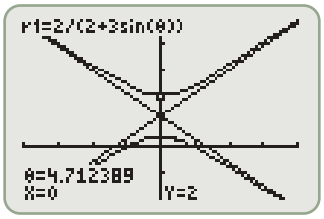
Use [Y =] to enter the polar equation $r = \dfrac{2}{2 + 3\sin\theta}$.

[r1 =] [2] [÷] [(] [2] [+] [3] [sin] [X, T, θ, n] [)] [)]



Use the [TRACE] key to trace the vertices of the hyperbola.





EXAMPLE 4 Identifying and Graphing a Conic from Its Equation

Identify and graph the conic defined by the equation $r = \dfrac{2}{2 + 3\sin\theta}$.

Solution:

Rewrite the equation in the form $r = \dfrac{ep}{1 + e\sin\theta}$.

$$r = \frac{2}{2 + 3\sin\theta} = \frac{\overbrace{\left(\frac{2}{3}\right)}^{p}\overbrace{\left(\frac{3}{2}\right)}^{e}}{1 + \underbrace{\left(\frac{3}{2}\right)}_{e}\sin\theta}$$

The conic is a *hyperbola* since $e = \frac{3}{2} > 1$.

The directrix is horizontal and $\frac{2}{3}$ unit above the pole (origin).

To find the vertices, let $\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{3\pi}{2}$.

$\theta = \dfrac{\pi}{2}$: $\qquad r = \dfrac{2}{2 + 3\sin\left(\frac{\pi}{2}\right)} = \dfrac{2}{5}$

$\theta = \dfrac{3\pi}{2}$: $\qquad r = \dfrac{2}{2 + 3\sin\left(\frac{3\pi}{2}\right)} = \dfrac{2}{-1} = -2$

The vertices in polar coordinates are $\left(\dfrac{2}{5}, \dfrac{\pi}{2}\right)$ and $\left(-2, \dfrac{3\pi}{2}\right)$.

The vertices in rectangular coordinates are $V_1 = \left(0, \frac{2}{5}\right)$ and $V_2 = (0, 2)$.

The center is the midpoint between the vertices: $\left(0, \frac{6}{5}\right)$.

The distance from the center to a focus is $c = \frac{6}{5}$.

Apply the formula $e = \dfrac{c}{a}$ with $c = \dfrac{6}{5}$ and $e = \dfrac{3}{2}$ to find a.

$$a = \frac{c}{e} = \frac{\frac{6}{5}}{\frac{3}{2}} = \frac{4}{5}$$

Let $a = \frac{4}{5}$ and $c = \frac{6}{5}$ in $b^2 = c^2 - a^2$.

$$b^2 = \left(\frac{6}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{20}{25}$$

Solve for b.

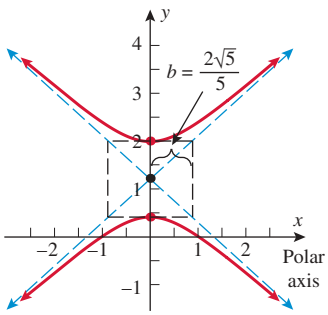
$$b = \frac{2\sqrt{5}}{5}$$

The asymptotes are given by

$y = \pm\dfrac{a}{b}(x - h) + k$, where $a = \dfrac{4}{5}$,

$b = \dfrac{2\sqrt{5}}{5}$, and $(h, k) = \left(0, \dfrac{6}{5}\right)$.

$$y = \pm\frac{2}{\sqrt{5}}x + \frac{6}{5}$$



It is important to note that although we relate specific points (vertices, foci, etc.) to rectangular coordinates, another approach to finding a rough sketch is to simply point-plot the equation in polar coordinates.

EXAMPLE 5 Graphing a Conic by Point-Plotting in Polar Coordinates

Sketch a graph of the conic $r = \dfrac{4}{1 - \sin\theta}$.

Solution:

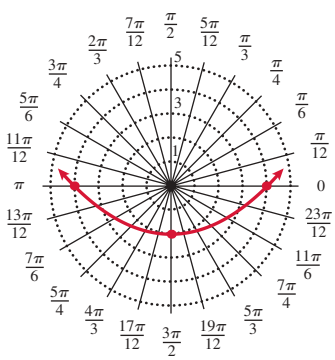
STEP 1 The conic is a parabola because the equation is in the form

$$r = \frac{(4)(1)}{1 - (1)\sin\theta}$$

Make a table with key values for θ and r.

θ	$r = \dfrac{4}{1 - \sin\theta}$	(r, θ)
0	$r = \dfrac{4}{1 - \sin 0} = \dfrac{4}{1} = 4$	$(4, 0)$
$\dfrac{\pi}{2}$	$r = \dfrac{4}{1 - \sin\frac{\pi}{2}} = \dfrac{4}{1 - 1} = \dfrac{4}{0}$	undefined
π	$r = \dfrac{4}{1 - \sin\pi} = \dfrac{4}{1} = 4$	$(4, \pi)$
$\dfrac{3\pi}{2}$	$r = \dfrac{4}{1 - \sin\frac{3\pi}{2}} = \dfrac{4}{1 - (-1)} = \dfrac{4}{2} = 2$	$\left(2, \dfrac{3\pi}{2}\right)$
2π	$r = \dfrac{4}{1 - \sin(2\pi)} = \dfrac{4}{1} = 4$	$(4, 2\pi)$

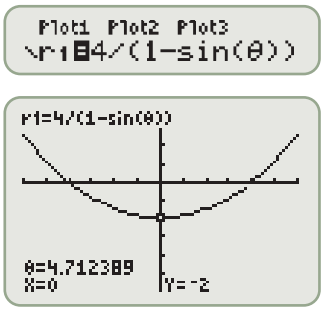
STEP 2 Plot the points on a polar graph and connect them with a smooth parabolic curve.



Technology Tip

Use $\boxed{Y =}$ to enter the polar equation $r = \dfrac{4}{1 - \sin\theta}$.

$\boxed{r1 =}$ $\boxed{4}$ $\boxed{\div}$ $\boxed{(}$ $\boxed{1}$ $\boxed{-}$ $\boxed{\sin}$ $\boxed{X, T, \theta, n}$ $\boxed{)}$ $\boxed{)}$



SECTION 9.8 SUMMARY

In this section, we found that we could graph polar equations of conics by identifying a single focus and the directrix. There are four possible equations in terms of eccentricity e:

EQUATION	DESCRIPTION
$r = \dfrac{ep}{1 + e\cos\theta}$	The directrix is *vertical* and p units to the *right* of the pole.
$r = \dfrac{ep}{1 - e\cos\theta}$	The directrix is *vertical* and p units to the *left* of the pole.
$r = \dfrac{ep}{1 + e\sin\theta}$	The directrix is *horizontal* and p units *above* the pole.
$r = \dfrac{ep}{1 - e\sin\theta}$	The directrix is *horizontal* and p units *below* the pole.

SECTION 9.8 EXERCISES

SKILLS

In Exercises 1–14, find the polar equation that represents the conic described (assume that a focus is at the origin).

	Conic	Eccentricity	Directrix		Conic	Eccentricity	Directrix
1.	Ellipse	$e = \frac{1}{2}$	$y = -5$	**2.**	Ellipse	$e = \frac{1}{3}$	$y = 3$
3.	Hyperbola	$e = 2$	$y = 4$	**4.**	Hyperbola	$e = 3$	$y = -2$
5.	Parabola	$e = 1$	$x = 1$	**6.**	Parabola	$e = 1$	$x = -1$
7.	Ellipse	$e = \frac{3}{4}$	$x = 2$	**8.**	Ellipse	$e = \frac{2}{3}$	$x = -4$
9.	Hyperbola	$e = \frac{4}{3}$	$x = -3$	**10.**	Hyperbola	$e = \frac{3}{2}$	$x = 5$
11.	Parabola	$e = 1$	$y = -3$	**12.**	Parabola	$e = 1$	$y = 4$
13.	Ellipse	$e = \frac{3}{5}$	$y = 6$	**14.**	Hyperbola	$e = \frac{8}{5}$	$y = 5$

In Exercises 15–26, identify the conic (parabola, ellipse, or hyperbola) that each polar equation represents.

15. $r = \dfrac{4}{1 + \cos\theta}$ **16.** $r = \dfrac{3}{2 - 3\sin\theta}$ **17.** $r = \dfrac{2}{3 + 2\sin\theta}$ **18.** $r = \dfrac{3}{2 - 2\cos\theta}$

19. $r = \dfrac{2}{4 + 8\cos\theta}$ **20.** $r = \dfrac{1}{4 - \cos\theta}$ **21.** $r = \dfrac{7}{3 + \cos\theta}$ **22.** $r = \dfrac{4}{5 + 6\sin\theta}$

23. $r = \dfrac{40}{5 + 5\sin\theta}$ **24.** $r = \dfrac{5}{5 - 4\sin\theta}$ **25.** $r = \dfrac{1}{1 - 6\cos\theta}$ **26.** $r = \dfrac{5}{3 - 3\sin\theta}$

In Exercises 27–42, for the given polar equations: (a) identify the conic as either a parabola, an ellipse, or a hyperbola; (b) find the eccentricity and vertex (or vertices); and (c) graph.

27. $r = \dfrac{2}{1 + \sin\theta}$ **28.** $r = \dfrac{4}{1 - \cos\theta}$ **29.** $r = \dfrac{4}{1 - 2\sin\theta}$ **30.** $r = \dfrac{3}{3 + 8\cos\theta}$

31. $r = \dfrac{2}{2 + \sin\theta}$ **32.** $r = \dfrac{1}{3 - \sin\theta}$ **33.** $r = \dfrac{1}{2 - 2\sin\theta}$ **34.** $r = \dfrac{1}{1 - 2\sin\theta}$

35. $r = \dfrac{4}{3 + \cos\theta}$

36. $r = \dfrac{2}{5 + 4\sin\theta}$

37. $r = \dfrac{6}{2 + 3\sin\theta}$

38. $r = \dfrac{6}{1 + \cos\theta}$

39. $r = \dfrac{2}{5 + 5\cos\theta}$

40. $r = \dfrac{10}{6 - 3\cos\theta}$

41. $r = \dfrac{6}{3\cos\theta + 1}$

42. $r = \dfrac{15}{3\sin\theta + 5}$

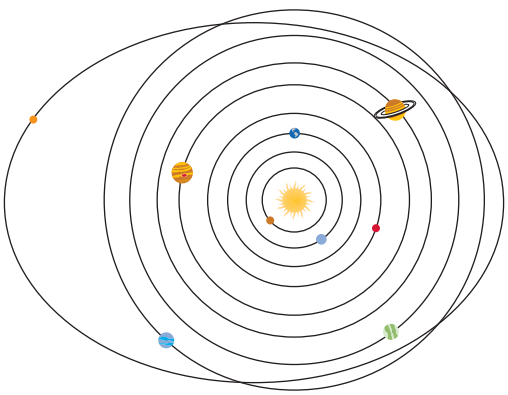
APPLICATIONS

For Exercises 43 and 44, refer to the following:

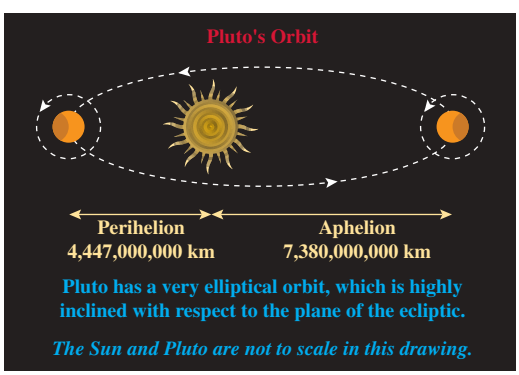
Planets travel in elliptical orbits around a single focus, the Sun. Pluto (orange), the dwarf planet furthest from the Sun, has a pronounced elliptical orbit, whereas Earth (royal blue) has an almost circular orbit. The polar equation of a planet's orbit can be expressed as

$$r = \frac{a(1 - e^2)}{(1 - e\cos\theta)}$$

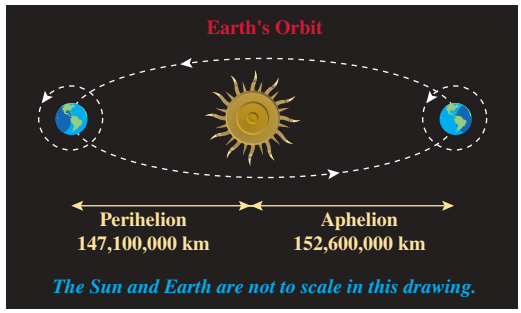
where e is the eccentricity and $2a$ is the length of the major axis. It can also be shown that the perihelion distance (minimum distance from the Sun to a planet) and the aphelion distance (maximum distance from the Sun to the planet) can be represented by $r = a(1 - e)$ and $r = a(1 + e)$, respectively.



43. Planetary Orbits. Pluto's orbit is summarized in the picture below. Find the eccentricity of Pluto's orbit. Find the polar equation that governs Pluto's orbit.



44. Planetary Orbits. Earth's orbit is summarized in the picture below. Find the eccentricity of Earth's orbit. Find the polar equation that governs Earth's orbit.



For Exercises 45 and 46, refer to the following:

Asteroids, meteors, and comets all orbit the Sun in elliptical patterns and often cross paths with Earth's orbit, making life a little tense now and again. Asteroids are large rocks (bodies under 1000 kilometers across), meteors range from sand particles to rocks, and comets are masses of debris. A few asteroids have orbits that cross Earth's orbits—called Apollos or Earth-crossing asteroids. In recent years, asteroids have passed within 100,000 kilometers of Earth!

45. Asteroids. The asteroid 433 or Eros is the second largest near-Earth asteroid. The semimajor axis of its orbit is 150 million kilometers and the eccentricity is 0.223. Find the polar equation of Eros's orbit.

46. Asteroids. The asteroid Toutatis is the largest near-Earth asteroid. The semimajor axis of its orbit is 350 million kilometers and the eccentricity is 0.634. On September 29, 2004, it missed Earth by 961,000 miles. Find the polar equation of Toutatis's orbit.

47. Earth's Orbit. A simplified model of Earth's orbit around the Sun is given by $r = \dfrac{1}{1 + 0.0167\cos\theta}$. Find the center of the orbit in

a. rectangular coordinates
b. polar coordinates

48. Uranus's Orbit. A simplified model of Uranus's orbit around the Sun is given by $r = \dfrac{1}{1 + 0.0461\cos\theta}$. Find the center of the orbit in

a. rectangular coordinates
b. polar coordinates

49. Orbit of Halley's Comet. A simplified model of the orbit of Halley's Comet around the Sun is given by $r = \dfrac{1}{1 + 0.967\sin\theta}$. Find the center of the orbit in rectangular coordinates.

50. Orbit of the Hale–Bopp Comet. A simplified model of the orbit of the Hale–Bopp Comet around the Sun is given by $r = \dfrac{1}{1 + 0.995\sin\theta}$. Find the center of the orbit in rectangular coordinates.

CONCEPTUAL

51. When $0 < e < 1$, the conic is an ellipse. Does the conic become more elongated or elliptical as e approaches 1 or as e approaches 0?

52. Show that $r = \dfrac{ep}{1 - e\sin\theta}$ is the polar equation of a conic with a horizontal directrix that is p units *below* the pole.

53. Convert from rectangular to polar coordinates to show that the equation of a hyperbola, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, in polar form is $r^2 = -\dfrac{b^2}{1 - e^2\cos^2\theta}$.

54. Convert from rectangular to polar coordinates to show that the equation of an ellipse, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, in polar form is $r^2 = \dfrac{b^2}{1 - e^2\cos^2\theta}$.

CHALLENGE

55. Find the major diameter of the ellipse with polar equation $r = \dfrac{ep}{1 + e\cos\theta}$ in terms of e and p.

56. Find the minor diameter of the ellipse with polar equation $r = \dfrac{ep}{1 + e\cos\theta}$ in terms of e and p.

57. Find the center of the ellipse with polar equation $r = \dfrac{ep}{1 + e\cos\theta}$ in terms of e and p.

58. Find the length of the latus rectum of the parabola with polar equation $r = \dfrac{p}{1 + \cos\theta}$. Assume that the focus is at the origin.

TECHNOLOGY

59. Let us consider the polar equations $r = \dfrac{ep}{1 + e\cos\theta}$ and $r = \dfrac{ep}{1 - e\cos\theta}$ with eccentricity $e = 1$. With a graphing utility, explore the equations with $p = 1, 2$, and 6. Describe the behavior of the graphs as $p \to \infty$ and also the difference between the two equations.

60. Let us consider the polar equations $r = \dfrac{ep}{1 + e\sin\theta}$ and $r = \dfrac{ep}{1 - e\sin\theta}$ with eccentricity $e = 1$. With a graphing utility, explore the equations with $p = 1, 2$, and 6. Describe the behavior of the graphs as $p \to \infty$ and also the difference between the two equations.

61. Let us consider the polar equations $r = \dfrac{ep}{1 + e\cos\theta}$ and $r = \dfrac{ep}{1 - e\cos\theta}$ with $p = 1$. With a graphing utility, explore the equations with $e = 1.5, 3$, and 6. Describe the behavior of the graphs as $e \to \infty$ and also the difference between the two equations.

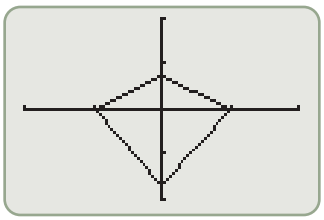
62. Let us consider the polar equations $r = \dfrac{ep}{1 + e\sin\theta}$ and $r = \dfrac{ep}{1 - e\sin\theta}$ with $p = 1$. With a graphing utility, explore the equations with $e = 1.5, 3$, and 6. Describe the behavior of the graphs as $e \to \infty$ and also the difference between the two equations.

63. Let us consider the polar equations $r = \dfrac{ep}{1 + e\cos\theta}$ and $r = \dfrac{ep}{1 - e\cos\theta}$ with $p = 1$. With a graphing utility, explore the equations with $e = 0.001, 0.5, 0.9$, and 0.99. Describe the behavior of the graphs as $e \to 1$ and also the difference between the two equations. Be sure to set the window parameters properly.

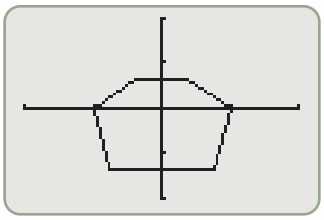
64. Let us consider the polar equations $r = \dfrac{ep}{1 + e\sin\theta}$ and $r = \dfrac{ep}{1 - e\sin\theta}$ with $p = 1$. With a graphing utility, explore the equations with $e = 0.001, 0.5, 0.9$, and 0.99. Describe the behavior of the graphs as $e \to 1$ and also the difference between the two equations. Be sure to set the window parameters properly.

65. Let us consider the polar equation $r = \dfrac{5}{5 + 2\sin\theta}$. Explain why the graphing utility gives the following graphs with the specified window parameters:

a. $[-2, 2]$ by $[-2, 2]$ with θ step $= \dfrac{\pi}{2}$

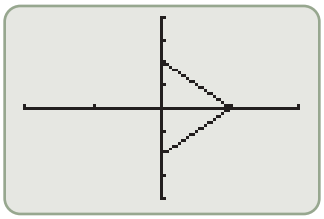


b. $[-2, 2]$ by $[-2, 2]$ with θ step $= \dfrac{\pi}{3}$

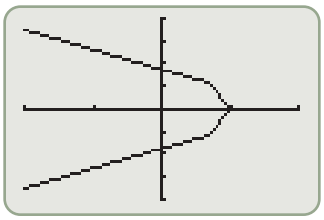


66. Let us consider the polar equation $r = \dfrac{2}{1 + \cos\theta}$. Explain why a graphing utility gives the following graphs with the specified window parameters:

a. $[-2, 2]$ by $[-4, 4]$ with θ step $= \dfrac{\pi}{2}$

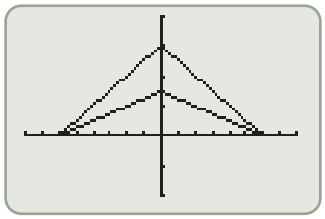


b. $[-2, 2]$ by $[-4, 4]$ with θ step $= \dfrac{\pi}{3}$

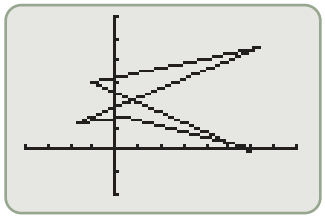


67. Let us consider the polar equation $r = \dfrac{6}{1 + 3\sin\theta}$. Explain why a graphing utility gives the following graphs with the specified window parameters:

a. $[-8, 8]$ by $[-2, 4]$ with θ step $= \dfrac{\pi}{2}$

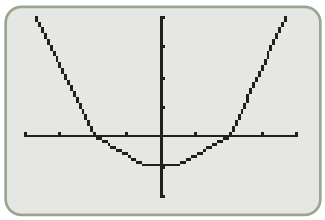


b. $[-4, 8]$ by $[-2, 6]$ with θ step $= 0.4\pi$

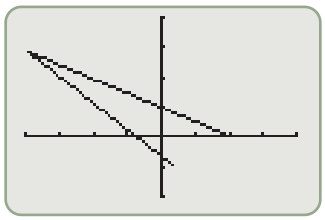


68. Let us consider the polar equation $r = \dfrac{2}{1 - \sin\theta}$. Explain why a graphing utility gives the following graphs with the specified window parameters:

a. $[-4, 4]$ by $[-2, 4]$ with θ step $= \dfrac{\pi}{3}$



b. $[-4, 4]$ by $[-2, 6]$ with θ step $= 0.8\pi$



PREVIEW TO CALCULUS

In calculus, when finding the area between two polar curves, we need to find the points of intersection of the two curves. In Exercises 69–72, find the values of θ where the two conic sections intersect on $[0, 2\pi]$.

69. $r = \dfrac{2}{2 + \sin\theta}$, $r = \dfrac{2}{2 + \cos\theta}$

70. $r = \dfrac{1}{3 + 2\sin\theta}$, $r = \dfrac{1}{3 - 2\sin\theta}$

71. $r = \dfrac{1}{4 - 3\sin\theta}$, $r = \dfrac{1}{-1 + 7\sin\theta}$

72. $r = \dfrac{1}{5 + 2\cos\theta}$, $r = \dfrac{1}{10 - 8\cos\theta}$

SECTION 9.9

PARAMETRIC EQUATIONS AND GRAPHS

SKILLS OBJECTIVES

- Graph parametric equations.
- Find an equation (in rectangular form) that corresponds to a graph defined parametrically.
- Find parametric equations for a graph that is defined by an equation in rectangular form.

CONCEPTUAL OBJECTIVES

- Understand that the results of increasing the value of the parameter reveal the orientation of a curve or the direction of motion along it.
- Use time as a parameter in parametric equations.

Parametric Equations of a Curve

Thus far we have talked about graphs in planes. For example, the equation $x^2 + y^2 = 1$ when graphed in a plane is the unit circle. Similarly, the function $f(x) = \sin x$ when graphed in a plane is a sinusoidal curve. Now, we consider the **path along a curve**. For example, if a car is being driven on a circular racetrack, we want to see the movement along the circle. We can determine where (position) along the circle the car is at some time t using *parametric equations*. Before we define *parametric equations* in general, let us start with a simple example.

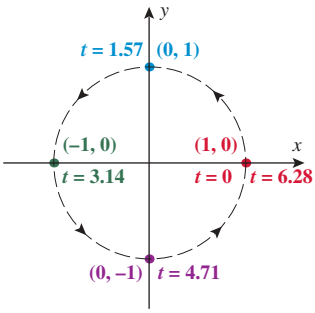
Let $x = \cos t$ and $y = \sin t$ and $t \geq 0$. We then can make a table of some corresponding values.

t SECONDS	$x = \cos t$	$y = \sin t$	(x, y)
0	$x = \cos 0 = 1$	$y = \sin 0 = 0$	$(1, 0)$
$\frac{\pi}{2}$	$x = \cos\left(\frac{\pi}{2}\right) = 0$	$y = \sin\left(\frac{\pi}{2}\right) = 1$	$(0, 1)$
π	$x = \cos \pi = -1$	$y = \sin \pi = 0$	$(-1, 0)$
$\frac{3\pi}{2}$	$x = \cos\left(\frac{3\pi}{2}\right) = 0$	$y = \sin\left(\frac{3\pi}{2}\right) = -1$	$(0, -1)$
2π	$x = \cos(2\pi) = 1$	$y = \sin(2\pi) = 0$	$(1, 0)$

If we plot these points and note the correspondence to time (by converting all numbers to decimals), we will be tracing a *path* counterclockwise along the unit circle.

TIME (SECONDS)	$t = 0$	$t = 1.57$	$t = 3.14$	$t = 4.71$
POSITION	$(1, 0)$	$(0, 1)$	$(-1, 0)$	$(0, -1)$

Notice that at time $t = 6.28$ seconds we are back to the point $(1, 0)$.



We can see that the path represents the unit circle, since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$.

DEFINITION **Parametric Equations**

Let $x = f(t)$ and $y = g(t)$ be functions defined for t on some interval. The set of points $(x, y) = [f(t), g(t)]$ represents a **plane curve**. The equations

$$x = f(t) \qquad \text{and} \qquad y = g(t)$$

are called **parametric equations** of the curve. The variable t is called the **parameter**.

Parametric equations are useful for showing movement along a curve. We insert arrows in the graph to show **direction**, or **orientation**, along the curve as t increases.

EXAMPLE 1 Graphing a Curve Defined by Parametric Equations

Graph the curve defined by the parametric equations

$$x = t^2 \qquad y = (t - 1) \qquad t \text{ in } [-2, 2]$$

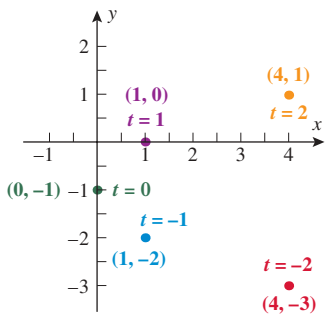
Indicate the orientation with arrows.

Solution:

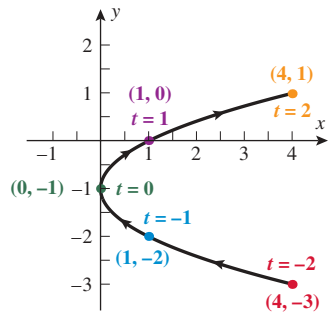
STEP 1 Make a table and find values for t, x, and y.

t	$x = t^2$	$y = (t - 1)$	(x, y)
$t = -2$	$x = (-2)^2 = 4$	$y = (-2 - 1) = -3$	$(4, -3)$
$t = -1$	$x = (-1)^2 = 1$	$y = (-1 - 1) = -2$	$(1, -2)$
$t = 0$	$x = 0^2 = 0$	$y = (0 - 1) = -1$	$(0, -1)$
$t = 1$	$x = 1^2 = 1$	$y = (1 - 1) = 0$	$(1, 0)$
$t = 2$	$x = 2^2 = 4$	$y = (2 - 1) = 1$	$(4, 1)$

STEP 2 Plot the points in the xy-plane.



STEP 3 Connect the points with a smooth curve and use arrows to indicate direction.



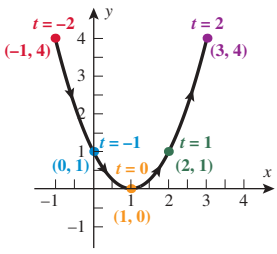
The shape of the graph appears to be a parabola. The parametric equations are $x = t^2$ and $y = (t - 1)$. If we solve the second equation for t, getting $t = y + 1$, and substitute this expression into $x = t^2$, the result is $x = (y + 1)^2$. The graph of $x = (y + 1)^2$ is a parabola with vertex at the point $(0, -1)$ and opening to the right.

■ **YOUR TURN** Graph the curve defined by the parametric equations

$$x = t + 1 \qquad y = t^2 \qquad t \text{ in } [-2, 2]$$

Indicate the orientation with arrows.

■ **Answer:**



Sometimes it is easier to show the rectangular equivalent of the curve and eliminate the parameter.

EXAMPLE 2 Graphing a Curve Defined by Parametric Equations by First Finding an Equivalent Rectangular Equation

Graph the curve defined by the parametric equations

$$x = 4\cos t \qquad y = 3\sin t \qquad t \text{ is any real number}$$

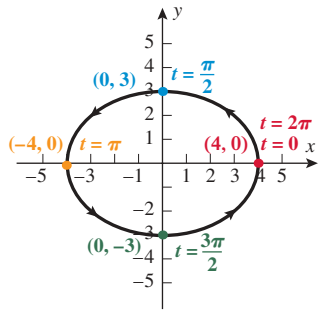
Indicate the orientation with arrows.

Solution:

One approach is to point-plot as in Example 1. A second approach is to find the equivalent rectangular equation that represents the curve.

We apply the Pythagorean identity.	$\sin^2 t + \cos^2 t = 1$
Find $\sin^2 t$ from the parametric equation for y.	$y = 3\sin t$
Square both sides.	$y^2 = 9\sin^2 t$
Divide by 9.	$\sin^2 t = \dfrac{y^2}{9}$
Similarly, find $\cos^2 t$.	$x = 4\cos t$
Square both sides.	$x^2 = 16\cos^2 t$
Divide by 16.	$\cos^2 t = \dfrac{x^2}{16}$
Substitute $\sin^2 t = \dfrac{y^2}{9}$ and $\cos^2 t = \dfrac{x^2}{16}$ into $\sin^2 t + \cos^2 t = 1$.	$\dfrac{y^2}{9} + \dfrac{x^2}{16} = 1$

The curve is an ellipse centered at the origin and elongated horizontally.



The orientation is counterclockwise. For example, when $t = 0$, the position is (4, 0); when $t = \frac{\pi}{2}$, the position is (0, 3); and when $t = \pi$, the position is $(-4, 0)$.

Study Tip

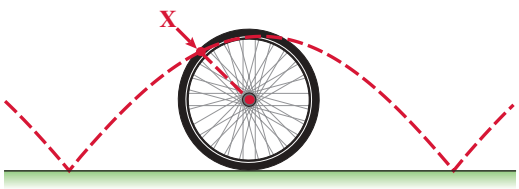
For open curves the orientation can be determined from two values of t. However, for closed curves three points should be chosen to ensure clockwise or counterclockwise orientation.

Applications of Parametric Equations

Parametric equations can be used to describe motion in many applications. Two that we will discuss are the *cycloid* and a *projectile*. Suppose that you paint a red **X** on a bicycle tire. As the bicycle moves in a straight line, if you watch the motion of the red **X**, you will see that it follows the path of a **cycloid**.

Study Tip

A cycloid is a curve that does not have a simple rectangular equation. The only convenient way to describe its path is with parametric equations.



The parametric equations that define a cycloid are

$$\boxed{x = a(t - \sin t)} \quad \text{and} \quad \boxed{y = a(1 - \cos t)}$$

where t is any real number.

EXAMPLE 3 Graphing a Cycloid

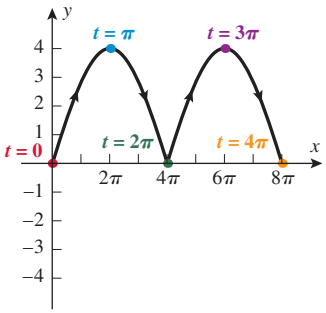
Graph the cycloid given by $x = 2(t - \sin t)$ and $y = 2(1 - \cos t)$ for t in $[0, 4\pi]$.

Solution:

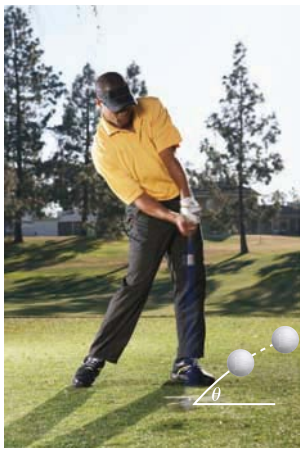
STEP 1 Make a table and find key values for t, x, and y.

t	$x = 2(t - \sin t)$	$y = 2(1 - \cos t)$	(x, y)
$t = 0$	$x = 2(0 - 0) = 0$	$y = 2(1 - 1) = 0$	$(0, 0)$
$t = \pi$	$x = 2(\pi - 0) = 2\pi$	$y = 2[1 - (-1)] = 4$	$(2\pi, 4)$
$t = 2\pi$	$x = 2(2\pi - 0) = 4\pi$	$y = 2(1 - 1) = 0$	$(4\pi, 0)$
$t = 3\pi$	$x = 2(3\pi - 0) = 6\pi$	$y = 2[1 - (-1)] = 4$	$(6\pi, 4)$
$t = 4\pi$	$x = 2(4\pi - 0) = 8\pi$	$y = 2(1 - 1) = 0$	$(8\pi, 0)$

STEP 2 Plot points in a plane and connect them with a smooth curve.



Another example of parametric equations describing real-world phenomena is *projectile motion*. The accompanying photo of a golfer hitting a golf ball presents an example of a projectile.



Joshua Dalsimer/© Corbis; iStockphoto (golf ball)

Let v_0 be the initial velocity of an object, θ be the initial angle of inclination with the horizontal, and h be the initial height above the ground. Then the parametric equations describing the **projectile motion** (which will be developed in calculus) are

$$\boxed{x = (v_0 \cos\theta)t} \quad \text{and} \quad \boxed{y = -\tfrac{1}{2}gt^2 + (v_0 \sin\theta)t + h}$$

where t is the time and g is the constant acceleration due to gravity (9.8 meters per square second or 32 feet per square second).

EXAMPLE 4 Graphing Projectile Motion

Suppose a golfer hits his golf ball with an initial velocity of 160 feet per second at an angle of 30° with the ground. How far is his drive, assuming the length of the drive is from the tee to where the ball first hits the ground? Graph the curve representing the path of the golf ball. Assume that he hits the ball straight off the tee and down the fairway.

Solution:

STEP 1 Find the parametric equations that describe the golf ball that the golfer drove.

First, write the parametric equations for projectile motion.

$$x = (v_0 \cos\theta)t \quad \text{and} \quad y = -\tfrac{1}{2}gt^2 + (v_0 \sin\theta)t + h$$

Let $g = 32\,\text{ft/sec}^2$, $v_0 = 160\,\text{ft/sec}$, $h = 0$, and $\theta = 30°$.

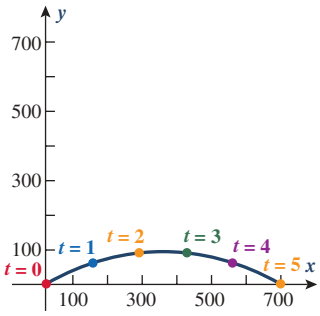
$$x = (160 \cdot \cos 30°)t \quad \text{and} \quad y = -16t^2 + (160 \cdot \sin 30°)t$$

Evaluate the sine and cosine functions and simplify.

$$x = 80\sqrt{3}t \quad \text{and} \quad y = -16t^2 + 80t$$

STEP 2 Graph the projectile motion.

t	$x = 80\sqrt{3}t$	$y = -16t^2 + 80t$	(x, y)
$t = 0$	$x = 80\sqrt{3}(0) = 0$	$y = -16(0)^2 + 80(0) = 0$	(0, 0)
$t = 1$	$x = 80\sqrt{3}(1) \approx 139$	$y = -16(1)^2 + 80(1) = 64$	(139, 64)
$t = 2$	$x = 80\sqrt{3}(2) \approx 277$	$y = -16(2)^2 + 80(2) = 96$	(277, 96)
$t = 3$	$x = 80\sqrt{3}(3) \approx 416$	$y = -16(3)^2 + 80(3) = 96$	(416, 96)
$t = 4$	$x = 80\sqrt{3}(4) \approx 554$	$y = -16(4)^2 + 80(4) = 64$	(554, 64)
$t = 5$	$x = 80\sqrt{3}(5) \approx 693$	$y = -16(5)^2 + 80(5) = 0$	(693, 0)



We can see that we selected our time increments well (the last point, (693, 0), corresponds to the ball hitting the ground 693 feet from the tee).

STEP 3 Identify the horizontal distance from the tee to where the ball first hits the ground.

Algebraically, we can determine the distance of the tee shot by setting the height y equal to zero. $y = -16t^2 + 80t = 0$

Factor (divide) the common, $-16t$. $-16t(t - 5) = 0$

Solve for t. $t = 0$ or $t = 5$

The ball hits the ground after 5 seconds.

Let $t = 5$ in the horizontal distance, $x = 80\sqrt{3}t$. $x = 80\sqrt{3}(5) \approx 693$

The ball hits the ground 693 feet from the tee.

With parametric equations, we can also determine when the ball lands (5 seconds).

Technology Tip

Graph $x = 80\sqrt{3}t$ and $y = -16t^2 + 80t$.

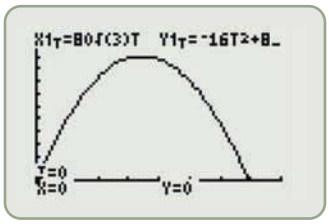
```
Plot1 Plot2 Plot3
\X1T=80√(3)T
 Y1T=-16T²+80T
```

T	X1T	Y1T
0	0	0
1	138.56	64
2	277.13	96
3	415.69	96
4	554.26	64
5	692.82	0
6	831.38	-96

T=0

```
WINDOW
 Tmin=0
 Tmax=5
 Tstep=.1
 Xmin=0
 Xmax=800
 Xscl=100
↓Ymin=-10
```

```
WINDOW
↑Tstep=.1
 Xmin=0
 Xmax=800
 Xscl=100
 Ymin=-10
 Ymax=120
 Yscl=10
```



SECTION 9.9 SUMMARY

Parametric equations are a way of describing as a function of t, the parameter, the path an object takes along a curve in the xy-plane. Parametric equations have equivalent rectangular equations. Typically, the method of graphing a set of parametric equations is to eliminate t and graph the corresponding rectangular equation. Once the curve is found, orientation along the curve can be determined by finding points corresponding to different t-values. Two important applications are cycloids and projectiles, whose paths we can trace using parametric equations.

SECTION 9.9 EXERCISES

SKILLS

In Exercises 1–30, graph the curve defined by the parametric equations.

1. $x = t + 1, y = \sqrt{t}, t \geq 0$
2. $x = 3t, y = t^2 - 1, t \text{ in } [0, 4]$
3. $x = -3t, y = t^2 + 1, t \text{ in } [0, 4]$
4. $x = t^2 - 1, y = t^2 + 1, t \text{ in } [-3, 3]$
5. $x = t^2, y = t^3, t \text{ in } [-2, 2]$
6. $x = t^3 + 1, y = t^3 - 1, t \text{ in } [-2, 2]$
7. $x = \sqrt{t}, y = t, t \text{ in } [0, 10]$
8. $x = t, y = \sqrt{t^2 + 1}, t \text{ in } [0, 10]$
9. $x = (t + 1)^2, y = (t + 2)^3, t \text{ in } [0, 1]$
10. $x = (t - 1)^3, y = (t - 2)^2, t \text{ in } [0, 4]$
11. $x = e^t, y = e^{-t}, -\ln 3 \leq t \leq \ln 3$
12. $x = e^{-2t}, y = e^{2t} + 4, -\ln 2 \leq t \leq \ln 3$
13. $x = 2t^4 - 1, y = t^8 + 1, 0 \leq t \leq 4$
14. $x = 3t^6 - 1, y = 2t^3, -1 \leq t \leq 1$
15. $x = t(t - 2)^3, y = t(t - 2)^3, 0 \leq t \leq 4$
16. $x = -t\sqrt[3]{t}, y = -5t^8 - 2, -3 \leq t \leq 3$
17. $x = 3 \sin t, y = 2 \cos t, t \text{ in } [0, 2\pi]$
18. $x = \cos(2t), y = \sin t, t \text{ in } [0, 2\pi]$
19. $x = \sin t + 1, y = \cos t - 2, t \text{ in } [0, 2\pi]$
20. $x = \tan t, y = 1, t \text{ in } \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
21. $x = 1, y = \sin t, t \text{ in } [-2\pi, 2\pi]$
22. $x = \sin t, y = 2, t \text{ in } [0, 2\pi]$
23. $x = \sin^2 t, y = \cos^2 t, t \text{ in } [0, 2\pi]$
24. $x = 2 \sin^2 t, y = 2 \cos^2 t, t \text{ in } [0, 2\pi]$
25. $x = 2 \sin(3t), y = 3 \cos(2t), t \text{ in } [0, 2\pi]$
26. $x = 4 \cos(2t), y = t, t \text{ in } [0, 2\pi]$
27. $x = \cos\left(\frac{t}{2}\right) - 1, y = \sin\left(\frac{t}{2}\right) + 1, -2\pi \leq t \leq 2\pi$
28. $x = \sin\left(\frac{t}{3}\right) + 3, y = \cos\left(\frac{t}{3}\right) - 1, 0 \leq t \leq 6\pi$
29. $x = 2 \sin\left(t + \frac{\pi}{4}\right), y = -2 \cos\left(t + \frac{\pi}{4}\right), -\frac{\pi}{4} \leq t \leq \frac{7\pi}{4}$
30. $x = -3 \cos^2(3t), y = 2 \cos(3t), -\frac{\pi}{3} \leq t \leq \frac{\pi}{3}$

In Exercises 31–40, the given parametric equations define a plane curve. Find an equation in rectangular form that also corresponds to the plane curve.

31. $x = \frac{1}{t}, y = t^2$
32. $x = t^2 - 1, y = t^2 + 1$
33. $x = t^3 + 1, y = t^3 - 1$
34. $x = 3t, y = t^2 - 1$
35. $x = t, y = \sqrt{t^2 + 1}$
36. $x = \sin^2 t, y = \cos^2 t$
37. $x = 2 \sin^2 t, y = 2 \cos^2 t$
38. $x = \sec^2 t, y = \tan^2 t$
39. $x = 4(t^2 + 1), y = 1 - t^2$
40. $x = \sqrt{t - 1}, y = \sqrt{t}$

APPLICATIONS

For Exercises 41–50, recall that the flight of a projectile can be modeled with the parametric equations

$$x = (v_0 \cos \theta)t \qquad y = -16t^2 + (v_0 \sin \theta)t + h$$

where t is in seconds, v_0 is the initial velocity, θ is the angle with the horizontal, and x and y are in feet.

41. **Flight of a Projectile.** A projectile is launched from the ground at a speed of 400 feet per second at an angle of 45° with the horizontal. After how many seconds does the projectile hit the ground?
42. **Flight of a Projectile.** A projectile is launched from the ground at a speed of 400 feet per second at an angle of 45° with the horizontal. How far does the projectile travel (what is the horizontal distance), and what is its maximum altitude?

43. Flight of a Baseball. A baseball is hit at an initial speed of 105 miles per hour and an angle of 20° at a height of 3 feet above the ground. If home plate is 420 feet from the back fence, which is 15 feet tall, will the baseball clear the back fence for a home run?

44. Flight of a Baseball. A baseball is hit at an initial speed of 105 miles per hour and an angle of 20° at a height of 3 feet above the ground. If there is no back fence or other obstruction, how far does the baseball travel (horizontal distance), and what is its maximum height?

45. Bullet Fired. A gun is fired from the ground at an angle of 60°, and the bullet has an initial speed of 700 feet per second. How high does the bullet go? What is the horizontal (ground) distance between the point where the gun is fired and the point where the bullet hits the ground?

46. Bullet Fired. A gun is fired from the ground at an angle of 60°, and the bullet has an initial speed of 2000 feet per second. How high does the bullet go? What is the horizontal (ground) distance between the point where the gun is fired and the point where the bullet hits the ground?

47. Missile Fired. A missile is fired from a ship at an angle of 30°, an initial height of 20 feet above the water's surface, and a speed of 4000 feet per second. How long will it be before the missile hits the water?

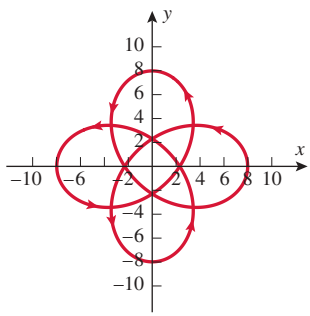
48. Missile Fired. A missile is fired from a ship at an angle of 40°, an initial height of 20 feet above the water's surface, and a speed of 5000 feet per second. Will the missile be able to hit a target that is 2 miles away?

49. Path of a Projectile. A projectile is launched at a speed of 100 feet per second at an angle of 35° with the horizontal. Plot the path of the projectile on a graph. Assume that $h = 0$.

50. Path of a Projectile. A projectile is launched at a speed of 150 feet per second at an angle of 55° with the horizontal. Plot the path of the projectile on a graph. Assume that $h = 0$.

For Exercises 51 and 52, refer to the following:

Modern amusement park rides are often designed to push the envelope in terms of speed, angle, and ultimately g's, and usually take the form of gargantuan roller coasters or skyscraping towers. However, even just a couple of decades ago, such creations were depicted only in fantasy-type drawings, with their creators never truly believing their construction would become a reality. Nevertheless, thrill rides still capable of nauseating any would-be rider were still able to be constructed; one example is the *Calypso*. This ride is a not-too-distant cousin of the more well-known *Scrambler*. It consists of four rotating arms (instead of three like the Scrambler), and on each of these arms, four cars (equally spaced around the circumference of a circular frame) are attached. Once in motion, the main piston to which the four arms are connected rotates clockwise, while each of the four arms themselves rotates counterclockwise. The combined motion appears as a blur to any onlooker from the crowd, but the motion of a single rider is much less chaotic. In fact, a single rider's path can be modeled by the following graph:



The equation of this graph is defined parametrically by

$$x(t) = A\cos t + B\cos(-3t)$$
$$y(t) = A\sin t + B\sin(-3t) \quad 0 \le t \le 2\pi$$

51. Amusement Rides. What is the location of the rider at $t = 0$, $t = \frac{\pi}{2}$, $t = \pi$, $t = \frac{3\pi}{2}$, and $t = 2\pi$?

52. Amusement Rides. Suppose that the ride conductor was rather sinister and speeded up the ride to twice the speed. How would you modify the parametric equations to model such a change? Now vary the values of A and B. What do you think these parameters are modeling in this problem?

CATCH THE MISTAKE

In Exercises 53 and 54, explain the mistake that is made.

53. Find the rectangular equation that corresponds to the plane curve defined by the parametric equations $x = t + 1$ and $y = \sqrt{t}$. Describe the plane curve.

Solution:

Square $y = \sqrt{t}$. $\quad y^2 = t$

Substitute $t = y^2$ into $x = t + 1$. $\quad x = y^2 + 1$

The graph of $x = y^2 + 1$ is a parabola opening to the right with its vertex at (1, 0).

This is incorrect. What mistake was made?

54. Find the rectangular equation that corresponds to the plane curve defined by the parametric equations $x = \sqrt{t}$ and $y = t - 1$. Describe the plane curve.

Solution:

Square $x = \sqrt{t}$. $\quad x^2 = t$

Substitute $t = x^2$ into $y = t - 1$. $\quad y = x^2 - 1$

The graph of $y = x^2 - 1$ is a parabola opening up with its vertex at $(0, -1)$.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 55 and 56, determine whether each statement is true or false.

55. Curves given by equations in rectangular form have orientation.

56. Curves given by parametric equations have orientation.

57. Determine what type of curve the parametric equations $x = \sqrt{t}$ and $y = \sqrt{1 - t}$ define.

58. Determine what type of curve the parametric equations $x = \ln t$ and $y = t$ define.

CHALLENGE

59. Prove that $x = a\tan t$, $y = b\sec t$, $0 \le t \le 2\pi$, $t \ne \frac{\pi}{2}, \frac{3\pi}{2}$ are parametric equations for a hyperbola. Assume that a and b are nonzero constants.

60. Prove that $x = a\csc\left(\frac{t}{2}\right)$, $y = b\cot\left(\frac{t}{2}\right)$, $0 \le t \le 4\pi$, $t \ne \pi, 3\pi$ are parametric equations for a hyperbola. Assume that a and b are nonzero constants.

61. Consider the parametric curve $x = a\sin^2 t - b\cos^2 t$, $y = b\cos^2 t + a\sin^2 t$, $0 \le t \le \frac{\pi}{2}$. Assume that a and b are nonzero constants. Find the Cartesian equation for this curve.

62. Consider the parametric curve $x = a\sin t + a\cos t$, $y = a\cos t - a\sin t$, $0 \le t \le 2\pi$. Assume that a is not zero. Find the Cartesian equation for this curve.

63. Consider the parametric curve $x = e^{at}$, $y = be^t$, $t > 0$. Assume that a is a positive integer and b is a positive real number. Determine the Cartesian equation.

64. Consider the parametric curve $x = a\ln t$, $y = \ln(bt)$, $t > 0$. Assume that b is a positive integer and a is a positive real number. Determine the Cartesian equation.

TECHNOLOGY

65. Consider the parametric equations: $x = a\sin t - \sin(at)$ and $y = a\cos t + \cos(at)$. With a graphing utility, explore the graphs for $a = 2, 3,$ and 4.

66. Consider the parametric equations: $x = a\cos t - b\cos(at)$ and $y = a\sin t + \sin(at)$. With a graphing utility, explore the graphs for $a = 3$ and $b = 1$, $a = 4$ and $b = 2$, and $a = 6$ and $b = 2$. Find the t-interval that gives one cycle of the curve.

67. Consider the parametric equations: $x = \cos(at)$ and $y = \sin(bt)$. With a graphing utility, explore the graphs for $a = 2$ and $b = 4$, $a = 4$ and $b = 2$, $a = 1$ and $b = 3$, and $a = 3$ and $b = 1$. Find the t-interval that gives one cycle of the curve.

68. Consider the parametric equations: $x = a\sin(at) - \sin t$ and $y = a\cos(at) - \cos t$. With a graphing utility, explore the graphs for $a = 2$ and 3. Describe the t-interval for each case.

69. Consider the parametric equations $x = a\cos(at) - \sin t$ and $y = a\sin(at) - \cos t$. With a graphing utility, explore the graphs for $a = 2$ and 3. Describe the t-interval for each case.

70. Consider the parametric equations $x = a\sin(at) - \cos t$ and $y = a\cos(at) - \sin t$. With a graphing utility, explore the graphs for $a = 2$ and 3. Describe the t-interval for each case.

PREVIEW TO CALCULUS

In calculus, some operations can be simplified by using parametric equations. Finding the points of intersection (if they exist) of two curves given by parametric equations is a standard procedure.

In Exercises 71–74, find the points of intersection of the given curves given s and t are any real numbers.

71. Curve I: $x = t$, $y = t^2 - 1$
Curve II: $x = s + 1$, $y = 4 - s$

72. Curve I: $x = t^2 + 3$, $y = t$
Curve II: $x = s + 2$, $y = 1 - s$

73. Curve I: $x = 100t$, $y = 80t - 16t^2$
Curve II: $x = 100 - 200t$, $y = -16t^2 + 144t - 224$

74. Curve I: $x = t^2$, $y = t + 1$
Curve II: $x = 2 + s$, $y = 1 - s$

CHAPTER 9 INQUIRY-BASED LEARNING PROJECT

"And the Rockets Red Glare . . ."

Scientists at Vandenberg Air Force Base are interested in tracing the path of some newly designed rockets. They will launch two rockets at 100 feet per second. One will depart at 45°, the other at 60°. From vector analysis and gravity, you determine the following coordinates (x, y) as a function of time t where y stands for height in feet above the ground and x stands for lateral distance traveled.

45°	60°
$x = 100\cos(45°)t$	$x = 100\cos(60°)t$
$y = -16t^2 + 100\sin(45°)t$	$y = -16t^2 + 100\sin(60°)t$

1. For each angle (45°, 60°), fill in the chart (round to one decimal place). You can do this by hand (very slowly) or use the table capabilities of a calculator or similar device.

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
X_{45}											
Y_{45}											
X_{60}											
Y_{60}											

2. What are a few things worth noting when looking at your table values? Think about the big picture and the fact that you are dealing with projectiles.

Show your work for the following:

3. Which rocket traveled higher and by how much? Recall that the y variable is the height. (List the heights of each rocket.)

4. When each rocket first hits the ground, which one has traveled farther laterally and by how much? (List distances of each.)

5. Which rocket was in the air longer and by how much? (List the times of each.)

6. For each rocket, write t in terms of x. Then substitute this value into the y equation. This is called "eliminating the parameter" and puts y as a function of x. Simplify completely. Use exact values. Reduce the fractions to their lowest terms.

MODELING OUR WORLD

In the Modeling Our World (MOW) features in Chapters 1–3, you used the average yearly temperature in degrees Fahrenheit (°F) and carbon dioxide emissions in parts per million (ppm) collected by NOAA in Mauna Loa, Hawaii, to develop linear (Chapter 1) and nonlinear (Chapters 2 and 3) models. In the following exercises, you will determine when these different models actually predict the same temperatures and carbon emissions. It is important to realize that not only can different models be used to predict trends, but also the choice of data those models are fitted to also affects the models and hence the predicted values.

YEAR	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
TEMPERATURE	44.45	43.29	43.61	43.35	46.66	45.71	45.53	47.53	45.86	46.23
CO_2 EMISSIONS (PPM)	316.9	320.0	325.7	331.1	338.7	345.9	354.2	360.6	369.4	379.7

1. Solve the system of nonlinear equations governing mean temperature that was found by using two data points:

Equation (1): Use the linear model developed in MOW Chapter 1, Exercise 2(a).

Equation (2): Use the quadratic model found in MOW Chapter 2, Exercise 2(a).

2. For what year do the models used in Exercise 1 agree? Compare the value given by the models that year to the actual data for the year.

3. Solve the system of nonlinear equations governing mean temperature that was found by applying regression (all data points):

Equation (1): Use the linear model developed in MOW Chapter 1, Exercise 2(c).

Equation (2): Use the quadratic model found in MOW Chapter 2, Exercise 2(c).

4. For what year do the models used in Exercise 3 agree? Compare the value given by the models that year to the actual data for the year.

5. Solve the system of nonlinear equations governing carbon dioxide emissions that was found by using two data points:

Equation (1): Use the linear model developed in MOW Chapter 1, Exercise 7(a).

Equation (2): Use the quadratic model found in MOW Chapter 2, Exercise 7(a).

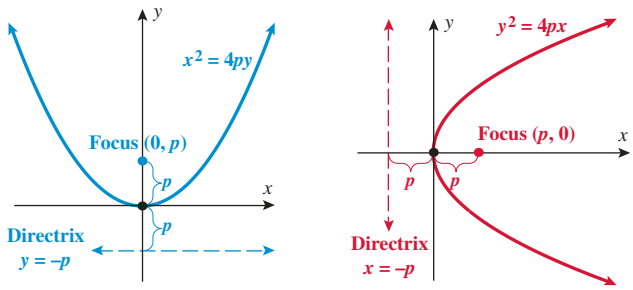
6. For what year do the models used in Exercise 5 agree? Compare the value given by the models that year to the actual data for the year.

7. Solve the system of nonlinear equations governing carbon emissions that was found by applying regression (all data points):

Equation (1): Use the linear model developed in MOW Chapter 1, Exercise 7(c).

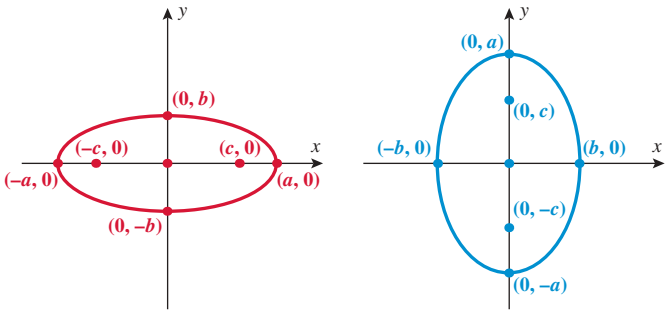
Equation (2): Use the quadratic model found in MOW Chapter 2, Exercise 7(c).

8. For what year do the models used in Exercise 7 agree? Compare the value given by the models that year to the actual data for the year.

SECTION	CONCEPT	KEY IDEAS/FORMULAS
9.1	**Conic basics**	
	Three types of conics	**Parabola, ellipse, and hyperbola:** **Parabola:** Distances from a point to a reference point (focus) and a reference line (directrix) are equal. **Ellipse:** Sum of the distances between the point and two reference points (foci) is constant. **Hyperbola:** Difference of the distances between the point and two reference points (foci) is constant.
9.2	**The parabola**	
	Parabola with a vertex at the origin	 Up: $p > 0$ Down: $p < 0$ Right: $p > 0$ Left: $p < 0$

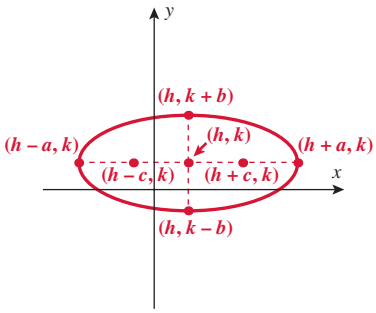
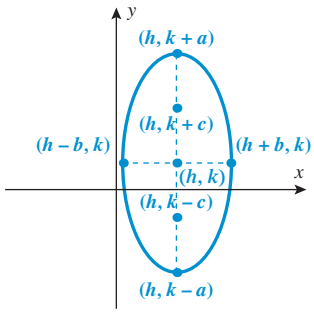
Parabola with a vertex at the point (h, k)

EQUATION	$(y - k)^2 = 4p(x - h)$	$(x - h)^2 = 4p(y - k)$
VERTEX	(h, k)	(h, k)
FOCUS	$(p + h, k)$	$(h, p + k)$
DIRECTRIX	$x = -p + h$	$y = -p + k$
AXIS OF SYMMETRY	$y = k$	$x = h$
$p > 0$	opens to the right	opens upward
$p < 0$	opens to the left	opens downward

9.3	**The ellipse**	
	Ellipse centered at the origin	 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $c^2 = a^2 - b^2$ $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $c^2 = a^2 - b^2$

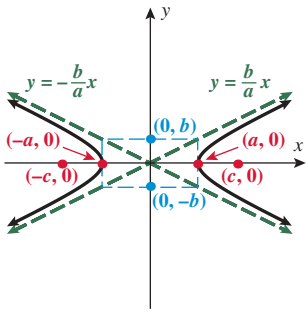
Section	Concept	Key Ideas/Formulas

Ellipse centered at the point (h, k)

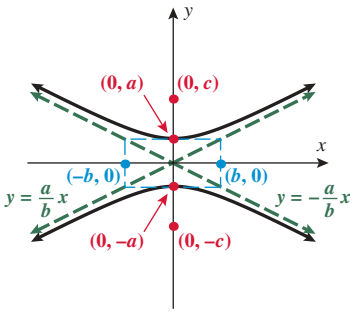
Orientation of Major Axis	Horizontal (parallel to the x-axis)	Vertical (parallel to the y-axis)
Equation	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$	$\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$
Graph		
Foci	$(h - c, k)$ and $(h + c, k)$	$(h, k - c)$ and $(h, k + c)$
Vertices	$(h - a, k)$ and $(h + a, k)$	$(h, k - a)$ and $(h, k + a)$

9.4 The hyperbola

Hyperbola centered at the origin



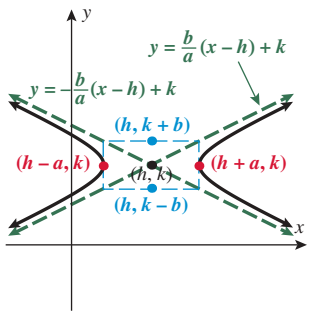
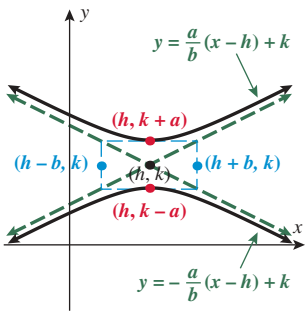
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$c^2 = a^2 + b^2$$



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$
$$c^2 = a^2 + b^2$$

Section	Concept	Key Ideas/Formulas

Hyperbola centered at the point (h, k)

Orientation of Transverse Axis	Horizontal (parallel to the x-axis)	Vertical (parallel to the y-axis)
Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Vertices	$(h-a, k)$ and $(h+a, k)$	$(h, k-a)$ and $(h, k+a)$
Foci	$(h-c, k)$ and $(h+c, k)$ where $c^2 = a^2 + b^2$	$(h, k-c)$ and $(h, k+c)$ where $c^2 = a^2 + b^2$
Graph		

9.5	**Systems of nonlinear equations**	There is no procedure guaranteed to solve nonlinear equations.
	Solving a system of nonlinear equations	**Elimination** Eliminate a variable by either adding one equation to or subtracting one equation from the other. **Substitution** Solve for one variable in terms of the other and substitute into the second equation.
9.6	**Systems of nonlinear inequalities**	Solutions are determined graphically by finding the common shaded regions. ■ $\leq$ or $\geq$ use solid curves. ■ $<$ or $>$ use dashed curves.
	Nonlinear inequalities in two variables	Step 1: Rewrite the inequality as an equation. Step 2: Graph the equation. Step 3: Test points. Step 4: Shade.
	Systems of nonlinear inequalities	Graph the individual inequalities and the solution in the common (overlapping) shaded region.
9.7	**Rotation of axes**	
	Rotation of axes formulas	$x = X\cos\theta - Y\sin\theta$ $y = X\sin\theta + Y\cos\theta$
	The angle of rotation necessary to transform a general second-degree equation into a familiar equation of a conic	$\cot(2\theta) = \frac{A-C}{B}$ or $\tan(2\theta) = \frac{B}{A-C}$

Section	Concept	Key Ideas/Formulas
9.8	**Polar equations of conics**	All three conics (parabolas, ellipses, and hyperbolas) are defined in terms of a single focus and a directrix.
	Equations of conics in polar coordinates	The directrix is *vertical* and p units to the *right* of the pole. $$r = \frac{ep}{1 + e\cos\theta}$$ The directrix is *vertical* and p units to the *left* of the pole. $$r = \frac{ep}{1 - e\cos\theta}$$ The directrix is *horizontal* and p units *above* the pole. $$r = \frac{ep}{1 + e\sin\theta}$$ The directrix is *horizontal* and p units *below* the pole. $$r = \frac{ep}{1 - e\sin\theta}$$
9.9	**Parametric equations and graphs**	
	Parametric equations of a curve	Parametric equations: $x = f(t)$ and $y = g(t)$ Plane curve: $(x, y) = (f(t), g(t))$
	Applications of parametric equations	Cycloids and projectiles

9.1 Conic Basics

Determine whether each statement is true or false.

1. The focus is a point on the graph of the parabola.
2. The graph of $y^2 = 8x$ is a parabola that opens upward.
3. $\frac{x^2}{9} - \frac{y^2}{1} = 1$ is the graph of a hyperbola that has a horizontal transverse axis.
4. $\frac{(x+1)^2}{9} + \frac{(y-3)^2}{16} = 1$ is a graph of an ellipse whose center is (1, 3).

9.2 The Parabola

Find an equation for the parabola described.

5. Vertex at (0, 0); Focus at (3, 0)
6. Vertex at (0, 0); Focus at (0, 2)
7. Vertex at (0, 0); Directrix at $x = 5$
8. Vertex at (0, 0); Directrix at $y = 4$
9. Vertex at (2, 3); Focus at (2, 5)
10. Vertex at (−1, −2); Focus at (1, −2)
11. Focus at (1, 5); Directrix at $y = 7$
12. Focus at (2, 2); Directrix at $x = 0$

Find the focus, vertex, directrix, and length of the latus rectum, and graph the parabola.

13. $x^2 = -12y$
14. $x^2 = 8y$
15. $y^2 = x$
16. $y^2 = -6x$
17. $(y+2)^2 = 4(x-2)$
18. $(y-2)^2 = -4(x+1)$
19. $(x+3)^2 = -8(y-1)$
20. $(x-3)^2 = -8(y+2)$
21. $x^2 + 5x + 2y + 25 = 0$
22. $y^2 + 2y - 16x + 1 = 0$

Applications

23. **Satellite Dish.** A satellite dish measures 10 feet across its opening and 2 feet deep at its center. The receiver should be placed at the focus of the parabolic dish. Where should the receiver be placed?
24. **Clearance Under a Bridge.** A bridge with a parabolic shape reaches a height of 40 feet in the center of the road, and the width of the bridge opening at ground level is 30 feet combined (both lanes). If an RV is 14 feet tall and 8 feet wide, will it make it through the tunnel?

9.3 The Ellipse

Graph each ellipse.

25. $\frac{x^2}{9} + \frac{y^2}{64} = 1$
26. $\frac{x^2}{81} + \frac{y^2}{49} = 1$
27. $25x^2 + y^2 = 25$
28. $4x^2 + 8y^2 = 64$

Find the standard form of an equation of the ellipse with the given characteristics.

29. Foci: (−3, 0) and (3, 0) Vertices: (−5, 0) and (5, 0)
30. Foci: (0, −2) and (0, 2) Vertices: (0, −3) and (0, 3)
31. Major axis vertical with length of 16, minor axis length of 6, and centered at (0, 0)
32. Major axis horizontal with length of 30, minor axis length of 20, and centered at (0, 0)

Graph each ellipse.

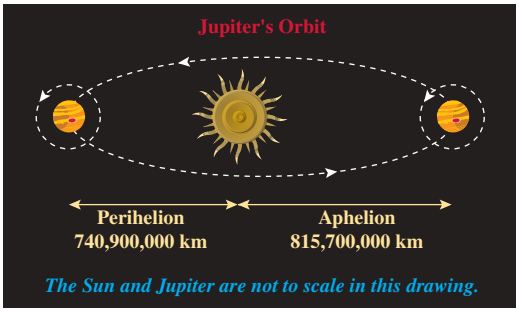
33. $\frac{(x-7)^2}{100} + \frac{(y+5)^2}{36} = 1$
34. $20(x+3)^2 + (y-4)^2 = 120$
35. $4x^2 - 16x + 12y^2 + 72y + 123 = 0$
36. $4x^2 - 8x + 9y^2 - 72y + 147 = 0$

Find the standard form of an equation of the ellipse with the given characteristics.

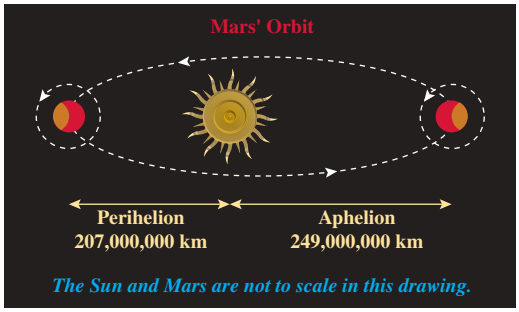
37. Foci: (−1, 3) and (7, 3) Vertices: (−2, 3) and (8, 3)
38. Foci: (1, −3) and (1, −1) Vertices: (1, −4) and (1, 0)

Applications

39. **Planetary Orbits.** Jupiter's orbit is summarized in the picture. Utilize the fact that the Sun is a focus to determine an equation for Jupiter's elliptical orbit around the Sun. Round to the nearest hundred thousand kilometers.



40. Planetary Orbits. Mars's orbit is summarized in the picture that follows. Utilize the fact that the Sun is a focus to determine an equation for Mars's elliptical orbit around the Sun. Round to the nearest million kilometers.



9.4 The Hyperbola

Graph each hyperbola.

41. $\frac{x^2}{9} - \frac{y^2}{64} = 1$

42. $\frac{x^2}{81} - \frac{y^2}{49} = 1$

43. $x^2 - 25y^2 = 25$

44. $8y^2 - 4x^2 = 64$

Find the standard form of an equation of the hyperbola with the given characteristics.

45. Vertices: $(-3, 0)$ and $(3, 0)$ Foci: $(-5, 0)$ and $(5, 0)$

46. Vertices: $(0, -1)$ and $(0, 1)$ Foci: $(0, -3)$ and $(0, 3)$

47. Center: $(0, 0)$; Transverse: y-axis; Asymptotes: $y = 3x$ and $y = -3x$

48. Center: $(0, 0)$; Transverse axis: y-axis; Asymptotes: $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$

Graph each hyperbola.

49. $\frac{(y-1)^2}{36} - \frac{(x-2)^2}{9} = 1$

50. $3(x+3)^2 - 12(y-4)^2 = 72$

51. $8x^2 - 32x - 10y^2 - 60y - 138 = 0$

52. $2x^2 + 12x - 8y^2 + 16y + 6 = 0$

Find the standard form of an equation of the hyperbola with the given characteristics.

53. Vertices: $(0, 3)$ and $(8, 3)$ Foci: $(-1, 3)$ and $(9, 3)$

54. Vertices: $(4, -2)$ and $(4, 0)$ Foci: $(4, -3)$ and $(4, 1)$

Applications

55. Ship Navigation. Two loran stations are located 220 miles apart along a coast. If a ship records a time difference of 0.00048 second and continues on the hyperbolic path corresponding to that difference, where would it reach shore? Assume that the speed of radio signals is 186,000 miles per second.

56. Ship Navigation. Two loran stations are located 400 miles apart along a coast. If a ship records a time difference of 0.0008 second and continues on the hyperbolic path corresponding to that difference, where would it reach shore?

9.5 Systems of Nonlinear Equations

Solve the system of equations with the elimination method.

57. $x^2 + y = -3$
$x - y = 5$

58. $x^2 + y^2 = 4$
$x^2 + y = 2$

59. $x^2 + y^2 = 5$
$2x^2 - y = 0$

60. $x^2 + y^2 = 16$
$6x^2 + y^2 = 16$

Solve the system of equations with the substitution method.

61. $x + y = 3$
$x^2 + y^2 = 4$

62. $xy = 4$
$x^2 + y^2 = 16$

63. $x^2 + xy + y^2 = -12$
$x - y = 2$

64. $3x + y = 3$
$x - y^2 = -9$

Solve the system of equations by applying any method.

65. $x^3 - y^3 = -19$
$x - y = -1$

66. $2x^2 + 4xy = 9$
$x^2 - 2xy = 0$

67. $\frac{2}{x^2} + \frac{1}{y^2} = 15$
$\frac{1}{x^2} - \frac{1}{y^2} = -3$

68. $x^2 + y^2 = 2$
$x^2 + y^2 = 4$

9.6 Systems of Nonlinear Inequalities

Graph the nonlinear inequality.

69. $y \geq x^2 + 3$

70. $x^2 + y^2 > 16$

71. $y \leq e^x$

72. $y < -x^3 + 2$

73. $y \geq \ln(x - 1)$

74. $9x^2 + 4y^2 \leq 36$

Solve each system of inequalities and shade the region on a graph, or indicate that the system has no solution.

75. $y \geq x^2 - 2$
$y \leq -x^2 + 2$

76. $x^2 + y^2 \leq 4$
$y \leq x$

77. $y \geq (x+1)^2 - 2$
$y \leq 10 - x$

78. $3x^2 + 3y^2 \leq 27$
$y \geq x - 1$

79. $4y^2 - 9x^2 \leq 36$
$y \geq x + 1$

80. $9x^2 + 16y^2 \leq 144$
$y \geq 1 - x^2$

9.7 Rotation of Axes

The coordinates of a point in the xy-coordinate system are given. Assuming the X- and Y-axes are found by rotating the x- and y-axes by the indicated angle θ, find the corresponding coordinates for the point in the XY-system.

81. $(-3, 2)$, $\theta = 60°$

82. $(4, -3)$, $\theta = 45°$

Transform the equation of the conic into an equation in X and Y (without an XY-term) by rotating the x- and y-axes through the indicated angle θ. Then graph the resulting equation.

83. $2x^2 + 4\sqrt{3}xy - 2y^2 - 16 = 0$, $\theta = 30°$

84. $25x^2 + 14xy + 25y^2 - 288 = 0$, $\theta = \frac{\pi}{4}$

REVIEW EXERCISES

Determine the angle of rotation necessary to transform the equation in x and y into an equation in X and Y with no XY-term.

85. $4x^2 + 2\sqrt{3}xy + 6y^2 - 9 = 0$

86. $4x^2 + 5xy + 4y^2 - 11 = 0$

Graph the second-degree equation.

87. $x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y + 8 = 0$

88. $76x^2 + 48\sqrt{3}xy + 28y^2 - 100 = 0$

9.8 Polar Equations of Conics

Find the polar equation that represents the conic described.

89. An ellipse with eccentricity $e = \frac{3}{7}$ and directrix $y = -7$

90. A parabola with directrix $x = 2$

Identify the conic (parabola, ellipse, or hyperbola) that each polar equation represents.

91. $r = \dfrac{6}{4 - 5\cos\theta}$

92. $r = \dfrac{2}{5 + 3\sin\theta}$

For the given polar equations, find the eccentricity and vertex (or vertices), and graph the curve.

93. $r = \dfrac{4}{2 + \cos\theta}$

94. $r = \dfrac{6}{1 - \sin\theta}$

9.9 Parametric Equations and Graphs

Graph the curve defined by the parametric equations.

95. $x = \sin t$, $y = 4\cos t$ for t in $[-\pi, \pi]$

96. $x = 5\sin^2 t$, $y = 2\cos^2 t$ for t in $[-\pi, \pi]$

97. $x = 4 - t^2$, $y = t^2$ for t in $[-3, 3]$

98. $x = t + 3$, $y = 4$ for t in $[-4, 4]$

The given parametric equations define a plane curve. Find an equation in rectangular form that also corresponds to the plane curve.

99. $x = 4 - t^2$, $y = t$

100. $x = 5\sin^2 t$, $y = 2\cos^2 t$

101. $x = 2\tan^2 t$, $y = 4\sec^2 t$

102. $x = 3t^2 + 4$, $y = 3t^2 - 5$

Technology Exercises

Section 9.2

103. In your mind, picture the parabola given by $(x - 0.6)^2 = -4(y + 1.2)$. Where is the vertex? Which way does this parabola open? Now plot the parabola with a graphing utility.

104. In your mind, picture the parabola given by $(y - 0.2)^2 = 3(x - 2.8)$. Where is the vertex? Which way does this parabola open? Now plot the parabola with a graphing utility.

105. Given is the parabola $y^2 + 2.8y + 3x - 6.85 = 0$.
a. Solve the equation for y, and use a graphing utility to plot the parabola.
b. Transform the equation into the form $(y - k)^2 = 4p(x - h)$. Find the vertex. Which way does the parabola open?
c. Do (a) and (b) agree with each other?

106. Given is the parabola $x^2 - 10.2x - y + 24.8 = 0$.
a. Solve the equation for y, and use a graphing utility to plot the parabola.
b. Transform the equation into the form $(x - h)^2 = 4p(y - k)$. Find the vertex. Which way does the parabola open?
c. Do (a) and (b) agree with each other?

Section 9.3

107. Graph the following three ellipses: $4x^2 + y^2 = 1$, $4(2x)^2 + y^2 = 1$, and $4(3x)^2 + y^2 = 1$. What can be said to happen to ellipse $4(cx)^2 + y^2 = 1$ as c increases?

108. Graph the following three ellipses: $x^2 + 4y^2 = 1$, $x^2 + 4(2y)^2 = 1$, and $x^2 + 4(3y)^2 = 1$. What can be said to happen to ellipse $x^2 + 4(cy)^2 = 1$ as c increases?

Section 9.4

109. Graph the following three hyperbolas: $4x^2 - y^2 = 1$, $4(2x)^2 - y^2 = 1$, and $4(3x)^2 - y^2 = 1$. What can be said to happen to hyperbola $4(cx)^2 - y^2 = 1$ as c increases?

110. Graph the following three hyperbolas: $x^2 - 4y^2 = 1$, $x^2 - 4(2y)^2 = 1$, and $x^2 - 4(3y)^2 = 1$. What can be said to happen to hyperbola $x^2 - 4(cy)^2 = 1$ as c increases?

Section 9.5

With a graphing utility, solve the following systems of equations:

111. $7.5x^2 + 1.5y^2 = 12.25$
$x^2y = 1$

112. $4x^2 + 2xy + 3y^2 = 12$
$x^3y = 3 - 3x^3$

Section 9.6

With a graphing utility, graph the following systems of nonlinear inequalities:

113. $y \geq 10^x - 1$
$y \leq 1 - x^2$

114. $x^2 + 4y^2 \leq 36$
$y \geq e^x$

Section 9.7

115. With a graphing utility, explore the second-degree equation $Ax^2 + Bxy + Cy^2 + 10x - 8y - 5 = 0$ for the following values of A, B, and C:
a. $A = 2, B = -3, C = 5$
b. $A = 2, B = 3, C = -5$

Show the angle of rotation to one decimal place. Explain the differences.

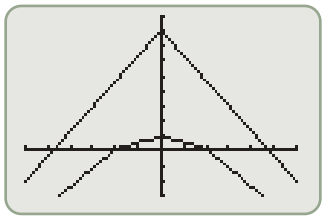
116. With a graphing utility, explore the second-degree equation $Ax^2 + Bxy + Cy^2 + 2x - y = 0$ for the following values of A, B, and C:
a. $A = 1, B = -2, C = -1$
b. $A = 1, B = 2, D = 1$

Show the angle of rotation to the nearest degree. Explain the differences.

Section 9.8

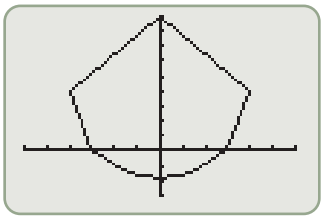
117. Let us consider the polar equation $r = \dfrac{8}{4 + 5\sin\theta}$. Explain why a graphing utility gives the following graph with the specified window parameters:

$$[-6, 6] \text{ by } [-3, 9] \text{ with } \theta \text{ step } = \frac{\pi}{4}$$



118. Let us consider the polar equation $r = \dfrac{9}{3 - 2\sin\theta}$. Explain why a graphing utility gives the following graph with the specified window parameters:

$$[-6, 6] \text{ by } [-3, 9] \text{ with } \theta \text{ step } = \frac{\pi}{2}$$



Section 9.9

119. Consider the parametric equations $x = a\cos(at) + b\sin(bt)$ and $y = a\sin(at) + b\cos(bt)$. Use a graphing utility to explore the graphs for $(a, b) = (2, 3)$ and $(a, b) = (3, 2)$. Describe the t-interval for each case.

120. Consider the parametric equations $x = a\sin(at) - b\cos(bt)$ and $y = a\cos(at) - b\sin(bt)$. Use a graphing utility to explore the graphs for $(a, b) = (1, 2)$ and $(a, b) = (2, 1)$. Describe the t-interval for each case.

Match the equation to the graph.

1. $x = 16y^2$

2. $y = 16x^2$

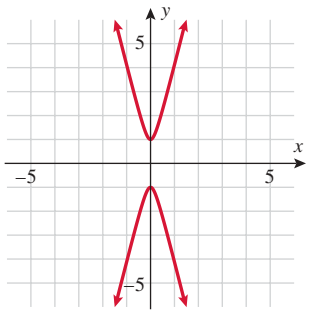
3. $x^2 + 16y^2 = 1$

4. $x^2 - 16y^2 = 1$

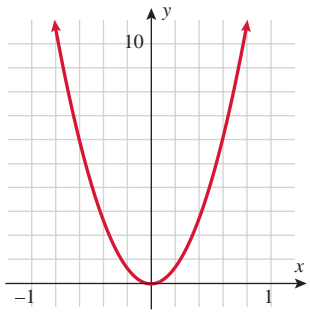
5. $16x^2 + y^2 = 1$

6. $16y^2 - x^2 = 1$

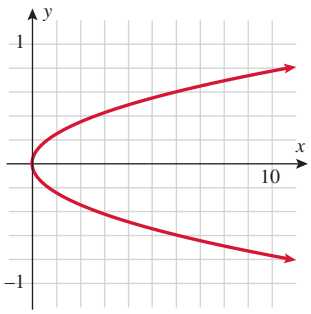
a.



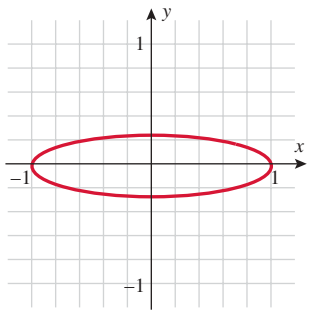
b.



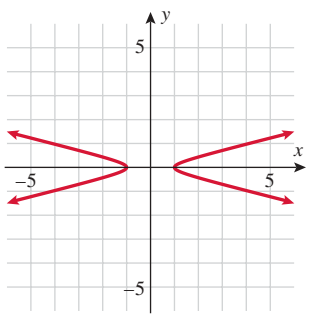
c.



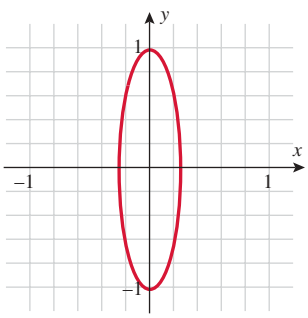
d.



e.



f.



Find the equation of the conic with the given characteristics.

7. Parabola — vertex: (0, 0) — focus: (−4, 0)

8. Parabola — vertex: (0, 0) — directrix: $y = 2$

9. Parabola — vertex: (−1, 5) — focus: (−1, 2)

10. Parabola — vertex: (2, −3) — directrix: $x = 0$

11. Ellipse — center: (0, 0)
vertices: (0, −4), (0, 4)
foci: (0, −3), (0, 3)

12. Ellipse — center: (0, 0)
vertices: (−3, 0), (3, 0)
foci: (−1, 0), (1, 0)

13. Ellipse — vertices: (2, −6), (2, 6)
foci: (2, −4), (2, 4)

14. Ellipse — vertices: (−7, −3), (−4, −3)
foci: (−6, −3), (−5, −3)

15. Hyperbola — vertices: (−1, 0) and (1, 0)
asymptotes: $y = -2x$ and $y = 2x$

16. Hyperbola — vertices: (0, −1) and (0, 1)
asymptotes: $y = -\frac{1}{3}x$ and $y = \frac{1}{3}x$

17. Hyperbola — foci: (2, −6), (2, 6)
vertices: (2, −4), (2, 4)

18. Hyperbola — foci: (−7, −3), (−4, −3)
vertices: (−6, −3), (−5, −3)

Graph the following equations:

19. $9x^2 + 18x - 4y^2 + 16y - 43 = 0$

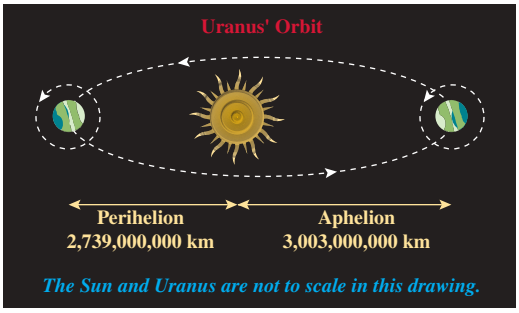
20. $4x^2 - 8x + y^2 + 10y + 28 = 0$

21. $y^2 + 4y - 16x + 20 = 0$

22. $x^2 - 4x + y + 1 = 0$

23. **Eyeglass Lens.** Eyeglass lenses can be thought of as very wide parabolic curves. If the focus occurs 1.5 centimeters from the center of the lens, and the lens at its opening is 4 centimeters across, find an equation that governs the shape of the lens.

24. **Planetary Orbits.** The planet Uranus's orbit is described in the following picture with the Sun as a focus of the elliptical orbit. Write an equation for the orbit.



Graph the following nonlinear inequalities:

25. $y < x^3 + 1$

26. $y^2 \geq 16x$

Graph the following systems of nonlinear inequalities:

27. $y \le 4 - x^2$
$16x^2 + 25y^2 \le 400$

28. $y \le e^{-x}$
$y \ge x^2 - 4$

29. Identify the conic represented by the equation $r = \dfrac{12}{3 + 2\sin\theta}$. State the eccentricity.

30. Use rotation of axes to transform the equation in x and y into an equation in X and Y that has no XY-term: $6\sqrt{3}x^2 + 6xy + 4\sqrt{3}y^2 = 21\sqrt{3}$. State the rotation angle.

31. A golf ball is hit with an initial speed of 120 feet per second at an angle of 45° with the ground. How long will the ball stay in the air? How far will the ball travel (horizontal distance) before it hits the ground?

32. Describe (classify) the plane curve defined by the parametric equations $x = \sqrt{1 - t}$ and $y = \sqrt{t}$ for t in [0, 1].

33. Use a graphing utility to graph the following nonlinear inequality:

$$x^2 + 4xy - 9y^2 - 6x + 8y + 28 \le 0$$

34. Use a graphing utility to solve the following systems of equations:

$$0.1225x^2 + 0.0289y^2 = 1$$
$$y^3 = 11x$$

Round your answers to three decimal places.

35. Given is the parabola $x^2 + 4.2x - y + 5.61 = 0$.

a. Solve the equation for y and use a graphing utility to plot the parabola.

b. Transform the equation into the form $(x - h)^2 = 4p(y - k)$. Find the vertex. Which way does the parabola open?

c. Do (a) and (b) agree with each other?

36. With a graphing utility, explore the second-degree equation $Ax^2 + Bxy + Cy^2 + 10x - 8y - 5 = 0$ for the following values of A, B, and C:

a. $A = 2, B = -\sqrt{3}, C = 1$

b. $A = 2, B = \sqrt{3}, C = -1$

Show the angle of rotation to one decimal place. Explain the differences.

PRACTICE TEST

1. Solve for x: $(x + 2)^2 - (x + 2) - 20 = 0$.

2. Find an equation of a circle centered at (5, 1) and passing through the point (6, −2).

3. Evaluate the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for the function $f(x) = 8 - 7x$.

4. Write an equation that describes the following variation: I is directly proportional to both P and t, and $I = 90$ when $P = 1500$ and $t = 2$.

5. Find the quadratic function that has vertex (7, 7) and goes through the point (10, 10).

6. **Compound Interest.** How much money should you put in a savings account now that earns 4.7% interest a year compounded weekly if you want to have $65,000 in 17 years?

7. Solve the logarithmic equation exactly: $\log x^2 - \log 16 = 0$.

8. In a 30°-60°-90° triangle, if the shortest leg has length 8 inches, what are the lengths of the other leg and the hypotenuse?

9. Use a calculator to evaluate $\cot(-27°)$. Round your answer to four decimal places.

10. **Sound Waves.** If a sound wave is represented by $y = 0.007\sin(850\pi t)$ cm, what are its amplitude and frequency?

11. For the trigonometric expression $\tan\theta(\csc\theta + \cos\theta)$, perform the operations and simplify. Write the answer in terms of $\sin\theta$ and $\cos\theta$.

12. Find the exact value of $\cos\left(-\dfrac{11\pi}{12}\right)$.

13. Solve the trigonometric equation $4\cos^2 x + 4\cos 2x + 1 = 0$ exactly over the interval $0 \le \theta \le 2\pi$.

14. **Airplane Speed.** A plane flew due north at 450 miles per hour for 2 hours. A second plane, starting at the same point and at the same time, flew southeast at an angle of 135° clockwise from due north at 375 miles per hour for 2 hours. At the end of 2 hours, how far apart were the two planes? Round to the nearest mile.

15. Find the vector with magnitude $|\mathbf{u}| = 15$ and direction angle $\theta = 110°$.

16. Given $z_1 = 5(\cos 15° + i\sin 15°)$ and $z_2 = 2(\cos 75° + i\sin 75°)$, find the product $z_1 z_2$ and express it in rectangular form.

17. At a food court, 3 medium sodas and 2 soft pretzels cost $6.77. A second order of 5 medium sodas and 4 soft pretzels costs $12.25. Find the cost of a soda and the cost of a soft pretzel.

18. Find the partial-fraction decomposition for the rational expression $\dfrac{3x + 5}{(x - 3)(x^2 + 5)}$.

19. Graph the system of inequalities or indicate that the system has no solution.

$$\begin{aligned} y &\ge 3x - 2 \\ y &\le 3x + 2 \end{aligned}$$

20. Solve the system using Gauss–Jordan elimination.

$$\begin{aligned} x - 2y + z &= 7 \\ -3x + y + 2z &= -11 \end{aligned}$$

21. Given $A = \begin{bmatrix} 3 & 4 & -7 \\ 0 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -2 & 6 \\ 9 & 0 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix}$, find $2B - 3A$.

22. Use Cramer's rule to solve the system of equations.

$$\begin{aligned} 25x + 40y &= -12 \\ 75x - 105y &= 69 \end{aligned}$$

23. Find the standard form of the equation of an ellipse with foci (6, 2) and (6, −6) and vertices (6, 3) and (6, −7).

24. Find the standard form of the equation of a hyperbola with vertices (5, −2) and (5, 0) and foci (5, −3) and (5, 1).

25. Solve the system of equations.

$$\begin{aligned} x + y &= 6 \\ x^2 + y^2 &= 20 \end{aligned}$$

26. Use a graphing utility to graph the following equation:

$$x^2 - 3xy + 10y^2 - 1 = 0$$

27. Use a graphing utility to graph the following system of nonlinear inequalities:

$$\begin{aligned} y &\ge e^{-0.3x} - 3.5 \\ y &\le 4 - x^2 \end{aligned}$$

10

Sequences and Series

Gaillardia Flower (55), Kenneth M. Highfil/Photo Researchers, Inc.; Michaelmas Daisy (89), Maxine Adcock/Photo Researchers, Inc.; Yellow Iris (3), Edward Kinsman/Photo Researchers, Inc.; Blue Columbine (5), Jeffrey Lepore/Photo Researchers, Inc.; Gerbera Daisy (34), Bonnie Sue Rauch/Photo Researchers, Inc.; Erect Dayflower (2), Michael Lustbader/Photo Researchers, Inc.; Calla Lilies (1), Adam Jones/Photo Researchers, Inc.; Cosmos Flower (8), Maria Mosolova/Photo Researchers, Inc.; Lemon Symphony (21), Bonnie Sue Rauch/Photo Researchers, Inc.; Black-Eyed Susan (13), Rod Planck/Photo Researchers, Inc.

A famous sequence that appears throughout nature is the *Fibonacci sequence*, where each term in the sequence is the sum of the previous two numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

The number of petals in certain flowers are numbers in the Fibonacci sequence.

IN THIS CHAPTER we will first define a sequence and a series and then discuss two particular kinds of sequences and series called arithmetic and geometric. We will then discuss mathematical proof by induction. Lastly, we will discuss the Binomial theorem, which allows us an efficient way to perform binomial expansions.

SEQUENCES AND SERIES

10.1 Sequences and Series
- Sequences
- Factorial Notation
- Recursion Formulas
- Sums and Series

10.2 Arithmetic Sequences and Series
- Arithmetic Sequences
- The General (nth) Term of an Arithmetic Sequence
- The Sum of an Arithmetic Sequence

10.3 Geometric Sequences and Series
- Geometric Sequences
- The General (nth) Term of a Geometric Sequence
- Geometric Series

10.4 Mathematical Induction
- Proof by Mathematical Induction

10.5 The Binomial Theorem
- Binomial Coefficients
- Binomial Expansion
- Pascal's Triangle
- Finding a Particular Term of a Binomial Expansion

LEARNING OBJECTIVES

- Use sigma notation to represent a series.
- Find the sum of an arithmetic sequence.
- Determine whether an infinite geometric series converges.
- Prove mathematical statements using mathematical induction.
- Perform binomial expansions.

SECTION 10.1

SEQUENCES AND SERIES

SKILLS OBJECTIVES

- Find terms of a sequence given the general term.
- Look for a pattern in a sequence and find the general term.
- Apply factorial notation.
- Apply recursion formulas.
- Use summation (sigma) notation to represent a series.
- Evaluate a series.

CONCEPTUAL OBJECTIVES

- Understand the difference between a sequence and a series.
- Understand the difference between a finite series and an infinite series.

Sequences

The word *sequence* means an order in which one thing follows another in succession. A sequence is an ordered list. For example, if we write $x, 2x^2, 3x^3, 4x^4, 5x^5, ?$, what would the next term in the *sequence* be, the one where the question mark now stands? The answer is $6x^6$.

Study Tip

A sequence is a set of terms written in a specific order

$$a_1, a_2, a_3, \ldots a_n, \ldots$$

where a_1 is called the first term, a_2 is called the second term, and a_n is called the nth term.

DEFINITION **A Sequence**

A **sequence** is a function whose domain is a set of positive integers. The function values, or **terms**, of the sequence are written as

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Rather than using function notation, sequences are usually written with subscript (or index) notation, $a_{\text{subscript}}$.

A **finite sequence** has the domain $\{1, 2, 3, \ldots, n\}$ for some positive integer n. An **infinite sequence** has the domain of all positive integers $\{1, 2, 3, \ldots\}$. There are times when it is convenient to start the indexing at 0 instead of 1:

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

Sometimes a pattern in the sequence can be obtained and the sequence can be written using a *general term*. In the previous example, $x, 2x^2, 3x^3, 4x^4, 5x^5, 6x^6, \ldots$, each term has the same exponent and coefficient. We can write this sequence as $a_n = nx^n$, $n = 1, 2, 3, 4, 5, 6, \ldots$, where a_n is called the **general** or ***n*th term**.

EXAMPLE 1 **Finding Several Terms of a Sequence, Given the General Term**

Find the first four ($n = 1, 2, 3, 4$) terms of each of the following sequences, given the general term.

a. $a_n = 2n - 1$

b. $b_n = \dfrac{(-1)^n}{n + 1}$

Solution (a): $a_n = 2n - 1$

Find the first term, $n = 1$. $a_1 = 2(1) - 1 = 1$

Find the second term, $n = 2$. $a_2 = 2(2) - 1 = 3$

Find the third term, $n = 3$. $a_3 = 2(3) - 1 = 5$

Find the fourth term, $n = 4$. $a_4 = 2(4) - 1 = 7$

The first four terms of the sequence are $\boxed{1, 3, 5, 7}$.

Solution (b): $b_n = \dfrac{(-1)^n}{n+1}$

Find the first term, $n = 1$. $b_1 = \dfrac{(-1)^1}{1+1} = -\dfrac{1}{2}$

Find the second term, $n = 2$. $b_2 = \dfrac{(-1)^2}{2+1} = \dfrac{1}{3}$

Find the third term, $n = 3$. $b_3 = \dfrac{(-1)^3}{3+1} = -\dfrac{1}{4}$

Find the fourth term, $n = 4$. $b_4 = \dfrac{(-1)^4}{4+1} = \dfrac{1}{5}$

The first four terms of the sequence are $\boxed{-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}}$.

■ **YOUR TURN** Find the first four terms of the sequence $a_n = \dfrac{(-1)^n}{n^2}$.

■ **Answer:** $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}$

EXAMPLE 2 Finding the General Term, Given Several Terms of the Sequence

Find the general term of the sequence, given the first five terms.

a. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$ **b.** $-1, 4, -9, 16, -25, \ldots$

Solution (a):

Write 1 as $\frac{1}{1}$. $\dfrac{1}{1}, \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \dfrac{1}{25}, \ldots$

Notice that each denominator is an integer squared. $\dfrac{1}{1^2}, \dfrac{1}{2^2}, \dfrac{1}{3^2}, \dfrac{1}{4^2}, \dfrac{1}{5^2}, \ldots$

Identify the general term. $\boxed{a_n = \dfrac{1}{n^2} \qquad n = 1, 2, 3, 4, 5, \ldots}$

Solution (b):

Notice that each term includes an integer squared. $-1^2, 2^2, -3^2, 4^2, -5^2, \ldots$

Identify the general term. $\boxed{b_n = (-1)^n n^2 \qquad n = 1, 2, 3, 4, 5, \ldots}$

■ **YOUR TURN** Find the general term of the sequence, given the first five terms.

a. $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{6}, \frac{1}{8}, -\frac{1}{10}, \ldots$ **b.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Study Tip

$(-1)^n$ or $(-1)^{n+1}$ is a way to represent an alternating sequence.

■ **Answer:**

a. $a_n = \dfrac{(-1)^n}{2n}$ **b.** $a_n = \dfrac{1}{2^n}$

Parts (b) in both Example 1 and Example 2 are called **alternating** sequences, because the terms alternate signs (positive and negative). If the odd-indexed terms, $a_1, a_3, a_5, \ldots$, are negative and the even-indexed terms, $a_2, a_4, a_6, \ldots$, are positive, we include $(-1)^n$ in the general term. If the opposite is true, and the odd-indexed terms are positive and the even-indexed terms are negative, we include $(-1)^{n+1}$ in the general term.

Factorial Notation

Many important sequences that arise in mathematics involve terms that are defined with products of consecutive positive integers. The products are expressed in *factorial notation.*

DEFINITION **Factorial**

If n is a positive integer, then $n!$ (stated as "n factorial") is the product of all positive integers from n down to 1.

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \qquad n \geq 2$$

and $0! = 1$ and $1! = 1$.

The values of $n!$ for the first six nonnegative integers are

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= 2\cdot 1 = 2 \\ 3! &= 3\cdot 2\cdot 1 = 6 \\ 4! &= 4\cdot 3\cdot 2\cdot 1 = 24 \\ 5! &= 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120 \end{aligned}$$

Notice that $4! = 4\cdot 3\cdot 2\cdot 1 = 4\cdot 3!$. In general, we can apply the formula $n! = n[(n-1)!]$. Often the brackets are not used, and the notation $n! = n(n-1)!$ implies calculating the factorial $(n-1)!$ and then multiplying that quantity by n. For example, to find 6!, we employ the relationship $n! = n(n-1)!$ and set $n = 6$:

$$6! = 6\cdot 5! = 6\cdot 120 = 720$$

EXAMPLE 3 Finding the Terms of a Sequence Involving Factorials

Find the first four terms of the sequence, given the general term $a_n = \dfrac{x^n}{n!}$.

Solution:

Find the first term, $n = 1$. $\quad a_1 = \dfrac{x^1}{1!} = x$

Find the second term, $n = 2$. $\quad a_2 = \dfrac{x^2}{2!} = \dfrac{x^2}{2\cdot 1} = \dfrac{x^2}{2}$

Find the third term, $n = 3$. $\quad a_3 = \dfrac{x^3}{3!} = \dfrac{x^3}{3\cdot 2\cdot 1} = \dfrac{x^3}{6}$

Find the fourth term, $n = 4$. $\quad a_4 = \dfrac{x^4}{4!} = \dfrac{x^4}{4\cdot 3\cdot 2\cdot 1} = \dfrac{x^4}{24}$

The first four terms of the sequence are $\boxed{x, \dfrac{x^2}{2}, \dfrac{x^3}{6}, \dfrac{x^4}{24}}$.

Technology Tip

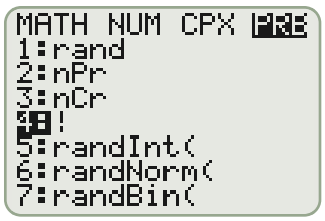
Find 0!, 1!, 2!, 3!, 4!, and 5!.

Scientific calculators:

	Press		Display
0	!	=	1
1	!	=	1
2	!	=	2
3	!	=	6
4	!	=	24
5	!	=	120

Graphing calculators:

	Press				Display
0	MATH	PRB	4:!	ENTER	1
1	MATH	PRB	4:!	ENTER	1
2	MATH	PRB	4:!	ENTER	2
3	MATH	PRB	4:!	ENTER	6
4	MATH	PRB	4:!	ENTER	24
5	MATH	PRB	4:!	ENTER	120



EXAMPLE 4 Evaluating Expressions with Factorials

Evaluate each factorial expression.

a. $\dfrac{6!}{2!\cdot 3!}$ **b.** $\dfrac{(n+1)!}{(n-1)!}$

Solution (a):

Expand each factorial in the numerator and denominator. $\dfrac{6!}{2!\cdot 3!} = \dfrac{6\cdot 5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{2\cdot 1\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$

Cancel the $3\cdot 2\cdot 1$ in both the numerator and denominator. $= \dfrac{6\cdot 5\cdot 4}{2\cdot 1}$

Simplify. $= \dfrac{6\cdot 5\cdot 2}{1} = 60$

$$\boxed{\frac{6!}{2!\cdot 3!} = 60}$$

Solution (b):

Expand each factorial in the numerator and denominator. $\dfrac{(n+1)!}{(n-1)!} = \dfrac{(n+1)(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$

Cancel the $(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ in both the numerator and denominator. $\dfrac{(n+1)!}{(n-1)!} = (n+1)(n)$

Alternatively, $\dfrac{(n+1)!}{(n-1)!} = \dfrac{(n+1)(n)\cancel{(n-1)!}}{\cancel{(n-1)!}}$

$$\boxed{\frac{(n+1)!}{(n-1)!} = n^2 + n}$$

COMMON MISTAKE

In Example 4 we found $\dfrac{6!}{2!\cdot 3!} = 60$. It is important to note that $2!\cdot 3! \neq 6!$

YOUR TURN Evaluate each factorial expression.

a. $\dfrac{3!\cdot 4!}{2!\cdot 6!}$ **b.** $\dfrac{(n+2)!}{n!}$

Technology Tip

Evaluate $\dfrac{6!}{2!3!}$.

Scientific calculators:

Press	*Display*
6 [!] [÷] [(] 2 [!] [x] 3 [!] [)] [=]	60

Graphing calculators:

6 [MATH] [▶] [*PRB*] [▼] [4: !] [ENTER] [÷] [(] [2] [MATH] [▶] [*PRB*] [▼] [4: !] [ENTER] [x] [3] [MATH] [▶] [*PRB*] [▼] [4: !] [ENTER] [)] [ENTER].

```
6!/(2!*3!)
                60
```

Study Tip

In general, $m!n! \neq (mn)!$

Answer:

a. $\frac{1}{10}$ **b.** $(n+2)(n+1)$

Recursion Formulas

Another way to define a sequence is **recursively**, or using a **recursion formula**. The first few terms are listed, and the recursion formula determines the remaining terms based on previous terms. For example, the famous Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... Each term in the Fibonacci sequence is found by adding the previous two terms.

$$1 + 1 = \mathbf{2} \qquad 1 + 2 = \mathbf{3} \qquad 2 + 3 = \mathbf{5}$$
$$3 + 5 = \mathbf{8} \qquad 5 + 8 = \mathbf{13} \qquad 8 + 13 = \mathbf{21}$$
$$13 + 21 = \mathbf{34} \qquad 21 + 34 = \mathbf{55} \qquad 34 + 55 = \mathbf{89}$$

We can define the Fibonacci sequence using a general term:

$$a_1 = 1,\ a_2 = 1, \quad \text{and} \quad a_n = a_{n-2} + a_{n-1} \qquad n \geq 3$$

The Fibonacci sequence is found in places we least expect it (e.g., pineapples, broccoli, and flowers). The number of petals in certain flowers is a Fibonacci number. For example, a wild rose has 5 petals, lilies and irises have 3 petals, and daisies have 34, 55, or even 89 petals. The number of spirals in an Italian broccoli is a Fibonacci number (13).

Study Tip

If $a_n = a_{n-1} + a_{n-3}$, then $a_{100} = a_{99} + a_{98}$.

EXAMPLE 5 Using a Recursion Formula to Find a Sequence

Find the first four terms of the sequence a_n if $a_1 = 2$ and $a_n = 2a_{n-1} - 1$, $n \geq 2$.

Solution:

Write the first term, $n = 1$.	$a_1 = 2$
Find the second term, $n = 2$.	$a_2 = 2a_1 - 1 = 2(2) - 1 = 3$
Find the third term, $n = 3$.	$a_3 = 2a_2 - 1 = 2(3) - 1 = 5$
Find the fourth term, $n = 4$.	$a_4 = 2a_3 - 1 = 2(5) - 1 = 9$

The first four terms of the sequence are $\boxed{2, 3, 5, 9}$.

■ **Answer:** $1, \frac{1}{2}, \frac{1}{12}, \frac{1}{288}$

■ **YOUR TURN** Find the first four terms of the sequence

$$a_1 = 1 \quad \text{and} \quad a_n = \frac{a_{n-1}}{n!} \quad n \geq 2$$

Sums and Series

When we add the terms in a sequence, the result is a *series*.

DEFINITION **Series**

Given the infinite sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$, the sum of all of the terms in the infinite sequence is called an **infinite series** and is denoted by

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

and the sum of only the first n terms is called a **finite series**, or ***n*th partial sum**, and is denoted by

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

Study Tip

Σ is a regular Greek letter, but when used to represent the mathematical sum operation, we oversize it.

The capital Greek letter Σ (sigma) corresponds to the capital S in our alphabet. Therefore, we use Σ as a shorthand way to represent a sum (series). For example, the sum of the first five terms of the sequence $1, 4, 9, 16, 25, \ldots, n^2, \ldots$ can be represented using **sigma (or summation) notation**:

$$\sum_{n=1}^{5} n^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2$$
$$= 1 + 4 + 9 + 16 + 25$$

Study Tip

We often use the following notation in running text:

$$\sum_{n=1}^{5} = \sum_{n=1}^{5}$$

This is read "the sum as n goes from 1 to 5 of n^2." The letter n is called the **index of summation**, and often other letters are used instead of n. It is important to note that the sum can start at other integers besides 1.

If we wanted the sum of all of the terms in the sequence, we would represent that infinite series using summation notation as

$$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + 25 + \cdots$$

Study Tip

The index of a summation (series) can start at any integer (not just 1).

EXAMPLE 6 Writing a Series Using Sigma Notation

Represent each of the following series using sigma notation:

a. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$ **b.** $8 + 27 + 64 + 125 + \cdots$

Solution (a):

Write 1 as $\frac{1}{1}$. $\frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$

Notice that we can write the denominators using factorials. $= \frac{1}{1} + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$

Recall that $0! = 1$ and $1! = 1$. $= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$

Identify the general term. $a_n = \frac{1}{n!} \qquad n = 0, 1, 2, 3, 4, 5$

Write the finite series using sigma notation. $\boxed{\sum_{n=0}^{5} \frac{1}{n!}}$

Solution (b):

Write the infinite series as a sum of terms cubed. $8 + 27 + 64 + 125 + \cdots$
$= 2^3 + 3^3 + 4^3 + 5^3 + \cdots$

Identify the general term of the series. $a_n = n^3 \qquad n \geq 2$

Write the infinite series using sigma notation. $\boxed{\sum_{n=2}^{\infty} n^3}$

■ **YOUR TURN** Represent each of the following series using sigma notation:

a. $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \cdots$

b. $4 + 8 + 16 + 32 + 64 + \cdots$

■ **Answer:**

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ **b.** $\sum_{n=2}^{\infty} 2^n$

Now that we are comfortable with sigma (summation) notation, let's turn our attention to evaluating a series (calculating the sum). You can always evaluate a finite series. However, you cannot always evaluate an infinite series.

EXAMPLE 7 Evaluating a Finite Series

Evaluate the series $\sum_{i=0}^{4} (2i + 1)$.

Solution:

Write out the partial sum.

$$\sum_{i=0}^{4}(2i + 1) = \underset{(i\,=\,0)}{1} + \underset{(i\,=\,1)}{3} + \underset{(i\,=\,2)}{5} + \underset{(i\,=\,3)}{7} + \underset{(i\,=\,4)}{9}$$

Simplify. $= 25$

$$\boxed{\sum_{i=0}^{4}(2i + 1) = 25}$$

■ **YOUR TURN** Evaluate the series $\sum_{n=1}^{5} (-1)^n n$.

Technology Tip

2nd [LIST] [▶] [MATH] [▼] 5:*sum*([ENTER] 2nd [LIST] [▶] *OPS* [▼] 5:*seq*([ENTER] 2 [ALPHA] [*I*] [+] 1 [,] [ALPHA] [*I*] [,] 0 [,] 4 [,] 1 [)] [)] [ENTER].

```
sum(seq(2I+1,I,0
,4,1))
                25
```

■ **Answer:** -3

Study Tip

The sum of a finite series always exists. The sum of an infinite series may or may not exist.

Infinite series may or may not have a finite sum. For example, if we keep adding $1 + 1 + 1 + 1 + \cdots$, then there is no single real number that the series sums to because the sum continues to grow without bound. However, if we add $0.9 + 0.09 + 0.009 + 0.0009 + \cdots$, this sum is $0.9999\ldots = 0.\overline{9}$, which is a rational number, and it can be proven that $0.\overline{9} = 1$.

Technology Tip

a. [2nd] [*LIST*] [▶] [MATH] [▼] [5:*sum*(] [ENTER] [2nd] [*LIST*] [▶] [*OPS*] [▼] [5:*seq*(] [ENTER] 3 [÷] 10 [^] [ALPHA] [*N*] [,] [ALPHA] [*N*] [,] 1 [,] 10 [,] 1 [)] [)] [ENTER].

```
sum(seq(3/10^N,N
,1,10,1))
      .3333333333
```

b. [2nd] [*LIST*] [▶] [MATH] [▼] [5:*sum*(] [ENTER] [2nd] [*LIST*] [▶] [*OPS*] [▼] [5:*seq*(] [ALPHA] [*N*] [x^2] [,] [^] [ALPHA] [*N*] [,] 1 [,] 100 [,] 1 [)] [)] [ENTER].

```
sum(seq(N²,N,1,1
00,1))
            338350
```

■ **Answer:** **a.** Series diverges.
b. Series converges to $\frac{2}{3}$.

EXAMPLE 8 Evaluating an Infinite Series, If Possible

Evaluate the following infinite series, if possible.

a. $\sum_{n=1}^{\infty} \frac{3}{10^n}$ **b.** $\sum_{n=1}^{\infty} n^2$

Solution (a):

Expand the series. $$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10{,}000} + \cdots$$

Write in decimal form. $$\sum_{n=1}^{\infty} \frac{3}{10^n} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$$

Calculate the sum. $$\sum_{n=1}^{\infty} \frac{3}{10^n} = 0.333333\overline{3} = \frac{1}{3}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{1}{3}}$$

Solution (b):

Expand the series. $$\sum_{n=1}^{\infty} n^2 = 1 + 4 + 9 + 16 + 25 + 36 + \cdots$$

This sum is infinite since it continues to grow without any bound.

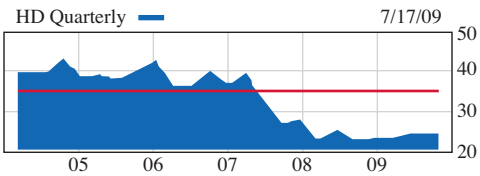
In part (a) we say that the series **converges** to $\frac{1}{3}$ and in part (b) we say that the series **diverges**.

■ **YOUR TURN** Evaluate the following infinite series, if possible.

a. $\sum_{n=1}^{\infty} 2n$ **b.** $\sum_{n=1}^{\infty} 6\left(\frac{1}{10}\right)^n$

Applications

The average stock price for Home Depot Inc (HD) was \$37 in 2007, \$28 in 2008, and \$23 in 2009. If a_n is the yearly average stock price, where $n = 0$ corresponds to 2007, then $\frac{1}{3}\sum_{n=0}^{2} a_n$ tells us the average yearly stock price over the three-year period 2007 to 2009.



SECTION 10.1 SUMMARY

In this section, we discussed finite and infinite sequences and series. When the terms of a sequence are added together, the result is a series.

Finite sequence: $a_1, a_2, a_3, \ldots, a_n$
Infinite sequence: $a_1, a_2, a_3, \ldots, a_n, \ldots$

Finite series: $a_1 + a_2 + a_3 + \cdots + a_n$
Infinite series: $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Factorial notation was also introduced:

$n! = n(n-1)\cdots 3 \cdot 2 \cdot 1 \quad n \geq 2$

and $0! = 1$ and $1! = 1$.

The sum of a finite series is always finite.

The sum of an infinite series is either convergent or divergent.

Sigma notation is used to express a series.

- **Finite series:** $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$
- **Infinite series:** $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

SECTION 10.1 EXERCISES

SKILLS

In Exercises 1–12, write the first four terms of each sequence. Assume *n* starts at 1.

1. $a_n = n$ **2.** $a_n = n^2$ **3.** $a_n = 2n - 1$ **4.** $a_n = x^n$

5. $a_n = \dfrac{n}{(n+1)}$ **6.** $a_n = \dfrac{(n+1)}{n}$ **7.** $a_n = \dfrac{2^n}{n!}$ **8.** $a_n = \dfrac{n!}{(n+1)!}$

9. $a_n = (-1)^n x^{n+1}$ **10.** $a_n = (-1)^{n+1} n^2$ **11.** $a_n = \dfrac{(-1)^n}{(n+1)(n+2)}$ **12.** $a_n = \dfrac{(n-1)^2}{(n+1)^2}$

In Exercises 13–20, find the indicated term of each sequence given.

13. $a_n = \left(\dfrac{1}{2}\right)^n \quad a_9 = ?$ **14.** $a_n = \dfrac{n}{(n+1)^2} \quad a_{15} = ?$ **15.** $a_n = \dfrac{(-1)^n n!}{(n+2)!} \quad a_{19} = ?$

16. $a_n = \dfrac{(-1)^{n+1}(n-1)(n+2)}{n} \quad a_{13} = ?$ **17.** $a_n = \left(1 + \dfrac{1}{n}\right)^2 \quad a_{100} = ?$ **18.** $a_n = 1 - \dfrac{1}{n^2} \quad a_{10} = ?$

19. $a_n = \log 10^n \quad a_{23} = ?$ **20.** $a_n = e^{\ln n} \quad a_{49} = ?$

In Exercises 21–28, write an expression for the *n*th term of the given sequence. Assume *n* starts at 1.

21. 2, 4, 6, 8, 10, . . . **22.** 3, 6, 9, 12, 15, . . . **23.** $\dfrac{1}{2\cdot 1}, \dfrac{1}{3\cdot 2}, \dfrac{1}{4\cdot 3}, \dfrac{1}{5\cdot 4}, \dfrac{1}{6\cdot 5}, \ldots$ **24.** $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \dfrac{1}{32}, \ldots$

25. $-\dfrac{2}{3}, \dfrac{4}{9}, -\dfrac{8}{27}, \dfrac{16}{81}, \ldots$ **26.** $\dfrac{1}{2}, \dfrac{3}{4}, \dfrac{9}{8}, \dfrac{27}{16}, \dfrac{81}{32}, \ldots$ **27.** $1, -1, 1, -1, 1, \ldots$ **28.** $\dfrac{1}{3}, -\dfrac{2}{4}, \dfrac{3}{5}, -\dfrac{4}{6}, \dfrac{5}{7}, \ldots$

In Exercises 29–40, simplify each ratio of factorials.

29. $\dfrac{9!}{7!}$ **30.** $\dfrac{4!}{6!}$ **31.** $\dfrac{29!}{27!}$ **32.** $\dfrac{32!}{30!}$ **33.** $\dfrac{75!}{77!}$ **34.** $\dfrac{100!}{103!}$

35. $\dfrac{97!}{93!}$ **36.** $\dfrac{101!}{98!}$ **37.** $\dfrac{(n-1)!}{(n+1)!}$ **38.** $\dfrac{(n+2)!}{n!}$ **39.** $\dfrac{(2n+3)!}{(2n+1)!}$ **40.** $\dfrac{(2n+2)!}{(2n-1)!}$

In Exercises 41–50, write the first four terms of the sequence defined by each recursion formula. Assume the sequence begins at $n = 1$.

41. $a_1 = 7 \quad a_n = a_{n-1} + 3$

42. $a_1 = 2 \quad a_n = a_{n-1} + 1$

43. $a_1 = 1 \quad a_n = n \cdot a_{n-1}$

44. $a_1 = 2 \quad a_n = (n + 1) \cdot a_{n-1}$

45. $a_1 = 100 \quad a_n = \dfrac{a_{n-1}}{n!}$

46. $a_1 = 20 \quad a_n = \dfrac{a_{n-1}}{n^2}$

47. $a_1 = 1,\ a_2 = 2 \quad a_n = a_{n-1} \cdot a_{n-2}$

48. $a_1 = 1,\ a_2 = 2 \quad a_n = \dfrac{a_{n-2}}{a_{n-1}}$

49. $a_1 = 1,\ a_2 = -1 \quad a_n = (-1)^n\left[a_{n-1}^2 + a_{n-2}^2\right]$

50. $a_1 = 1,\ a_2 = -1 \quad a_n = (n - 1)\, a_{n-1} + (n - 2)a_{n-2}$

In Exercises 51–64, evaluate each finite series.

51. $\sum_{n=1}^{5} 2$

52. $\sum_{n=1}^{5} 7$

53. $\sum_{n=0}^{4} n^2$

54. $\sum_{n=1}^{4} \frac{1}{n}$

55. $\sum_{n=1}^{6} (2n - 1)$

56. $\sum_{n=1}^{6} (n + 1)$

57. $\sum_{n=0}^{4} 1^n$

58. $\sum_{n=0}^{4} 2^n$

59. $\sum_{n=0}^{3} (-x)^n$

60. $\sum_{n=0}^{3} (-x)^{n+1}$

61. $\sum_{k=0}^{5} \frac{2^k}{k!}$

62. $\sum_{k=0}^{5} \frac{(-1)^k}{k!}$

63. $\sum_{k=0}^{4} \frac{x^k}{k!}$

64. $\sum_{k=0}^{4} \frac{(-1)^k x^k}{k!}$

In Exercises 65–68, evaluate each infinite series, if possible.

65. $\sum_{j=0}^{\infty} 2 \cdot (0.1)^j$

66. $\sum_{j=0}^{\infty} 5 \cdot \left(\frac{1}{10}\right)^j$

67. $\sum_{j=0}^{\infty} n^j \quad n \geq 1$

68. $\sum_{j=0}^{\infty} 1^j$

In Exercises 69–76, use sigma notation to represent each sum.

69. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{1}{64}$

70. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{64} + \cdots$

71. $1 - 2 + 3 - 4 + 5 - 6 + \cdots$

72. $1 + 2 + 3 + 4 + 5 + \cdots + 21 + 22 + 23$

73. $\frac{2 \cdot 1}{1} + \frac{3 \cdot 2 \cdot 1}{1} + \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$

74. $1 + \frac{2}{1} + \frac{2^2}{2 \cdot 1} + \frac{2^3}{3 \cdot 2 \cdot 1} + \frac{2^4}{4 \cdot 3 \cdot 2 \cdot 1} + \cdots$

75. $1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \cdots$

76. $x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24} + \frac{x^6}{120}$

APPLICATIONS

77. Money. Upon graduation Jessica receives a commission from the U.S. Navy to become an officer and a \$20,000 signing bonus for selecting aviation. She puts the entire bonus in an account that earns 6% interest compounded monthly. The balance in the account after n months is

$$A_n = 20{,}000\left(1 + \frac{0.06}{12}\right)^n \qquad n = 1, 2, 3, \ldots$$

Her commitment to the Navy is 6 years. Calculate A_{72}. What does A_{72} represent?

78. Money. Dylan sells his car during his freshman year and puts \$7000 in an account that earns 5% interest compounded quarterly. The balance in the account after n quarters is

$$A_n = 7000\left(1 + \frac{0.05}{4}\right)^n \qquad n = 1, 2, 3, \ldots$$

Calculate A_{12}. What does A_{12} represent?

79. Salary. An attorney is trying to calculate the costs associated with going into private practice. If she hires a paralegal to assist her, she will have to pay the paralegal \$20 per hour. To be competitive with most firms, she will have to give her paralegal a \$2 per hour raise each year. Find a general term of a sequence a_n that would represent the hourly salary of a paralegal with n years of experience. What will be the paralegal's salary with 20 years of experience?

80. NFL Salaries. A player in the NFL typically has a career that lasts 3 years. The practice squad makes the league minimum of \$275,000 (2004) in the first year, with a \$75,000 raise per year. Write the general term of a sequence a_n that represents the salary of an NFL player making the league minimum during his entire career. Assuming $n = 1$ corresponds to the first year, what does $\sum_{n=1}^{3} a_n$ represent?

81. Salary. Upon graduation Sheldon decides to go to work for a local police department. His starting salary is \$30,000 per year, and he expects to get a 3% raise per year. Write the recursion formula for a sequence that represents his annual salary after n years on the job. Assume $n = 0$ represents his first year making \$30,000.

82. *Escherichia coli.* A single cell of bacteria reproduces through a process called binary fission. *Escherichia coli* cells divide into two every 20 minutes. Suppose the same rate of division is maintained for 12 hours after the original cell enters the body. How many *E. coli* bacteria cells would be in the body 12 hours later? Suppose there is an infinite nutrient source so that the *E. coli* bacteria cells maintain the same rate of division for 48 hours after the original cell enters the body. How many *E. coli* bacteria cells would be in the body 48 hours later?

83. AIDS/HIV. A typical person has 500 to 1500 T cells per drop of blood in the body. HIV destroys the T cell count at a rate of 50–100 cells per drop of blood per year, depending on how aggressive it is in the body. Generally, the onset of AIDS occurs once the body's T cell count drops below 200. Write a sequence that represents the total number of T cells in a person infected with HIV. Assume that before infection the person has a 1000 T cell count ($a_0 = 1000$) and the rate at which the infection spreads corresponds to a loss of 75 T cells per drop of blood per year. How much time will elapse until this person has full-blown AIDS?

84. Company Sales. Lowe's reported total sales from 2003 through 2004 in the billions. The sequence $a_n = 3.8 + 1.6n$ represents the total sales in billions of dollars. Assuming $n = 3$ corresponds to 2003, what were the reported sales in 2003 and 2004? What does $\frac{1}{2} \cdot \sum_{n=3}^{4} a_n$ represent?

85. Cost of Eating Out. A college student tries to save money by bringing a bag lunch instead of eating out. He will be able to save \$100 per month. He puts the money into his savings account, which draws 1.2% interest and is compounded monthly. The balance in his account after n months of bagging his lunch is

$$A_n = 100{,}000[(1.001)^n - 1] \qquad n = 1, 2, \ldots$$

Calculate the first four terms of this sequence. Calculate the amount after 3 years (36 months).

86. Cost of Acrylic Nails. A college student tries to save money by growing her own nails out and not spending \$50 per month on acrylic fills. She will be able to save \$50 per month. She puts the money into her savings account, which draws 1.2% interest and is compounded monthly. The balance in her account after n months of natural nails is

$$A_n = 50{,}000[(1.001)^n - 1] \qquad n = 1, 2, \ldots$$

Calculate the first four terms of this sequence. Calculate the amount after 4 years (48 months).

87. Math and Engineering. The formula $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ can be used to approximate the function $y = e^x$. Estimate e^2 by using the sum of the first five terms and compare this result with the calculator value of e^2.

88. Home Prices. If the inflation rate is 3.5% per year and the average price of a home is \$195,000, the average price of a home after n years is given by $A_n = 195{,}000(1.035)^n$. Find the average price of the home after 6 years.

89. Approximating Functions. Polynomials can be used to approximate transcendental functions such as $\ln(x)$ and e^x, which are found in advanced mathematics and engineering. For example, $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$ can be used to approximate $\ln(x)$, when x is close to 1. Use the first five terms of the series to approximate $\ln(1.1)$ and compare with the value indicated by your calculator for $\ln(1.1)$.

90. Future Value of an Annuity. The future value of an ordinary annuity is given by the formula $FV = PMT[((1 + i)^n - 1)/i]$, where PMT = amount paid into the account at the end of each period, i = interest rate per period, and n = number of compounding periods. If you invest \$5000 at the end of each year for 5 years, you will have an accumulated value of FV as given in the above formula at the end of the nth year. Determine how much is in the account at the end of each year for the next 5 years if $i = 0.06$.

CATCH THE MISTAKE

In Exercises 91–94, explain the mistake that is made.

91. Simplify the ratio of factorials $\dfrac{(3!)(5!)}{6!}$.

Solution:

Express 6! in factored form. $\dfrac{(3!)(5!)}{(3!)(2!)}$

Cancel the 3! in the numerator and denominator. $\dfrac{(5!)}{(2!)}$

Write out the factorials. $\dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1}$

Simplify. $5 \cdot 4 \cdot 3 = 60$

$\dfrac{(3!)(5!)}{(3!)(2!)} \neq 60$. What mistake was made?

92. Simplify the factorial expression $\dfrac{2n(2n-2)!}{(2n+2)!}$.

Solution:

Express factorials in factored form.

$$\frac{2n(2n-2)(2n-4)(2n-6)\cdots}{(2n+2)(2n)(2n-2)(2n-4)(2n-6)\cdots}$$

Cancel common terms. $\dfrac{1}{2n+2}$

This is incorrect. What mistake was made?

93. Find the first four terms of the sequence defined by $a_n = (-1)^{n+1}n^2$.

Solution:

Find the $n = 1$ term. $a_1 = -1$

Find the $n = 2$ term. $a_2 = 4$

Find the $n = 3$ term. $a_3 = -9$

Find the $n = 4$ term. $a_4 = 16$

The sequence $-1, 4, -9, 16, \ldots$ is incorrect. What mistake was made?

94. Evaluate the series $\sum_{k=0}^{3}(-1)^{k+1}k^2$.

Solution:

Write out the sum. $\displaystyle\sum_{k=0}^{3}(-1)^{k+1}k^2 = -1 + 4 - 9$

Simplify the sum. $\displaystyle\sum_{k=0}^{3}(-1)^{k+1}k^2 = -6$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 95–100, determine whether each statement is true or false.

95. $\displaystyle\sum_{k=0}^{n} cx^k = c\sum_{k=0}^{n} x^k$

96. $\displaystyle\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

97. $\displaystyle\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} b_k$

98. $(a!)(b!) = (ab)!$

99. $\displaystyle\sum_{k=1}^{\infty} a_k = \infty$

100. If $m! < n!$, then $m < n$.

CHALLENGE

101. Write the first four terms of the sequence defined by the recursion formula

$$a_1 = C \qquad a_n = a_{n-1} + D \qquad D \neq 0$$

102. Write the first four terms of the sequence defined by the recursion formula

$$a_1 = C \qquad a_n = Da_{n-1} \qquad D \neq 0$$

103. Fibonacci Sequence. An explicit formula for the nth term of the Fibonacci sequence is

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

Apply algebra (not your calculator) to find the first two terms of this sequence and verify that these are indeed the first two terms of the Fibonacci sequence.

104. Let $a_n = \sqrt{a_{n-1}}$ for $n \geq 2$ and $a_1 = 7$. Find the first five terms of this sequence and make a generalization for the nth term.

TECHNOLOGY

105. The sequence defined by $a_n = \left(1 + \frac{1}{n}\right)^n$ approaches the number e as n gets large. Use a graphing calculator to find a_{100}, a_{1000}, $a_{10,000}$, and keep increasing n until the terms in the sequence approach 2.7183.

106. The Fibonacci sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-2} + a_{n-1}$ for $n \geq 3$. The ratio $\frac{a_{n+1}}{a_n}$ is an approximation of the golden ratio. The ratio approaches a constant ϕ (phi) as n gets large. Find the golden ratio using a graphing utility.

107. Use a graphing calculator "SUM" to sum $\sum_{k=0}^{5} \frac{2^k}{k!}$. Compare it with your answer to Exercise 61.

108. Use a graphing calculator "SUM" to sum $\sum_{k=0}^{5} \frac{(-1)^k}{k!}$. Compare it with your answer to Exercise 62.

PREVIEW TO CALCULUS

In calculus, we study the convergence of sequences. A sequence is *convergent* when its terms approach a limiting value. For example, $a_n = \frac{1}{n}$ is convergent because its terms approach zero. If the terms of a sequence satisfy $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots$, the sequence is *monotonic nondecreasing*. If $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$, the sequence is *monotonic nonincreasing*.

In Exercises 109–112, classify each sequence as monotonic or not monotonic. If the sequence is monotonic, determine whether it is nondecreasing or nonincreasing.

109. $a_n = \frac{4n}{n+5}$

110. $a_n = \sin\left(\frac{n\pi}{4}\right)$

111. $a_n = \frac{2 + (-1)^n}{n+4}$

112. $a_n = \frac{3n^2}{5n^2 + 1}$

SECTION 10.2 ARITHMETIC SEQUENCES AND SERIES

SKILLS OBJECTIVES

- Recognize an arithmetic sequence.
- Find the general, or nth, term of an arithmetic sequence.
- Evaluate a finite arithmetic series.
- Use arithmetic sequences and series to model real-world problems.

CONCEPTUAL OBJECTIVE

- Understand the difference between an arithmetic sequence and an arithmetic series.

Arithmetic Sequences

The word *arithmetic* (with emphasis on the third syllable) often implies adding or subtracting of numbers. *Arithmetic sequences* are sequences whose terms are found by adding a constant to each previous term. The sequence 1, 3, 5, 7, 9, . . . is arithmetic because each successive term is found by adding 2 to the previous term.

DEFINITION **Arithmetic Sequences**

A sequence is **arithmetic** if the difference of any two consecutive terms is constant: $a_{n+1} - a_n = d$, where the number d is called the **common difference**. Each term in the sequence is found by adding the same real number d to the previous term, so that $a_{n+1} = a_n + d$.

EXAMPLE 1 Identifying the Common Difference in Arithmetic Sequences

Determine whether each sequence is arithmetic. If so, find the common difference for each of the arithmetic sequences.

a. $5, 9, 13, 17, \ldots$ **b.** $18, 9, 0, -9, \ldots$ **c.** $\frac{1}{2}, \frac{5}{4}, 2, \frac{11}{4}, \ldots$

Solution (a):

Label the terms. $a_1 = 5, a_2 = 9, a_3 = 13, a_4 = 17, \ldots$

Find the difference $d = a_{n+1} - a_n$. $d = a_2 - a_1 = 9 - 5 = \boxed{4}$

Check that the difference of the next two successive pairs of terms is also 4.

$d = a_3 - a_2 = 13 - 9 = 4$

$d = a_4 - a_3 = 17 - 13 = 4$

There is a common difference of $\boxed{4}$. Therefore, this sequence is arithmetic and each successive term is found by adding 4 to the previous term.

Study Tip

We check several terms to confirm that a series is arithmetic.

Solution (b):

Label the terms. $a_1 = 18, a_2 = 9, a_3 = 0, a_4 = -9, \ldots$

Find the difference $d = a_{n+1} - a_n$. $d = a_2 - a_1 = 9 - 18 = \boxed{-9}$

Check that the difference of the next two successive pairs of terms is also -9.

$d = a_3 - a_2 = 0 - 9 = -9$

$d = a_4 - a_3 = -9 - 0 = -9$

There is a common difference of $\boxed{-9}$. Therefore, this sequence is arithmetic and each successive term is found by subtracting 9 from (i.e., adding -9 to) the previous term.

Solution (c):

Label the terms. $a_1 = \frac{1}{2}, a_2 = \frac{5}{4}, a_3 = 2, a_4 = \frac{11}{4}, \ldots$

Find the difference $d = a_{n+1} - a_n$. $d = a_2 - a_1 = \frac{5}{4} - \frac{1}{2} = \boxed{\frac{3}{4}}$

Check that the difference of the next two successive pairs of terms is also $\frac{3}{4}$.

$d = a_3 - a_2 = 2 - \frac{5}{4} = \frac{3}{4}$

$d = a_4 - a_3 = \frac{11}{4} - 2 = \frac{3}{4}$

There is a common difference of $\boxed{\frac{3}{4}}$. Therefore, this sequence is arithmetic and each successive term is found by adding $\frac{3}{4}$ to the previous term.

■ **Answer:** **a.** -5 **b.** $\frac{2}{3}$

■ **YOUR TURN** Find the common difference for each of the arithmetic sequences.

a. $7, 2, -3, -8, \ldots$ **b.** $1, \frac{5}{3}, \frac{7}{3}, 3, \ldots$

The General (nth) Term of an Arithmetic Sequence

To find a formula for the general, or nth, term of an arithmetic sequence, write out the first several terms and look for a pattern.

First term, $n = 1$.	a_1
Second term, $n = 2$.	$a_2 = a_1 + d$
Third term, $n = 3$.	$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$
Fourth term, $n = 4$.	$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$

In general, the nth term is given by $a_n = a_1 + (n - 1)d$.

THE nTH TERM OF AN ARITHMETIC SEQUENCE

The ***n*th term** of an arithmetic sequence with common difference d is given by

$$a_n = a_1 + (n - 1)d \qquad \text{for } n \geq 1$$

EXAMPLE 2 Finding the nth Term of an Arithmetic Sequence

Find the 13th term of the arithmetic sequence 2, 5, 8, 11, ...

Solution:

Identify the common difference.	$d = 5 - 2 = 3$
Identify the first ($n = 1$) term.	$a_1 = 2$
Substitute $a_1 = 2$ and $d = 3$ into $a_n = a_1 + (n - 1)d$.	$a_n = 2 + 3(n - 1)$
Substitute $n = 13$ into $a_n = 2 + 3(n - 1)$.	$a_{13} = 2 + 3(13 - 1) = \boxed{38}$

■ **YOUR TURN** Find the 10th term of the arithmetic sequence 3, 10, 17, 24, ...

■ **Answer:** 66

Technology Tip

2nd | LIST | ▶ | OPS | ▼ | 5:seq(| ENTER | 2 | + | 3 | (| ALPHA | N | − | 1 |) | , | ALPHA | N | , | 13 | , | 1 |) | ENTER |.

```
seq(2+3(N-1),N,1
3,13,1)
                {38}
```

EXAMPLE 3 Finding the Arithmetic Sequence

The 4th term of an arithmetic sequence is 16, and the 21st term is 67. Find a_1 and d and construct the sequence.

Solution:

Write the 4th and 21st terms.	$a_4 = 16$ and $a_{21} = 67$
Adding d 17 times to a_4 results in a_{21}.	$a_{21} = a_4 + 17d$
Substitute $a_4 = 16$ and $a_{21} = 67$.	$67 = 16 + 17d$
Solve for d.	$\boxed{d = 3}$
Substitute $d = 3$ into $a_n = a_1 + (n - 1)d$.	$a_n = a_1 + 3(n - 1)$
Let $a_4 = 16$.	$16 = a_1 + 3(4 - 1)$
Solve for a_1.	$\boxed{a_1 = 7}$

The arithmetic sequence that starts at 7 and has a common difference of 3 is $\boxed{7, 10, 13, 16, \ldots}$.

■ **YOUR TURN** Construct the arithmetic sequence whose 7th term is 26 and whose 13th term is 50.

■ **Answer:** 2, 6, 10, 14, . . .

The Sum of an Arithmetic Sequence

What is the sum of the first 100 counting numbers

$$1 + 2 + 3 + 4 + \cdots + 99 + 100 = ?$$

If we write this sum twice (one in ascending order and one in descending order) and add, we get 100 pairs of 101.

$$\begin{array}{rcrcrcrcrcrcr} 1 & + & 2 & + & 3 & + & 4 & + \cdots + & 99 & + & 100 & & \\ 100 & + & 99 & + & 98 & + & 97 & + \cdots + & 2 & + & 1 & & \\ \hline 101 & + & 101 & + & 101 & + & 101 & + \cdots + & 101 & + & 101 & = & 100\,(101) \end{array}$$

Since we added twice the sum, we divide by 2.

$$1 + 2 + 3 + 4 + \cdots + 99 + 100 = \frac{(101)(100)}{2} = 5050$$

Now, let us develop the sum of a general arithmetic series.

The sum of the first n terms of an arithmetic sequence is called the ***n*th partial sum**, or **finite arithmetic series**, and is denoted by S_n. An arithmetic sequence can be found by starting at the first term and adding the common difference to each successive term, and so the nth partial sum, or finite series, can be found the same way, but terminating the sum at the nth term:

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots$$
$$\boldsymbol{S_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + (a_n)}$$

Similarly, we can start with the nth term and find terms going backward by subtracting the common difference until we arrive at the first term:

$$S_n = a_n + a_{n-1} + a_{n-2} + a_{n-3} + \cdots$$
$$\boldsymbol{S_n = a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \cdots + (a_1)}$$

Add these two representations of the nth partial sum. Notice that the d terms are eliminated:

$$\begin{array}{rcl} \boldsymbol{S_n} & \boldsymbol{=} & \boldsymbol{a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + (a_n)} \\ \boldsymbol{S_n} & \boldsymbol{=} & \boldsymbol{a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \cdots + (a_1)} \\ \hline 2S_n & = & \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)}_{n(a_1 + a_n)} \end{array}$$

$$2S_n = n(a_1 + a_n) \qquad \text{or} \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

DEFINITION **Evaluating a Finite Arithmetic Series**

The sum of the first n terms of an arithmetic sequence (*nth partial sum*), called a **finite arithmetic series**, is given by the formula

$$S_n = \frac{n}{2}(a_1 + a_n) \quad n \geq 2$$

Study Tip

S_n can also be written as

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d]$$

For an arithmetic sequence, let $a_n = a_1 + (n - 1)d$.

$$\begin{aligned} S_n &= \frac{n}{2}[a_1 + a_1 + (n - 1)d] \\ &= \frac{n}{2}[2a_1 + (n - 1)d] = na_1 + \frac{n(n - 1)d}{2} \end{aligned}$$

EXAMPLE 4 Evaluating a Finite Arithmetic Series

Evaluate the finite arithmetic series $\sum_{k=1}^{100} k$.

Solution:

Expand the arithmetic series. $\sum_{k=1}^{100} k = 1 + 2 + 3 + \cdots + 99 + 100$

This is the sum of an arithmetic sequence of numbers with a common difference of 1.

Identify the parameters of the arithmetic sequence. $a_1 = 1$, $a_n = 100$, and $n = 100$

Substitute these values into $S_n = \frac{n}{2}(a_1 + a_n)$. $S_{100} = \frac{100}{2}(1 + 100)$

Simplify. $\boxed{S_{100} = 5050}$

The sum of the first 100 natural numbers is 5050.

YOUR TURN Evaluate the following finite arithmetic series:

a. $\sum_{k=1}^{30} k$ **b.** $\sum_{k=1}^{20} (2k + 1)$

Technology Tip

To find the sum of the series $\sum_{k=1}^{100} k$, press

2nd [LIST] ▶ [MATH] ▼ 5:*sum*([ENTER] 2nd [LIST] ▶ [OPS] ▼ 5:*seq*([ENTER] [ALPHA] [K] , [ALPHA] [K] , 1 , 100 , 1)) [ENTER].

```
sum(seq(K,K,1,10
0,1))
                5050
```

Answer: a. 465 **b.** 440

EXAMPLE 5 Finding the *n*th Partial Sum of an Arithmetic Sequence

Find the sum of the first 20 terms of the arithmetic sequence 3, 8, 13, 18, 23, ...

Solution:

Recall the partial sum formula. $S_n = \frac{n}{2}(a_1 + a_n)$

Find the 20th partial sum of this arithmetic sequence. $S_{20} = \frac{20}{2}(a_1 + a_{20})$

Recall that the general *n*th term of an arithmetic sequence is given by $a_n = a_1 + (n - 1)d$

Note that the first term of the arithmetic sequence is 3. $a_1 = 3$

This is an arithmetic sequence with a common difference of 5. $d = 5$

Substitute $a_1 = 3$ and $d = 5$ into $a_n = a_1 + (n - 1)d$. $a_n = 3 + (n - 1)5$

Substitute $n = 20$ to find the 20th term. $a_{20} = 3 + (20 - 1)5 = 98$

Substitute $a_1 = 3$ and $a_{20} = 98$ into the partial sum. $\boxed{S_{20} = 10(3 + 98) = 1010}$

The sum of the first 20 terms of this arithmetic sequence is 1010.

YOUR TURN Find the sum of the first 25 terms of the arithmetic sequence 2, 6, 10, 14, 18, ...

Technology Tip

To find the sum of the series $\sum_{n=1}^{20} 3 + (n - 1)5$, press

2nd [LIST] ▶ [MATH] ▼ 5:*sum*([ENTER] 2nd [LIST] ▶ [OPS] ▼ 5:*seq*([ENTER] 3 + ([ALPHA] [N] − 1) 5 , [ALPHA] [N] , 1 , 20 , 1)) [ENTER].

```
sum(seq(3+(N-1)5
,N,1,20,1))
                1010
```

Answer: 1250

Applications

EXAMPLE 6 Marching Band Formation

Suppose a band has 18 members in the first row, 22 members in the second row, and 26 members in the third row and continues with that pattern for a total of nine rows. How many marchers are there all together?

Brian Bahr/Staff/Getty Images

The Ohio State University marching band

Solution:

The number of members in each row forms an arithmetic sequence with a common difference of 4, and the first row has 18 members.

$$a_1 = 18 \quad d = 4$$

Calculate the nth term of the sequence $a_n = a_1 + (n - 1)d$.

$$a_n = 18 + (n - 1)4$$

Find the 9th term, $n = 9$.

$$a_9 = 18 + (9 - 1)4 = 50$$

Calculate the sum $S_n = \frac{n}{2}(a_1 + a_n)$ of the nine rows.

$$S_9 = \frac{9}{2}(a_1 + a_9)$$

$$= \frac{9}{2}(18 + 50)$$

$$= \frac{9}{2}(68)$$

$$= \boxed{306}$$

There are 306 members in the marching band.

■ **Answer:** 328

■ **YOUR TURN** Suppose a bed of tulips is arranged in a garden so that there are 20 tulips in the first row, 26 tulips in the second row, and 32 tulips in the third row and the rows continue with that pattern for a total of 8 rows. How many tulips are there all together?

SECTION 10.2 SUMMARY

In this section, arithmetic sequences were defined as sequences of which each successive term is found by adding the same constant d to the previous term. Formulas were developed for the general, or nth, term of an arithmetic sequence, and for the nth partial sum of an arithmetic sequence, also called a finite arithmetic series.

$$a_n = a_1 + (n - 1)d \qquad n \geq 1$$

$$S_n = \frac{n}{2}(a_1 + a_n) = na_1 + \frac{n(n-1)}{2}d$$

SECTION 10.2 EXERCISES

SKILLS

In Exercises 1–10, determine whether each sequence is arithmetic. If it is, find the common difference.

1. $2, 5, 8, 11, 14, \ldots$

2. $9, 6, 3, 0, -3, -6, \ldots$

3. $1^2 + 2^2 + 3^2 + \cdots$

4. $1! + 2! + 3! + \cdots$

5. $3.33, 3.30, 3.27, 3.24, \ldots$

6. $0.7, 1.2, 1.7, 2.2, \ldots$

7. $4, \frac{14}{3}, \frac{16}{3}, 6, \ldots$

8. $2, \frac{7}{3}, \frac{8}{3}, 3, \ldots$

9. $10^1, 10^2, 10^3, 10^4, \ldots$

10. $120, 60, 30, 15, \ldots$

In Exercises 11–20, find the first four terms of each sequence described. Determine whether the sequence is arithmetic, and if so, find the common difference.

11. $a_n = -2n + 5$

12. $a_n = 3n - 10$

13. $a_n = n^2$

14. $a_n = \dfrac{n^2}{n!}$

15. $a_n = 5n - 3$

16. $a_n = -4n + 5$

17. $a_n = 10(n - 1)$

18. $a_n = 8n - 4$

19. $a_n = (-1)^n n$

20. $a_n = (-1)^{n+1} 2n$

In Exercises 21–28, find the general, or nth, term of each arithmetic sequence given the first term and the common difference.

21. $a_1 = 11 \quad d = 5$

22. $a_1 = 5 \quad d = 11$

23. $a_1 = -4 \quad d = 2$

24. $a_1 = 2 \quad d = -4$

25. $a_1 = 0 \quad d = \frac{2}{3}$

26. $a_1 = -1 \quad d = -\frac{3}{4}$

27. $a_1 = 0 \quad d = e$

28. $a_1 = 1.1 \quad d = -0.3$

In Exercises 29–32, find the specified term for each arithmetic sequence given.

29. The 10th term of the sequence $7, 20, 33, 46, \ldots$

30. The 19th term of the sequence $7, 1, -5, -11, \ldots$

31. The 100th term of the sequence $9, 2, -5, -12, \ldots$

32. The 90th term of the sequence $13, 19, 25, 31, \ldots$

33. The 21st term of the sequence $\frac{1}{3}, \frac{7}{12}, \frac{5}{6}, \frac{13}{12}, \ldots$

34. The 33rd term of the sequence $\frac{1}{5}, \frac{8}{15}, \frac{13}{15}, \frac{6}{5}, \ldots$

In Exercises 35–40, for each arithmetic sequence described, find a_1 and d and construct the sequence by stating the general, or nth, term.

35. The 5th term is 44 and the 17th term is 152.

36. The 9th term is -19 and the 21st term is -55.

37. The 7th term is -1 and the 17th term is -41.

38. The 8th term is 47 and the 21st term is 112.

39. The 4th term is 3 and the 22nd term is 15.

40. The 11th term is -3 and the 31st term is -13.

In Exercises 41–52, find each sum given.

41. $\sum_{k=1}^{23} 2k$ **42.** $\sum_{k=0}^{20} 5k$ **43.** $\sum_{n=1}^{30} (-2n + 5)$ **44.** $\sum_{n=0}^{17} (3n - 10)$ **45.** $\sum_{j=3}^{14} 0.5j$ **46.** $\sum_{j=1}^{33} \frac{j}{4}$

47. $2 + 7 + 12 + 17 + \cdots + 62$ **48.** $1 - 3 - 7 - \cdots - 75$ **49.** $4 + 7 + 10 + \cdots + 151$

50. $2 + 0 - 2 - \cdots - 56$ **51.** $\frac{1}{6} - \frac{1}{6} - \frac{1}{2} - \cdots - \frac{13}{2}$ **52.** $\frac{11}{12} + \frac{7}{6} + \frac{17}{12} + \cdots + \frac{14}{3}$

In Exercises 53–58, find the indicated partial sum of each arithmetic series.

53. The first 18 terms of $1 + 5 + 9 + 13 + \cdots$

54. The first 21 terms of $2 + 5 + 8 + 11 + \cdots$

55. The first 43 terms of $1 + \frac{1}{2} + 0 - \frac{1}{2} - \cdots$

56. The first 37 terms of $3 + \frac{3}{2} + 0 - \frac{3}{2} - \cdots$

57. The first 18 terms of $-9 + 1 + 11 + 21 + 31 + \cdots$

58. The first 21 terms of $-2 + 8 + 18 + 28 + \cdots$

APPLICATIONS

59. Comparing Salaries. Colin and Camden are twin brothers graduating with B.S. degrees in biology. Colin takes a job at the San Diego Zoo making $28,000 for his first year with a $1500 raise per year every year after that. Camden accepts a job at Florida Fish and Wildlife making $25,000 with a guaranteed $2000 raise per year. How much will each of the brothers have made in a total of 10 years?

60. Comparing Salaries. On graduating with a Ph.D. in optical sciences, Jasmine and Megan choose different career paths. Jasmine accepts a faculty position at the University of Arizona making $80,000 with a guaranteed $2000 raise every year. Megan takes a job with the Boeing Corporation making $90,000 with a guaranteed $5000 raise each year. Calculate how much each woman will have made after 15 years.

61. Theater Seating. You walk into the premiere of Brad Pitt's new movie, and the theater is packed, with almost every seat filled. You want to estimate the number of people in the theater. You quickly count to find that there are 22 seats in the front row, and there are 25 rows in the theater. Each row appears to have one more seat than the row in front of it. How many seats are in that theater?

62. Field of Tulips. Every spring the Skagit County Tulip Festival plants more than 100,000 bulbs. In honor of the Tri-Delta sorority that has sent 120 sisters from the University of Washington to volunteer for the festival, Skagit County has planted tulips in the shape of ΔΔΔ. In each of the triangles there are 20 rows of tulips, each row having one less than the row before. How many tulips are planted in each delta if there is one tulip in the first row?

63. World's Largest Champagne Fountain. From December 28 to 30, 1999, Luuk Broos, director of Maison Luuk-Chalet Fontain, constructed a 56-story champagne fountain at the Steigenberger Kurhaus Hotel, Scheveningen, Netherlands. The fountain consisted of 30,856 champagne glasses. Assuming there was one glass at the top and the number of glasses in each row forms an arithmetic sequence, how many were on the bottom row (story)? How many glasses less did each successive row (story) have? Assume each story is one row.

64. Stacking of Logs. If 25 logs are laid side by side on the ground, and 24 logs are placed on top of those, and 23 logs are placed on the 3rd row, and the pattern continues until there is a single log on the 25th row, how many logs are in the stack?

65. Falling Object. When a skydiver jumps out of an airplane, she falls approximately 16 feet in the 1st second, 48 feet during the 2nd second, 80 feet during the 3rd second, 112 feet during the 4th second, and 144 feet during the 5th second, and this pattern continues. If she deploys her parachute after 10 seconds have elapsed, how far will she have fallen during those 10 seconds?

66. Falling Object. If a penny is dropped out of a plane, it falls approximately 4.9 meters during the 1st second, 14.7 meters during the 2nd second, 24.5 meters during the 3rd second, and 34.3 meters during the 4th second. Assuming this pattern continues, how many meters will the penny have fallen after 10 seconds?

67. Grocery Store. A grocer has a triangular display of oranges in a window. There are 20 oranges in the bottom row and the number of oranges decreases by one in each row above this row. How many oranges are in the display?

68. Salary. Suppose your salary is $45,000 and you receive a $1500 raise for each year you work for 35 years.
a. How much will you earn during the 35th year?
b. What is the total amount you earned over your 35-year career?

69. Theater Seating. At a theater, seats are arranged in a triangular pattern of rows with each succeeding row having one more seat than the previous row. You count the number of seats in the fourth row and determine that there are 26 seats.
a. How many seats are in the first row?
b. Now, suppose there are 30 rows of seats. How many total seats are there in the theater?

70. Mathematics. Find the exact sum of

$$\frac{1}{e} + \frac{3}{e} + \frac{5}{e} + \cdots + \frac{23}{e}$$

CATCH THE MISTAKE

In Exercises 71–74, explain the mistake that is made.

71. Find the general, or nth, term of the arithmetic sequence 3, 4, 5, 6, 7, ...

Solution:

The common difference of this sequence is 1. $d = 1$

The first term is 3. $a_1 = 3$

The general term is $a_n = a_1 + nd$. $a_n = 3 + n$

This is incorrect. What mistake was made?

72. Find the general, or nth, term of the arithmetic sequence 10, 8, 6, ...

Solution:

The common difference of this sequence is 2. $d = 2$

The first term is 10. $a_1 = 10$

The general term is $a_n = a_1 + (n - 1)d$. $a_n = 10 + 2(n - 1)$

This is incorrect. What mistake was made?

73. Find the sum $\sum_{k=0}^{10}(2n + 1)$.

Solution:

The sum is given by $S_n = \frac{n}{2}(a_1 + a_n)$, where $n = 10$.

Identify the 1st and 10th terms. $a_1 = 1$, $a_{10} = 21$

Substitute $a_1 = 1$, $a_{10} = 21$, and $n = 10$ into $S_n = \frac{n}{2}(a_1 + a_n)$. $S_{10} = \frac{10}{2}(1 + 21) = 110$

This is incorrect. What mistake was made?

74. Find the sum $3 + 9 + 15 + 21 + 27 + 33 + \cdots + 87$.

Solution:

This is an arithmetic sequence with common difference of 6. $d = 6$

The general term is given by $a_n = a_1 + (n - 1)d$. $a_n = 3 + (n - 1)6$

87 is the 15th term of the series. $a_{15} = 3 + (15 - 1)6 = 87$

The sum of the series is $S_n = \frac{n}{2}(a_n - a_1)$. $S_{15} = \frac{15}{2}(87 - 3) = 630$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 75–78, determine whether each statement is true or false.

75. An arithmetic sequence and a finite arithmetic series are the same.

76. The sum of all infinite and finite arithmetic series can always be found.

77. An alternating sequence cannot be an arithmetic sequence.

78. The common difference of an arithmetic sequence is always positive.

CHALLENGE

79. Find the sum $a + (a + b) + (a + 2b) + \cdots + (a + nb)$.

80. Find the sum $\sum_{k=-29}^{30} \ln e^k$.

81. The wave number λ (reciprocal of wave length) of certain light waves in the spectrum of light emitted by hydrogen is given by $\lambda = R\left(\frac{1}{k^2} - \frac{1}{n^2}\right)$, $n > k$, where $R = 109{,}678$. A series of lines is given by holding k constant and varying the value of n. Suppose $k = 2$ and $n = 3, 4, 5, \ldots$. Find what value the wave number of the series approaches as n increases.

82. In a certain arithmetic sequence $a_1 = -4$ and $d = 6$. If $S_n = 570$, find the value of n.

TECHNOLOGY

83. Use a graphing calculator "SUM" to sum the natural numbers from 1 to 100.

84. Use a graphing calculator to sum the even natural numbers from 1 to 100.

85. Use a graphing calculator to sum the odd natural numbers from 1 to 100.

86. Use a graphing calculator to find $\sum_{n=1}^{30}(-2n + 5)$. Compare it with your answer to Exercise 43.

87. Use a graphing calculator to find $\sum_{n=1}^{100}[-59 + 5(n - 1)]$.

88. Use a graphing calculator to find $\sum_{n=1}^{200}\left[-18 + \frac{4}{5}(n - 1)\right]$.

PREVIEW TO CALCULUS

In calculus, when estimating certain integrals, we use sums of the form $\sum_{i=1}^{n} f(x_i)\Delta x$, where f is a function and Δx is a constant.

In Exercises 89–92, find the indicated sum.

89. $\sum_{i=1}^{100} f(x_i)\Delta x$, where $f(x_i) = 2i$ and $\Delta x = 0.1$

90. $\sum_{i=1}^{50} f(x_i)\Delta x$, where $f(x_i) = 4i - 2$ and $\Delta x = 0.01$

91. $\sum_{i=1}^{43} f(x_i)\Delta x$, where $f(x_i) = 6 + i$ and $\Delta x = 0.001$

92. $\sum_{i=1}^{85} f(x_i)\Delta x$, where $f(x_i) = 6 - 7i$ and $\Delta x = 0.2$

SECTION 10.3 GEOMETRIC SEQUENCES AND SERIES

SKILLS OBJECTIVES

- Recognize a geometric sequence.
- Find the general, or nth, term of a geometric sequence.
- Evaluate a finite geometric series.
- Evaluate an infinite geometric series, if it exists.
- Use geometric sequences and series to model real-world problems.

CONCEPTUAL OBJECTIVES

- Understand the difference between a geometric sequence and a geometric series.
- Distinguish between an arithmetic sequence and a geometric sequence.
- Understand why it is not possible to evaluate all infinite geometric series.

Geometric Sequences

In Section 10.2, we discussed *arithmetic* sequences, where successive terms had a *common difference*. In other words, each term was found by adding the same constant to the previous term. In this section, we discuss *geometric* sequences, where successive terms have a *common ratio*. In other words, each term is found by multiplying the previous term by the same constant. The sequence 4, 12, 36, 108, . . . is geometric because each successive term is found by multiplying the previous term by 3.

DEFINITION **Geometric Sequences**

A sequence is **geometric** if each term in the sequence is found by multiplying the previous term by a number r, so that $a_{n+1} = r \cdot a_n$. Because $\frac{a_{n+1}}{a_n} = r$, the number r is called the **common ratio**.

EXAMPLE 1 Identifying the Common Ratio in Geometric Sequences

Find the common ratio for each of the geometric sequences.

a. 5, 20, 80, 320, . . . **b.** $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$ **c.** \$5000, \$5500, \$6050, \$6655, . . .

Solution (a):

Label the terms. $a_1 = 5, a_2 = 20, a_3 = 80, a_4 = 320, \ldots$

Find the ratio $r = \dfrac{a_{n+1}}{a_n}$.

$$r = \frac{a_2}{a_1} = \frac{20}{5} = 4$$

$$r = \frac{a_3}{a_2} = \frac{80}{20} = 4$$

$$r = \frac{a_4}{a_3} = \frac{320}{80} = 4$$

$\boxed{\text{The common ratio is 4.}}$

Solution (b):

Label the terms. $a_1 = 1, a_2 = -\dfrac{1}{2}, a_3 = \dfrac{1}{4}, a_4 = -\dfrac{1}{8}, \ldots$

Find the ratio $r = \dfrac{a_{n+1}}{a_n}$.

$$r = \frac{a_2}{a_1} = \frac{-1/2}{1} = -\frac{1}{2}$$

$$r = \frac{a_3}{a_2} = \frac{1/4}{-1/2} = -\frac{1}{2}$$

$$r = \frac{a_4}{a_3} = \frac{-1/8}{1/4} = -\frac{1}{2}$$

$\boxed{\text{The common ratio is } -\frac{1}{2}.}$

Solution (c):

Label the terms. $a_1 = \$5000, a_2 = \$5500, a_3 = \$6050, a_4 = \$6655, \ldots$

Find the ratio $r = \dfrac{a_{n+1}}{a_n}$.

$$r = \frac{a_2}{a_1} = \frac{\$5500}{\$5000} = 1.1$$

$$r = \frac{a_3}{a_2} = \frac{\$6050}{\$5500} = 1.1$$

$$r = \frac{a_4}{a_3} = \frac{\$6655}{\$6050} = 1.1$$

$\boxed{\text{The common ratio is 1.1.}}$

YOUR TURN Find the common ratio of each geometric series.

a. $1, -3, 9, -27, \ldots$ **b.** 320, 80, 20, 5, . . .

Answer: a. -3 **b.** $\frac{1}{4}$ or 0.25

The General (*n*th) Term of a Geometric Sequence

To find a formula for the general, or *n*th, term of a geometric sequence, write out the first several terms and look for a pattern.

WORDS	MATH
First term, $n = 1$.	a_1
Second term, $n = 2$.	$a_2 = a_1 \cdot r$
Third term, $n = 3$.	$a_3 = a_2 \cdot r = (a_1 \cdot r) \cdot r = a_1 \cdot r^2$
Fourth term, $n = 4$.	$a_4 = a_3 \cdot r = (a_1 \cdot r^2) \cdot r = a_1 \cdot r^3$

In general, the nth term is given by $a_n = a_1 \cdot r^{n-1}$.

THE nTH TERM OF A GEOMETRIC SEQUENCE

The ***n*th term** of a geometric sequence with common ratio r is given by

$$a_n = a_1 \cdot r^{n-1} \qquad \text{for } n \geq 1$$

Technology Tip

Use [seq] to find the nth term of the sequence by setting the initial index value equal to the final index value. To find the 7th term of the geometric sequence $a_n = 2 \cdot 5^{n-1}$, press

[2nd] [LIST] [▶] [OPS] [▼] [5:seq(] [ENTER] 2 [×] 5 [^] [(] [ALPHA] [N] [−] 1 [)] [,] [ALPHA] [N] [,] 7 [,] 7 [,] 1 [)] [ENTER].

```
seq(2*5^(N-1),N,
7,7,1)
              {31250}
```

■ **Answer:** 49,152

EXAMPLE 2 Finding the *n*th Term of a Geometric Sequence

Find the 7th term of the sequence 2, 10, 50, 250, ...

Solution:

Identify the common ratio. $r = \frac{10}{2} = \frac{50}{10} = \frac{250}{50} = 5$

Identify the first ($n = 1$) term. $a_1 = 2$

Substitute $a_1 = 2$ and $r = 5$ into $a_n = a_1 \cdot r^{n-1}$. $a_n = 2 \cdot 5^{n-1}$

Substitute $n = 7$ into $a_n = 2 \cdot 5^{n-1}$. $a_7 = 2 \cdot 5^{7-1} = 2 \cdot 5^6 = 31{,}250$

The 7th term of the geometric sequence is 31,250.

■ **YOUR TURN** Find the 8th term of the sequence 3, 12, 48, 192, ...

EXAMPLE 3 Finding the Geometric Sequence

Find the geometric sequence whose 5th term is 0.01 and whose common ratio is 0.1.

Solution:

Label the common ratio and 5th term. $a_5 = 0.01$ and $r = 0.1$

Substitute $a_5 = 0.01$, $n = 5$, and $r = 0.1$ into $a_n = a_1 \cdot r^{n-1}$. $0.01 = a_1 \cdot (0.1)^{5-1}$

Solve for a_1. $a_1 = \frac{0.01}{(0.1)^4} = \frac{0.01}{0.0001} = 100$

The geometric sequence that starts at 100 and has a common ratio of 0.1 is 100, 10, 1, 0.1, 0.01, ...

■ **Answer:** 81, 27, 9, 3, 1, . . .

■ **YOUR TURN** Find the geometric sequence whose 4th term is 3 and whose common ratio is $\frac{1}{3}$.

Geometric Series

The sum of the terms of a geometric sequence is called a **geometric series**.

$$a_1 + a_1 \cdot r + a_1 \cdot r^2 + a_1 \cdot r^3 + \cdots$$

If we only sum the first n terms of a geometric sequence, the result is a **finite geometric series** given by

$$S_n = a_1 + a_1 \cdot r + a_1 \cdot r^2 + a_1 \cdot r^3 + \cdots + a_1 \cdot r^{n-1}$$

To develop a formula for this nth partial sum, we multiply the above equation by r:

$$r \cdot S_n = a_1 \cdot r + a_1 \cdot r^2 + a_1 \cdot r^3 + \cdots + a_1 \cdot r^{n-1} + a_1 \cdot r^n$$

Subtracting the **second** equation from the **first** equation, we find that all of the terms on the right side drop out except the *first* term in the **first** equation and the *last* term in the **second** equation:

$$\begin{array}{rl} S_n = & a_1 + a_1 \cdot r + a_1 \cdot r^2 + \cdots + a_1 r^{n-1} \\ -rS_n = & \quad - a_1 \cdot r - a_1 \cdot r^2 - \cdots - a_1 r^{n-1} - a_1 r^n \\ \hline S_n - rS_n = & a_1 \qquad\qquad\qquad\qquad\qquad\qquad -a_1 r^n \end{array}$$

Factor the S_n out of the left side and the a_1 out of the right side:

$$S_n(1 - r) = a_1(1 - r^n)$$

Divide both sides by $(1 - r)$, assuming $r \neq 1$. The result is a general formula for the sum of a finite geometric series:

$$S_n = a_1 \frac{(1 - r^n)}{(1 - r)} \qquad r \neq 1$$

Evaluating a finite geometric series

The sum of the first n terms of a geometric sequence, called a **finite geometric series**, is given by the formula

$$S_n = a_1 \frac{(1 - r^n)}{(1 - r)} \qquad r \neq 1$$

It is important to note that a finite geometric series can also be written in sigma (summation) notation:

$$S_n = \sum_{k=1}^{n} a_1 \cdot r^{k-1} = a_1 + a_1 \cdot r + a_1 \cdot r^2 + a_1 \cdot r^3 + \cdots + a_1 \cdot r^{n-1}$$

Study Tip

The underscript $k = 1$ applies only when the summation starts at the a_1 term. It is important to note which term is the starting term.

EXAMPLE 4 Evaluating a Finite Geometric Series

Evaluate each finite geometric series.

a. $\sum_{k=1}^{13} 3 \cdot (0.4)^{k-1}$

b. The first nine terms of the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$

Solution (a):

Identify a_1, n, and r. $\quad a_1 = 3$, $n = 13$, and $r = 0.4$

Substitute $a_1 = 3$, $n = 13$, and $r = 0.4$ into $S_n = a_1 \dfrac{(1 - r^n)}{(1 - r)}$. $\quad S_{13} = 3\dfrac{(1 - 0.4^{13})}{(1 - 0.4)}$

Simplify. $\quad \boxed{S_{13} \approx 4.99997}$

Solution (b):

Identify the first term and common ratio. $\quad a_1 = 1$ and $r = 2$

Substitute $a_1 = 1$ and $r = 2$ into $S_n = a_1 \dfrac{(1 - r^n)}{(1 - r)}$. $\quad S_n = \dfrac{(1 - 2^n)}{(1 - 2)}$

To sum the first nine terms, let $n = 9$. $\quad S_9 = \dfrac{(1 - 2^9)}{(1 - 2)}$

Simplify. $\quad \boxed{S_9 = 511}$

Technology Tip

a. To find the sum of the series $\sum_{k=1}^{13} 3 \cdot (0.4)^{k-1}$, press

[2nd] [*LIST*] [▶] [MATH] [▼] [5:*sum*(] [ENTER] [2nd] [*LIST*] [▶] [*OPS*] [▼] [5:*seq*(] [ENTER] 3 [×] [(] 0.4 [)] [^] [(] [ALPHA] [*K*] [−] 1 [)] [,] [ALPHA] [*K*] [,] 1 [,] 13 [,] 1 [)] [)] [ENTER].

```
sum(seq(3*(0.4)^
(K-1),K,1,13,1))
        4.999966446
```

b. To find the sum of the first nine terms of the series $1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$, press

[2nd] [*LIST*] [▶] [MATH] [▼] [5:*sum*(] [ENTER] [2nd] [*LIST*] [▶] [*OPS*] [▼] [5:*seq*(] [ENTER] 2 [^] [(] [ALPHA] [*K*] [−] 1 [)] [,] [ALPHA] [*K*] [,] 1 [,] 9 [,] 1 [)] [)] [ENTER].

```
sum(seq(2^(K-1),
K,1,9,1))
              511
```

The sum of an infinite geometric sequence is called an **infinite geometric series**. Some infinite geometric series *converge* (yield a finite sum) and some *diverge* (do not have a finite sum). For example,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots = 1 \text{ (converges)}$$

$$2 + 4 + 8 + 16 + 32 + \cdots + 2^n + \cdots \text{ (diverges)}$$

For infinite geometric series that converge, the partial sum S_n approaches a single number as n gets large. The formula used to evaluate a finite geometric series

$$S_n = a_1 \frac{(1 - r^n)}{(1 - r)}$$

can be extended to an infinite geometric series for certain values of r. If $|r| < 1$, then when r is raised to a power, it continues to get smaller, approaching 0. For those values of r, the infinite geometric series converges to a finite sum.

$$\text{Let } n \to \infty\text{; then } a_1 \frac{(1 - r^n)}{(1 - r)} \to a_1 \frac{(1 - 0)}{(1 - r)} = \frac{a_1}{1 - r}\text{, if } |r| < 1.$$

EVALUATING AN INFINITE GEOMETRIC SERIES

The **sum of an infinite geometric series** is given by the formula

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = \sum_{n=0}^{\infty} a_1 \cdot r^n = a_1 \frac{1}{(1-r)} \qquad |r| < 1$$

Study Tip

The formula used to evaluate an infinite geometric series is

$$\sum_{n=0}^{\infty} a_1 r^n = \frac{\text{First term}}{1 - \text{Ratio}}$$

EXAMPLE 5 Determining Whether the Sum of an Infinite Series Exists

Determine whether the sum exists for each of the geometric series.

a. $3 + 15 + 75 + 375 + \cdots$ **b.** $8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Solution (a):

Identify the common ratio. $r = 5$

Since 5 is greater than 1, the $\boxed{\text{sum does not exist}}$. $|r| = 5 > 1$

Solution (b):

Identify the common ratio. $r = \frac{1}{2}$

Since $\frac{1}{2}$ is less than 1, the $\boxed{\text{sum exists}}$. $|r| = \frac{1}{2} < 1$

■ **YOUR TURN** Determine whether the sum exists for each of the geometric series.

a. $81 + 9 + 1 + \frac{1}{9} + \cdots$ **b.** $1 + 5 + 25 + 125 + \cdots$

■ **Answer: a.** yes **b.** no

Do you expect $\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{64} + \cdots$ and $\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{64} + \cdots$ to sum to the same number? The answer is no, because the second series is an alternating series and terms are both added and subtracted. Hence, we would expect the second series to sum to a smaller number than the first series sums to.

EXAMPLE 6 Evaluating an Infinite Geometric Series

Evaluate each infinite geometric series.

a. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ **b.** $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

Solution (a):

Identify the first term and the common ratio. $a_1 = 1 \qquad r = \frac{1}{3}$

Since $|r| = \frac{1}{3} < 1$, the sum of the series exists.

Substitute $a_1 = 1$ and $r = \frac{1}{3}$ into $\sum_{n=0}^{\infty} a_1 \cdot r^n = \frac{a_1}{(1-r)}$. $\qquad \frac{1}{1 - \frac{1}{3}}$

Simplify. $\qquad = \frac{1}{\frac{2}{3}} = \frac{3}{2}$

$$\boxed{1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{3}{2}}$$

Solution (b):

Identify the first term and the common ratio. $a_1 = 1 \qquad r = -\dfrac{1}{3}$

Since $|r| = |-\frac{1}{3}| = \frac{1}{3} < 1$, the sum of the series exists.

Substitute $a_1 = 1$ and $r = -\frac{1}{3}$ into $\sum_{n=0}^{\infty} a_1 \cdot r^n = \dfrac{a_1}{(1 - r)}$.

Simplify.
$$= \frac{1}{1 - (-1/3)} = \frac{1}{1 + (1/3)} = \frac{1}{4/3} = \frac{3}{4}$$

$$\boxed{1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots = \frac{3}{4}}$$

Notice that the alternating series summed to $\frac{3}{4}$, whereas the positive series summed to $\frac{3}{2}$.

■ **Answer: a.** $\frac{3}{8}$ **b.** $\frac{3}{16}$

■ **YOUR TURN** Find the sum of each infinite geometric series.

a. $\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots$ **b.** $\frac{1}{4} - \frac{1}{12} + \frac{1}{36} - \frac{1}{108} + \cdots$

It is important to note the restriction on the common ratio r. The absolute value of the common ratio has to be strictly less than 1 for an infinite geometric series to converge. Otherwise, the infinite geometric series diverges.

EXAMPLE 7 Evaluating an Infinite Geometric Series

Evaluate the infinite geometric series, if possible.

a. $\displaystyle\sum_{n=0}^{\infty} 2\left(-\frac{1}{4}\right)^n$ **b.** $\displaystyle\sum_{n=1}^{\infty} 3 \cdot 2^{n-1}$

Solution (a):

Identify a_1 and r.
$$\sum_{n=0}^{\infty} 2\left(-\frac{1}{4}\right)^n = \underbrace{2}_{a_1} - \underbrace{\frac{1}{2} + \frac{1}{8}}_{r = -\frac{1}{4}} - \underbrace{\frac{1}{32} + \frac{1}{128}}_{r = -\frac{1}{4}} - \cdots \qquad r = -\frac{1}{4}$$

Since $|r| = |-\frac{1}{4}| = \frac{1}{4} < 1$, the infinite geometric series converges.
$$\sum_{n=0}^{\infty} a_1 \cdot r^n = \frac{a_1}{(1 - r)}$$

Let $a_1 = 2$ and $r = -\frac{1}{4}$.
$$= \frac{2}{[1 - (-1/4)]}$$

Simplify.
$$= \frac{2}{1 + (1/4)} = \frac{2}{5/4} = \frac{8}{5}$$

This infinite geometric series converges.
$$\boxed{\sum_{n=0}^{\infty} 2\left(-\frac{1}{4}\right)^n = \frac{8}{5}}$$

Solution (b):

Identify a_1 and r.
$$\sum_{n=1}^{\infty} 3 \cdot (2)^{n-1} = \underbrace{3}_{a_1} + \overbrace{6 + 12}^{r = 2} + \underbrace{24 + 48}_{r = 2} + \cdots$$

$$\boxed{\text{Since } r = 2 > 1, \text{ this infinite geometric series diverges.}}$$

Applications

Suppose you are given a job offer with a guaranteed percentage raise per year. What will your annual salary be 10 years from now? That answer can be obtained using a geometric sequence. Suppose you want to make voluntary contributions to a retirement account directly debited from your paycheck every month. Suppose the account earns a fixed percentage rate: How much will you have in 30 years if you deposit \$50 a month? What is the difference in the total you will have in 30 years if you deposit \$100 a month instead? These important questions about your personal finances can be answered using geometric sequences and series.

EXAMPLE 8 Future Salary: Geometric Sequence

Suppose you are offered a job as an event planner for the PGA Tour. The starting salary is \$45,000, and employees are given a 5% raise per year. What will your annual salary be during the 10th year with the PGA Tour?

Solution:

Every year the salary is 5% more than the previous year.

Label the year 1 salary.	$a_1 = 45{,}000$
Calculate the year 2 salary.	$a_2 = 1.05 \cdot a_1$
Calculate the year 3 salary.	$a_3 = 1.05 \cdot a_2$
	$= 1.05(1.05 \cdot a_1) = (1.05)^2 a_1$
Calculate the year 4 salary.	$a_4 = 1.05 \cdot a_3$
	$= 1.05(1.05)^2 a_1 = (1.05)^3 a_1$
Identify the year n salary.	$a_n = 1.05^{n-1} a_1$
Substitute $n = 10$ and $a_1 = 45{,}000$.	$a_{10} = (1.05)^9 \cdot 45{,}000$
Simplify.	$a_{10} \approx 69{,}809.77$

During your 10th year with the company, your salary will be \$69,809.77.

■ **YOUR TURN** Suppose you are offered a job with AT&T at \$37,000 per year with a guaranteed raise of 4% every year. What will your annual salary be after 15 years with the company?

■ **Answer:** \$64,072.03

EXAMPLE 9 Savings Growth: Geometric Series

Karen has maintained acrylic nails by paying for them with money earned from a part-time job. After hearing a lecture from her economics professor on the importance of investing early in life, she decides to remove the acrylic nails, which cost \$50 per month, and do her own manicures. She has that \$50 automatically debited from her checking account on the first of every month and put into a money market account that earns 3% interest compounded monthly. What will the balance be in the money market account exactly 2 years from the day of her initial \$50 deposit?

Solution:

Recall the compound interest formula. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Substitute $r = 0.03$ and $n = 12$ into the compound interest formula.

$$A = P\left(1 + \frac{0.03}{12}\right)^{12t}$$
$$= P(1.0025)^{12t}$$

Let $t = \frac{n}{12}$, where n is the number of months of the investment. $A_n = P(1.0025)^n$

The first deposit of \$50 will gain interest for 24 months. $A_{24} = 50(1.0025)^{24}$

The second deposit of \$50 will gain interest for 23 months. $A_{23} = 50(1.0025)^{23}$

The third deposit of \$50 will gain interest for 22 months. $A_{22} = 50(1.0025)^{22}$

The last deposit of \$50 will gain interest for 1 month. $A_1 = 50(1.0025)^1$

Sum the amounts accrued from the 24 deposits.

$$A_1 + A_2 + \cdots + A_{24} = 50(1.0025) + 50(1.0025)^2 + 50(1.0025)^3 + \cdots + 50(1.0025)^{24}$$

Identify the first term and common ratio. $a_1 = 50(1.0025)$ and $r = 1.0025$

Sum the first n terms of a geometric series. $S_n = a_1\frac{(1 - r^n)}{(1 - r)}$

Substitute $n = 24$, $a_1 = 50(1.0025)$, and $r = 1.0025$. $S_{24} = 50(1.0025)\frac{(1 - 1.0025^{24})}{(1 - 1.0025)}$

Simplify. $S_{24} \approx 1238.23$

Karen will have \$1238.23 saved in her money market account in 2 years.

YOUR TURN Repeat Example 9 with Karen putting \$100 (instead of \$50) in the same money market account. Assume she does this for 4 years (instead of 2 years).

Answer: \$5105.85

Technology Tip

Use a calculator to find $S_{24} = 50(1.0025)\frac{(1 - 1.0025^{24})}{(1 - 1.0025)}$.

Scientific calculators:

Press	*Display*
50 [x] 1.0025 [x] [(] 1 [−] 1.0025 [x^y] 24 [)] [÷] [(] 1 [−] 1.0025 [)] [=]	1238.23

Graphing calculators:

50 [x] 1.0025 [x] [(] 1 [−] 1.0025 [^] 24 [)] [÷] [(] 1 [−] 1.0025 [)] [ENTER]

```
50*1.0025*(1-1.0
025^24)/(1-1.002
5)
     1238.228737
```

SECTION 10.3 SUMMARY

In this section, we discussed geometric sequences, in which each successive term is found by multiplying the previous term by a constant, so that $a_{n+1} = r \cdot a_n$. That constant, r, is called the common ratio. The nth term of a geometric sequence is given by $a_n = a_1 r^{n-1}$, $n \geq 1$ or $a_{n+1} = a_1 r^n$, $n \geq 0$. The sum of the terms of a geometric sequence is called a geometric series. Finite geometric series converge to a number. Infinite geometric series converge to a number if the absolute value of the common ratio is less than 1. If the absolute value of the common ratio is greater than or equal to 1, the infinite geometric series diverges and the sum does not exist. Many real-world applications involve geometric sequences and series, such as growth of salaries and annuities through percentage increases.

Finite Geometric Series: $\sum_{k=1}^{n} a_1 r^{k-1} = a_1\frac{(1 - r^n)}{(1 - r)}$ $\quad r \neq 1$

Infinite Geometric Series: $\sum_{k=1}^{\infty} a_1 r^{k-1} = a_1\frac{1}{(1 - r)}$ $\quad |r| < 1$

SECTION 10.3 EXERCISES

SKILLS

In Exercises 1–8, determine whether each sequence is geometric. If it is, find the common ratio.

1. $1, 3, 9, 27, \ldots$ **2.** $2, 4, 8, 16, \ldots$ **3.** $1, 4, 9, 16, 25, \ldots$ **4.** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

5. $8, 4, 2, 1, \ldots$ **6.** $8, -4, 2, -1, \ldots$ **7.** $800, 1360, 2312, 3930.4, \ldots$ **8.** $7, 15.4, 33.88, 74.536, \ldots$

In Exercises 9–16, write the first five terms of each geometric series.

9. $a_1 = 6 \quad r = 3$ **10.** $a_1 = 17 \quad r = 2$ **11.** $a_1 = 1 \quad r = -4$ **12.** $a_1 = -3 \quad r = -2$

13. $a_1 = 10{,}000 \quad r = 1.06$ **14.** $a_1 = 10{,}000 \quad r = 0.8$ **15.** $a_1 = \frac{2}{3} \quad r = \frac{1}{2}$ **16.** $a_1 = \frac{1}{10} \quad r = -\frac{1}{5}$

In Exercises 17–24, write the formula for the *n*th term of each geometric series.

17. $a_1 = 5 \quad r = 2$ **18.** $a_1 = 12 \quad r = 3$ **19.** $a_1 = 1 \quad r = -3$ **20.** $a_1 = -4 \quad r = -2$

21. $a_1 = 1000 \quad r = 1.07$ **22.** $a_1 = 1000 \quad r = 0.5$ **23.** $a_1 = \frac{16}{3} \quad r = -\frac{1}{4}$ **24.** $a_1 = \frac{1}{200} \quad r = 5$

In Exercises 25–30, find the indicated term of each geometric sequence.

25. 7th term of the sequence $-2, 4, -8, 16, \ldots$

26. 10th term of the sequence $1, -5, 25, -225, \ldots$

27. 13th term of the sequence $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

28. 9th term of the sequence $100, 20, 4, 0.8, \ldots$

29. 15th term of the sequence $1000, 50, 2.5, 0.125, \ldots$

30. 8th term of the sequence $1000, -800, 640, -512, \ldots$

In Exercises 31–40, find the sum of each finite geometric series.

31. $\frac{1}{3} + \frac{2}{3} + \frac{2^2}{3} + \cdots + \frac{2^{12}}{3}$

32. $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{10}}$

33. $2 + 6 + 18 + 54 + \cdots + 2(3^9)$

34. $1 + 4 + 16 + 64 + \cdots + 4^9$

35. $\sum_{n=0}^{10} 2(0.1)^n$ **36.** $\sum_{n=0}^{11} 3(0.2)^n$ **37.** $\sum_{n=1}^{8} 2(3)^{n-1}$ **38.** $\sum_{n=1}^{9} \frac{2}{3}(5)^{n-1}$ **39.** $\sum_{k=0}^{13} 2^k$ **40.** $\sum_{k=0}^{13} \left(\frac{1}{2}\right)^k$

In Exercises 41–54, find the sum of each infinite geometric series, if possible.

41. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ **42.** $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ **43.** $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$ **44.** $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ **45.** $\sum_{n=0}^{\infty} 1^n$

46. $\sum_{n=0}^{\infty} 1.01^n$ **47.** $\sum_{n=0}^{\infty} -9\left(\frac{1}{3}\right)^n$ **48.** $\sum_{n=0}^{\infty} -8\left(-\frac{1}{2}\right)^n$ **49.** $\sum_{n=0}^{\infty} 10{,}000(0.05)^n$ **50.** $\sum_{n=0}^{\infty} 200(0.04)^n$

51. $\sum_{n=1}^{\infty} 0.4^n$ **52.** $0.3 + 0.03 + 0.003 + 0.0003 + \cdots$ **53.** $\sum_{n=0}^{\infty} 0.99^n$ **54.** $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$

APPLICATIONS

55. Salary. Jeremy is offered a government job with the Department of Commerce. He is hired on the "GS" scale at a base rate of \$34,000 with a 2.5% increase in his salary per year. Calculate what his salary will be after he has been with the Department of Commerce for 12 years.

56. Salary. Alison is offered a job with a small start-up company that wants to promote loyalty to the company with incentives for employees to stay with the company. The company offers her a starting salary of \$22,000 with a guaranteed 15% raise per year. What will her salary be after she has been with the company for 10 years?

57. Depreciation. Brittany, a graduating senior in high school, receives a laptop computer as a graduation gift from her Aunt Jeanine so that she can use it when she gets to the University of Alabama. If the laptop costs $2000 new and depreciates 50% per year, write a formula for the value of the laptop n years after it was purchased. How much will the laptop be worth when Brittany graduates from college (assuming she will graduate in 4 years)? How much will it be worth when she finishes graduate school? Assume graduate school is another 3 years.

58. Depreciation. Derek is deciding between a new Honda Accord and the BMW 325 series. The BMW costs $35,000 and the Honda costs $25,000. If the BMW depreciates at 20% per year and the Honda depreciates at 10% per year, find formulas for the value of each car n years after it is purchased. Which car is worth more in 10 years?

59. Bungee Jumping. A bungee jumper rebounds 70% of the height jumped. Assuming the bungee jump is made with a cord that stretches to 100 feet, how far will the bungee jumper travel upward on the fifth rebound?

Laurence Fordyce/Eye Ubiquitous/Corbis Images

60. Bungee Jumping. A bungee jumper rebounds 65% of the height jumped. Assuming the bungee cord stretches 200 feet, how far will the bungee jumper travel upward on the eighth rebound?

61. Population Growth. One of the fastest-growing universities in the country is the University of Central Florida. The student populations each year starting in 2000 were 36,000, 37,800, 39,690, 41,675, ... Assuming this rate has continued, how many students are enrolled at UCF in 2010?

62. Web Site Hits. The Web site for Matchbox 20 (www.matchboxtwenty.com) has noticed that every week the number of hits to its Web site increases 5%. If there were 20,000 hits this week, how many will there be exactly 52 weeks from now if this rate continues?

63. Rich Man's Promise. A rich man promises that he will give you $1000 on January 1, and every day after that, he will pay you 90% of what he paid you the day before. How many days will it take before you are making less than $1? How much will the rich man pay out for the entire month of January? Round to the nearest dollar.

64. Poor Man's Clever Deal. A poor man promises to work for you for $0.01 the first day, $0.02 on the second day, $0.04 on the third day; his salary will continue to double each day. If he started on January 1 how much would he be paid to work on January 31? How much in total would he make during the month? Round to the nearest dollar.

65. Investing Lunch. A newlywed couple decides to stop going out to lunch every day and instead bring their lunch to work. They estimate it will save them $100 per month. They invest that $100 on the first of every month in an account that is compounded monthly and pays 5% interest. How much will be in the account at the end of 3 years?

66. Pizza as an Investment. A college freshman decides to stop ordering late-night pizzas (for both health and cost reasons). He realizes that he has been spending $50 a week on pizzas. Instead, he deposits $50 into an account that compounds weekly and pays 4% interest. (Assume 52 weeks annually.) How much money will be in the account after 52 weeks?

67. Tax-Deferred Annuity. Dr. Schober contributes $500 from her paycheck (weekly) to a tax-deferred investment account. Assuming the investment earns 6% and is compounded weekly, how much will be in the account after 26 weeks? 52 weeks?

68. Saving for a House. If a new graduate decides she wants to save for a house and she is able to put $300 every month into an account that earns 5% compounded monthly, how much will she have in the account after 5 years?

69. House Values. In 2008 you buy a house for $195,000. The value of the house appreciates 6.5% per year, on the average. How much is the house worth after 15 years?

70. The Bouncing Ball Problem. A ball is dropped from a height of 9 feet. Assume that on each bounce, the ball rebounds to one-third of its previous height. Find the total distance that the ball travels.

71. Probability. A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is given by the function $p(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 1$. Show that

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1.0$$

72. Salary. Suppose you work for a supervisor who gives you two different options to choose from for your monthly pay. Option 1: The company pays you 1 cent for the first day of work, 2 cents the second day, 4 cents for the third day, 8 cents for the fourth day, and so on for 30 days. Option 2: You can receive a check right now for $10,000,000. Which pay option is better? How much better is it?

CATCH THE MISTAKE

In Exercises 73–76, explain the mistake that is made.

73. Find the nth term of the geometric sequence

$$-1, \tfrac{1}{3}, -\tfrac{1}{9}, \tfrac{1}{27}, \ldots.$$

Solution:

Identify the first term and common ratio. $a_1 = -1$ and $r = \dfrac{1}{3}$

Substitute $a_1 = -1$ and $r = \frac{1}{3}$ into $a_n = a_1 \cdot r^{n-1}$. $a_n = (-1) \cdot \left(\dfrac{1}{3}\right)^{n-1}$

Simplify. $a_n = \dfrac{-1}{3^{n-1}}$

This is incorrect. What mistake was made?

74. Find the sum of the first n terms of the finite geometric series

$$2, 4, 8, 16, \ldots.$$

Solution:

Write the sum in sigma notation. $\displaystyle\sum_{k=1}^{n} (2)^k$

Identify the first term and common ratio. $a_1 = 1$ and $r = 2$

Substitute $a_1 = 1$ and $r = 2$ into $S_n = a_1 \dfrac{(1 - r^n)}{(1 - r)}$. $S_n = 1\dfrac{(1 - 2^n)}{(1 - 2)}$

Simplify. $S_n = 2^n - 1$

This is incorrect. What mistake was made?

75. Find the sum of the finite geometric series $\displaystyle\sum_{n=1}^{8} 4(-3)^n$.

Solution:

Identify the first term and common ratio. $a_1 = 4$ and $r = -3$

Substitute $a_1 = 4$ and $r = -3$ into $S_n = a_1 \dfrac{(1 - r^n)}{(1 - r)}$. $S_n = 4\dfrac{[1 - (-3)^n]}{[1 - (-3)]}$

$= 4\dfrac{[1 - (-3)^n]}{4}$

Simplify. $S_n = [1 - (-3)^n]$

Substitute $n = 8$. $S_8 = [1 - (-3)^8] = -6560$

This is incorrect. What mistake was made?

76. Find the sum of the infinite geometric series $\displaystyle\sum_{n=1}^{\infty} 2 \cdot 3^{n-1}$.

Solution:

Identify the first term and common ratio. $a_1 = 2$ and $r = 3$

Substitute $a_1 = 2$ and $r = 3$ into $S_\infty = a_1 \dfrac{1}{(1 - r)}$. $S_\infty = 2\dfrac{1}{(1 - 3)}$

Simplify. $S_\infty = -1$

This is incorrect. The series does not sum to -1. What mistake was made?

CONCEPTUAL

In Exercises 77–80, determine whether each statement is true or false.

77. An alternating sequence cannot be a geometric sequence.

78. All finite and infinite geometric series can always be evaluated.

79. The common ratio of a geometric sequence can be positive or negative.

80. An infinite geometric series can be evaluated if the common ratio is less than or equal to 1.

CHALLENGE

81. State the conditions for the sum

$$a + a \cdot b + a \cdot b^2 + \cdots + a \cdot b^n + \cdots$$

to exist. Assuming those conditions are met, find the sum.

82. Find the sum of $\sum_{k=0}^{20} \log 10^{2^k}$.

83. Represent the repeating decimal 0.474747. . . as a fraction (ratio of two integers).

84. Suppose the sum of an infinite geometric series is $S = \dfrac{2}{1 - x}$, where x is a variable.

a. Write out the first five terms of the series.

b. For what values of x will the series converge?

TECHNOLOGY

85. Sum the series $\sum_{k=1}^{50}(-2)^{k-1}$. Apply a graphing utility to confirm your answer.

86. Does the sum of the infinite series $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ exist? Use a graphing calculator to find it.

87. Apply a graphing utility to plot $y_1 = 1 + x + x^2 + x^3 + x^4$ and $y_2 = \dfrac{1}{1-x}$. Based on what you see, what do you expect the geometric series $\sum_{n=0}^{\infty} x^n$ to sum to?

88. Apply a graphing utility to plot $y_1 = 1 - x + x^2 - x^3 + x^4$ and $y_2 = \dfrac{1}{1+x}$. Based on what you see, what do you expect the geometric series $\sum_{n=0}^{\infty}(-1)^n x^n$ to sum to?

PREVIEW TO CALCULUS

In calculus, we study the convergence of geometric series. A geometric series with ratio r diverges if $|r| \geq 1$. If $|r| < 1$, then the geometric series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

In Exercises 89–92, determine the convergence or divergence of the series. If the series is convergent, find its sum.

89. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$ **90.** $1 + \frac{5}{4} + \frac{25}{16} + \frac{125}{64} + \cdots$ **91.** $\frac{3}{8} + \frac{3}{32} + \frac{3}{128} + \frac{3}{512} + \cdots$ **92.** $\sum_{n=0}^{\infty}\frac{\pi}{3}\left(-\frac{8}{9}\right)^n$

SECTION 10.4 MATHEMATICAL INDUCTION

SKILLS OBJECTIVES

- Know the steps required to prove a statement by mathematical induction.
- Prove mathematical statements using mathematical induction.

CONCEPTUAL OBJECTIVES

- Understand that just because there appears to be a pattern, the pattern is not necessarily true for all values.
- Understand that when mathematical ideas are accepted, it is because they can be proved.

Proof by Mathematical Induction

n	$n^2 - n + 41$	PRIME?
1	41	Yes
2	43	Yes
3	47	Yes
4	53	Yes
5	61	Yes

Is the expression $n^2 - n + 41$ *always* a prime number if n is a natural number? Your instinct may lead you to try a few values for n.

It appears that the statement might be true for all natural numbers. However, what about when $n = 41$?

$$n^2 - n + 41 = (41)^2 - 41 + 41 = 41^2$$

We find that when $n = 41$, $n^2 - n + 41$ is not prime. The moral of the story is that just because a pattern seems to exist for *some* values, the pattern is not necessarily true for *all* values. We must look for a way to show whether a statement is true for all values. In this section, we talk about *mathematical induction*, which is a way to show a statement is true for all values.

Mathematics is based on logic and proof (not assumptions or belief). One of the most famous mathematical statements was Fermat's Last Theorem. Pierre de Fermat (1601–1665) conjectured that there are no positive integer values for x, y, and z such that $x^n + y^n = z^n$, if $n \geq 3$. Although mathematicians *believed* that this theorem was true, no one was able to

prove it until 350 years after the assumption was made. Professor Andrew Wiles at Princeton University received a $50,000 prize for successfully proving Fermat's Last Theorem in 1994.

Mathematical induction is a technique used in precalculus and even in very advanced mathematics to prove many kinds of mathematical statements. In this section, you will use it to prove statements like "If $x > 1$, then $x^n > 1$ for all natural numbers n."

The principle of mathematical induction can be illustrated by a row of standing dominos, as in the image here. We make two assumptions:

1. The first domino is knocked down.
2. If a domino is knocked down, then the domino immediately following it will also be knocked down.

If both of these assumptions are true, then it is also true that all of the dominos will fall.

PRINCIPLE OF MATHEMATICAL INDUCTION

Let S_n be a statement involving the positive integer n. To prove that S_n is true for all positive integers, the following steps are required:

Step 1: Show that S_1 is true.
Step 2: Assume S_k is true and show that S_{k+1} is true ($k = 1, 2, 3, \ldots$).

Combining Steps 1 and 2 proves the statement is true for all positive integers (natural numbers).

EXAMPLE 1 Using Mathematical Induction

Apply the principle of mathematical induction to prove this statement:

If $x > 1$, then $x^n > 1$ for all natural numbers n.

Solution:

STEP 1	Show the statement is true for $n = 1$.	$x^1 > 1$ because $x > 1$
STEP 2	Assume the statement is true for $n = k$.	$x^k > 1$
	Show the statement is true for $k + 1$.	
	Multiply both sides by x.	$x^k \cdot x > 1 \cdot x$
	(Since $x > 1$, this step does not reverse the inequality sign.)	
	Simplify.	$x^{k+1} > x$
	Recall that $x > 1$.	$x^{k+1} > x > 1$

Therefore, we have shown that $x^{k+1} > 1$.

This completes the induction proof. Thus, the following statement is true:

"If $x > 1$, then $x^n > 1$ for **all** natural numbers n."

EXAMPLE 2 Using Mathematical Induction

Use mathematical induction to prove that $n^2 + n$ is divisible by 2 for all natural numbers (positive integers) n.

Solution:

STEP 1	Show the statement we are testing is true for $n = 1$.	$1^2 + 1 = 2$
	2 is divisible by 2.	$\frac{2}{2} = 1$

STEP 2 Assume the statement is true for $n = k$. $\quad \dfrac{k^2 + k}{2} = \text{an integer}$

Show it is true for $k + 1$ where $k \geq 1$. $\quad \dfrac{(k+1)^2 + (k+1)}{2} \stackrel{?}{=} \text{an integer}$

$$\frac{k^2 + 2k + 1 + k + 1}{2} \stackrel{?}{=} \text{an integer}$$

Regroup terms. $\quad \dfrac{(k^2 + k) + 2(k+1)}{2} \stackrel{?}{=} \text{an integer}$

$$\frac{(k^2 + k)}{2} + \frac{2(k+1)}{2} \stackrel{?}{=} \text{an integer}$$

We assumed $\dfrac{k^2 + k}{2} = \text{an integer}$. $\quad \text{an integer} + (k + 1) \stackrel{?}{=} \text{an integer}$

Since k is a natural number (integer). $\quad \text{an integer} + \text{an integer} = \text{an integer}$

This completes the induction proof. The following statement is true:

"$n^2 + n$ is divisible by 2 for all natural numbers n."

Study Tip

Mathematical induction is not always the best approach. Example 3 can be shown via arithmetic series.

Mathematical induction is often used to prove formulas for partial sums.

EXAMPLE 3 Proving a Partial–Sum Formula with Mathematical Induction

Apply mathematical induction to prove the following partial–sum formula:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \text{ for all positive integers } n$$

Solution:

STEP 1 Show the formula is true for $n = 1$. $\quad 1 = \dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1$

STEP 2 Assume the formula is true for $n = k$. $\quad 1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$

Show it is true for $n = k + 1$.

$$1 + 2 + 3 + \cdots + k + (k+1) \stackrel{?}{=} \frac{(k+1)(k+2)}{2}$$

$$\underbrace{1 + 2 + 3 + \cdots + k}_{\frac{k(k+1)}{2}} + (k+1) \stackrel{?}{=} \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2} + (k+1) \stackrel{?}{=} \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1) + 2(k+1)}{2} \stackrel{?}{=} \frac{(k+1)(k+2)}{2}$$

$$\frac{k^2 + 3k + 2}{2} \stackrel{?}{=} \frac{(k+1)(k+2)}{2}$$

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

This completes the induction proof. The following statement is true:

"$1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all positive integers n."

Technology Tip

To visualize what needs to be proved in the partial–sum formula, use the [sum] command to find the sum of the series $\sum_{k=1}^{n} k$ on the left side for an arbitrary n value, say, $n = 100$. Press [2nd] [*LIST*] [▶] [MATH] [▼] [5:*sum*(] [ENTER] [2nd] [*LIST*] [▶] [*OPS*] [▼] [5:*seq*(] [ENTER] [ALPHA] [*N*] [,] [ALPHA] [*N*] [,] 1 [,] 100 [,] 1 [)] [)] [ENTER].

```
sum(seq(N,N,1,10
0,1))
                5050
```

Now calculate the sum by substituting $n = 100$ into $n(n + 1)/2$ on the right side.

```
sum(seq(N,N,1,10
0,1))
                5050
100(100+1)/2
                5050
```

Note: The Solutions to the left and right side of the formula agree with each other.

SECTION 10.4 SUMMARY

Just because we believe something is true does not mean that it is. In mathematics we rely on proof. In this section, we discussed *mathematical induction*, a process of proving some kinds of mathematical statements. The two-step procedure for mathematical induction is to (1) show the statement is true for $n = 1$, then (2) assume the statement is true for $n = k$ (any positive integer) and show the statement must be true for $n = k + 1$. The combination of Steps 1 and 2 proves the statement.

SECTION 10.4 EXERCISES

SKILLS

In Exercises 1–24, prove each statement using mathematical induction for all positive integers *n*.

1. $n^2 \le n^3$

2. If $0 < x < 1$, then $0 < x^n < 1$.

3. $2n \le 2^n$

4. $5^n < 5^{n+1}$

5. $n! > 2^n \qquad n \ge 4$ (Show it is true for $n = 4$, instead of $n = 1$.)

6. $(1 + c)^n \ge nc \qquad c > 1$

7. $n(n + 1)(n - 1)$ is divisible by 3.

8. $n^3 - n$ is divisible by 3.

9. $n^2 + 3n$ is divisible by 2.

10. $n(n + 1)(n + 2)$ is divisible by 6.

11. $2 + 4 + 6 + 8 + \cdots + 2n = n(n + 1)$

12. $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

13. $1 + 3 + 3^2 + 3^3 + \cdots + 3^n = \dfrac{3^{n+1} - 1}{2}$

14. $2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 2$

15. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

16. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n + 1)^2}{4}$

17. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$

18. $\dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{(n + 1)(n + 2)} = \dfrac{n}{2(n + 2)}$

19. $(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + n(n + 1) = \dfrac{n(n + 1)(n + 2)}{3}$

20. $(1 \cdot 3) + (2 \cdot 4) + (3 \cdot 5) + \cdots + n(n + 2) = \dfrac{n(n + 1)(2n + 7)}{6}$

21. $1 + x + x^2 + x^3 + \cdots + x^{n-1} = \dfrac{1 - x^n}{1 - x} \qquad x \ne 1$

22. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

23. The sum of an arithmetic sequence: $a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d] = \dfrac{n}{2}[2a_1 + (n - 1)d]$.

24. The sum of a geometric sequence: $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = a_1\left(\dfrac{1 - r^n}{1 - r}\right), r \ne 1$.

APPLICATIONS

The Tower of Hanoi. This is a game with three pegs and n disks (largest on the bottom and smallest on the top). The goal is to move this entire tower of disks to another peg (in the same order). The challenge is that you may move only one disk at a time, and at no time can a larger disk be resting on a smaller disk. You may want to first go online to www.mazeworks.com/hanoi/index/htm and play the game.

Andy Washnik

Tower of Hanoi

25. What is the smallest number of moves needed if there are three disks?

26. What is the smallest number of moves needed if there are four disks?

27. What is the smallest number of moves needed if there are five disks?

28. What is the smallest number of moves needed if there are n disks? Prove it by mathematical induction.

29. **Telephone Infrastructure.** Suppose there are n cities that are to be connected with telephone wires. Apply mathematical induction to prove that the number of telephone wires required to connect the n cities is given by $\frac{n(n-1)}{2}$. Assume each city has to connect directly with any other city.

30. **Geometry.** Prove, with mathematical induction, that the sum of the measures of the interior angles in degrees of a regular polygon of n sides is given by the formula $(n-2)(180°)$ for $n \geq 3$. *Hint:* Divide a polygon into triangles. For example, a four-sided polygon can be divided into two triangles. A five-sided polygon can be divided into three triangles. A six-sided polygon can be divided into four triangles, and so on.

CONCEPTUAL

In Exercises 31 and 32, determine whether each statement is true or false.

31. Assume S_k is true. If it can be shown that S_{k+1} is true, then S_n is true for all n, where n is any positive integer.

32. Assume S_1 is true. If it can be shown that S_2 and S_3 are true, then S_n is true for all n, where n is any positive integer.

CHALLENGE

33. Apply mathematical induction to prove

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

34. Apply mathematical induction to prove

$$\sum_{k=1}^{n} k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

35. Apply mathematical induction to prove

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$$

36. Apply mathematical induction to prove that $x + y$ is a factor of $x^{2n} - y^{2n}$.

37. Apply mathematical induction to prove that $x - y$ is a factor of $x^{2n} - y^{2n}$.

38. Apply mathematical induction to prove

$$\ln(c_1 \cdot c_2 \cdot c_3 \cdots c_n) = \ln c_1 + \ln c_2 + \cdots + \ln c_n$$

39. Use a graphing calculator to sum the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$ and evaluate the expression $1 - \frac{1}{2^n}$ for $n = 8$. Do they agree with each other? Do your answers confirm the proof for Exercise 22?

TECHNOLOGY

40. Use a graphing calculator to sum the series $(1 \cdot 2) + (2 \cdot 3) + (3 \cdot 4) + \cdots + n(n + 1)$ and evaluate the expression $\dfrac{n(n + 1)(n + 2)}{3}$ for $n = 200$. Do they agree with each other? Do your answers confirm the proof for Exercise 19?

PREVIEW TO CALCULUS

Several of the results studied in calculus must be proved by mathematical induction. In Exercises 41–44, apply mathematical induction to prove each formula.

41. $(\pi + 1) + (\pi + 2) + (\pi + 3) + \cdots + (\pi + n) = \dfrac{n(2\pi + n + 1)}{2}$

42. $(1 + 1) + (2 + 4) + (3 + 9) + \cdots + (n + n^2) = \dfrac{n(n + 1)(n + 2)}{3}$

43. $(1 + 1) + (2 + 8) + (3 + 27) + \cdots + (n + n^3) = \dfrac{n(n + 1)(n^2 + n + 2)}{4}$

44. $(1 + 1) + (4 + 8) + (9 + 27) + \cdots + (n^2 + n^3) = \dfrac{n(n + 1)(n + 2)(3n + 1)}{12}$

SECTION 10.5 THE BINOMIAL THEOREM

SKILLS OBJECTIVES

- Evaluate a binomial coefficient with the Binomial theorem.
- Evaluate a binomial coefficient with Pascal's triangle.
- Expand a binomial raised to a positive integer power.
- Find a particular term of a binomial expansion.

CONCEPTUAL OBJECTIVE

- Recognize patterns in binomial expansions.

Binomial Coefficients

A **binomial** is a polynomial that has two terms. The following are all examples of binomials:

$$x^2 + 2y \qquad a + 3b \qquad 4x^2 + 9$$

In this section, we will develop a formula for the expression for raising a binomial to a power n, where n is a positive integer.

$$(x^2 + 2y)^6 \qquad (a + 3b)^4 \qquad (4x^2 + 9)^5$$

To begin, let's start by writing out the expansions of $(a + b)^n$ for several values of n.

$$\begin{aligned}
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{aligned}$$

There are several *patterns* that all of the **binomial expansions** have:

1. The number of terms in each resulting polynomial is always *one more* than the power of the binomial n. Thus, there are $n + 1$ terms in each expansion.

$$n = 3: \quad (a + b)^3 = \underbrace{a^3 + 3a^2b + 3ab^2 + b^3}_{\text{four terms}}$$

2. Each expansion has symmetry. For example, a and b can be interchanged and you will arrive at the same expansion. Furthermore, the powers of ***a* decrease** by 1 in each successive term, and the powers of ***b* increase** by 1 in each successive term.

$$(a + b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

3. The sum of the powers of each term in the expansion is n.

$$n = 3: \quad (a + b)^3 = \overbrace{a^3b^0}^{3+0=3} + 3\overbrace{a^2b^1}^{2+1=3} + 3\overbrace{a^1b^2}^{1+2=3} + \overbrace{a^0b^3}^{0+3=3}$$

4. The coefficients **increase** and **decrease** in a symmetric manner.

$$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

Using these patterns, we can develop a generalized formula for $(a + b)^n$.

$$(a + b)^n = \square a^n + \square a^{n-1}b + \square a^{n-2}b^2 + \cdots + \square a^2b^{n-2} + \square ab^{n-1} + \square b^n$$

We know that there are $n + 1$ terms in the expansion. We also know that the sum of the powers of each term must equal n. The powers increase and decrease by 1 in each successive term, and if we interchanged a and b, the result would be the same expansion. The question that remains is, what coefficients go in the blanks?

We know that the coefficients must increase and then decrease in a symmetric order (similar to walking up and then down a hill). It turns out that the *binomial coefficients* are represented by a symbol that we will now define.

DEFINITION **Binomial Coefficients**

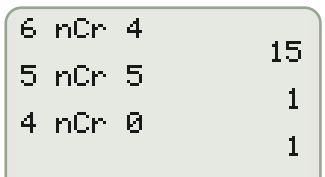
For nonnegative integers n and k, where $n \geq k$, the symbol $\binom{n}{k}$ is called the **binomial coefficient** and is defined by

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \qquad \binom{n}{k} \text{ is read "}n\text{ choose }k\text{."}$$

You will see in the following sections that "n choose k" comes from counting combinations of n things taken k at a time.

Technology Tip

Press	*Display*
6 nCr 4 =	15
5 nCr 5 =	1
4 nCr 0 =	1
10 nCr 9 =	10



EXAMPLE 1 **Evaluating a Binomial Coefficient**

Evaluate the following binomial coefficients:

a. $\binom{6}{4}$ **b.** $\binom{5}{5}$ **c.** $\binom{4}{0}$ **d.** $\binom{10}{9}$

Solution:

Select the top number as n and the bottom number as k and substitute into the binomial coefficient formula $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

a. $\binom{6}{4} = \frac{6!}{(6-4)!4!} = \frac{6!}{2!4!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{(2\cdot1)(4\cdot3\cdot2\cdot1)} = \frac{6\cdot5}{2} = \boxed{15}$

b. $\binom{5}{5} = \frac{5!}{(5-5)!5!} = \frac{5!}{0!5!} = \frac{1}{0!} = \frac{1}{1} = \boxed{1}$

c. $\binom{4}{0} = \frac{4!}{(4-0)!0!} = \frac{4!}{4!0!} = \frac{1}{0!} = \boxed{1}$

d. $\binom{10}{9} = \frac{10!}{(10-9)!9!} = \frac{10!}{1!9!} = \frac{10\cdot9!}{9!} = \boxed{10}$

■ **YOUR TURN** Evaluate the following binomial coefficients:

a. $\binom{9}{6}$ **b.** $\binom{8}{6}$

■ **Answer: a.** 84 **b.** 28

Parts (b) and (c) of Example 1 lead to the general formulas

$$\binom{n}{n} = 1 \quad \text{and} \quad \binom{n}{0} = 1$$

Binomial Expansion

Let's return to the question of the binomial expansion and how to determine the coefficients:

$$(a+b)^n = \square a^n + \square a^{n-1}b + \square a^{n-2}b^2 + \cdots + \square a^2b^{n-2} + \square ab^{n-1} + \square b^n$$

The symbol $\binom{n}{k}$ is called a binomial coefficient because the coefficients in the blanks in the binomial expansion are equivalent to this symbol.

THE BINOMIAL THEOREM

Let a and b be real numbers; then for any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

or in sigma (summation) notation as

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

EXAMPLE 2 Applying the Binomial Theorem

Expand $(x+2)^3$ with the Binomial theorem.

Solution:

Substitute $a = x$, $b = 2$, and $n = 3$ into the equation of the Binomial theorem. $(x+2)^3 = \sum_{k=0}^{3}\binom{3}{k}x^{3-k}2^k$

Expand the summation. $= \binom{3}{0}x^3 + \binom{3}{1}x^2\cdot 2 + \binom{3}{2}x\cdot 2^2 + \binom{3}{3}2^3$

Find the binomial coefficients. $= x^3 + 3x^2\cdot 2 + 3x\cdot 2^2 + 2^3$

Simplify. $= \boxed{x^3 + 6x^2 + 12x + 8}$

■ **YOUR TURN** Expand $(x+5)^4$ with the Binomial theorem.

■ **Answer:** $x^4 + 20x^3 + 150x^2 + 500x + 625$

Study Tip

The Binomial theorem can be used to expand any two-termed expression raised to a non-negative integer power, like $\left(\frac{1}{x} + 1\right)^3$ or $(\sqrt{x} + y)^n$.

EXAMPLE 3 Applying the Binomial Theorem

Expand $(2x - 3)^4$ with the Binomial theorem.

Solution:

Substitute $a = 2x$, $b = -3$, and $n = 4$ into the equation of the Binomial theorem.

$$(2x - 3)^4 = \sum_{k=0}^{4} \binom{4}{k} (2x)^{4-k}(-3)^k$$

Expand the summation.

$$= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-3) + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)(-3)^3 + \binom{4}{4}(-3)^4$$

Find the binomial coefficients.

$$= (2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4$$

Simplify.

$$\boxed{= 16x^4 - 96x^3 + 216x^2 - 216x + 81}$$

■ **YOUR TURN** Expand $(3x - 2)^4$ with the Binomial theorem.

■ **Answer:** $81x^4 - 216x^3 + 216x^2 - 96x + 16$

Pascal's Triangle

Instead of writing out the Binomial theorem and calculating the binomial coefficients using factorials every time you want to do a binomial expansion, we now present an alternative, more convenient way of remembering the binomial coefficients, called **Pascal's triangle**.

Notice that the first and last number in every row is 1. Each of the other numbers is found by adding the two numbers directly above it. For example,

$$3 = 2 + 1 \qquad 4 = 1 + 3 \qquad 10 = 6 + 4$$

Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Let's arrange values of $\binom{n}{k}$ in a triangular pattern. Notice that the *value* of the binomial coefficients below are given in the margin.

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

It turns out that the numbers in Pascal's triangle are exactly the coefficients in a binomial expansion.

$$1$$

$$1a + 1b$$

$$1a^2 + 2ab + 1b^2$$

$$1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

The top row is called the *zero row* because it corresponds to the binomial raised to the zero power, $n = 0$. Since each row in Pascal's triangle starts and ends with a 1 and all other values are found by adding the two numbers directly above it, we can now easily calculate the sixth row.

$$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

$$(a + b)^6 = 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$$

Study Tip

Since the top row of Pascal's triangle is called the zero row, the fifth row is the row with 6 coefficients. The nth row is the row with $n + 1$ coefficients.

EXAMPLE 4 Applying Pascal's Triangle in a Binomial Expansion

Use Pascal's triangle to determine the binomial expansion of $(x + 2)^5$.

Solution:

Write the binomial expansion with blanks for coefficients.

$$(x + 2)^5 = \square x^5 + \square x^4 \cdot 2 + \square x^3 \cdot 2^2 + \square x^2 \cdot 2^3 + \square x \cdot 2^4 + \square 2^5$$

Write the binomial coefficients in the *fifth* row of Pascal's triangle.

$$1, 5, 10, 10, 5, 1$$

Substitute these coefficients into the blanks of the binomial expansion.

$$(x + 2)^5 = 1x^5 + 5x^4 \cdot 2 + 10x^3 \cdot 2^2 + 10x^2 \cdot 2^3 + 5x \cdot 2^4 + 1 \cdot 2^5$$

Simplify. $\boxed{(x + 2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32}$

■ **YOUR TURN** Apply Pascal's triangle to determine the binomial expansion of $(x + 3)^4$.

■ **Answer:** $x^4 + 12x^3 + 54x^2 + 108x + 81$

EXAMPLE 5 Applying Pascal's Triangle in a Binomial Expansion

Use Pascal's triangle to determine the binomial expansion of $(2x + 5)^4$.

Solution:

Write the binomial expansion with blanks for coefficients.

$$(2x + 5)^4 = \square(2x)^4 + \square(2x)^3 \cdot 5 + \square(2x)^2 \cdot 5^2 + \square(2x) \cdot 5^3 + \square 5^4$$

Write the binomial coefficients in the *fourth* row of Pascal's triangle.

$$1, 4, 6, 4, 1$$

Substitute these coefficients into the blanks of the binomial expansion.

$$(2x + 5)^4 = \mathbf{1}(2x)^4 + \mathbf{4}(2x)^3 \cdot 5 + \mathbf{6}(2x)^2 \cdot 5^2 + \mathbf{4}(2x) \cdot 5^3 + \mathbf{1} \cdot 5^4$$

Simplify. $\boxed{(2x + 5)^4 = 16x^4 + 160x^3 + 600x^2 + 1000x + 625}$

■ **YOUR TURN** Use Pascal's triangle to determine the binomial expansion of

a. $(3x + 2)^3$ **b.** $(3x - 2)^5$

■ **Answer:**
a. $27x^3 + 54x^2 + 36x + 8$
b. $243x^5 - 810x^4 + 1080x^3 - 720x^2 + 240x - 32$

Finding a Particular Term of a Binomial Expansion

What if we don't want to find the entire expansion, but instead want just a single term? For example, what is the fourth term of $(a + b)^5$?

WORDS	MATH
Recall the sigma notation.	$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
Let $n = 5$.	$(a + b)^5 = \sum_{k=0}^{5} \binom{5}{k} a^{5-k} b^k$
Expand.	$(a + b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \underbrace{\binom{5}{3}a^2b^3}_{\text{fourth term}} + \binom{5}{4}ab^4 + \binom{5}{5}b^5$
Simplify the fourth term.	$10a^2b^3$

FINDING A PARTICULAR TERM OF A BINOMIAL EXPANSION

The $(r + 1)$ term of the expansion $(a + b)^n$ is $\binom{n}{r}a^{n-r}b^r$.

EXAMPLE 6 Finding a Particular Term of a Binomial Expansion

Find the fifth term of the binomial expansion of $(2x - 7)^6$.

Solution:

Recall that the $r + 1$ term of $(a + b)^n$ is $\binom{n}{r}a^{n-r}b^r$.

For the fifth term, let $r = 4$. $\binom{n}{4}a^{n-4}b^4$

For this expansion, let $a = 2x$, $b = -7$, and $n = 6$. $\binom{6}{4}(2x)^{6-4}(-7)^4$

Note that $\binom{6}{4} = 15$. $15(2x)^2(-7)^4$

Simplify. $\boxed{144{,}060x^2}$

★ **YOUR TURN** What is the third term of the binomial expansion of $(3x - 2)^5$?

Technology Tip

6 [MATH] [▶] [PRB] [▼] [3:nCr] [ENTER] 4 [×] 2 [x^2] [×] [(] [(–)] 7 [)] [^] 4 [ENTER].

```
6 nCr 4*2²*(-7)^
4
              144060
```

■ **Answer:** $1080x^3$

SECTION 10.5 SUMMARY

In this section, we developed a formula for expanding a binomial raised to a non-negative integer power, n. The patterns that surfaced were

- that the expansion displays symmetry between the two terms
- every expansion has $n + 1$ terms
- the powers sum to n
- the coefficients, called binomial coefficients, are ratios of factorials

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Also, Pascal's triangle, a shortcut method for evaluating the binomial coefficients, was discussed. The patterns in the triangle are that every row begins and ends with 1 and all other numbers are found by adding the two numbers in the row above the entry.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Lastly, a formula was given for finding a particular term of a binomial expansion; the $(r + 1)$ term of $(a + b)^n$ is $\binom{n}{r} a^{n-r} b^r$.

SECTION 10.5 EXERCISES

SKILLS

In Exercises 1–10, evaluate each binomial coefficient.

1. $\binom{7}{3}$ **2.** $\binom{8}{2}$ **3.** $\binom{10}{8}$ **4.** $\binom{23}{21}$ **5.** $\binom{17}{0}$

6. $\binom{100}{0}$ **7.** $\binom{99}{99}$ **8.** $\binom{52}{52}$ **9.** $\binom{48}{45}$ **10.** $\binom{29}{26}$

In Exercises 11–32, expand each expression using the Binomial theorem.

11. $(x + 2)^4$ **12.** $(x + 3)^5$ **13.** $(y - 3)^5$ **14.** $(y - 4)^4$

15. $(x + y)^5$ **16.** $(x - y)^6$ **17.** $(x + 3y)^3$ **18.** $(2x - y)^3$

19. $(5x - 2)^3$ **20.** $(a - 7b)^3$ **21.** $\left(\frac{1}{x} + 5y\right)^4$ **22.** $\left(2x + \frac{3}{y}\right)^4$

23. $(x^2 + y^2)^4$ **24.** $(r^3 - s^3)^3$ **25.** $(ax + by)^5$ **26.** $(ax - by)^5$

27. $(\sqrt{x} + 2)^6$ **28.** $(3 + \sqrt{y})^4$ **29.** $(a^{3/4} + b^{1/4})^4$ **30.** $(x^{2/3} + y^{1/3})^3$

31. $(x^{1/4} + 2\sqrt{y})^4$ **32.** $(\sqrt{x} - 3y^{1/4})^8$

In Exercises 33–36, expand each expression using Pascal's triangle.

33. $(r - s)^4$ **34.** $(x^2 + y^2)^7$ **35.** $(ax + by)^6$ **36.** $(x + 3y)^4$

In Exercises 37–44, find the coefficient C of the given term in each binomial expansion.

Binomial	Term	Binomial	Term	Binomial	Term
37. $(x + 2)^{10}$	Cx^6	**38.** $(3 + y)^9$	Cy^5	**39.** $(y - 3)^8$	Cy^4
40. $(x - 1)^{12}$	Cx^5	**41.** $(2x + 3y)^7$	Cx^3y^4	**42.** $(3x - 5y)^9$	Cx^2y^7
43. $(x^2 + y)^8$	Cx^8y^4	**44.** $(r - s^2)^{10}$	Cr^6s^8		

APPLICATIONS

45. Lottery. In a state lottery in which 6 numbers are drawn from a possible 40 numbers, the number of possible 6-number combinations is equal to $\binom{40}{6}$. How many possible combinations are there?

46. Lottery. In a state lottery in which 6 numbers are drawn from a possible 60 numbers, the number of possible 6-number combinations is equal to $\binom{60}{6}$. How many possible combinations are there?

47. Poker. With a deck of 52 cards, 5 cards are dealt in a game of poker. There are a total of $\binom{52}{5}$ different 5-card poker hands that can be dealt. How many possible hands are there?

48. Canasta. In the card game Canasta, two decks of cards including the jokers are used and 11 cards are dealt to each person. There are a total of $\binom{108}{11}$ different 11-card Canasta hands that can be dealt. How many possible hands are there?

CATCH THE MISTAKE

In Exercises 49 and 50, explain the mistake that is made.

49. Evaluate the expression $\binom{7}{5}$.

Solution:

Write out the binomial coefficient in terms of factorials. $\binom{7}{5} = \frac{7!}{5!}$

Write out the factorials. $\binom{7}{5} = \frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

Simplify. $\binom{7}{5} = \frac{7!}{5!} = \frac{7 \cdot 6}{1} = 42$

This is incorrect. What mistake was made?

50. Expand $(x + 2y)^4$.

Solution:

Write out the expansion with blanks.

$$(x + 2y)^4 = \square x^4 + \square x^3y + \square x^2y^2 + \square xy^3 + \square y^4$$

Write out the terms from the fifth row of Pascal's triangle.

1, 4, 6, 4, 1

Substitute these coefficients into the expansion.

$$(x + 2y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 51–56, determine whether each statement is true or false.

51. The binomial expansion of $(x + y)^{10}$ has 10 terms.

52. The binomial expansion of $(x^2 + y^2)^{15}$ has 16 terms.

53. $\binom{n}{n} = 1$

54. $\binom{n}{-n} = -1$

55. The coefficient of x^8 in the expansion of $(2x - 1)^{12}$ is 126,720.

56. The sixth term of the binomial expansion of $(x^2 + y)^{10}$ is $252x^5y^5$.

CHALLENGE

57. Show that $\binom{n}{k} = \binom{n}{n-k}$, if $0 \le k \le n$.

58. Show that if n is a positive integer, then

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Hint: Let $2^n = (1 + 1)^n$ and use the Binomial theorem to expand.

59. Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $k < n$ and $n \ge 2$.

60. Show that $(n - 2k)\binom{n}{k} = n\left[\binom{n-1}{k} - \binom{n-1}{k-1}\right]$ for $n > k$ and $n \ge 2$.

TECHNOLOGY

61. With a graphing utility, plot $y_1 = 1 - 3x + 3x^2 - x^3$, $y_2 = -1 + 3x - 3x^2 + x^3$, and $y_3 = (1 - x)^3$ in the same viewing screen. Which is the binomial expansion of $(1 - x)^3$, y_1 or y_2?

62. With a graphing utility, plot $y_1 = (x + 3)^4$, $y_2 = x^4 + 4x^3 + 6x^2 + 4x + 1$, and $y_3 = x^4 + 12x^3 + 54x^2 + 108x + 81$. Which is the binomial expansion of $(x + 3)^4$, y_2 or y_3?

63. With a graphing utility, plot $y_1 = 1 - 3x$, $y_2 = 1 - 3x + 3x^2$, $y_3 = 1 - 3x + 3x^2 - x^3$, and $y_4 = (1 - x)^3$ for $-1 < x < 1$. What do you notice happening each time an additional term is added to the series? Now, let $1 < x < 2$. Does the same thing happen?

64. With a graphing utility, plot $y_1 = 1 - \frac{3}{x}$, $y_2 = 1 - \frac{3}{x} + \frac{3}{x^2}$, $y_3 = 1 - \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$, and $y_4 = \left(1 - \frac{1}{x}\right)^3$ for $1 < x < 2$. What do you notice happening each time an additional term is added to the series? Now, let $0 < x < 1$. Does the same thing happen?

65. With a graphing utility, plot $y_1 = 1 + \frac{3}{x}$, $y_2 = 1 + \frac{3}{x} + \frac{3}{x^2}$, $y_3 = 1 + \frac{3}{x} + \frac{3}{x^2} - \frac{1}{x^3}$, and $y_4 = \left(1 + \frac{1}{x}\right)^3$ for $1 < x < 2$. What do you notice happening each time an additional term is added to the series? Now, let $0 < x < 1$. Does the same thing happen?

66. With a graphing utility, plot $y_1 = 1 + \frac{x}{1!}$, $y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!}$, $y_3 = 1 + \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!}$, and $y_4 = e^x$ for $-1 < x < 1$. What do you notice happening each time an additional term is added to the series? Now, let $1 < x < 2$. Does the same thing happen?

PREVIEW TO CALCULUS

In calculus, the difference quotient $\frac{f(x + h) - f(x)}{h}$ of a function f is used to find the derivative of the function f.

In Exercises 67 and 68, use the Binomial theorem to find the difference quotient of each function.

67. $f(x) = x^n$

68. $f(x) = (2x)^n$

69. In calculus, we learn that the derivative of $f(x) = (x + 1)^n$ is $f'(x) = n(x + 1)^{n-1}$. Using the Binomial theorem, find an expression for f' using sigma notation.

70. In calculus, we learn that the antiderivative of $f(x) = (x + 1)^n$ is $F(x) = \frac{(x + 1)^{n+1}}{n + 1}$ for $n \neq -1$. Using the Binomial theorem, find an expression for $F(x)$ using sigma notation.

CHAPTER 10 INQUIRY-BASED LEARNING PROJECT

Pascal's triangle, which you first learned about in Section 10.5, is useful for expanding binomials. As you shall see next, it can also help you compute probabilities of outcomes and events in a certain type of multistage experiment.

1. Suppose a math teacher gives her students a pop quiz, including 2 true/false questions. Since Cameron didn't study, he decides to randomly guess for the true/false portion of the quiz. Each of his answers will either be right (R) or wrong (W). There are four possible outcomes for his answers: RR, RW, WR, and WW. These outcomes may be organized according to the number of correct answers given in each possible outcome, as shown below.

Randomly Guessing on 2 True/False Quiz Questions			
Number of correct answers given	2	1	0
Outcomes	RR	RW, WR	WW
Number of ways	1	2	1

a. Cameron's math teacher will give several more pop quizzes this term, with various numbers of true/false questions on each quiz. Complete the following tables for randomly guessing on true/false quizzes with 1 question, 3 questions, and 4 questions.

Randomly Guessing on 1 True/False Quiz Question		
Number of correct answers given	1	0
Outcomes		
Number of ways		

Randomly Guessing on 3 True/False Quiz Questions	
Number of correct answers given	
Outcomes	
Number of ways	

Randomly Guessing on 4 True/False Quiz Questions	
Number of correct answers given	
Outcomes	
Number of ways	

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

b. Notice that the "Number of ways" row of each table in part (a) are the rows of Pascal's triangle, shown above. Notice that the first and last entry of each row is a 1. Describe how to find the other entries.

c. Extend Pascal's triangle for four more rows, to the row that begins 1 8 .... How can you interpret each of the entries in the 1 8 ... row, in the context of randomly guessing on a true/false quiz?

d. What is the sum of the entries in each of the rows of Pascal's triangle? Try to find a formula for the sum of the entries in the *n*th row. What do these numbers represent, in terms of randomly guessing on a true/false quiz?

2. Use Pascal's triangle to find the probability of each event *E* given in parts (a) and (b) below.

a. *E* is the event of guessing 4 answers correctly on a 6-question true/false quiz.

b. *E* is the event of guessing *at least* 5 answers correctly on an 8-question true/false quiz.

3. Randomly guessing on a true/false quiz is an example of a multistage experiment in which there are *two equally likely outcomes* at each stage. Pascal's triangle may also be used to solve similar probability problems, as with flipping a coin or births of boys and girls.

Show how to use Pascal's triangle to find the probability of each of the following events.

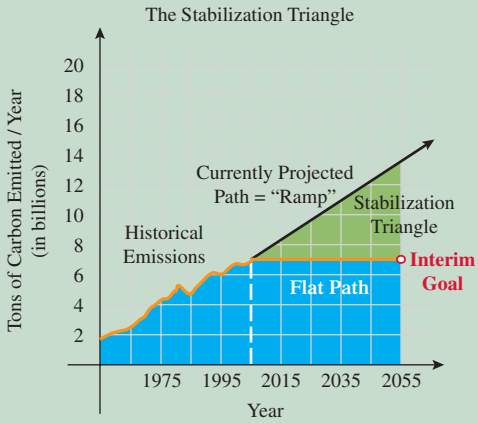
a. No boys in a family with 4 children.

b. At least one boy in a family with 6 children.

c. All 10 heads when flipping a coin 10 times.

MODELING OUR WORLD

In 2005 the world was producing 7 billion tons of carbon emissions per year. In 2055 this number is projected to double with the worldwide production of carbon emissions equaling 14 billion tons per year.



1. Determine the equation of the line for the projected path: an increase of 7 gigatons of carbon (GtC) over 50 years (2005–2055). Calculate the slope of the line.
2. What is the increase (per year) in the rate of carbon emissions per year based on the projected path model?
3. Develop a model in terms of a finite series that yields the total additional billions of tons of carbon emitted over the 50-year period (2005–2055) for the projected path over the flat path.
4. Calculate the total additional billions of tons of carbon of the projected path over the flat path [i.e., sum the series in (3)].
5. Discuss possible ways to provide the reduction between the projected path and the flat path based on the proposals given by Pacala and Socolow (professors at Princeton).*

*S. Pacala and R. Socolow, "Stabilization Wedges: Solving the Climate Problem for the Next 50 Years with Current Technologies," *Science*, Vol. 305 (2004).

Section	Concept	Key Ideas/Formulas
10.1	**Sequences and series**	
	Sequences	$a_1, a_2, a_3, \ldots, a_n, \ldots$ $\quad a_n$ is the general term.
	Factorial notation	$n! = n(n-1)(n-2)\cdot \cdots 3\cdot 2\cdot 1 \quad n \geq 2$ $0! = 1$ and $1! = 1$
	Recursion formulas	When a_n is defined by previous terms a_{n-i} $(i = 1, 2, \ldots)$.
	Sums and series	Infinite series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ Finite series, or nth partial sum, S_n: $S_n = a_1 + a_2 + a_3 + \cdots + a_n$
10.2	**Arithmetic sequences and series**	
	Arithmetic sequences	$a_{n+1} = a_n + d$ or $a_{n+1} - a_n = d$ d is called the **common difference**.
	The general (nth) term of an arithmetic sequence	$a_n = a_1 + (n-1)d$ for $n \geq 1$
	The sum of an arithmetic sequence	$S_n = \frac{n}{2}(a_1 + a_n)$
10.3	**Geometric sequences and series**	
	Geometric sequences	$a_{n+1} = r\cdot a_n$ or $\frac{a_{n+1}}{a_n} = r$ r is called the **common ratio**.
	The general (nth) term of a geometric sequence	$a_n = a_1 \cdot r^{n-1}$ for $n \geq 1$
	Geometric series	Finite series: $S_n = a_1\frac{(1-r^n)}{(1-r)} \quad r \neq 1$ Infinite series: $\sum_{n=0}^{\infty} a_1 r^n = a_1\frac{1}{(1-r)} \quad \lvert r\rvert < 1$
10.4	**Mathematical induction**	
	Proof by mathematical induction	Prove that S_n is true for all positive integers: Step 1: Show that S_1 is true. Step 2: Assume S_n is true for S_k and show it is true for S_{k+1} (k = positive integer).

Section	Concept	Key Ideas/Formulas
10.5	**The Binomial theorem**	
	Binomial coefficients	$\binom{n}{k} = \frac{n!}{(n-k)!k!}$ $\binom{n}{n} = 1$ and $\binom{n}{0} = 1$
	Binomial expansion	$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
	Pascal's triangle	Shortcut way of remembering binomial coefficients. Each term is found by adding the two numbers in the row directly above it.
	Finding a particular term of a binomial expansion	The $(r+1)$ term of the expansion $(a+b)^n$ is $\binom{n}{r} a^{n-r} b^r$.

CHAPTER REVIEW

10.1 Sequences and Series

Write the first four terms of each sequence. Assume n starts at 1.

1. $a_n = n^3$

2. $a_n = \dfrac{n!}{n}$

3. $a_n = 3n + 2$

4. $a_n = (-1)^n x^{n+2}$

Find the indicated term of each sequence.

5. $a_n = \left(\dfrac{2}{3}\right)^n \qquad a_5 = ?$

6. $a_n = \dfrac{n^2}{3^n} \qquad a_8 = ?$

7. $a_n = \dfrac{(-1)^n(n-1)!}{n(n+1)!} \qquad a_{15} = ?$

8. $a_n = 1 + \dfrac{1}{n} \qquad a_{10} = ?$

Write an expression for the nth term of each given sequence.

9. $3, -6, 9, -12, \ldots$

10. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots$

11. $-1, 1, -1, 1, \ldots$

12. $1, 10, 10^2, 10^3, \ldots$

Simplify each ratio of factorials.

13. $\dfrac{8!}{6!}$

14. $\dfrac{20!}{23!}$

15. $\dfrac{n(n-1)!}{(n+1)!}$

16. $\dfrac{(n-2)!}{n!}$

Write the first four terms of each sequence defined by the recursion formula.

17. $a_1 = 5 \qquad a_n = a_{n-1} - 2$

18. $a_1 = 1 \qquad a_n = n^2 \cdot a_{n-1}$

19. $a_1 = 1, a_2 = 2 \qquad a_n = (a_{n-1})^2 \cdot (a_{n-2})$

20. $a_1 = 1, a_2 = 2 \qquad a_n = \dfrac{a_{n-2}}{(a_{n-1})^2}$

Evaluate each finite series.

21. $\displaystyle\sum_{n=1}^{5} 3$

22. $\displaystyle\sum_{n=1}^{4} \frac{1}{n^2}$

23. $\displaystyle\sum_{n=1}^{6} (3n + 1)$

24. $\displaystyle\sum_{k=0}^{5} \frac{2^{k+1}}{k!}$

Use sigma (summation) notation to represent each sum.

25. $-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \cdots - \frac{1}{64}$

26. $2 + 4 + 6 + 8 + 10 + \cdots + 20$

27. $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \cdots$

28. $x - x^2 + \dfrac{x^3}{2} - \dfrac{x^4}{6} + \dfrac{x^5}{24} - \dfrac{x^6}{120} + \cdots$

Applications

29. A Marine's Investment. With the prospect of continued fighting in Iraq, in December 2004, the Marine Corps offered bonuses of as much as \$30,000—in some cases, tax-free—to persuade enlisted personnel with combat experience and training to reenlist. Suppose a Marine put his entire \$30,000 reenlistment bonus in an account that earned 4% interest compounded monthly. The balance in the account after n months would be

$$A_n = 30{,}000\left(1 + \frac{0.04}{12}\right)^n \qquad n = 1, 2, 3, \ldots$$

His commitment to the Marines is 5 years. Calculate A_{60}. What does A_{60} represent?

30. Sports. The NFL minimum salary for a rookie is \$180,000. Suppose a rookie enters the league making the minimum and gets a \$30,000 raise each year. Write the general term a_n of a sequence that represents the salary of an NFL player making the league minimum during his entire career. Assuming $n = 1$ corresponds to the first year, what does $\sum_{n=1}^{4} a_n$ represent?

10.2 Arithmetic Sequences and Series

Determine whether each sequence is arithmetic. If it is, find the common difference.

31. $7, 5, 3, 1, -1, \ldots$

32. $1^3 + 2^3 + 3^3 + \cdots$

33. $1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

34. $a_n = -n + 3$

35. $a_n = \dfrac{(n+1)!}{n!}$

36. $a_n = 5(n - 1)$

Find the general, or nth, term of each arithmetic sequence given the first term and the common difference.

37. $a_1 = -4 \qquad d = 5$

38. $a_1 = 5 \qquad d = 6$

39. $a_1 = 1 \qquad d = -\frac{2}{3}$

40. $a_1 = 0.001 \qquad d = 0.01$

For each arithmetic sequence described below, find a_1 and d and construct the sequence by stating the general, or nth, term.

41. The 5th term is 13 and the 17th term is 37.

42. The 7th term is -14 and the 10th term is -23.

43. The 8th term is 52 and the 21st term is 130.

44. The 11th term is -30 and the 21st term is -80.

Find each sum.

45. $\displaystyle\sum_{k=1}^{20} 3k$

46. $\displaystyle\sum_{n=1}^{15} (n + 5)$

47. $2 + 8 + 14 + 20 + \cdots + 68$

48. $\frac{1}{4} - \frac{1}{4} - \frac{3}{4} - \cdots - \frac{31}{4}$

Applications

49. Salary. Upon graduating with MBAs, Bob and Tania opt for different career paths. Bob accepts a job with the U.S. Department of Transportation making $45,000 with a guaranteed $2000 raise every year. Tania takes a job with Templeton Corporation making $38,000 with a guaranteed $4000 raise every year. Calculate how many total dollars both Bob and Tania will have each made after 15 years.

50. Gravity. When a skydiver jumps out of an airplane, she falls approximately 16 feet in the 1st second, 48 feet during the 2nd second, 80 feet during the 3rd second, 112 feet during the 4th second, and 144 feet during the 5th second, and this pattern continues. If she deploys her parachute after 5 seconds have elapsed, how far will she have fallen during those 5 seconds?

10.3 Geometric Sequences and Series

Determine whether each sequence is geometric. If it is, find the common ratio.

51. $2, -4, 8, -16, \ldots$

52. $1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \ldots$

53. $20, 10, 5, \frac{5}{2}, \ldots$

54. $\frac{1}{100}, \frac{1}{10}, 1, 10, \ldots$

Write the first five terms of each geometric series.

55. $a_1 = 3 \quad r = 2$

56. $a_1 = 10 \quad r = \frac{1}{4}$

57. $a_1 = 100 \quad r = -4$

58. $a_1 = -60 \quad r = -\frac{1}{2}$

Write the formula for the *n*th term of each geometric series.

59. $a_1 = 7 \quad r = 2$

60. $a_1 = 12 \quad r = \frac{1}{3}$

61. $a_1 = 1 \quad r = -2$

62. $a_1 = \frac{32}{5} \quad r = -\frac{1}{4}$

Find the indicated term of each geometric sequence.

63. 25th term of the sequence $2, 4, 8, 16, \ldots$

64. 10th term of the sequence $\frac{1}{2}, 1, 2, 4, \ldots$

65. 12th term of the sequence $100, -20, 4, -0.8, \ldots$

66. 11th term of the sequence $1000, -500, 250, -125, \ldots$

Evaluate each geometric series, if possible.

67. $\frac{1}{2} + \frac{3}{2} + \frac{3^2}{2} + \cdots + \frac{3^8}{2}$

68. $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}}$

69. $\sum_{n=1}^{8} 5(3)^{n-1}$

70. $\sum_{n=1}^{7} \frac{2}{3}(5)^n$

71. $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

72. $\sum_{n=1}^{\infty} \left(-\frac{1}{5}\right)^{n+1}$

Applications

73. Salary. Murad is fluent in four languages and is offered a job with the U.S. government as a translator. He is hired on the "GS" scale at a base rate of $48,000 with a 2% increase in his salary per year. Calculate what his salary will be *after* he has been with the U.S. government for 12 years.

74. Boat Depreciation. Upon graduating from Auburn University, Philip and Steve get jobs at Disney Ride and Show Engineering and decide to buy a ski boat together. If the boat costs $15,000 new, and depreciates 20% per year, write a formula for the value of the boat n years after it was purchased. How much will the boat be worth when Philip and Steve have been working at Disney for 3 years?

10.4 Mathematical Induction

Prove each statement using mathematical induction for all positive integers *n*.

75. $3n \le 3^n$

76. $4^n < 4^{n+1}$

77. $2 + 7 + 12 + 17 + \cdots + (5n - 3) = \frac{n}{2}(5n - 1)$

78. $2n^2 > (n + 1)^2 \quad n \ge 3$

10.5 The Binomial Theorem

Evaluate each binomial coefficient.

79. $\binom{11}{8}$

80. $\binom{10}{0}$

81. $\binom{22}{22}$

82. $\binom{47}{45}$

Expand each expression using the Binomial theorem.

83. $(x - 5)^4$

84. $(x + y)^5$

85. $(2x - 5)^3$

86. $(x^2 + y^3)^4$

87. $(\sqrt{x} + 1)^5$

88. $(x^{2/3} + y^{1/3})^6$

Expand each expression using Pascal's triangle.

89. $(r - s)^5$

90. $(ax + by)^4$

Find the coefficient *C* of the term in each binomial expansion.

	Binomial	Term
91.	$(x - 2)^8$	Cx^6
92.	$(3 + y)^7$	Cy^4
93.	$(2x + 5y)^6$	Cx^2y^4
94.	$(r^2 - s)^8$	Cr^8s^4

Applications

95. Lottery. In a state lottery in which 6 numbers are drawn from a possible 53 numbers, the number of possible 6-number combinations is equal to $\binom{53}{6}$. How many possible combinations are there?

96. Canasta. In the card game Canasta, two decks of cards including the jokers are used, and 13 cards are dealt to each person. A total of $\binom{108}{13}$ different 13-card Canasta hands can be dealt. How many possible hands are there?

Technology Exercises

Section 10.1

97. Use a graphing calculator "SUM" to find the sum of the series $\sum_{n=1}^{6} \frac{1}{n^2}$.

98. Use a graphing calculator "SUM" to find the sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$, if possible.

Section 10.2

99. Use a graphing calculator to sum $\sum_{n=1}^{75} \left[\frac{3}{2} + \frac{6}{7}(n - 1)\right]$.

100. Use a graphing calculator to sum $\sum_{n=1}^{264} \left[-19 + \frac{1}{3}(n - 1)\right]$.

Section 10.3

101. Apply a graphing utility to plot $y_1 = 1 - 2x + 4x^2 - 8x^3 + 16x^4$ and $y_2 = \frac{1}{1 + 2x}$, and let x range from $[-0.3, 0.3]$. Based on what you see, what do you expect the geometric series $\sum_{n=0}^{\infty} (-1)^n (2x)^n$ to sum to in this range of x-values?

102. Does the sum of the infinite series $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$ exist? Use a graphing calculator to find it and round to four decimal places.

Section 10.4

103. Use a graphing calculator to sum the series $2 + 7 + 12 + 17 + \cdots + (5n - 3)$ and evaluate the expression $\frac{n}{2}(5n - 1)$ for $n = 200$. Do they agree with each other? Do your answers confirm the proof for Exercise 77?

104. Use a graphing calculator to plot the graphs of $y_1 = 2x^2$ and $y_2 = (x + 1)^2$ in the [100, 1000] by [10,000, 2,500,000] viewing rectangle. Do your results confirm the proof for Exercise 78?

Section 10.5

105. With a graphing utility, plot $y_1 = 1 + 8x$, $y_2 = 1 + 8x + 24x^2$, $y_3 = 1 + 8x + 24x^2 + 32x^3$, $y_4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4$, and $y_5 = (1 + 2x)^4$ for $-0.1 < x < 0.1$. What do you notice happening each time an additional term is added to the series? Now, let $0.1 < x < 1$. Does the same thing happen?

106. With a graphing utility, plot $y_1 = 1 - 8x$, $y_2 = 1 - 8x + 24x^2$, $y_3 = 1 - 8x + 24x^2 - 32x^3$, $y_4 = 1 - 8x + 24x^2 - 32x^3 + 16x^4$, and $y_5 = (1 - 2x)^4$ for $-0.1 < x < 0.1$. What do you notice happening each time an additional term is added to the series? Now, let $0.1 < x < 1$. Does the same thing happen?

For Exercises 1–5, use the sequence $1, x, x^2, x^3, \ldots$

1. Write the nth term of the sequence.
2. Classify this sequence as arithmetic, geometric, or neither.
3. Find the nth partial sum of the series S_n.
4. Assuming this sequence is infinite, write its series using sigma notation.
5. Assuming this sequence is infinite, what condition would have to be satisfied in order for the sum to exist?
6. Find the following sum: $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$.
7. Find the following sum: $\sum_{n=1}^{10} 3 \cdot \left(\frac{1}{4}\right)^n$.
8. Find the following sum: $\sum_{k=1}^{50} (2k + 1)$.
9. Write the following series using sigma notation, then find its sum: $2 + 7 + 12 + 17 + \cdots + 497$.
10. Use mathematical induction to prove that $2 + 4 + 6 + \cdots + 2n = n^2 + n$.
11. Evaluate $\dfrac{7!}{2!}$.
12. Find the third term of $(2x + y)^5$.

In Exercises 13 and 14, evaluate each expression.

13. $\binom{15}{12}$
14. $\binom{k}{k}$
15. Simplify the expression $\dfrac{42!}{7! \cdot 37!}$.
16. Simplify the expression $\dfrac{(n + 2)!}{(n - 2)!}$ for $n \geq 2$.
17. Expand the expression $\left(x^2 + \dfrac{1}{x}\right)^5$.
18. Use the Binomial theorem to expand the binomial $(3x - 2)^4$.

Prove each statement using mathematical induction for all positive integers n.

19. $3 + 7 + 11 + \cdots + 4n - 1 = 2n^2 + n$
20. $2^1 + 1^3 + 2^2 + 2^3 + 2^3 + 3^3 + 2^4 + 4^3 + \cdots + 2^n + n^3$
$= 2^{n+1} - 2 + \dfrac{n^2(n + 1)^2}{4}$

Find the sum of each infinite geometric series, if possible.

21. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$
22. $1 + \dfrac{3}{4} + \dfrac{9}{16} + \dfrac{27}{64} + \cdots$

Apply sigma notation to represent each series.

23. $\dfrac{5}{8} + \dfrac{5}{10} + \dfrac{5}{12} + \dfrac{5}{14} + \cdots$
24. $\dfrac{1}{4} + \dfrac{5}{8} + \dfrac{7}{16} + \dfrac{17}{32} + \dfrac{31}{64} + \dfrac{65}{128} + \cdots$

In Exercises 25 and 26, expand the expression using the Binomial theorem.

25. $(2 - 3x)^7$
26. $(x^2 + y^3)^6$
27. Find the constant term in the expression $\left(x^3 + \dfrac{1}{x^3}\right)^{20}$.
28. Use a graphing calculator to sum $\sum_{n=1}^{125} \left[-\frac{11}{4} + \frac{5}{6}(n - 1)\right]$.

1. Find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ of the function $f(x) = \dfrac{x^2}{x+1}$.

2. Given $f(x) = x^2 + x$ and $g(x) = \dfrac{1}{x+3}$, find $(f \circ g)(x)$.

3. Write the polynomial function $f(x) = x^4 - 4x^3 - 4x^2 - 4x - 5$ as a product of linear factors.

4. Find the inverse of the function $f(x) = 5x - 4$.

5. Use long division to divide the polynomials: $(-6x^5 + 3x^3 + 2x^2 - 7) \div (x^2 + 3)$.

6. Write the logarithmic equation $\log 0.001 = -3$ in its equivalent exponential form.

7. Solve for x: $\ln(5x - 6) = 2$. Round to three decimal places.

8. **Sprinkler Coverage.** A sprinkler has a 21-foot spray and it rotates through an angle of 50º. What is the area that the sprinkler covers?

9. Find the exact value of $\sin\left(-\dfrac{7\pi}{4}\right)$.

10. If $\tan x = \dfrac{7}{24}$ and $\pi < x < \dfrac{3\pi}{2}$, find $\cos(2x)$.

11. Solve the trigonometric equation exactly $2\cos^2\theta - \cos\theta - 1 = 0$ over $0 \le \theta \le 2\pi$.

12. Given $\gamma = 53°$, $a = 18$, and $c = 17$, determine if a triangle (or two) exist and if so solve the triangles.

13. Express the complex number $\sqrt{2} - \sqrt{2}i$ in polar form.

14. Solve the system of linear equations.

$$\begin{aligned} 8x - 5y &= 15 \\ y &= \frac{8}{5}x + 10 \end{aligned}$$

15. Solve the system of linear equations.

$$\begin{aligned} 2x - y + z &= 1 \\ x - y + 4z &= 3 \end{aligned}$$

16. Maximize the objective function $z = 4x + 5y$, subject to the constraints $x + y \le 5$, $x \ge 1$, $y \ge 2$.

17. Solve the system using Gauss–Jordan elimination.

$$\begin{aligned} x + 5y - 2z &= 3 \\ 3x + y + 2z &= -3 \\ 2x - 4y + 4z &= 10 \end{aligned}$$

18. Given

$$A = \begin{bmatrix} 3 & 4 & -7 \\ 0 & 1 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 8 & -2 & 6 \\ 9 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 9 & 0 \\ 1 & 2 \end{bmatrix}$$

find $C(A + B)$.

19. Calculate the determinant.

$$\begin{vmatrix} 2 & 5 & -1 \\ 1 & 4 & 0 \\ -2 & 1 & 3 \end{vmatrix}$$

20. Find the equation of a parabola with vertex (3, 5) and directrix $x = 7$.

21. Graph $x^2 + y^2 < 4$.

22. The parametric equations $x = 2\sin t$, $y = 3\cos t$ define a plane curve. Find an equation in rectangular form that also corresponds to the plane curve.

23. Find the sum of the finite series $\sum_{n=1}^{4} \dfrac{2^{n-1}}{n!}$.

24. Classify the sequence as arithmetic, geometric, or neither.

$$5, 15, 45, 135, \ldots$$

25. The number of subsets with k elements of a set with n elements is given by $\dfrac{n!}{k!(n-k)!}$. Find the number of subsets with 3 elements of a set with 7 elements.

26. Find the binomial expansion of $(x - x^2)^5$.

Answers To Odd Numbered Exercises*

CHAPTER 0

Section 0.1

1. $m = 2$ **3.** $t = \frac{7}{5}$ **5.** $x = -10$

7. $n = 2$ **9.** $x = 12$ **11.** $t = -\frac{15}{2}$

13. $x = -1$ **15.** $p = -\frac{9}{2}$ **17.** $x = \frac{1}{4}$

19. $x = -\frac{3}{2}$ **21.** $a = -8$ **23.** $x = -15$

25. $c = -\frac{35}{13}$ **27.** $m = \frac{60}{11}$ **29.** $x = 36$

31. $p = 8$ **33.** $y = -2$ **35.** $p = 2$

37. no solution **39.** 12 mi **41.** 270 units

43. $r_1 = 3$ ft, $r_2 = 6$ ft

45. 5.25 ft **47.** \$20,000 at 4%, \$100,000 at 7%

49. \$3000 at 10%, \$5500 at 2%, \$5500 at 40%

51. 70 ml of 5% HCl, 30 ml of 15% HCl **53.** 9 min

55. \$3.07 per gallon **57.** 233 ml **59.** 2.3 mph

61. walker: 4 mph, jogger: 6 mph

63. bicyclist: 6 min, walker: 18 min **65.** 22.5 hr

67. 2.4 hr **69.** 2 field goals, 6 touchdowns

71. 3.5 ft from the center

73. Fulcrum is 0.4 unit from Maria and 0.6 unit from Max.

75. Should have subtracted $4x$ and added 7 to both sides; $x = 5$

77. $x = \dfrac{c - b}{a}$ **79.** $\dfrac{P - 2l}{2} = w$ **81.** $\dfrac{2A}{b} = h$

83. $\dfrac{A}{l} = w$ **85.** $\dfrac{V}{lw} = h$

87. Janine's average speed is 58 mph, Tricia's average speed is 70 mph.

89. $x = 2$ **91.** all real numbers **93.** \$191,983.35

95. Option B: better for 5 or few plays/mo
Option A: better for 6 or more plays/mo

Section 0.2

1. $x = 3$ or $x = 2$ **3.** $p = 5$ or $p = 3$

5. $x = -4$ or $x = 3$ **7.** $x = -\frac{1}{4}$

9. $y = \frac{1}{3}$ **11.** $y = 0$ or $y = 2$

13. $p = \frac{2}{3}$ **15.** $x = -3$ or $x = 3$

17. $x = -6$ or $x = 2$ **19.** $p = -5$ or $p = 5$

21. $x = -2$ or $x = 2$ **23.** $p = \pm 2\sqrt{2}$

25. $x = \pm 3i$ **27.** $x = -3, 9$

29. $x = \dfrac{-3 \pm 2i}{2}$ **31.** $x = \dfrac{2 \pm 3\sqrt{3}}{5}$

33. $x = -2, 4$ **35.** $x = -3, 1$

37. $t = 1, 5$ **39.** $y = 1, 3$

41. $p = \dfrac{-4 \pm \sqrt{10}}{2}$ **43.** $x = \frac{1}{2}, 3$

45. $x = \dfrac{4 \pm 3\sqrt{2}}{2}$ **47.** $t = \dfrac{-3 \pm \sqrt{13}}{2}$

49. $s = \dfrac{-1 \pm i\sqrt{3}}{2}$ **51.** $x = \dfrac{3 \pm \sqrt{57}}{6}$

53. $x = 1 \pm 4i$ **55.** $x = \dfrac{-7 \pm \sqrt{109}}{10}$

57. $x = \dfrac{-4 \pm \sqrt{34}}{3}$ **59.** $v = -2, 10$

61. $t = -6, 1$ **63.** $x = -7, 1$

65. $p = 4 \pm 2\sqrt{3}$ **67.** $w = \dfrac{-1 \pm i\sqrt{167}}{8}$

69. $p = \dfrac{9 \pm \sqrt{69}}{6}$ **71.** $t = \dfrac{10 \pm \sqrt{130}}{10}$

73. $x = -0.3, 0.4$

75. $t = 8$ (Aug. 2003) and $t = 12$ (Dec. 2003)

77. 31,000 units **79.** \$1 per bottle

81. 3 days

83. **a.** 55.25 sq in.
b. $4x^2 + 30x + 55.25$
c. $4x^2 + 30x$ represents the increase in usable area of the paper.
d. $x \approx 0.3$ in.

85. 20 in. **87.** 17, 18

89. Length: 15 ft, width: 9 ft **91.** Base: 6, height: 20

93. Impact with ground in 2.5 sec

95. 21.2 ft **97.** 5 ft $\times$ 5 ft

99. 2.3 ft **101.** 10 days

103. The problem is factored incorrectly. The correction would be $t = -1, 6$.

105. When taking the square root of both sides, the i is missing from the right side. The correction would be $a = \pm\frac{3}{4}i$.

107. false **109.** true

*Answers that require a proof, graph, or otherwise lengthy solution are not included.

111. $x^2 - 2ax + a^2 = 0$

113. $x^2 - 7x + 10 = 0$

115. $t = \pm\sqrt{\dfrac{2s}{g}}$

117. $c = \pm\sqrt{a^2 + b^2}$

119. $x = 0, \pm 2$

121. $x = -1, \pm 2$

123. $\dfrac{-b}{2a} + \dfrac{\sqrt{b^2 - 4ac}}{2a} - \dfrac{b}{2a} - \dfrac{\sqrt{b^2 - 4ac}}{2a} = \dfrac{-2b}{2a} = \dfrac{-b}{a}$

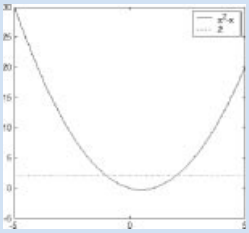
125. $x^2 - 6x + 4 = 0$

127. 250 mph

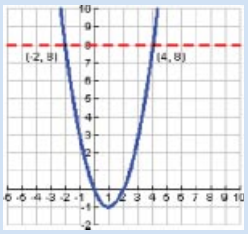
129. $ax^2 - bx + c = 0$

131. Small jet: 300 mph, 757: 400 mph

133. $x = -1, 2$

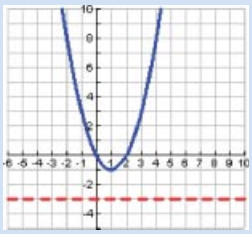


135. a. $x = -2, 4$

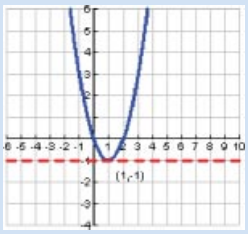


$b = -1$: $x = 1$

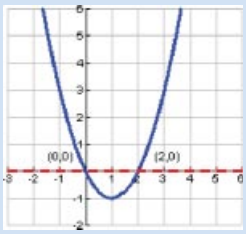
b. $b = -3$: $x = 1 \pm i\sqrt{2}$

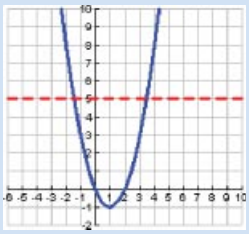


$b = 0$: $x = 0, 2$



$b = 5$: $x = 1 \pm \sqrt{6}$





Section 0.3

1. $x \neq 2$, no solution

3. $p \neq 1$, no solution

5. $x \neq -2, x = -10$

7. $n \neq -1, 0$, no solution

9. $a \neq 0, -3$, no solution

11. $n \neq 1, n = \frac{53}{11}$

13. $x \neq -\frac{1}{5}, \frac{1}{2}, x = -3$

15. $t \neq 1$, no solution

17. $x = 3$ or $x = 4$

19. $x = -\frac{3}{4}$ or $x = 2$

21. no solution

23. $x = 5$

25. $y = -\frac{1}{2}$

27. $x = 5$

29. $x = -9$ or $x = 7$

31. $x = 4$

33. $y = 0, 25$

35. $s = 3, 6$

37. $x = -3, -1$

39. $x = 0$

41. $x = 1$ and $x = 5$

43. $x = 7$

45. $x = -3$ and $x = -\frac{15}{4}$

47. $x = \frac{5}{2}$

49. no solution

51. $x = 1$

53. $x = 4$ and $x = -8$

55. $x = 1, 5$

57. $x = 7$

59. $x = 4$

61. $x = 0, x = -8$

63. $x = \pm 1, x = \pm\sqrt{2}$

65. $x = \dfrac{\pm i\sqrt{6}}{2}, x = \pm i\sqrt{2}$

67. $t = \frac{5}{4}, t = 3$

69. $x = \pm 1, \pm i, x = \pm\frac{1}{2}, \pm\frac{1}{2}i$

71. $y = -\frac{3}{4}, y = 1$

73. $z = 1$

75. $t = -27, t = 8$

77. $x = -\frac{4}{3}, x = 0$

79. $u = \pm 8, u = \pm 1$

81. $x = 0, -3, 4$

83. $p = 0, \pm\frac{3}{2}$

85. $u = 0, \pm 2, \pm 2i$

87. $x = \pm 3, 5$

89. $y = -2, 5, 7$

91. $x = 0, 3, -1$

93. $t = \pm 5$

95. $y = 2, 3$

97. $p = 10$ or $p = 4$

99. $y = 5$ or $y = 3$

101. $t = 4$ or $t = 2$

103. $x = 8$ or $x = -1$

105. $y = 0$ or $y = \frac{2}{3}$

107. $x = -\frac{23}{14}$ or $x = \frac{47}{14}$

109. $x = 13$ or $x = -3$

111. $p = 7$ or $p = -13$

113. $y = 9$ or $y = -5$

115. $x = \pm\sqrt{5}$ or $x = \pm\sqrt{3}$

117. $x = \pm 2$

119. January and September

121. 162 cm

123. 7.5 cm in front of lens

125. Object distance = 6 cm
image distance = 3 cm

127. 132 ft

129. 25 cm

131. 80% of the speed of light

133. no solution

135. Cannot cross multiply—must multiply by LCD first; $p = \frac{6}{5}$

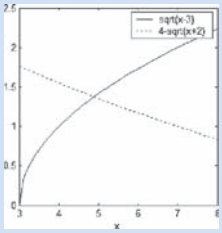
137. false

139. $x = \dfrac{a - b}{c}$

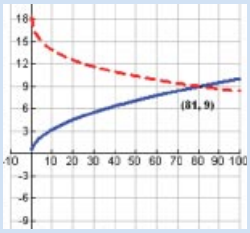
141. $x = -2$

143. $x = \dfrac{by}{a - y - cy}, x \neq 0, -\dfrac{b}{c + 1}$

145. $x = \frac{313}{64} \cong 4.89$

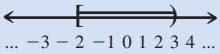


147. $x = 81$

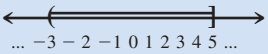


Section 0.4

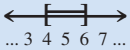
1. $[-2, 3)$



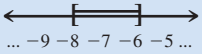
3. $(-3, 5]$



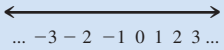
5. $[4, 6]$



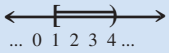
7. $[-8, -6]$



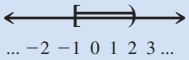
9. $\varnothing$



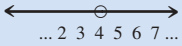
11. $[1, 4)$



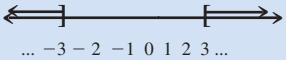
13. $[-1, 2)$



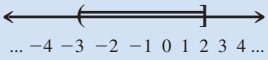
15. $(-\infty, 4) \cup (4, \infty)$



17. $(-\infty, -3] \cup [3, \infty)$



19. $(-3, 2]$



21. $(-3, \infty)$

23. $(-\infty, 6)$

25. $(-\infty, 1)$

27. $(-\infty, -0.5)$

29. $[-8, 4)$

31. $(-6, 6)$

33. $\left[\frac{1}{2}, \frac{5}{4}\right]$

35. $[-4.5, 0.5]$

37. $\left[-1, \frac{3}{2}\right]$

39. $\left(\frac{1}{3}, \frac{1}{2}\right)$

41. $\left(-\infty, -\frac{1}{2}\right] \cup [3, \infty)$

43. $\left(-\infty, -1 - \sqrt{5}\right] \cup \left[-1 + \sqrt{5}, \infty\right)$

45. $\left(2 - \sqrt{10}, 2 + \sqrt{10}\right)$

47. $(-\infty, 0] \cup [3, \infty)$

49. $(-\infty, -3) \cup (3, \infty)$

51. $(-\infty, -2] \cup [0, 1]$

53. $(0, 1) \cup (1, \infty)$

55. $(-\infty, -2) \cup [-1, 2)$

57. $(-\infty, -5] \cup (-2, 0]$

59. $(-2, 2)$

61. $\mathbb{R}$ (consistent)

63. $[-3, 3) \cup (3, \infty)$

65. $(-3, -1] \cup (3, \infty)$

67. $(-\infty, -4) \cup (2, 5]$

69. $(-\infty, -2) \cup (2, \infty)$

71. $(-\infty, 2) \cup (6, \infty)$

73. $[3, 5]$

75. $\mathbb{R}$

77. $(-\infty, 2] \cup [5, \infty)$

79. $\mathbb{R}$

81. $\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$

83. $(-\infty, -3) \cup (3, \infty)$

85. $[-3, 3]$

87. $0.9\, r_T \le r_R \le 1.1\, r_T$

89. $4{,}386.25 \le T \le 15{,}698.75$

91. 285,700 units

93. Between 33% and 71% intensities

95. Between 30 and 100 orders

97. For years 3–5, the car is worth more than you owe. In the first 3 years you owe more than the car is worth.

99. 75 sec

101. A price increase less than \$1 per bottle or greater than \$20 per bottle

103. Win: $d < 4$, tie: $d = 4$

105. When the number of units sold was between 25 and 75 units.

107. Forgot to flip the sign when dividing by -3. Answer should be $[2, \infty)$.

109. Cannot divide by x $(-\infty, 0) \cup (3, \infty)$

111. true

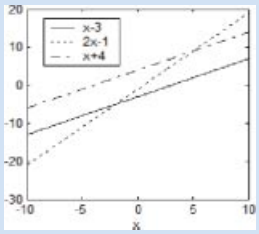
113. false

115. $\mathbb{R}$

117. no solution

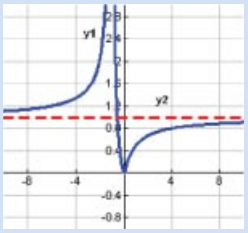
119. **a.** $(-2, 5)$

b.



c. agree

121. $\left(-\frac{1}{2}, \infty\right)$



Section 0.5

1. $d = 4$, $(3, 3)$

3. $d = 4\sqrt{2}$, $(1, 2)$

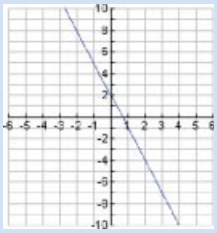
5. $d = 3\sqrt{10}$, $\left(-\frac{17}{2}, \frac{7}{2}\right)$

7. $d = 5$, $\left(-5, \frac{1}{2}\right)$

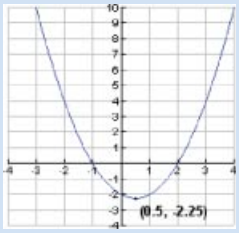
9. $d = 4\sqrt{2}, (-4, -6)$

11. $d = 5, \left(\frac{3}{2}, \frac{11}{6}\right)$

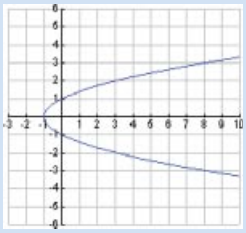
13.



15.



17.



19. (3, 0), (0, −6)

21. (4, 0), no y-intercept

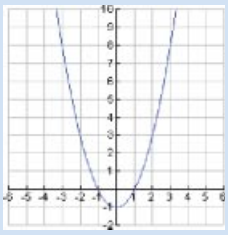
23. $(\pm 2, 0), (0, \pm 4)$

25. x-axis

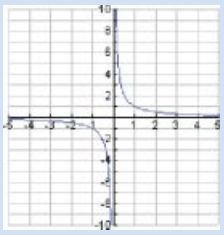
27. x-axis

29. y-axis

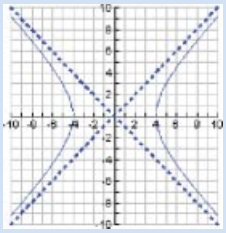
31.



33.



35.



37. $(x - 5)^2 + (y - 7)^2 = 81$

39. $(x + 11)^2 + (y - 12)^2 = 169$

41. $(x - 5)^2 + (y + 3)^2 = 12$

43. $\left(x - \frac{2}{3}\right)^2 + \left(y + \frac{3}{5}\right)^2 = \frac{1}{16}$

45. $(2, -5), r = 7$

47. $(4, 9), r = 2\sqrt{5}$

49. $\left(\frac{2}{5}, \frac{1}{7}\right), r = \frac{2}{3}$

51. $(5, 7), r = 9$

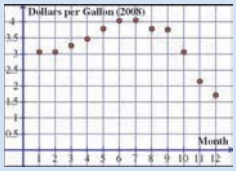
53. $(1, 3), r = 3$

55. $(5, -3), r = 2\sqrt{3}$

57. $(3, 2), r = 2\sqrt{3}$

59. $\left(\frac{1}{2}, -\frac{1}{2}\right), r = \frac{1}{2}$

61.



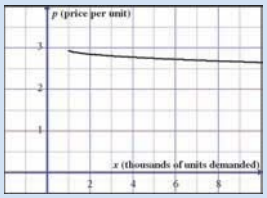
63. 268 mi

65. \$330 million

67. $x^2 + y^2 = 2{,}250{,}000$

69. $x^2 + y^2 = 40{,}000$

71. $x \geq 1$ or $[1, \infty)$; the demand model is defined when at least 1000 units per day are demanded.



73. The equation is not linear—you need more than two points to plot the graph.

75. The center should be (4, −3).

77. false

79. true

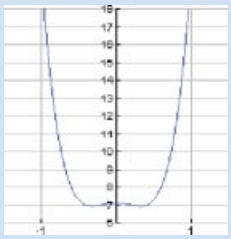
81. single point (−5, 3)

83. origin

85. $(x - 3)^2 + (y + 2)^2 = 20$

87. $4c = a^2 + b^2$

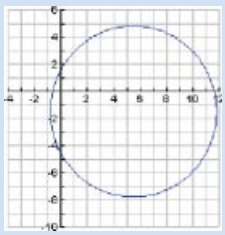
89. y-axis



91. **a.** $(5.5, -1.5), r = 6.3$

b. $y = -1.5 \pm \sqrt{39.69 - (x - 5.5)^2}$

c.



Section 0.6

1. 3

3. -2

5. $-\frac{19}{10}$

7. 2.379

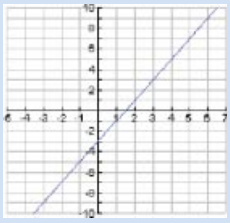
9. -3

11. (0.5, 0), (0, -1), $m = 2$, increasing

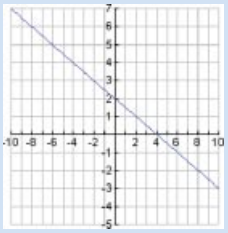
13. (1, 0), (0, 1), $m = -1$, decreasing

15. none, (0, 1), $m = 0$, horizontal

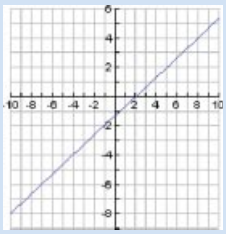
17. $\left(\frac{3}{2}, 0\right)$, (0, -3)



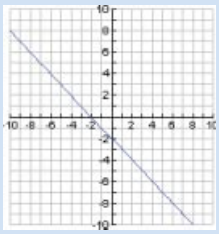
19. (4, 0), (0, 2)



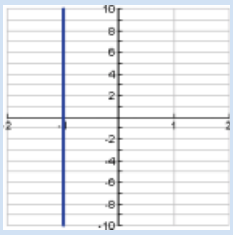
21. (2, 0), $\left(0, -\frac{4}{3}\right)$



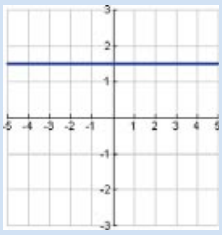
23. (-2, 0), (0, -2)



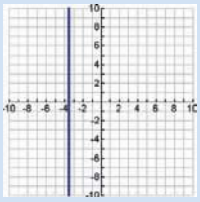
25. (-1, 0), none



27. none, (0, 1.5)



29. $\left(-\frac{7}{2}, 0\right)$, none



31. $y = \frac{2}{5}x - 2$ $\quad m = \frac{2}{5}$ $\quad y$-intercept: (0, -2)

33. $y = -\frac{1}{3}x + 2$ $\quad m = -\frac{1}{3}$ $\quad y$-intercept: (0, 2)

35. $y = 4x - 3$ $\quad m = 4$ $\quad y$-intercept: (0, -3)

37. $y = -2x + 4$ $\quad m = -2$ $\quad y$-intercept: (0, 4)

39. $y = \frac{2}{3}x - 2$ $\quad m = \frac{2}{3}$ $\quad y$-intercept: (0, -2)

41. $y = -\frac{3}{4}x + 6$ $\quad m = -\frac{3}{4}$ $\quad y$-intercept: (0, 6)

43. $y = 2x + 3$ **45.** $y = -\frac{1}{3}x$ **47.** $y = 2$

49. $x = \frac{3}{2}$ **51.** $y = 5x + 2$ **53.** $y = -3x - 4$

55. $y = \frac{3}{4}x - \frac{7}{4}$ **57.** $y = 4$ **59.** $x = -1$

61. $y = \frac{3}{5}x + \frac{1}{5}$ **63.** $y = -5x - 16$ **65.** $y = \frac{1}{6}x - \frac{121}{3}$

67. $y = -3x + 1$ **69.** $y = \frac{3}{2}x$ **71.** $x = 3$

73. $y = 7$ **75.** $y = \frac{6}{5}x + 6$ **77.** $x = -6$

79. $x = \frac{2}{5}$ **81.** $y = x - 1$ **83.** $y = -2x + 3$

85. $y = -\frac{1}{2}x + 1$ **87.** $y = 2x + 7$ **89.** $y = \frac{3}{2}x$

91. $y = 5$ **93.** $y = 2$ **95.** $y = \frac{3}{2}x - 4$

97. $y = \frac{5}{4}x + \frac{3}{2}$ **99.** $y = \frac{3}{7}x + \frac{5}{2}$

101. $C(h) = 1200 + 25h$; \$2000 **103.** \$375

105. 347 units **107.** $F = \frac{9}{5}C + 32$, $-40°\text{C} = -40°\text{F}$

109. $\frac{1}{50}$ in./yr **111.** 0.06 oz/yr, 6 lb 12.4 oz

113. y-intercept is the flat monthly fee of \$35.

115. -0.35 in./yr, 2.75 in.

117. 2.4 plastic bags per year (in billions), 404 billion

119. **a.** (1, 31.93) (2, 51.18) (5, 111.83)

b. $m = 25.59$. This means that when you buy one bottle of Hoisin it costs \$31.93 per bottle.

c. $m = 31.93$. This means that when you buy two bottles of Hoisin it costs \$25.59 per bottle.

d. $m = 22.366$. This means that when you buy five bottles of Hoisin it costs \$22.37 per bottle.

121. The computations used to calculate the x- and y-intercepts should be reversed. So, the x-intercept is (3, 0) and the y-intercept is (0, -2).

123. The denominator and numerator in the slope computation should be switched, resulting in the slope being undefined.

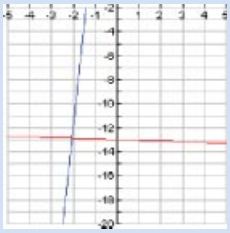
125. true **127.** false

129. Any vertical line is perpendicular to a line with slope 0.

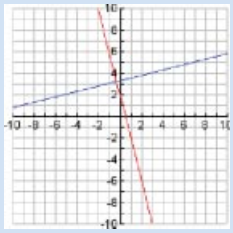
131. $y = -\dfrac{A}{B}x + 1$ **133.** $y = \dfrac{B}{A}x + (2B - 1)$

135. $b_1 = b_2$

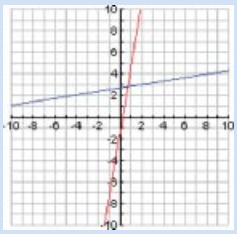
137. perpendicular



139. perpendicular



141. neither



Section 0.7

1. $y = kx$ **3.** $V = kx^3$ **5.** $z = km$

7. $f = \dfrac{k}{\lambda}$ **9.** $F = \dfrac{kw}{L}$ **11.** $v = kgt$

13. $R = \dfrac{k}{PT}$ **15.** $y = k\sqrt{x}$ **17.** $d = rt$

19. $V = lwh$ **21.** $A = \pi r^2$ **23.** $V = \dfrac{\pi}{16}hr^2$

25. $V = \dfrac{400{,}000}{P}$ **27.** $F = \dfrac{2\pi}{\lambda L}$ **29.** $t = \dfrac{19.2}{s}$

31. $R = \dfrac{4.9}{I^2}$ **33.** $R = \dfrac{0.01L}{A}$ **35.** $F = \dfrac{0.025m_1m_2}{d^2}$

37. $W = 7.5H$ **39.** 1292 mph **41.** $F = 1.618H$

43. 24 cm **45.** \$37.50 **47.** 20,000

49. 600 w/m^2

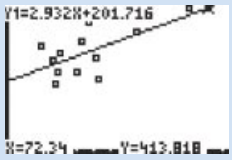
51. Bank of America: 1.5%; Navy Federal Credit Union: 3%

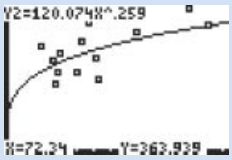
53. $\frac{11}{12}$ or 0.92 atm

55. Should be y is <u>inversely</u> proportional to x

57. true **59.** b **61.** $\sigma_{p_1}^2 = 1.23C_n^2k^{7/6}L^{11/6}$

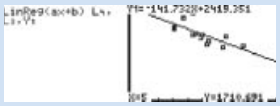
63. a. $y = 2.93x + 201.72$ **b.** 120.07, $y = 120.074x^{0.259}$



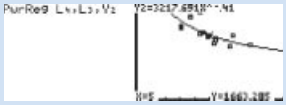


c. When the oil price is \$72.70 per barrel in September 2006, the predicted stock index obtained from the least squares regression line is 415, and from the equation of direct variation it is 364. The least squares regression line provides a closer approximation to the actual value, 417.

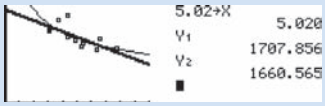
65. a. $y = -141.73x + 2{,}419.35$



b. 3,217.69, $y = \dfrac{3217.69}{x^{0.41}}$



c. When the 5-year maturity rate is 5.02% in September 2006, the predicted number of housing units obtained from the least squares regression line is 1708, and the equation of inverse variation is 1661. The equation of the least squares regression line provides a closer approximation to the actual value, 1861. The picture of the least squares line, with the scatterplot, as well as the computations using the TI-8* is as follows:



67. a. $y = 0.218x + 0.898$ **b.** about \$2.427 per gallon, yes
c. \$3.083

Section 0.8

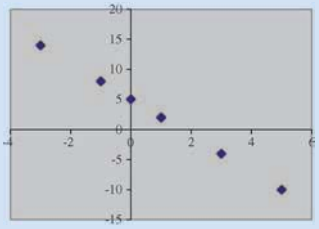
1. Negative linear association because the data closely cluster around what is reasonably described as a linear with negative slope.

3. Although the data seem to be comprised of two *linear* segments, the overall data set cannot be described as having a positive or negative direction of association. Moreover, the pattern of the data is not linear, per se; rather, it is nonlinear and conforms to an identifiable curve (an upside down V called the *absolute value* function).

5. **B** because the association is positive, thereby eliminating choices A and C. And, since the data are closely clustered around a linear curve, the bigger of the two correlation coefficients, 0.80 and 0.20, is more appropriate.

7. **C** because the association is negative, thereby eliminating choices B and D. And, the data are more loosely clustered around a linear curve than are those pictured in #6. So, the correlation coefficient is the negative choice closer to 0.

9. a.

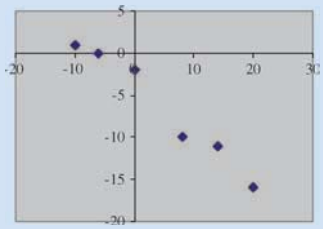


b. The data seem to be nearly perfectly aligned to a line with negative slope. So, it is reasonable to guess that the correlation coefficient is very close to -1.

c. The equation of the best fit line is $y = -3x + 5$ with a correlation coefficient of $r = -1$.

d. There is a perfect negative linear association between x and y.

11. a.

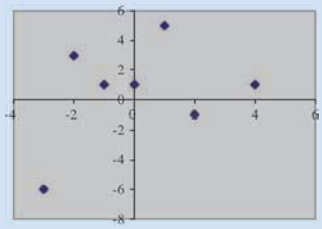


b. The data tends to fall from left to right, so that the correlation coefficient should be negative. Also, the data do not seem to stray too far from a linear curve, so the r value should be reasonably close to -1, but not equal to it. A reasonable guess would be around -0.90.

c. The equation of the best fit line is approximately $y = -0.5844x - 3.801$ with a correlation coefficient of about $r = 0.9833$.

d. There is a strong (but not perfect) negative linear relationship between x and y.

13. a.

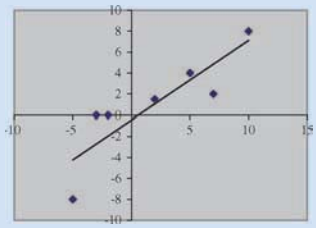


b. The data seems to rise from left to right, but it is difficult to be certain about this relationship since the data stray considerably away from an identifiable line. As such, it is reasonable to guess that r is a rather small value close to 0, say around 0.30.

c. The equation of the best fit line is approximately $y = 0.5x + 0.5$ with a correlation coefficient of about 0.349.

d. There is a very loose (bordering on unidentifiable) positive linear relationship between x and y.

15. a.

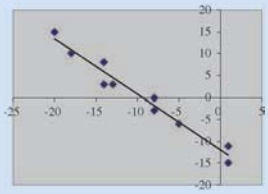


The equation of the best fit line is about $y = 0.7553x - 0.4392$ with a correlation coefficient of about $r = 0.868$.

b. The values $x = 0$ and $x = -6$ are within the range of the data set, so that using the best fit line for predictive purposes is reasonable. This is not the case for the values $x = 12$ and $x = -15$. The predicted value of y when $x = 0$ is approximately -0.4392, and the predicted value of y when $x = -6$ is -4.971.

c. Solve the equation $2 = 0.7553x - 0.4392$ for x to obtain: $2.4392 = 0.7553x$ so that $x = 3.229$. So, using the best fit line, you would expect to get a y-value of 2 when x is approximately 3.229.

17. a.

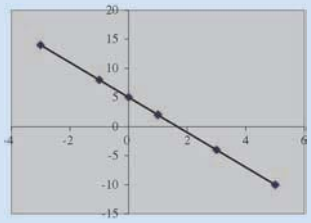


The equation of the best fit line is about $y = -1.2631x - 11.979$ with a correlation coefficient of about $r = -0.980$.

b. The values $x = -15, -6$, and 0 are within the range of the data set, so that using the best fit line for predictive purposes is reasonable. This is not the case for the value $x = 12$. The predicted value of y when $x = -15$ is approximately 6.9675, the predicted value of y when $x = -6$ is about -4.4004, and the predicted value of y when $x = 0$ is -11.979.

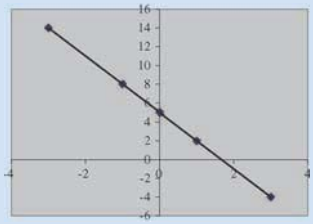
c. Solve the equation $2 = -1.2631x - 11.979$ for x to obtain: $13.979 = -1.2631x$ so that $x = -11.067$. So, using the best fit line, you would expect to get a y-value of 2 when x is approximately -11.067.

19. a. The scatterplot for the entire data set is:



The equation of the best fit line is $y = -3x + 5$ with a correlation coefficient of $r = -1$.

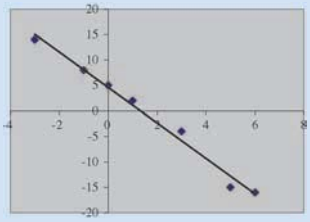
b. The scatterplot for the data set obtained by removing the starred data point $(5, -10)$ is:



The equation of the best fit line of this modified data set is $y = -3x + 5$ with a correlation coefficient of $r = -1$.

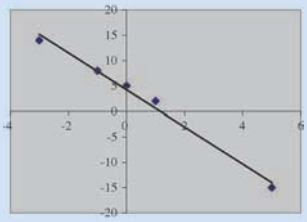
c. Removing the data point did not result in the slightest change in either the equation of the best fit line or the correlation coefficient. This is reasonable since the relationship between x and y in the original data set is perfectly linear, so that all of the points lie ON the same line. As such, removing one of them has no effect on the line itself.

21. a. The scatterplot for the entire data set is:



The equation of the best fit line is $y = -3.4776x + 4.6076$ with a correlation coefficient of about $r = -0.993$.

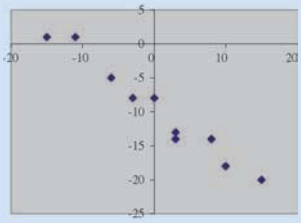
b. The scatterplot for the data set obtained by removing the starred data points $(3, -4)$ and $(6, -16)$ is:



The equation of the best fit line of this modified data set is $y = -3.6534x + 4.2614$ with a correlation coefficient of $r = -0.995$.

c. Removing the data point did change both the best fit line and the correlation coefficient, but only very slightly.

23. a.

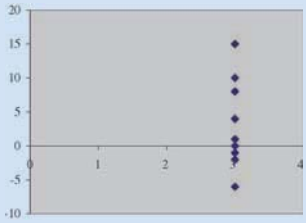


b. The correlation coefficient is approximately $r = -0.980$. This is identical to the r-value from Problem 17. This makes sense because simply interchanging the x and y-values does not change how the points cluster together in the xy-plane.

c. The equation of the best fit line for the paired data (y, x) is $x = -0.7607y - 9.4957$.

d. It is not reasonable to use the best fit line in (c) to find the predicted value of x when $y = 23$ because this value falls outside the range of the given data. However, it is okay to use the best fit line to find the predicted values of x when $y = 2$ or $y = -16$. Indeed, the predicted value of x when $y = 2$ is about -11.0171, and the predicted value of x when $y = -16$ is about 2.6755.

25. First, note that the scatterplot is given by



The paired data all lie identically on the vertical line $x = 3$. As such, you might think that the square of the correlation coefficient would be 1 and the best fit line is, in fact, $x = 3$. However, since there is absolutely no *variation* in the x-values for this data set, it turns out that in the formula for the correlation coefficient

$$r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\sum x^2 - \left(\sum x\right)^2} \cdot \sqrt{n\sum y^2 - \left(\sum y\right)^2}}$$

the quantity $\sqrt{n\sum x^2 - \left(\sum x\right)^2}$ turns out to be zero. (Check this on Excel for this data set!) As such, there is no meaningful r-value for this data set.

Also, the best fit line is definitely the vertical line $x = 3$, but the technology cannot provide it because its slope is undefined.

27. The y-intercept 1.257 is mistakenly interpreted as the slope. The correct interpretation is that for every unit increase in x, the y-value increases by about 5.175.

29. a. Here is a table listing all of the correlation coefficients between each of the events and the total points:

EVENT	r
100 m	−0.714
Long Jump	0.768
Shot Put	0.621
High Jump	0.627
400 m	−0.704
1500 m	−0.289
110 m hurdle	−0.653
Discus	0.505
Pole Vault	0.283
Javeline	0.421

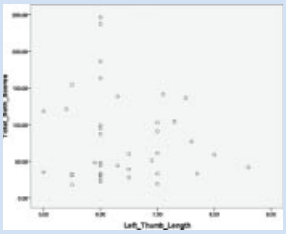
Long jump has the strongest relationship to the total points.

b. The correlation coefficient between *long jump* and *total events* is $r = 0.768$.

c. The equation of the best fit line between the two events in (b) is $y = 838.70x + 1957.77$.

d. Evaluate the equation in (c) at $x = 40$ to get the total points are about 35,506.

31. a. The following is a scatterplot illustrating the relationship between ***left thumb length*** and ***total both scores***.



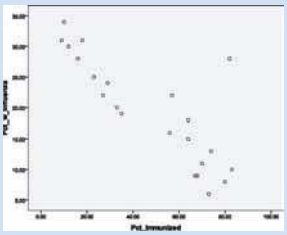
b. $r = -0.105$

c. The correlation coefficient ($r = -0.105$) indicates a weak relationship between left thumb length and total both scores.

d. The equation of the best fit line for these two events is $y = -7.62x + 127.73$.

e. No, given the weak correlation coefficient between total both scores and left thumb length, one could not use the best fit line to produce accurate predictions.

33. a. A scatterplot illustrating the relationship between ***% residents immunized*** and ***% residents with influenza*** is shown below.



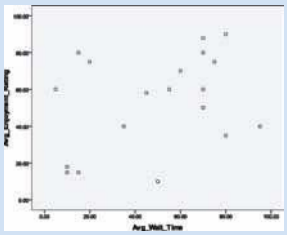
b. The correlation coefficient between ***% residents immunized*** and ***% residents with influenza*** is $r = -0.812$

c. Based on the correlation coefficient ($r = -0.812$), we would believe that there is a strong relationship between % residents immunized and % residents with influenza.

d. The equation of the best fit line that describes the relationship between ***% residents immunized*** and ***% residents with influenza*** is $y = -0.27x + 32.20$.

e. Since $r = -0.812$ indicates a strong relationship between % residents immunized and % residents with influenza, we can make a reasonably accurate prediction. However it will not be completely accurate.

35. a. A scatterplot illustrating the relationship between ***average wait times*** and ***average rating of enjoyment*** is shown below.



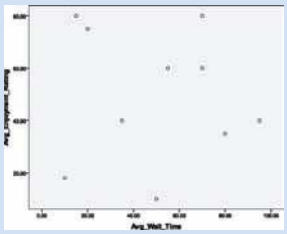
b. The correlation coefficient between ***average wait times*** and ***average rating of enjoyment*** is $r = 0.348$.

c. The correlation coefficient ($r = 0.348$) indicates a somewhat weak relationship between average wait times and average rating of enjoyment.

d. The equation of the best fit line that describes the relationship between ***average wait times*** and ***average rating of enjoyment*** is $y = 0.31x + 37.83$.

e. No, given the somewhat weak correlation coefficient between average wait times and average rating of enjoyment, one could not use the best fit line to produce accurate predictions.

37. a. A scatterplot illustrating the relationship between ***average wait times*** and ***average rating of enjoyment*** for ***Park 2*** is shown below.



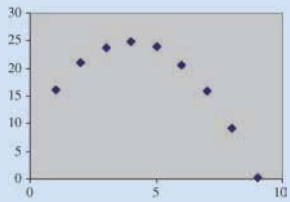
b. The correlation coefficient between ***average wait times*** and ***average rating of enjoyment*** is $r = -0.064$.

c. The correlation coefficient ($r = -0.064$) indicates a weak relationship between average wait times and average rating of enjoyment for Park 2.

d. The equation of the best fit line that describes the relationship between ***average wait times*** and ***average rating of enjoyment*** is $y = -0.05x + 52.53$.

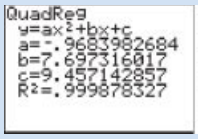
e. No, given the weak correlation coefficient between average wait times and average rating of enjoyment for Park 2, one could not use the best fit line to produce accurate predictions.

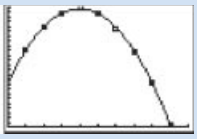
39. a. The scatterplot for this data set is given by



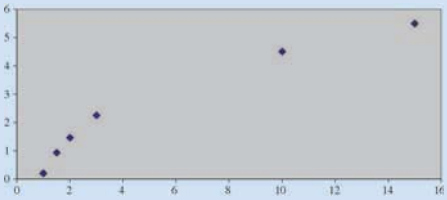
b. The equation of the best fit *line* is $y = -1.9867x + 27.211$ with a correlation coefficient of $r = -0.671$. This line does not seem to accurately describe the data because some of the points rise as you move left to right, while others fall as you move left to right; a line cannot capture both types of behavior simultaneously. Also, r being negative has no meaning here.

c. The best fit is provided by QuadReg. The associated equation of the best fit *quadratic* curve, the correlation coefficient, and associated scatterplot are:



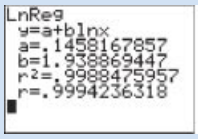


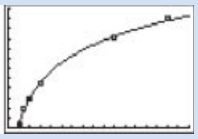
41. a. The scatterplot for this data set is given by



b. The equation of the best fit *line* is $y = 0.3537x + 0.5593$ with a correlation coefficient of $r = 0.971$. This line seems to provide a very good fit for this data, although not perfect.

c. The best fit is provided by LnReg. The associated equation of the best fit *logarithmic* curve, the correlation coefficient, and associated scatterplot are:





Chapter 0 Review

1. $x = \frac{16}{7}$ **3.** $p = -\frac{8}{25}$ **5.** $x = 27$

7. $y = -\frac{17}{5}$ **9.** $b = \frac{6}{7}$ **11.** $x = -\frac{6}{17}$

13. \$5000@20%; \$20,000@8%

15. 60 ml of 5%; 90 ml of 10%

17. $b = -3, 7$ **19.** $x = 0, 8$ **21.** $q = \pm 13$

23. $x = 2 \pm 4i$ **25.** $x = -2, 6$ **27.** $x = \frac{1 \pm \sqrt{33}}{2}$

29. $t = -1, \frac{7}{3}$ **31.** $f = \frac{1 \pm \sqrt{337}}{48}$ **33.** $q = \frac{3 \pm \sqrt{69}}{10}$

35. $x = -1, \frac{5}{2}$ **37.** $x = -3, \frac{2}{7}$ **39.** $h = 1$ ft, $b = 4$ ft

41. $x = \frac{6 \pm \sqrt{39}}{3}$ **43.** $t = -\frac{34}{5}, t \neq -4, 0$

45. $x = -\frac{1}{2}, x \neq 0$ **47.** $x = 6$ **49.** $x = 125$

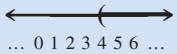
51. no solution **53.** $x \cong -0.6$ **55.** $y = \frac{1}{4}, 1$

57. $x = -\frac{125}{8}, 1$ **59.** $x = -\frac{1}{8}, -1$ **61.** $x = \pm 2, \pm 3i$

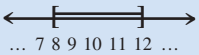
63. $x = 0, -8, 4$ **65.** $p = \pm 2, 3$ **67.** $p = -\frac{1}{2}, \frac{5}{2}, 3$

69. $y = \pm 9$ **71.** no solution **73.** $x = 0.9667, 1.7$

75. $(4, \infty)$



77. $[8, 12]$



79. $\left(-\infty, \frac{5}{3}\right)$ **81.** $\left(-\frac{3}{2}, \infty\right)$ **83.** $(4, 9]$

85. $\left[3, \frac{7}{2}\right]$ **87.** $[-6, 6]$ **89.** $(-\infty, 0] \cup [4, \infty)$

91. $\left(-\infty, -\frac{3}{4}\right) \cup (4, \infty)$

93. $(0, 3)$ **95.** $(-\infty, -6] \cup [9, \infty)$

97. $(-\infty, 2) \cup (4, 5]$ **99.** $(-\infty, -11) \cup (3, \infty)$

101. $(-\infty, -3) \cup (3, \infty)$ **103.** $\mathbb{R}$

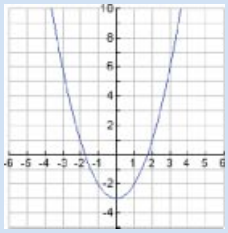
105. $3\sqrt{5}$ **107.** $\sqrt{205}$

109. $\left(\frac{5}{2}, 6\right)$ **111.** $(3.85, 5.3)$

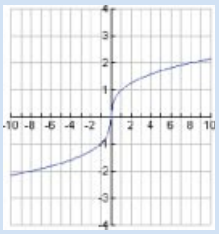
113. $(\pm 2, 0), (0, \pm 1)$ **115.** $(\pm 3, 0)$, no y-intercepts

117. y-axis **119.** origin

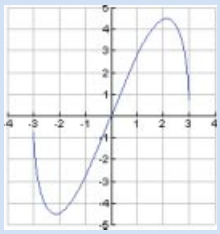
121.



123.



125.



127. $(-2, -3), r = 9$ **129.** not a circle

131. $y = -2x - 2$ **133.** $y = 6$

135. $y = \frac{5}{6}x + \frac{4}{3}$ **137.** $y = -2x - 1$

139. $y = \frac{2}{3}x + \frac{1}{3}$ **141.** $C = 2\pi r$

143. $A = \pi r^2$

Chapter 0 Practice Test

1. $p = -3$ **3.** $t = -4, 7$ **5.** $x = -\frac{1}{2}, \frac{8}{3}$

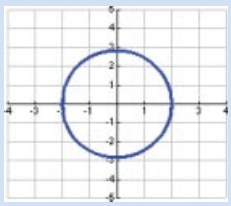
7. $y = -8$ **9.** $x = 4$ **11.** $y = 1$

13. $x = 0, 2, 6$ **15.** $(-\infty, 17]$ **17.** $\left(-\frac{32}{5}, -6\right]$

19. $(-\infty, -1] \cup \left[\frac{4}{3}, \infty\right)$ **21.** $\left(-\frac{1}{2}, 3\right]$

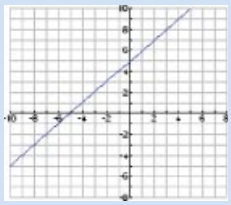
23. $\sqrt{82}$

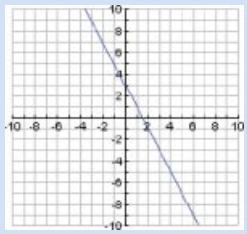
25.



27. $(6, 0), (0, -2)$ **29.** $y = \frac{8}{3}x - 8$

31. $y = x + 5$ **33.** $y = -2x + 3$





35. $F = \dfrac{30m}{P}$

CHAPTER 1

Section 1.1

1. not a function **3.** not a function

5. function **7.** not a function

9. not a function **11.** function

13. not a function **15.** function

17. not a function

19. **a.** 5 **b.** 1 **c.** -3 **21.** **a.** 3 **b.** 2 **c.** 5

23. **a.** -5 **b.** -5 **c.** -5 **25.** **a.** 2 **b.** -8 **c.** -5

27. 1 **29.** -3 and 1 **31.** $[-4, 4]$

33. 6 **35.** -7 **37.** 6

39. -1 **41.** -33 **43.** $-\frac{7}{6}$

45. $\frac{2}{3}$ **47.** 4 **49.** $8 - x - a$

51. $(-\infty, \infty)$ **53.** $(-\infty, \infty)$ **55.** $(-\infty, 5) \cup (5, \infty)$

57. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ **59.** $(-\infty, \infty)$

61. $(-\infty, 7]$ **63.** $\left[-\frac{5}{2}, \infty\right)$ **65.** $(-\infty, -2] \cup [2, \infty)$

67. $(3, \infty)$ **69.** $(-\infty, \infty)$ **71.** $(-\infty, -4) \cup (-4, \infty)$

73. $\left(-\infty, \frac{3}{2}\right)$ **75.** $(-\infty, -2) \cup (3, \infty)$

77. $(-\infty, -4] \cup [4, \infty)$ **79.** $\left(-\infty, \frac{3}{2}\right)$

81. $(-\infty, \infty)$ **83.** $x = -2, 4$ **85.** $x = -1, 5, 6$

87. $T(6) = 64.8°\text{F}$, $T(12) = 90°\text{F}$ **89.** 27 ft, [0, 2.8]

91. $V(x) = x(10 - 2x)^2$, $(0, 5)$

93. E(4) $\approx$ 84 yen **95.** 229 people
E(7) $\approx$ 84 yen
E(8) $\approx$ 83 yen

97. **a.** $A(x) = x(x - 3.375)$
b. $A(4.5) \approx 5$. The area in the window is approximately 5 sq in.
c. $A(8.5) \approx 44$. This is not possible because the window would be larger than the entire envelope (32 sq in.).

99. Yes, for every input (year), there corresponds a unique output (federal funds rate).

101. (1989, 4000), (1993, 6000), (1997, 6000), (2001, 8000), (2005, 11000)

103. **a.** $F(50) = 0$ **b.** $g(50) = 1000$ **c.** $H(50) = 2000$

105. Should apply the <u>vertical</u> line test to determine if the relationship describes a function, which it is a function.

107. $f(x + 1) \neq f(x) + f(1)$, in general.

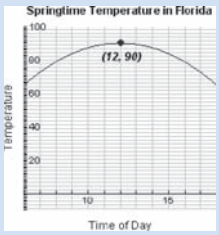
109. $G(-1 + h) \neq G(-1) + G(h)$, in general.

111. false **113.** true **115.** $A = 2$

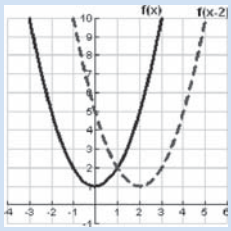
117. $C = -5, D = -2$

119. $(-\infty, -a) \cup (-a, a) \cup (a, \infty)$

121. Warmest: noon, 90°F. The values of T outside the interval [6, 18] are too small to be considered temperatures in Florida.



123. Shift graph of $f(x)$ two units to the right.



125. $f'(x) = 3x^2 + 1$ **127.** $f'(x) = \dfrac{8}{(x + 3)^2}$

Section 1.2

1. neither **3.** odd **5.** even

7. even **9.** neither **11.** neither

13. neither **15.** neither

17. **a.** $(-\infty, \infty)$ **b.** $[-1, \infty)$ **c.** increasing: $(-1, \infty)$, decreasing: $(-3, -2)$, constant: $(-\infty, -3) \cup (-2, -1)$ **d.** 0 **e.** -1 **f.** 2

19. **a.** $[-7, 2]$ **b.** $[-5, 4]$ **c.** increasing: $(-4, 0)$, decreasing: $(-7, -4) \cup (0, 2)$, constant: nowhere **d.** 4 **e.** 1 **f.** -5

21. **a.** $(-\infty, \infty)$ **b.** $(-\infty, \infty)$ **c.** increasing: $(-\infty, -3) \cup (4, \infty)$, decreasing: nowhere, constant: $(-3, 4)$ **d.** 2 **e.** 2 **f.** 2

23. **a.** $(-\infty, \infty)$ **b.** $[-4, \infty)$ **c.** increasing: $(0, \infty)$, decreasing: $(-\infty, 0)$, constant: nowhere **d.** -4 **e.** 0 **f.** 0

25. **a.** $(-\infty, 0) \cup (0, \infty)$ **b.** $(-\infty, 0) \cup (0, \infty)$ **c.** increasing: $(-\infty, 0) \cup (0, \infty)$, decreasing: nowhere, constant: nowhere **d.** undefined **e.** 3 **f.** -3

27. **a.** $(-\infty, 0) \cup (0, \infty)$ **b.** $(-\infty, 5) \cup [7]$ **c.** increasing: $(-\infty, 0)$, decreasing: $(5, \infty)$, constant: $(0, 5)$ **d.** undefined **e.** 3 **f.** 7

29. $2x + h - 1$ **31.** $2x + h + 3$

33. $2x + h - 3$ **35.** $-6x - 3h + 5$

37. $3x^2 + 3xh + h^2 + 2x + h$

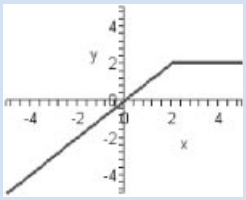
39. $\dfrac{-2}{(x + h - 2)(x - 2)}$

41. $\dfrac{-2}{\sqrt{1 - 2(x + h)} + \sqrt{1 - 2x}}$

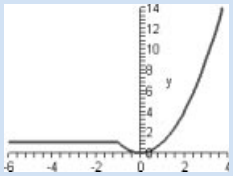
43. $\dfrac{-4}{\sqrt{x(x + h)}\left(\sqrt{x} + \sqrt{x + h}\right)}$

45. 13 **47.** 1 **49.** -2 **51.** -1

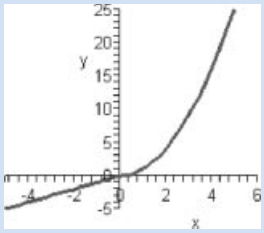
53. domain: $(-\infty, \infty)$ range: $(-\infty, 2]$
increasing: $(-\infty, 2)$ decreasing: nowhere
constant: $(2, \infty)$



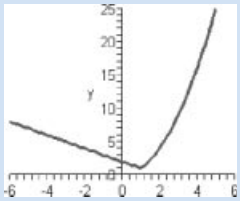
55. domain: $(-\infty, \infty)$ range: $[0, \infty)$
increasing: $(0, \infty)$ decreasing: $(-1, 0)$
constant: $(-\infty, -1)$



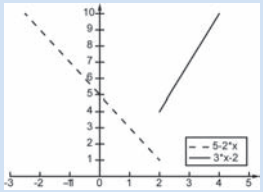
57. domain: $(-\infty, \infty)$ range: $(-\infty, \infty)$
increasing: $(-\infty, \infty)$ decreasing: nowhere
constant: nowhere



59. domain: $(-\infty, \infty)$ range: $[1, \infty)$
increasing: $(1, \infty)$ decreasing: $(-\infty, 1)$
constant: nowhere

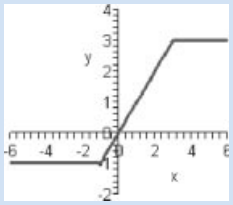


61.

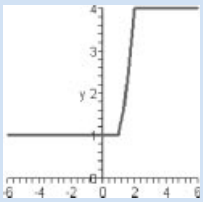


domain: $(-\infty, 2) \cup (2, \infty)$
range: $(1, \infty)$
increasing: $(2, \infty)$
decreasing: $(-\infty, 2)$

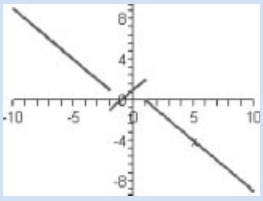
63. domain: $(-\infty, \infty)$ range: $[-1, 3]$
increasing: $(-1, 3)$ decreasing: nowhere
constant: $(-\infty, -1) \cup (3, \infty)$



65. domain: $(-\infty, \infty)$ range: $[1, 4]$
increasing: $(1, 2)$ decreasing: nowhere
constant: $(-\infty, 1) \cup (2, \infty)$

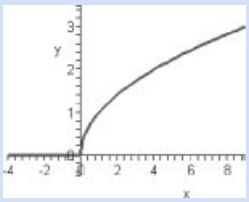


67. domain: $(-\infty, -2) \cup (-2, \infty)$
range: $(-\infty, \infty)$ increasing: $(-2, 1)$
decreasing: $(-\infty, -2) \cup (1, \infty)$
constant: nowhere

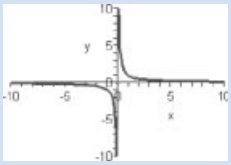


open holes $(-2, 1)$, $(-2, -1)$, $(1, 2)$, closed hole $(1, 0)$

69. domain: $(-\infty, \infty)$ range: $[0, \infty)$
increasing: $(0, \infty)$ decreasing: nowhere
constant: $(-\infty, 0)$

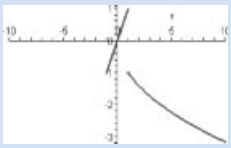


71. domain: $(-\infty, \infty)$ range: $(-\infty, \infty)$
increasing: nowhere
decreasing: $(-\infty, 0) \cup (0, \infty)$
constant: nowhere



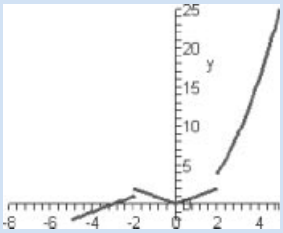
closed hole $(0, 0)$

73. domain: $(-\infty, 1) \cup (1, \infty)$
range: $(-\infty, -1) \cup (-1, \infty)$ increasing: $(-1, 1)$
decreasing: $(-\infty, -1) \cup (1, \infty)$
constant: nowhere



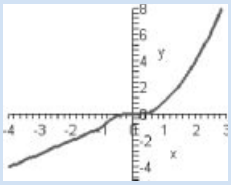
open holes
$(-1, -1)$, $(1, 1)$, $(1, -1)$
graph of $-\sqrt[3]{x}$ on $(-\infty, -1)$
closed hole $(-1, 1)$

75. domain: $(-\infty, \infty)$ range: $(-\infty, 2) \cup [4, \infty)$
increasing: $(-\infty, -2) \cup (0, 2) \cup (2, \infty)$
decreasing: $(-2, 0)$ constant: nowhere



open holes $(-2, 2)$, $(2, 2)$
closed holes $(-2, 1)$, $(2, 4)$

77. domain: $(-\infty, 1) \cup (1, \infty)$
range: $(-\infty, 1) \cup (1, \infty)$
increasing: $(-\infty, 1) \cup (1, \infty)$
decreasing: nowhere constant: nowhere



open hole $(1, 1)$

79. Profit is increasing from October through December and decreasing from January through October.

81. $C(x) = \begin{cases} 10x, & 0 \le x \le 50 \\ 9x, & 50 < x \le 100 \\ 8x, & x > 100 \end{cases}$

83. $C(x) = \begin{cases} 250x, & 0 \le x \le 10 \\ 175x + 750, & x > 10 \end{cases}$

85. $C(x) = \begin{cases} 1000 + 35x, & 0 \le x \le 100 \\ 2000 + 25x, & x > 100 \end{cases}$

87. $R(x) = \begin{cases} 50{,}000 + 3x, & 0 \le x \le 100{,}000 \\ -50{,}000 + 4x, & x > 100{,}000 \end{cases}$

89. $P(x) = 65x - 800$

91. $f(x) = 0.80 + 0.17[\![x]\!], x \ge 0$

93. $f(t) = 3(-1)^{[\![t]\!]}, t \ge 0$

95. **a.** 20 per yr **b.** 110 per yr

97. 0 ft/sec

99. Demand for the product is increasing at an approximate rate of 236 units over the first quarter.

101. Should exclude the origin since $x = 0$ is not in the domain. The range should be $(0, \infty)$.

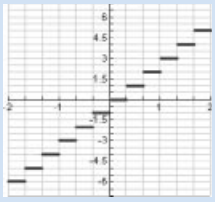
103. The portion of $C(x)$ for $x > 30$ should be:
$15 + \underbrace{x - 30}_{\text{Number miles beyond first 30}}$

105. false

107. yes, if $a = 2b$

109. yes, if $a = -4, b = -5$ **111.** odd **113.** odd

115. domain: $\mathbb{R}$
range: set of integers



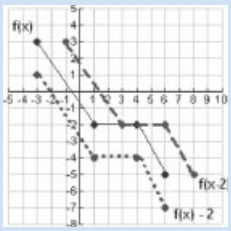
117. $f'(x) = 0$ **119.** $f'(x) = 2ax + b$

Section 1.3

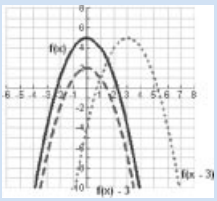
1. $y = |x| + 3$ **3.** $y = |-x| = |x|$ **5.** $y = 3|x|$

7. $y = x^3 - 4$ **9.** $y = (x + 1)^3 + 3$ **11.** $y = -x^3$

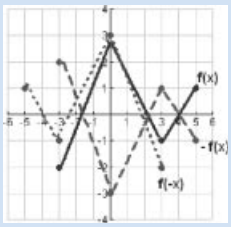
13.



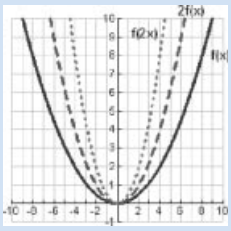
15.



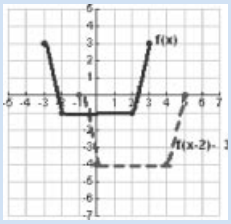
17.



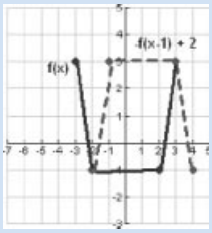
19.



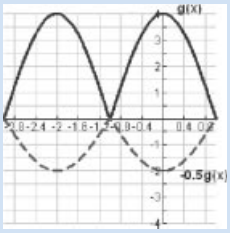
21.



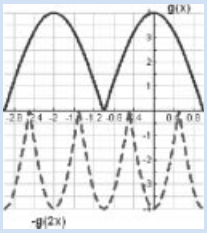
23.



25.

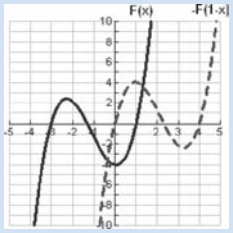


27.

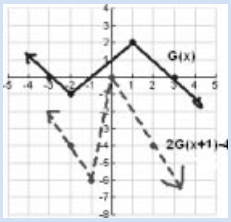


29.

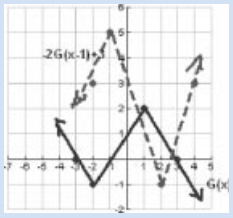
31.



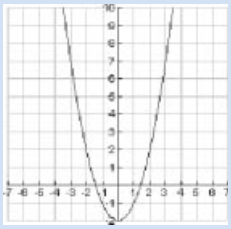
33.



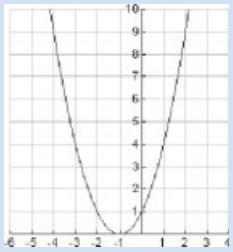
35.



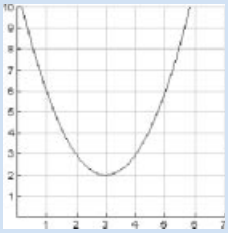
37.



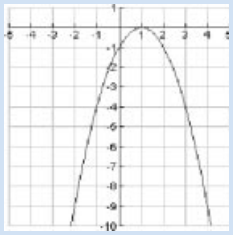
39.



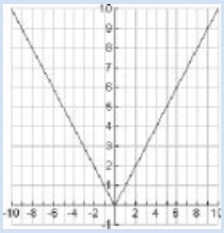
41.



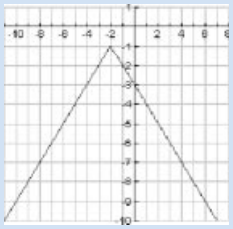
43.



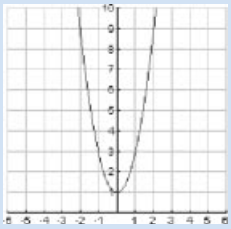
45.



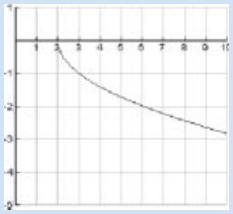
47.



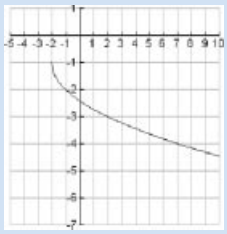
49.



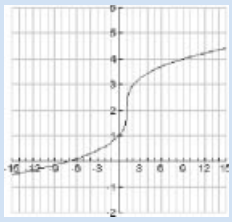
51.



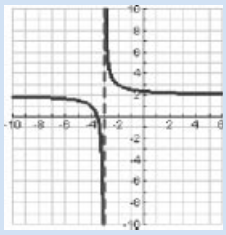
53.



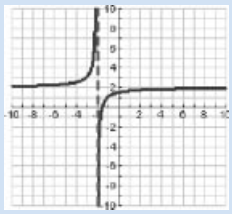
55.



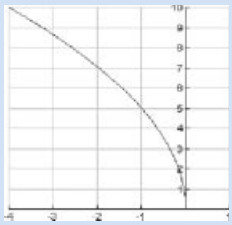
57.



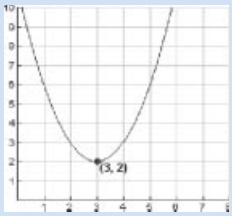
59.



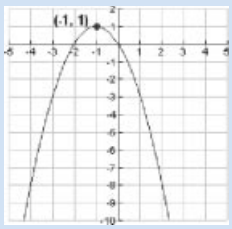
61.



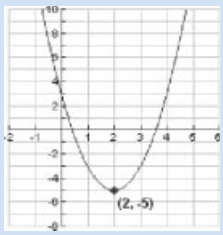
63. $f(x) = (x - 3)^2 + 2$



65. $f(x) = -(x + 1)^2 + 1$



67. $f(x) = 2(x - 2)^2 - 5$



69. $S(x) = 10x$ and $S(x) = 10x + 50$

71. $T(x) = 0.33(x - 6500)$

73. $f(x) = 7.00 + 0.30x$, $f(x + 2) = 7.60 + 0.30x$

75. $Q(t) = P(t + 50)$

77. **a.** $\text{BSA}(w) = \sqrt{\dfrac{9w}{200}}$

b. $\text{BSA}(w - 3) = \sqrt{\dfrac{9(w - 3)}{200}}$

79. (b) is wrong − shift right three units.

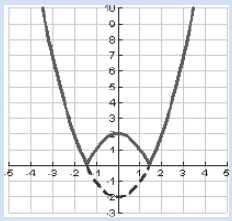
81. (b) should be deleted since $|3 - x| = |x - 3|$.
The correct sequence of steps would be:
$(a) \to (c)^* \to (d)$, where
(c)*: Shift to the right three units.

83. true **85.** true **87.** true

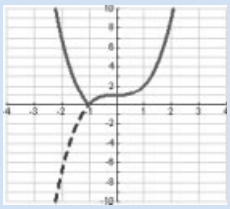
89. $(a + 3, b + 2)$ **91.** $(a - 1, 2b - 1)$

93. Any part of the graph of $y = f(x)$ that is below the x-axis is reflected above it for the graph of $y = |f(x)|$.

a.

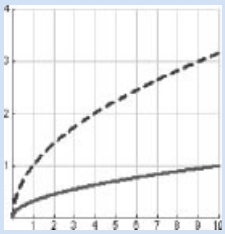


b.

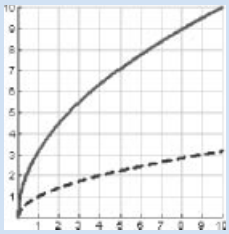


95. If $a > 1$, then the graph is a horizontal compression. If $0 < a < 1$, then the graph is a horizontal expansion.

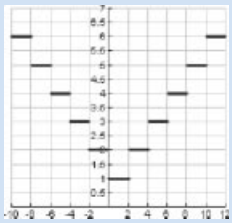
a.



b.



97. Each horizontal line in the graph of $y = [\![x]\!]$ is stretched by a factor of 2. Any portion of the graph that is below the x-axis is reflected above it. Also, there is a vertical shift up of one unit.



99. $f'(x) = 3x^2 + 1$, $g'(x) = 2(x - 1)$, g' is obtained by shifting f' left five units.

101. $f'(x) = 2$, $g'(x) = 2$, f' and g' are the same.

Section 1.4

1.

$$\left.\begin{aligned} f(x) + g(x) &= x + 2 \\ f(x) - g(x) &= 3x \\ f(x) \cdot g(x) &= -2x^2 + x + 1 \end{aligned}\right\} \text{domain: } (-\infty, \infty)$$

$$\frac{f(x)}{g(x)} = \frac{2x + 1}{1 - x} \quad \text{domain: } (-\infty, 1) \cup (1, \infty)$$

3. $\left.\begin{aligned} f(x) + g(x) &= 3x^2 - x - 4 \\ f(x) - g(x) &= x^2 - x + 4 \\ f(x)\cdot g(x) &= 2x^4 - x^3 - 8x^2 + 4x \end{aligned}\right\}$ domain: $(-\infty, \infty)$

$\dfrac{f(x)}{g(x)} = \dfrac{2x^2 - x}{x^2 - 4}$ domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

5.

$\left.\begin{aligned} f(x) + g(x) &= \dfrac{1 + x^2}{x} \\ f(x) - g(x) &= \dfrac{1 - x^2}{x} \\ f(x)\cdot g(x) &= 1 \\ \dfrac{f(x)}{g(x)} &= \dfrac{1}{x^2} \end{aligned}\right\}$ domain: $(-\infty, 0) \cup (0, \infty)$

7.

$\left.\begin{aligned} f(x) + g(x) &= 3\sqrt{x} \\ f(x) - g(x) &= -\sqrt{x} \\ f(x)\cdot g(x) &= 2x \end{aligned}\right\}$ domain $[0, \infty)$

$\dfrac{f(x)}{g(x)} = \dfrac{1}{2}$ domain: $(0, \infty)$

9.

$\left.\begin{aligned} f(x) + g(x) &= \sqrt{4 - x} + \sqrt{x + 3} \\ f(x) - g(x) &= \sqrt{4 - x} - \sqrt{x + 3} \\ f(x)\cdot g(x) &= \sqrt{4 - x}\cdot\sqrt{x + 3} \end{aligned}\right\}$ domain: $[-3, 4]$

$\dfrac{f(x)}{g(x)} = \dfrac{\sqrt{4 - x}\sqrt{x + 3}}{x + 3}$ domain: $(-3, 4]$

11. $(f\circ g)(x) = 2x^2 - 5$ domain: $(-\infty, \infty)$
$(g\circ f)(x) = 4x^2 + 4x - 2$ domain: $(-\infty, \infty)$

13. $(f\circ g)(x) = \dfrac{1}{x + 1}$ domain: $(-\infty, -1) \cup (-1, \infty)$

$(g\circ f)(x) = \dfrac{2x - 1}{x - 1}$ domain: $(-\infty, 1) \cup (1, \infty)$

15. $(f\circ g)(x) = \dfrac{1}{|x - 1|}$ domain: $(-\infty, 1) \cup (1, \infty)$

$(g\circ f)(x) = \dfrac{1}{|x| - 1}$ domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

17. $(f\circ g)(x) = \sqrt{x + 4}$ domain: $[-4, \infty)$
$(g\circ f)(x) = \sqrt{x - 1} + 5$ domain: $[1, \infty)$

19. $(f\circ g)(x) = x$ domain: $(-\infty, \infty)$
$(g\circ f)(x) = x$ domain: $(-\infty, \infty)$

21. 15 **23.** 13 **25.** $26\sqrt{3}$

27. $\frac{110}{3}$ **29.** 11 **31.** $3\sqrt{2}$

33. undefined **35.** undefined **37.** 13

39. $f(g(1)) = \frac{1}{3}$ $g(f(2)) = 2$

41. $f(g(1))$ = undefined $g(f(2))$ = undefined

43. $f(g(1)) = \frac{1}{3}$ $g(f(2)) = 4$

45. $f(g(1)) = \sqrt{5}$ $g(f(2)) = 6$

47. $f(g(1))$ = undefined $g(f(2))$ = undefined

49. $f(g(1)) = \sqrt[3]{3}$ $g(f(2)) = 4$

51. $f(g(x)) = \cancel{2}\left(\dfrac{x - 1}{\cancel{2}}\right) + 1 = x - 1 + 1 = x$

$g(f(x)) = \dfrac{(2x + 1) - 1}{2} = \dfrac{2x}{2} = x$

53. $f(g(x)) = \sqrt{(x^2 + 1) - 1} = \sqrt{x^2} = \underbrace{|x| = x}_{\text{Since } x \geq 1}$

$g(f(x)) = (\sqrt{x - 1})^2 + 1 = (x - 1) + 1 = x$

55. $f(g(x)) = \dfrac{1}{\frac{1}{x}} = x$ $g(f(x)) = \dfrac{1}{\frac{1}{x}} = x$

57. $f(g(x)) = 4\left(\dfrac{\sqrt{x + 9}}{2}\right)^2 - 9$

$= 4\left(\dfrac{x + 9}{4}\right) - 9 = x$

$g(f(x)) = \dfrac{\sqrt{(4x^2 - 9) + 9}}{2}$

$= \dfrac{\sqrt{4x^2}}{2} = \dfrac{2x}{2} = x$

59. $f(g(x)) = \dfrac{1}{\frac{x+1}{x} - 1} = \dfrac{1}{\frac{x+1-x}{x}} = \dfrac{1}{\frac{1}{x}} = x$

$g(f(x)) = \dfrac{\frac{1}{x-1} + 1}{\frac{1}{x-1}} = \dfrac{\frac{1+x-1}{x-1}}{\frac{1}{x-1}} = \dfrac{\frac{x}{\cancel{x-1}}}{\frac{1}{\cancel{x-1}}} = x$

61. $f(x) = 2x^2 + 5x$ $g(x) = 3x - 1$

63. $f(x) = \dfrac{2}{|x|}$ $g(x) = x - 3$

65. $f(x) = \dfrac{3}{\sqrt{x} - 2}$ $g(x) = x + 1$

67. $F(C(K)) = \frac{9}{5}(K - 273.15) + 32$

69. **a.** $A(x) = \left(\dfrac{x}{4}\right)^2$ **b.** $A(100) = 625 \text{ ft}^2$

c. $A(200) = 2500 \text{ ft}^2$

71. **a.** $C(p) = 62{,}000 - 20p$
b. $R(p) = 600{,}000 - 200p$
c. $P(p) = 538{,}000 - 180p$

73. **a.** $C(n(t)) = -10t^2 + 500t + 1375$

b. $C(n(16)) = 6815$

The cost of production on a day when the assembly line was running for 16 hours is \$6,815,000.

75. **a.** $A(r(t)) = \pi\left(10t - 0.2t^2\right)^2$

b. 11,385 sq mi

77. $A(t) = \pi\left[150\sqrt{t}\right]^2 = 22{,}500\pi t \text{ ft}^2$

79. $d(h) = \sqrt{h^2 + 4}$

81. Must exclude -2 from the domain

83. $(f \circ g)(x) = f(g(x))$, not $f(x) \cdot g(x)$

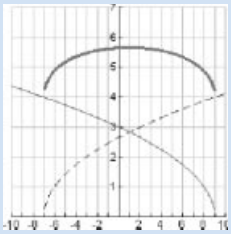
85. Function notation, not multiplication

87. false **89.** true

91. $(g \circ f)(x) = \dfrac{1}{x}$ domain: $x \neq 0, a$

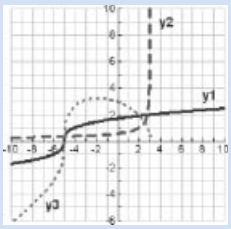
93. $(g \circ f)(x) = x$ domain: $[-a, \infty)$

95.



domain: $[-7, 9]$

97.



domain: $(-\infty, 3) \cup (-3, -1] \cup [4, 6) \cup (6, \infty)$

99. $H'(x) = F'(x) + G'(x)$

101. $H'(x) \neq F'(x)G'(x)$

Section 1.5

1. not a function

3. function, not one-to-one

5. function, not one-to-one

7. function, one-to-one

9. function, not one-to-one

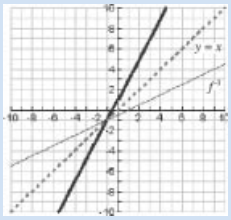
11. not one-to-one function

13. one-to-one function

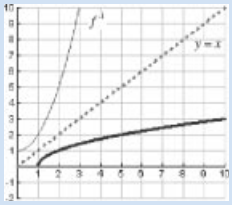
15. not one-to-one function

17. one-to-one function

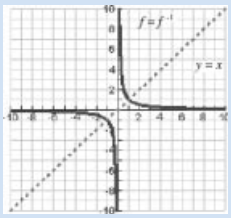
19.



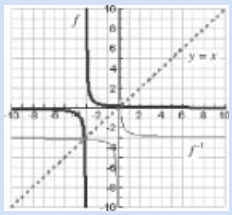
21.



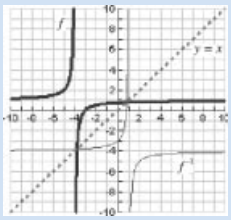
23.



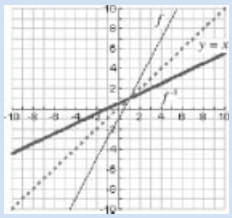
25.



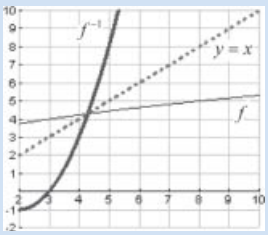
27.



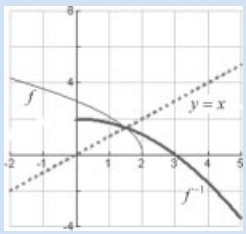
29.



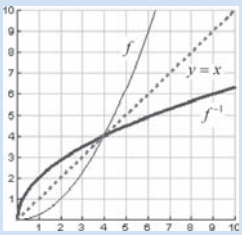
31.



33.



35.



37. $f^{-1}(x) = -\frac{1}{3}x + \frac{2}{3}$

domain f: $(-\infty, \infty)$ domain f^{-1}: $(-\infty, \infty)$

range f: $(-\infty, \infty)$ range f^{-1}: $(-\infty, \infty)$

39. $f^{-1}(x) = \sqrt[3]{x - 1}$

domain f: $(-\infty, \infty)$ domain f^{-1}: $(-\infty, \infty)$
range f: $(-\infty, \infty)$ range f^{-1}: $(-\infty, \infty)$

41. $f^{-1}(x) = x^2 + 3$

domain f: $[3, \infty)$ domain f^{-1}: $[0, \infty)$
range f: $[0, \infty)$ range f^{-1}: $[3, \infty)$

43. $f^{-1}(x) = \sqrt{x + 1}$

domain f: $[0, \infty)$ domain f^{-1}: $[-1, \infty)$
range f: $[-1, \infty)$ range f^{-1}: $[0, \infty)$

45. $f^{-1}(x) = -2 + \sqrt{x + 3}$

domain f: $[-2, \infty)$ domain f^{-1}: $[-3, \infty)$
range f: $[-3, \infty)$ range f^{-1}: $[-2, \infty)$

47. $f^{-1}(x) = \dfrac{2}{x}$

domain f: $(-\infty, 0) \cup (0, \infty)$
range f: $(-\infty, 0) \cup (0, \infty)$
domain f^{-1}: $(-\infty, 0) \cup (0, \infty)$
range f^{-1}: $(-\infty, 0) \cup (0, \infty)$

49. $f^{-1}(x) = \dfrac{3x - 2}{x} = 3 - \dfrac{2}{x}$

domain f: $(-\infty, 3) \cup (3, \infty)$
range f: $(-\infty, 0) \cup (0, \infty)$
domain f^{-1}: $(-\infty, 0) \cup (0, \infty)$
range f^{-1}: $(-\infty, 3) \cup (3, \infty)$

51. $f^{-1}(x) = \dfrac{5x - 1}{x + 7}$

domain f: $(-\infty, 5) \cup (5, \infty)$
range f: $(-\infty, -7) \cup (-7, \infty)$
domain f^{-1}: $(-\infty, -7) \cup (-7, \infty)$
range f^{-1}: $(-\infty, 5) \cup (5, \infty)$

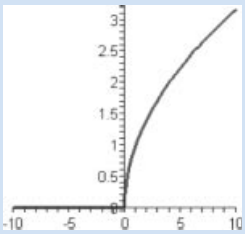
53. $f^{-1}(x) = \dfrac{1}{x^2}$

domain f = range f^{-1}: $(0, \infty)$
range f = domain f^{-1}: $(0, \infty)$

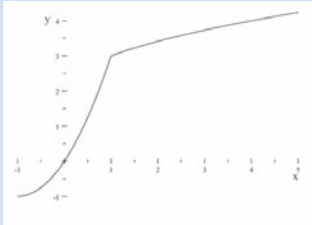
55. $f^{-1}(x) = \dfrac{2x^2 + 1}{x^2 - 1}$

domain f = range f^{-1}: $(-\infty, -1] \cup (2, \infty)$
range f = domain f^{-1}: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

57. not one-to-one.

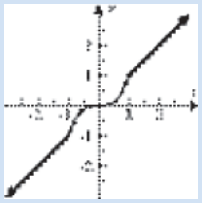


59. one-to-one



$$f^{-1}(x) = \begin{cases} x^3, & x \le -1, \\ -1 + \sqrt{x + 1}, & -1 < x \le 1, \\ (x - 2)^2, & x > 1 \end{cases}$$

61. one-to-one



$$f^{-1}(x) = \begin{cases} x & x \le -1 \\ \sqrt[3]{x} & -1 < x < 1 \\ x & x \ge 1 \end{cases}$$

63. $f^{-1}(x) = \frac{5}{9}(x - 32)$ The inverse function represents the conversion from degrees Fahrenheit to degrees Celsius.

65. $C(x) = \begin{cases} 250x, & 0 \le x \le 10 \\ 2500 + 175(x - 10), & x > 10 \end{cases}$

$$C^{-1}(x) = \begin{cases} \dfrac{x}{250}, & 0 \le x \le 2500 \\ \dfrac{x - 750}{175}, & x > 2500 \end{cases}$$

67. $E(x) = 5.25x$, $E^{-1}(x) = \dfrac{x}{5.25}$, $x \ge 0$ The inverse function tells you how many hours you need to work to attain a certain take home pay.

69. Domain: [0, 24] Range: [97.5528, 101.70]

71. Domain: [97.5528, 101.70] Range: [0, 24]

73. $M(x) = \begin{cases} 0.60x, & 0 \le x \le 15 \\ 0.60(15) + 0.90(x - 15), & x > 15 \end{cases}$

$$M^{-1}(x) = \begin{cases} \dfrac{5x}{3}, & 0 \le x \le 9 \\ \dfrac{10x}{9} + 5, & x > 9 \end{cases}$$

75. $V(x) = \begin{cases} 20{,}000 - 600x, & 0 \le x \le 5, \\ 21{,}500 - 900x, & x > 5 \end{cases}$

$$V^{-1}(x) = \begin{cases} \dfrac{21{,}500 - x}{900}, & 0 \le x \le 17{,}000 \\ \dfrac{20{,}000 - x}{600}, & 17{,}000 < x \le 20{,}000 \end{cases}$$

77. Not a function since the graph does not pass the vertical line test

79. Must restrict the domain to a portion on which f is one-to-one, say $x \geq 0$. Then, the calculation will be valid.

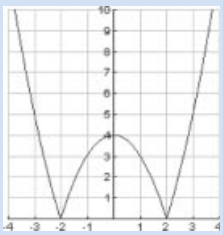
81. false **83.** false **85.** $(b, 0)$

87. $f(x) = \sqrt{1 - x^2}, \quad 0 \leq x \leq 1,$
$f^{-1}(x) = \sqrt{1 - x^2}, \quad 0 \leq x \leq 1$
Domain and range of both are [0, 1].

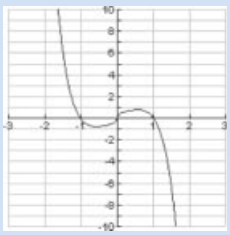
89. $m \neq 0$

91. $a = 4, f^{-1}(x) = \dfrac{1 - 2x}{x}, (-\infty, 0) \cup (0, \infty)$

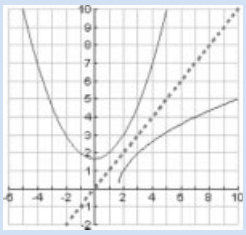
93. not one-to-one



95. not one-to-one

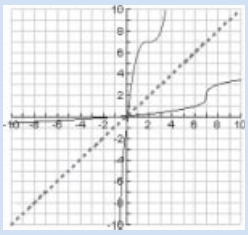


97. no



To be inverses, restrict the domain of the parabola to $[0, \infty)$.

99.



101. a. $f^{-1}(x) = \dfrac{x - 1}{2}$ **b.** $f'(x) = 2$ **c.** $(f^{-1})'(x) = \frac{1}{2}$

103. a. $f^{-1}(x) = x^2 - 2, x \geq 0$ **b.** $f'(x) = \dfrac{1}{2\sqrt{x + 2}}$
c. $\left(f^{-1}\right)'(x) = 2x$

Review Exercises

1. yes **3.** no **5.** yes **7.** no

9. a. 2 **b.** 4 **c.** $x = -3, 4$

11. a. 0 **b.** -2 **c.** $x \approx -5, 2$

13. 5 **15.** -665 **17.** -2 **19.** 4

21. $(-\infty, \infty)$ **23.** $(-\infty, -4) \cup (-4, \infty)$ **25.** $[4, \infty)$

27. $D = 18$ **29.** odd **31.** odd

33. a. $[-5, \infty)$ **b.** $[-3, \infty)$ **c.** increasing: $(-5, -3) \cup (3, \infty)$, decreasing: $(-1, 1)$, constant: $(-3, 1) \cup (1, 3)$ **d.** 2 **e.** 3 **f.** 1

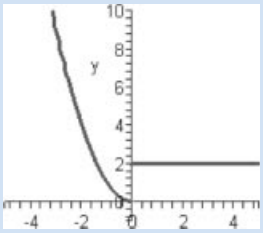
35. a. $[-6, 6]$ **b.** $[0, 3] \cup \{-3, -2, -1\}$
c. increasing: $(0, 3)$, decreasing: $(0, 3)$, contant: $(-6, -4) \cup (-4, -2) \cup (-2, 0)$
d. -1 **e.** -2 **f.** 3

37. $3x^2 + 3xh + h^2$

39. $1 - \dfrac{1}{x(x + h)}$

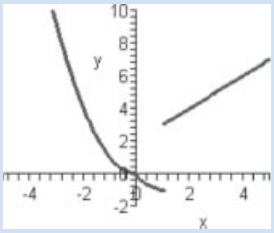
41. -2

43. domain: $(-\infty, \infty)$
range: $(0, \infty)$



open hole (0, 0), closed hole (0, 2)

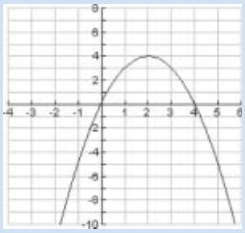
45. domain: $(-\infty, \infty)$
range: $[-1, \infty)$



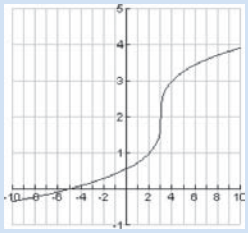
open hole (1, 3), closed hole (1, −1)

47. \$29,000 per year

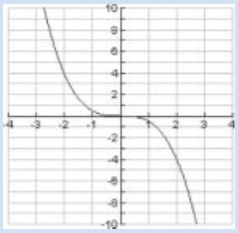
49.



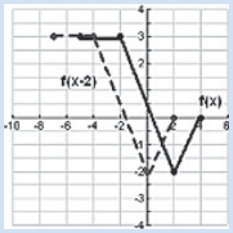
51.



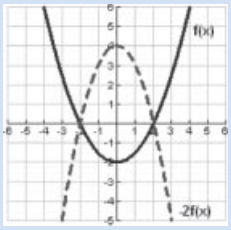
53.



55.



57.

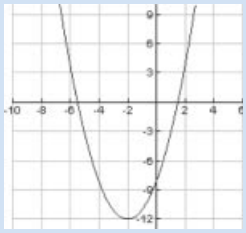


59. $y = \sqrt{x + 3}$ domain: $[-3, \infty)$

61. $y = \sqrt{x - 2} + 3$ domain: $[2, \infty)$

63. $y = 5\sqrt{x} - 6$ domain: $[0, \infty)$

65. $y = (x+2)^2 - 12$



67.

$$\left.\begin{aligned} g(x) + h(x) &= -2x - 7 \\ g(x) - h(x) &= -4x - 1 \\ g(x)\cdot h(x) &= -3x^2 + 5x + 12 \end{aligned}\right\} \text{domain: } (-\infty, \infty)$$

$$\frac{g(x)}{h(x)} = \frac{-3x - 4}{x - 3} \quad \text{domain: } (-\infty, 3) \cup (3, \infty)$$

69.

$$\left.\begin{aligned} g(x) + h(x) &= \frac{1}{x^2} + \sqrt{x} \\ g(x) - h(x) &= \frac{1}{x^2} - \sqrt{x} \\ g(x)\cdot h(x) &= \frac{1}{x^{3/2}} \\ \frac{g(x)}{h(x)} &= \frac{1}{x^{5/2}} \end{aligned}\right\} \text{domain: } (0, \infty)$$

71.

$$\left.\begin{aligned} g(x) + h(x) &= \sqrt{x-4} + \sqrt{2x+1} \\ g(x) - h(x) &= \sqrt{x-4} - \sqrt{2x+1} \\ g(x)\cdot h(x) &= \sqrt{x-4}\cdot\sqrt{2x+1} \\ \frac{g(x)}{h(x)} &= \frac{\sqrt{x-4}}{\sqrt{2x+1}} \end{aligned}\right\} \text{domain: } [4, \infty)$$

73. $(f \circ g)(x) = 6x - 1$ domain: $(-\infty, \infty)$
$(g \circ f)(x) = 6x - 7$ domain: $(-\infty, \infty)$

75. $(f \circ g)(x) = \dfrac{8 - 2x}{13 - 3x}$
domain: $(-\infty, 4) \cup \left(4, \frac{13}{3}\right) \cup \left(\frac{13}{3}, \infty\right)$
$(g \circ f)(x) = \dfrac{x + 3}{4x + 10}$
domain: $(-\infty, -3) \cup \left(-3, -\frac{5}{2}\right) \cup \left(-\frac{5}{2}, \infty\right)$

77. $(f \circ g)(x) = \sqrt{(x-3)(x+3)}$
domain: $(-\infty, -3] \cup [3, \infty)$
$(g \circ f)(x) = x - 9$ domain: $[5, \infty)$

79. $f(g(3)) = 857$, $g(f(-1)) = 51$

81. $f(g(3)) = \frac{17}{31}$, $g(f(-1)) = 1$

83. $f(g(3)) = 12$, $g(f(-1)) = 2$

85. $f(x) = 3x^2 + 4x + 7$, $g(x) = x - 2$

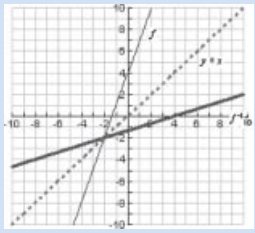
87. $f(x) = \dfrac{1}{\sqrt{x}}$, $g(x) = x^2 + 7$

89. $A(t) = 625\pi(t + 2)$ in^2

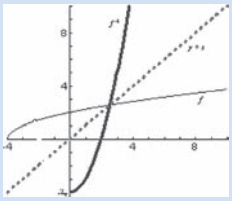
91. yes **93.** yes **95.** yes

97. not one-to-one **99.** one-to-one

101.



103.



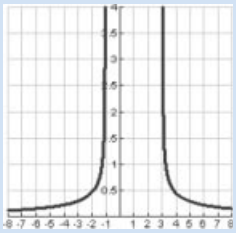
105. $f^{-1}(x) = \frac{1}{2}(x - 1) = \dfrac{x - 1}{2}$
domain f: $(-\infty, \infty)$ domain f^{-1}: $(-\infty, \infty)$
range f: $(-\infty, \infty)$ range f^{-1}: $(-\infty, \infty)$

107. $f^{-1}(x) = x^2 - 4$
domain f: $[-4, \infty)$ domain f^{-1}: $[0, \infty)$
range f: $[0, \infty)$ range f^{-1}: $[-4, \infty)$

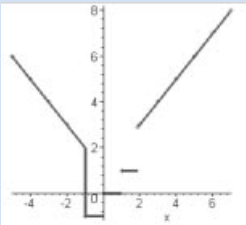
109. $f^{-1}(x) = \dfrac{6 - 3x}{x - 1}$
domain f: $(-\infty, -3) \cup (-3, \infty)$
range f: $(-\infty, 1) \cup (1, \infty)$
domain f^{-1}: $(-\infty, 1) \cup (1, \infty)$
range f^{-1}: $(-\infty, -3) \cup (-3, \infty)$

111. $S(x) = 22{,}000 + 0.08x$, $S^{-1}(x) = \dfrac{x - 22{,}000}{0.08}$,
sales required to earn a desired income

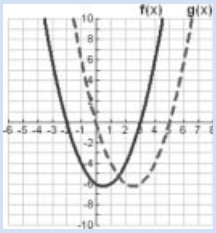
113. domain: $(-\infty, -1) \cup (3, \infty)$



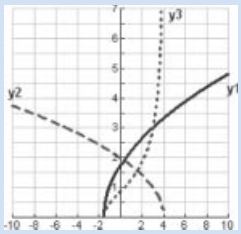
115. **a.** $(-\infty, 2) \cup (2, \infty)$ **b.** $\{-1, 0, 1\} \cup (2, \infty)$
c. increasing: $(2, \infty)$, decreasing: $(-\infty, -1)$, constant: $(-1, 2)$



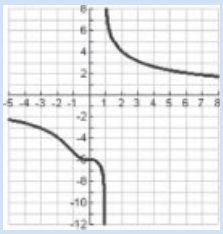
117. The graph of f can be obtained by shifting the graph of g two units to the left. That is, $f(x) = g(x + 2)$.



119. domain: $[-1.5, 4)$



121. yes

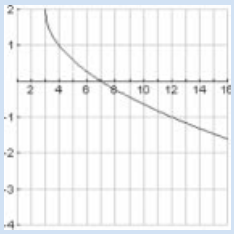


Practice Test

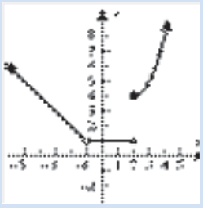
1. b **3.** c **5.** $\dfrac{\sqrt{x-2}}{x^2+11}$ domain: $[2, \infty)$

7. $x + 9$ domain: $(2, \infty)$ **9.** 4 **11.** neither

13. domain: $[3, \infty)$ range: $(-\infty, 2]$



15. domain: $(-\infty, -1) \cup (-1, \infty)$ range: $[1, \infty)$



17. **a.** -2 **b.** 4 **c.** -3 **d.** $x = -3, 2$

19. $6x + 3h - 4$ **21.** -32

23. $f^{-1}(x) = x^2 + 5$
domain f: $[5, \infty)$ domain f^{-1}: $[0, \infty)$
range f: $[0, \infty)$ range f^{-1}: $[5, \infty)$

25. $f^{-1}(x) = \dfrac{5x - 1}{x + 2}$
domain f: $(-\infty, 5) \cup (5, \infty)$
range f: $(-\infty, -2) \cup (-2, \infty)$
domain f^{-1}: $(-\infty, -2) \cup (-2, \infty)$
range f^{-1}: $(-\infty, 5) \cup (5, \infty)$

27. $[0, \infty)$ **29.** $P(t) = \frac{9}{10}t + 10$

31. quadrant III, "quarter of unit circle"

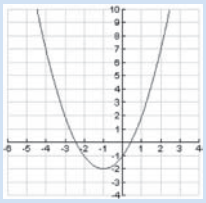
33. $C(x) = \begin{cases} 15, & 0 \le x \le 30 \\ x - 15, & x > 30 \end{cases}$

CHAPTER 2

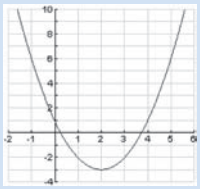
Section 2.1

1. b **3.** a **5.** b **7.** c

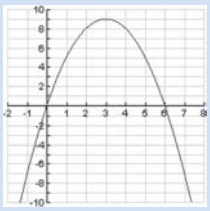
9.



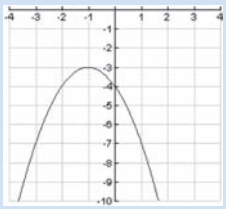
11.



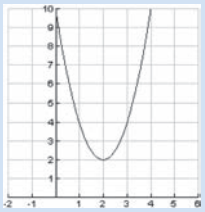
13.



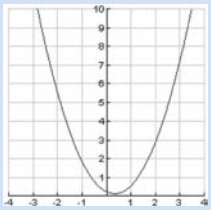
15.



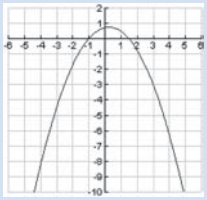
17.



19.



21.



23. $f(x) = (x + 3)^2 - 12$

25. $f(x) = -(x + 5)^2 + 28$

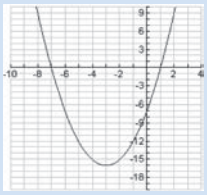
27. $f(x) = 2(x + 2)^2 - 10$

29. $f(x) = -4(x - 2)^2 + 9$

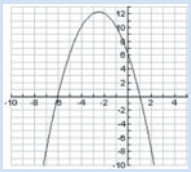
31. $f(x) = (x + 5)^2 - 25$

33. $f(x) = \frac{1}{2}(x - 4)^2 - 5$

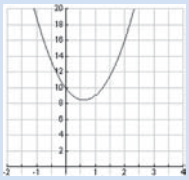
35.



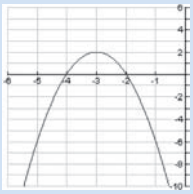
37.



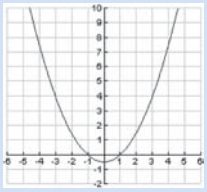
39.



41.



43.



45. $\left(\frac{1}{33}, \frac{494}{33}\right)$

47. $\left(7, -\frac{39}{2}\right)$

49. $\left(\frac{15}{28}, \frac{829}{392}\right)$

51. $(-75, 12.95)$

53. $(21.67, -24.65)$

55. $y = -2(x + 1)^2 + 4$

57. $y = -5(x - 2)^2 + 5$

59. $y = \frac{5}{9}(x + 1)^2 - 3$

61. $y = 10(x + 2)^2 - 4$

63. $y = 12\left(x - \frac{1}{2}\right)^2 - \frac{3}{4}$

65. $y = \frac{5}{4}(x - 2.5)^2 - 3.5$

67. **a.** 350,000 units **b.** \$12,262,500

69. He is gaining weight during January of 2010 and losing weight from February 2010 to June 2011.

71. **a.** 120 ft **b.** 50 yd

73. 2,083,333 sq ft

75. **a.** 1 sec, 116 ft **b.** 3.69 sec

77. **a.** 26,000 ft **b.** 8944 ft

79. **a.** 100 boards **b.** \$24,000

81. 15 to 16 or 64 to 65 units to break even.

83. **a.** $f(t) = \frac{28}{27}t^2 + 16$ **b.** 219 million

85. **a.** $y = -0.01(t - 225)^2 + 400$ **b.** 425 min

87. Step 2 is wrong: Vertex is $(-3, -1)$. Step 4 is wrong: The x-intercepts are $(-2, 0)$, $(-4, 0)$. Should graph $y = (x + 3)^2 - 1$.

89. Step 2 is wrong: $(-x^2 + 2x) = -(x^2 - 2x)$

91. true

93. false

95. $f(x) = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$

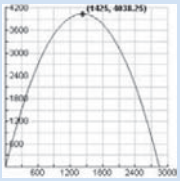
97. **a.** 62,500 sq ft **b.** 79,577 sq ft

99. $x = 5$

101. **a.** (1425, 4038.25) **b.** $(0, -23)$

c. (4.04, 0), (2845.96, 0)

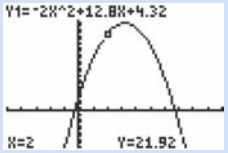
d. $x = 1425$



103. **a.** $y = -2x^2 + 12.8x + 4.32$

b. $y = -2(x - 3.2)^2 + 24.8$, (3.2, 24.8)

c. yes



105. Plot

107. $\dfrac{(x - 0)^2}{9} + \dfrac{(y - 2)^2}{4} = 1$ ellipse

109. $(x + 3)^2 = 20\left(y + \frac{1}{5}\right)$ parabola

Section 2.2

1. polynomial; degree 5

3. polynomial, degree 7

5. not a polynomial

7. not a polynomial

9. not a polynomial

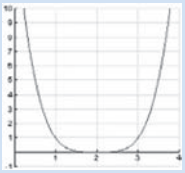
11. h

13. b

15. e

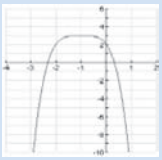
17. c

19.



21.

23.



25. 3 (multiplicity 1), -4 (multiplicity 3)

27. 0 (multiplicity 2), 7 (multiplicity 2), -4 (multiplicity 1)

29. 0 (multiplicity 2), 1 (multiplicity 2)

31. 0 (multiplicity 1), $\frac{3}{2}$ (multiplicity 1), $-\frac{9}{4}$ (multiplicity 1)

33. 0 (multiplicity 2), -3 (multiplicity 1)

35. 0 (multiplicity 4)

37. $P(x) = x(x + 3)(x - 1)(x - 2)$

39. $P(x) = x(x + 5)(x + 3)(x - 2)(x - 6)$

41. $P(x) = (2x + 1)(3x - 2)(4x - 3)$

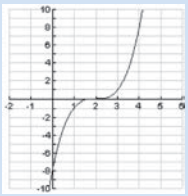
43. $P(x) = x^2 - 2x - 1$

45. $P(x) = x^2(x + 2)^3$

47. $P(x) = (x + 3)^2(x - 7)^5$

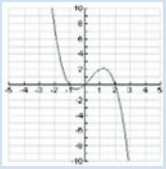
49. $P(x) = x^2(x + 1)\left(x + \sqrt{3}\right)^2\left(x - \sqrt{3}\right)^2$

51. $f(x) = (x - 2)^3$ **a.** 2 (multiplicity 3)
b. crosses at 2 **c.** $(0, -8)$ **d.** falls left, rises right
e.

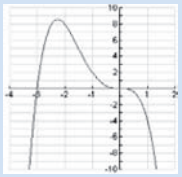


53. $f(x) = x(x - 3)(x + 3)$ **a.** 0, 3, -3 (multiplicity 1)
b. crosses at each zero **c.** (0, 0) **d.** falls left, rises right
e.

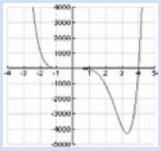
55. $f(x) = -x(x - 2)(x + 1)$ **a.** 0, 2, -1 (multiplicity 1)
b. crosses at each zero **c.** (0, 0) **d.** falls right, rises left
e.



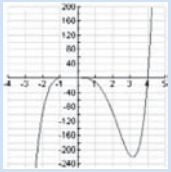
57. $f(x) = -x^3(x + 3)$ **a.** 0 (multiplicity 3), -3 (multiplicity 1)
b. crosses at both 0 and -3 **c.** (0, 0) **d.** falls left and right, without bound
e.



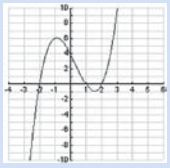
59. $f(x) = 12x^4(x - 4)(x + 1)$ **a.** 0 (multiplicity 4), 4 (multiplicity 1), -1 (multiplicity 1)
b. touches at 0 and crosses at 4 and -1
c. (0, 0) **d.** rises left and right, without bound
e.



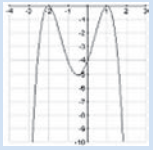
61. $f(x) = 2x^3(x - 4)(x + 1)$ **a.** 0 (multiplicity 3), 4 (multiplicity 1), -1 (multiplicity 1)
b. crosses at each zero **c.** (0, 0) **d.** falls left, rises right
e.



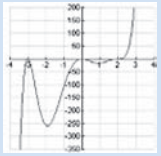
63. $f(x) = (x - 2)(x + 2)(x - 1)$ **a.** 1, 2, -2 (multiplicity 1)
b. crosses at each zero **c.** (0, 4) **d.** falls left, rises right
e.



65. $f(x) = -(x + 2)^2(x - 1)^2$ **a.** -2 (multiplicity 2), 1 (multiplicity 2) **b.** touches at both -2 and 1 **c.** $(0, -4)$
d. falls left and right, without bound
e.



67. $f(x) = x^2(x - 2)^3(x + 3)^2$ **a.** 0 (multiplicity 2), 2 (multiplicity 3), -3 (multiplicity 2)
b. touches at both 0 and -3, and crosses at 2
c. (0, 0) **d.** falls left, and rises right
e.

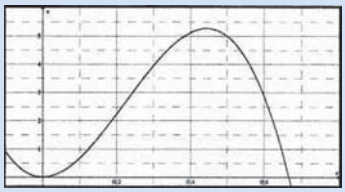


69. **a.** -3 (multiplicity 1), -1 (multiplicity 2), 2 (multiplicity 1)
b. even **c.** negative **d.** (0, 6)
e. $f(x) = -(x + 1)^2(x - 2)(x + 3)$

71. **a.** 0 (multiplicity 2), -2 (multiplicity 2), $\frac{3}{2}$ (multiplicity 1)
b. odd **c.** positive **d.** (0, 0)
e. $f(x) = x^2(2x - 3)(x + 2)^2$

73. **a.** Revenue for the company is increasing when advertising costs are less than \$400,000. Revenue for the company is decreasing when advertising costs are between \$400,000 and \$600,000.
b. The zeros of the revenue function occur when \$0 and \$600,000 are spent on advertising. When either \$0 or \$600,000 is spent on advertising, the company's revenue is \$0.

75. The velocity of air in the trachea is increasing when the radius of the trachea is between 0 and 0.45 cm and decreasing when the radius of the trachea is between 0.45 cm and 0.65 cm.



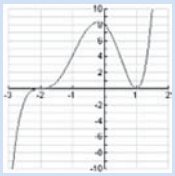
77. 6th degree polynomial **79.** down

81. 4th degree **83.** 4

85. between 35 and 39 degrees

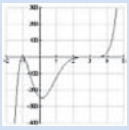
87. If h is a zero of a polynomial, then $(x - h)$ is a factor of it. So, in this case the function would be:
$P(x) = (x + 2)(x + 1)(x - 3)(x - 4)$

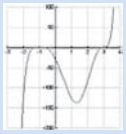
89. The zeros are correct. But it is a fifth degree polynomial. The graph should touch at 1 (since even multiplicity) and cross at -2. The graph should look like:



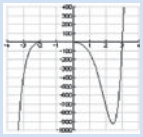
91. false **93.** true **95.** n

97. $f(x) = (x + 1)^2(x - 3)^5$, $g(x) = (x + 1)^4(x - 3)^3$





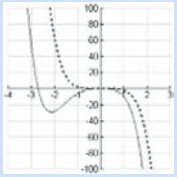
$h(x) = (x + 1)^6(x - 3)$



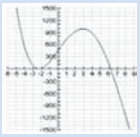
99. 0, a, $-b$

101. no x-intercepts

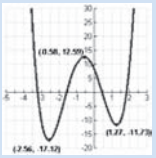
103. $y = -2x^5$, yes



105. x-intercepts: $(-2.25, 0)$, $(6.2, 0)$, $(14.2, 0)$ zeros: -2.25 (multiplicity 2), 6.2 (multiplicity 1), 14.2, (multiplicity 1)



107.



$(-2.56, -17.12)$
$(-0.58, 12.59)$
$(1.27, -11.73)$

109. $x = 1.154$ **111.** $x = -0.865$, $x = 1.363$

Section 2.3

1. $Q(x) = 3x - 3$, $r(x) = -11$

3. $Q(x) = 3x - 28$, $r(x) = 130$

5. $Q(x) = x - 4$, $r(x) = 12$

7. $Q(x) = 3x + 5$, $r(x) = 0$

9. $Q(x) = 2x - 3$, $r(x) = 0$

11. $Q(x) = 4x^2 + 4x + 1$, $r(x) = 0$

13. $Q(x) = 2x^2 - x - \frac{1}{2}$, $r(x) = \frac{15}{2}$

15. $Q(x) = 4x^2 - 10x - 6$, $r(x) = 0$

17. $Q(x) = -2x^2 - 3x - 9$,
$r(x) = -27x^2 + 3x + 9$

19. $Q(x) = x^2 + 1$, $r(x) = 0$

21. $Q(x) = x^2 + x + \frac{1}{6}$, $r(x) = -\frac{121}{6}x + \frac{121}{3}$

23. $Q(x) = -3x^3 + 5.2x^2 + 3.12x - 0.128$,
$r(x) = 0.9232$

25. $Q(x) = x^2 - 0.6x + 0.09$, $r(x) = 0$

27. $Q(x) = 3x + 1$, $r(x) = 0$

29. $Q(x) = 7x - 10$, $r(x) = 15$

31. $Q(x) = -x^3 + 3x - 2$, $r(x) = 0$

33. $Q(x) = x^3 - x^2 + x - 1$, $r(x) = 2$

35. $Q(x) = x^3 - 2x^2 + 4x - 8, \quad r(x) = 0$

37. $Q(x) = 2x^2 - 6x + 2, \quad r(x) = 0$

39. $Q(x) = 2x^3 - \frac{5}{3}x^2 + \frac{53}{9}x + \frac{106}{27}, \quad r(x) = -\frac{112}{81}$

41. $Q(x) = 2x^3 + 6x^2 - 18x - 54, \quad r(x) = 0$

43. $Q(x) = x^6 + x^5 + x^4 - 7x^3 - 7x^2 - 4x - 4, \quad r(x) = -3$

45. $Q(x) = x^5 + \sqrt{5}x^4 - 44x^3 - 44\sqrt{5}x^2 - 245x - 245\sqrt{5}$, $r(x) = 0$

47. $Q(x) = 2x - 7, \quad r(x) = 0$

49. $Q(x) = x^2 - 9, \quad r(x) = 0$

51. $Q(x) = x + 6, \quad r(x) = -x + 1$

53. $Q(x) = x^4 - 2x^3 - 4x + 7, \quad r(x) = 0$

55. $Q(x) = x^4 + 2x^3 + 8x^2 + 18x + 36, \quad r(x) = 71$

57. $Q(x) = x^2 + 1, \quad r(x) = -24$

59. $Q(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \quad r(x) = 0$

61. $3x^2 + 2x + 1$ ft

63. $x^2 + 1$ hr

65. Should have subtracted each term in the long division rather than adding them.

67. Forgot the "0" placeholder.

69. true

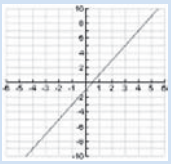
71. false

73. false

75. yes

77. $Q(x) = x^{2n} + 2x^n + 1, \quad r(x) = 0$

79. $2x - 1$



81. $x^3 - 1$

83. quadratic function

85. $(2x - 5) + \dfrac{10}{x + 2}$

87. $(2x^2 - 2x + 3) - \dfrac{x - 3}{x^2 + x + 1}$

Section 2.4

1. $-4, 1, 3$; $P(x) = (x - 1)(x + 4)(x - 3)$

3. $-3, \frac{1}{2}, 2$; $P(x) = (2x - 1)(x + 3)(x - 2)$

5. $-3, 5$; $P(x) = (x^2 + 4)(x - 5)(x + 3)$

7. $-3, 1$; $P(x) = (x - 1)(x + 3)(x^2 - 2x + 2)$

9. $-2, -1$ (both multiplicity 2);
$P(x) = (x + 2)^2(x + 1)^2$

11. $\pm 1, \pm 2, \pm 4$

13. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

15. $\pm\frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8$

17. $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}$

19. $\pm 1, \pm 2, \pm 4, \pm 8$; rational zeros: $-4, -1, 2, 1$

21. $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$; rational zeros: $\frac{1}{2}, 1, 3$

23.

Positive Real Zeros	Negative Real Zeros
1	1

25.

Positive Real Zeros	Negative Real Zeros
1	0

27.

Positive Real Zeros	Negative Real Zeros
2	1
0	1

29.

Positive Real Zeros	Negative Real Zeros
1	1

31.

Positive Real Zeros	Negative Real Zeros
2	2
0	2
2	0
0	0

33.

Positive Real Zeros	Negative Real Zeros
4	0
2	0
0	0

35. **a.** Number of sign variations for $P(x)$: 0
Number of sign variations for $P(-x)$: 3

Positive Real Zeros	Negative Real Zeros
0	3
0	1

b. possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$
c. rational zeros: $-1, -2, -3$
d. $P(x) = (x + 1)(x + 2)(x + 3)$

37. **a.** Number of sign variations for $P(x)$: 2
Number of sign variations for $P(-x)$: 1

Positive Real Zeros	Negative Real Zeros
2	1
0	1

b. possible rational zeros: $\pm 1, \pm 7$
c. rational zeros: $-1, 1, 7$
d. $P(x) = (x + 1)(x - 1)(x - 7)$

39. **a.** Number of sign variations for $P(x)$: 1
Number of sign variations for $P(-x)$: 2

Positive Real Zeros	Negative Real Zeros
1	2
1	0

b. possible rational zeros: $\pm 1, \pm 2, \pm 5, \pm 10$
c. rational zeros: $0, 1, -2, -5$
d. $P(x) = x(x - 1)(x + 2)(x + 5)$

41. **a.** Number of sign variations for $P(x)$: 4
Number of sign variations for $P(-x)$: 0

Positive Real Zeros	Negative Real Zeros
4	0
2	0
0	0

b. possible rational zeros: $\pm 1, \pm 2, \pm 13, \pm 26$
c. rational zeros: $1, 2$
d. $P(x) = (x - 1)(x - 2)(x^2 - 4x + 13)$

43. **a.** Number of sign variations for $P(x)$: 2
Number of sign variations for $P(-x)$: 1

Positive Real Zeros	Negative Real Zeros
2	1
0	1

b. possible rational zeros: $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{1}{10}$
c. rational zeros: $-1, -\frac{1}{2}, \frac{1}{5}$
d. $P(x) = (x - 1)(2x + 1)(5x - 1)$

45. **a.** Number of sign variations for $P(x)$: 1
Number of sign variations for $P(-x)$: 2

Positive Real Zeros	Negative Real Zeros
1	2
1	0

b. possible rational zeros: $\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm\frac{5}{2}, \pm\frac{5}{3}, \pm\frac{5}{6}, \pm\frac{10}{3}$
c. rational zeros: $-1, -\frac{5}{2}, \frac{2}{3}$
d. $P(x) = 6(x + 1)\left(x + \frac{5}{2}\right)\left(x - \frac{2}{3}\right)$

47. **a.** Number of sign variations for $P(x)$: 4
Number of sign variations for $P(-x)$: 0

Positive Real Zeros	Negative Real Zeros
4	0
2	0
0	0

b. possible rational zeros: $\pm 1, \pm 2, \pm 4$
c. rational zeros: 1
d. $P(x) = (x - 1)^2(x^2 + 4)$

49. **a.** Number of sign variations for $P(x)$: 1
Number of sign variations for $P(-x)$: 1

Positive Real Zeros	Negative Real Zeros
1	1

b. possible rational zeros:
$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$
c. rational zeros: $-1, 1$
d. $P(x) = (x + 1)(x - 1)(x^2 + 9)(x^2 + 4)$

51. **a.** Number of sign variations for $P(x)$: 4
Number of sign variations for $P(-x)$: 0

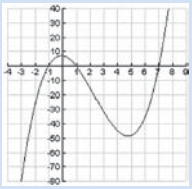
Positive Real Zeros	Negative Real Zeros
4	0
2	0
0	0

b. possible rational zeros:
$\pm 1, \pm 5, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{5}{2}, \pm\frac{5}{4}$

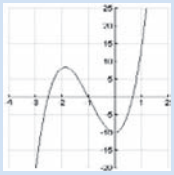
c. rational zeros: $\frac{1}{2}$

d. $P(x) = 4\left(x - \frac{1}{2}\right)^2(x^2 - 4x + 5)$

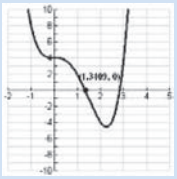
53.



55.

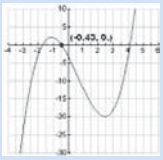


57. $x = 1.34$



59. $x = 0.22$

61. $x = -0.43$



63. $x = 2.88$ **65.** 6 in. $\times$ 8 in. **67.** 30 cows

69. $P(x) = -0.0002x^2 + 8x - 1500$; 0 or 2 positive real zeros

71. 18 hr

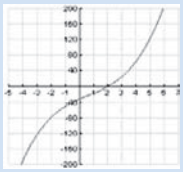
73. It is true that one can get 5 negative zeros here, but there may be just 1 or 3.

Positive Real Zeros	Negative Real Zeros
0	5
0	3
0	1

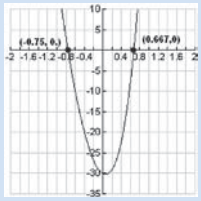
75. true **77.** false **79.** false

81. b, c **83.** a, c, and $-c$

85. possible rational zeros:
$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32$ zeros: 2



87. **a.** $-\frac{3}{4}, \frac{2}{3}$ **b.** $(3x - 2)(4x + 3)(x^2 + 2x + 5)$



89. $1, 5, -2$; $(-2, -1) \cup (5, \infty)$

91. $-\frac{1}{2}, 3, \pm\sqrt{2}$; $\left(-\sqrt{2}, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 3\right)$

Section 2.5

1. $x = \pm 2i$; $P(x) = (x + 2i)(x - 2i)$

3. $x = 1 \pm i$; $P(x) = (x - (1 - i))(x - (1 + i))$

5. $x = \pm 2, \pm 2i$; $P(x) = (x - 2)(x + 2)(x - 2i)(x + 2i)$

7. $x = \pm\sqrt{5}, \pm i\sqrt{5}$;
$P(x) = (x - \sqrt{5})(x + \sqrt{5})(x - i\sqrt{5})(x + i\sqrt{5})$

9. $-i$ **11.** $-2i, 3 + i$

13. $1 + 3i, 2 - 5i$ **15.** $i, 1 + i$

17. $P(x) = x^3 - 2x^2 + 5x$

19. $P(x) = x^3 - 3x^2 + 28x - 26$

21. $P(x) = x^4 - 2x^3 + 11x^2 - 18x + 18$

23. $\pm 2i, -3, 5$; $P(x) = (x - 2i)(x + 2i)(x - 5)(x + 3)$

25. $\pm i, 1, 3$; $P(x) = (x - i)(x + i)(x - 3)(x - 1)$

27. $\pm 3i, 1$ (multiplicity 2); $P(x) = (x - 3i)(x + 3i)(x - 1)^2$

29. $1 \pm i, -1 \pm 2\sqrt{2}$;
$P(x) = (x - (1 + i))(x - (1 - i)) \cdot$
$(x - (-1 - 2\sqrt{2}))(x - (-1 + 2\sqrt{2}))$

31. $3 \pm i, \pm 2$;
$P(x) = (x - (3 + i))(x - (3 - i))(x - 2)(x + 2)$

33. $2 \pm i, 1, 4$;
$P(x) = (x - (2 + i))(x - (2 - i))(x - 1)(x - 4)$

35. $P(x) = (x + 3i)(x - 3i)(x - 1)$

37. $P(x) = (x + i)(x - i)(x - 5)$

39. $P(x) = (x + 2i)(x - 2i)(x + 1)$

41. $P(x) = (x - 3)(x - (-1 + i\sqrt{5}))(x - (-1 - i\sqrt{5}))$

43. $P(x) = (x + 3)(x - 5)(x + 2i)(x - 2i)$

45. $P(x) = (x + 1)(x - 5)(x + 2i)(x - 2i)$

47. $P(x) = (x - 1)(x - 2)(x - (2 - 3i))(x - (2 + 3i))$

49. $P(x) = -(x + 1)(x - 2)(x - (2 - i))(x - (2 + i))$

51. $P(x) = (x - 1)^2(x + 2i)(x - 2i)$

53. $P(x) = (x - 1)(x + 1)(x - 2i)(x + 2i)(x - 3i)(x + 3i)$

55. $P(x) = (2x - 1)^2(x - (2 - i))(x - (2 + i))$

57. $P(x) = (x - 1)(x + 1)(3x - 2)(x - 2i)(x + 2i)$

59. Yes. In such case, $P(x)$ is always above the x-axis since the leading coefficient is positive, indicating that the end behavior should resemble that of $y = x^{2n}$, for some positive integer n. So, profit is always positive and increasing.

61. No. In such case, it crosses the x-axis and looks like $y = -x^3$. So, profit is decreasing.

63. Since the profit function is a third-degree polynomial, we know that the function has three zeros and at most two turning points. Looking at the graph, we can see there is one real zero where $t \leq 0$. There are no real zeros when $t > 0$, so the other two zeros must be complex conjugates. Therefore, the company always has a profit greater than approximately 5.1 million dollars and, in fact, the profit will increase toward infinity as t increases.

65. Since the concentration function is a third-degree polynomial, we know that the function has three zeros and at most two turning points. Looking at the graph, we can see there is one real zero sometimes $t \geq 8$. The remaining zeros are a complex conjugate pair. Therefore, the concentration of the drug in the bloodstream will decrease to zero as the hours go by. Note that the concentration will not approach negative infinity since concentration is a nonnegative quantity.

67. Step 2 is an error. In general, the additive inverse of a real root need not be a root. This is being confused with the fact that complex roots occur in conjugate pairs.

69. false **71.** true

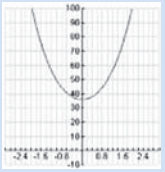
73. No. Complex zeros occur in conjugate pairs. So, the collection of complex solutions contributes an even number of zeros, thereby requiring there to be at least one real zero.

75. $P(x) = x^6 + 3b^2x^4 + 3b^4x^2 + b^6$

77. $P(x) = x^6 + (2a^2 + b^2)x^4 + (a^4 + 2a^2b^2)x^2 + b^2a^4$

79. All roots are complex.

Real Zeros	Complex Zeros
0	4
2	2
4	0



81. $\frac{3}{5}, \pm i, \pm 2i$;
$P(x) = -5(x - 0.6)(x - 2i)(x + 2i)(x - i)(x + i)$

83. **a.** $f(x) = (x + 1)(x - i)(x + i)$
b. $f(x) = (x + 1)(x^2 + 1)$

85. **a.** $f(x) = (x + 2i)(x - 2i)(x + i)(x - i)$
b. $f(x) = (x^2 + 4)(x^2 + 1)$

Section 2.6

1. $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$

3. $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

5. $(-\infty, \infty)$

7. $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

9. HA: $y = 0$ VA: $x = -2$

11. HA: none VA: $x = -5$

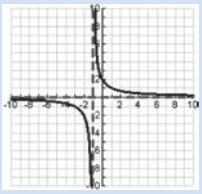
13. HA: none VA: $x = \frac{1}{2}, x = -\frac{4}{3}$

15. HA: $y = \frac{1}{3}$ VA: none

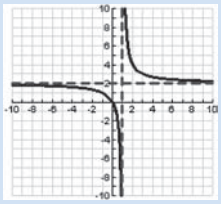
17. $y = x + 6$ **19.** $y = 2x + 24$ **21.** $y = 4x + \frac{11}{2}$

23. b **25.** a **27.** e

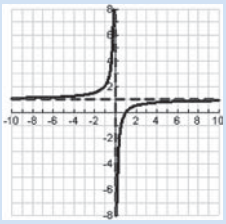
29.



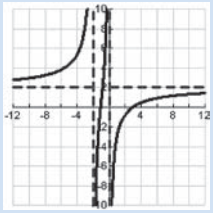
31.



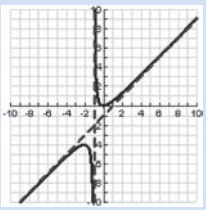
33.



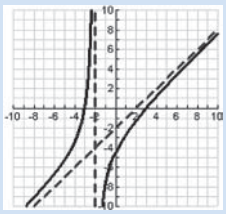
35.



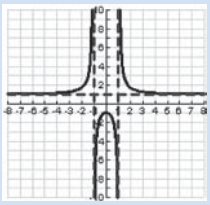
37.



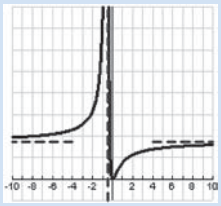
39.



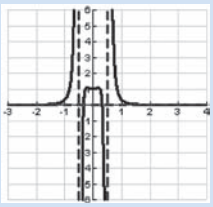
41.



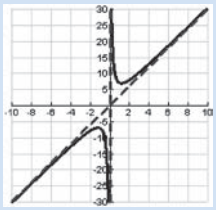
43.



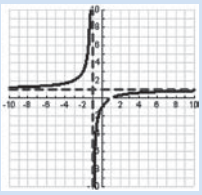
45.



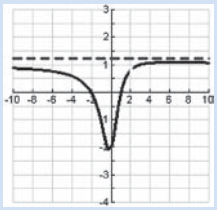
47.



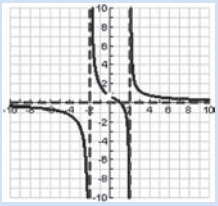
49.



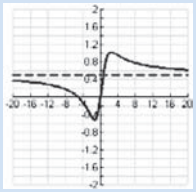
51.



53.



55. $y = \frac{1}{2}$



57. **a.** x-intercept: (2, 0); y-intercept: (0, 0.5)
b. HA: $y = 0$ VA: $x = -1, x = 4$
c. $f(x) = \dfrac{x - 2}{(x + 1)(x - 4)}$

59. **a.** x-intercept: (0, 0); y-intercept: (0, 0)
b. HA: $y = -3$ VA: $x = -4, x = 4$
c. $f(x) = \dfrac{-3x^2}{(x + 4)(x - 4)}$

61. **a.** 4500 people **b.** 6 mo **c.** stabilizes around 9500

63. **a.** $C(1) \cong 0.0198$ **b.** $C(60) \cong 0.0324$
c. $C(300) \cong 0$
d. $y = 0$; after several days, $C(t) \cong 0$

65. **a.** $N(0) = 52$ wpm **b.** $N(12) \cong 107$ wpm
c. $N(36) \cong 120$ wpm **d.** $y = 130$; 130 wpm

67. $y = 10$, 10 oz of food **69.** $\dfrac{2w^2 + 1000}{w}$

71. 2000 or 8000 units; average profit of \$16 per unit.

73. The concentration of the drug in the bloodstream 15 hours after taking the dose is approximately 25.4 μg/mL. There are two times, 1 hour and 15 hours, after taking the medication at which the concentration of the drug in the bloodstream is approximately 25.4 μg/mL. The first time, approximately 1 hour, occurs as the concentration of the drug is increasing to a level high enough that the body will be able to maintain a concentration of approximately 25 μg/mL throughout the day. The second time, approximately 15 hours, occurs many hours later in the day as the concentration of the medication in the bloodstream drops.

75. $f(x) = \dfrac{x - 1}{x^2 - 1} = \dfrac{\cancel{x - 1}}{\cancel{(x - 1)}(x + 1)} = \dfrac{1}{x + 1}$ with a hole at $x = 1$. So $x = 1$ is not a vertical asymptote.

77. In Step 2, the ratio of the leading coefficients should be $\frac{-1}{1}$. So the horizontal asymptote is $y = -1$.

79. true **81.** false

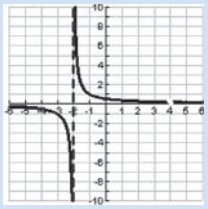
83. HA: $y = 1$ VA: $x = c, x = -d$

85. Two possibilities: $y = \dfrac{4x^2}{(x + 3)(x - 1)}$ and

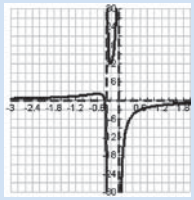
$y = \dfrac{4x^5}{(x + 3)^3(x - 1)^2}$

87. $f(x) = \dfrac{x^3+1}{x^2+1}$

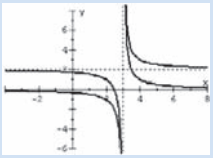
89. VA: $x = -2$, yes



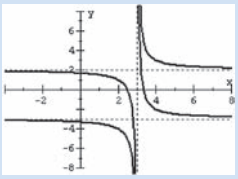
91. HA: $y = 0$ VA: $x = 0, x = -\frac{1}{3}$
Intercepts: $\left(-\frac{2}{5}, 0\right)$, yes



93. **a.** f: HA: $y = 0$ VA: $x = 3$
g: HA: $y = 2$ VA: $x = 3$
h: HA: $y = -3$ VA: $x = 3$
b. graphs of f and g: as
$x \to \pm\infty, f(x) \to 0$ and $g(x) \to 2$



c. graphs of g and h below:
as $x \to \pm\infty$, $g(x) \to 2$ and $h(x) \to -3$



d. $g(x) = \dfrac{2x - 5}{x - 3}$, $h(x) = \dfrac{-3x + 10}{x - 3}$ yes

95. $x = -2, x = -1, x = 5$

97. $x = \frac{2}{3}, x = -\frac{1}{2}$

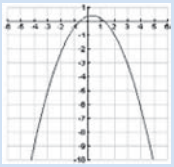
Review Exercises

1. b

3. a

5.

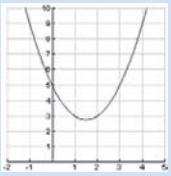
7.



9. $f(x) = \left(x - \frac{3}{2}\right)^2 - \frac{49}{4}$

11. $f(x) = 4(x + 1)^2 - 11$

13.



15.

17. $\left(\frac{5}{26}, \frac{599}{52}\right)$

19. $\left(-\frac{2}{15}, \frac{451}{125}\right)$

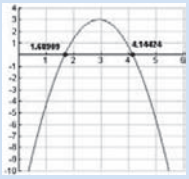
21. $y = \frac{1}{9}(x + 2)^2 + 3$

23. $y = 5.6(x - 2.7)^2 + 3.4$

25. **a.** $P(x) = -2x^2 + \frac{35}{3}x - 14$

b. $x \cong 4.1442433, 1.68909$

c.

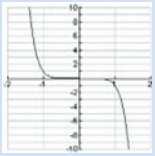


d. (1.6891, 4.144) or 1689 to 4144

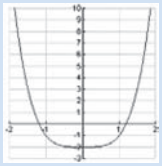
27. $A(x) = -\frac{1}{2}(x - 1)^2 + \frac{9}{2}$, maximum at $x = 1$
base: 3 units, height: 3 units

29. yes, 6 **31.** no **33.** d **35.** a

37.



39.



41. 6 (multiplicity 5), -4 (multiplicity 2)

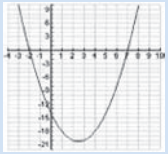
43. $0, -2, 2, 3, -3$, all multiplicity 1

45. $f(x) = x(x + 3)(x - 4)$

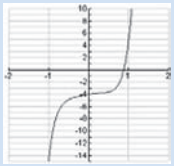
47. $f(x) = x(5x + 2)(4x - 3)$

49. $f(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$

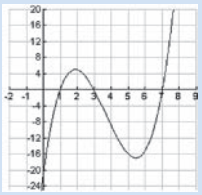
51. $f(x) = (x - 7)(x + 2)$ **a.** $-2, 7$ (both multiplicity 1)
b. crosses at $-2, 7$ **c.** $(0, -14)$ **d.** rises right and left
e.



53. $f(x) = 6x^7 + 3x^5 - x^2 + x - 4$ **a.** (0.8748, 0) with multiplicity 1 **b.** crosses at its only real zero **c.** $(0, -4)$
d. falls left and rises right
e.



55. **a.**



b. 1, 3, 7 (all with multiplicity 1)
c. between 1 and 3 hr, and more than 7 hr is financially beneficial

57. $Q(x) = x + 4, \quad r(x) = 2$

59. $Q(x) = 2x^3 - 4x^2 - 2x - \frac{7}{2}, \quad r(x) = -23$

61. $Q(x) = x^3 + 2x^2 + x - 4, \quad r(x) = 0$

63. $Q(x) = x^5 - 8x^4 + 64x^3 - 512x^2 + 4096x - 32{,}768$,
$r(x) = 262{,}080$

65. $Q(x) = x + 3, \quad r(x) = -4x - 8$

67. $Q(x) = x^2 - 5x + 7, \quad r(x) = -15$

69. $3x^3 + 2x^2 - x + 4$ ft

71. $f(-2) = -207$

73. $g(1) = 0$

75. no

77. yes

79. $P(x) = x(x + 2)(x - 4)^2$

81. $P(x) = x^2(x + 3)(x - 2)^2$

83.

Positive Real Zeros	Negative Real Zeros
1	1

85.

Positive Real Zeros	Negative Real Zeros
5	2
5	0
3	2
3	0
1	2
1	0

87. possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

89. possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32, \pm 64, \pm\frac{1}{2}$

91. possible rational zeros: $\pm 1, \pm\frac{1}{2}$; zeros: $\frac{1}{2}$

93. possible rational zeros: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$; zeros: $1, 2, 4, -2$

95. a.

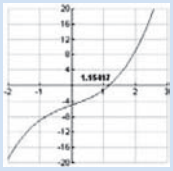
Positive Real Zeros	Negative Real Zeros
1	0

b. $\pm 1, \pm 5$

c. -1 is a lower bound, 5 is an upper bound.

d. none **e.** not possible

f.



97. a.

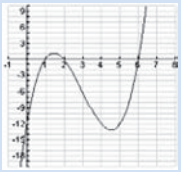
Positive Real Zeros	Negative Real Zeros
3	0
1	0

b. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

c. -4 is a lower bound, 12 is an upper bound.

d. $1, 2, 6$ **e.** $P(x) = (x - 1)(x - 6)(x - 2)$

f.



99. a.

Positive Real Zeros	Negative Real Zeros
0	0
0	2
2	2
2	0

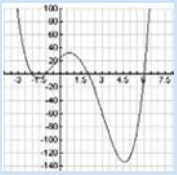
b. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8 \pm 12, \pm 24$

c. -4 is a lower bound, 8 is an upper bound.

d. $-2, -1, 1, 6$

e. $P(x) = (x - 2)(x + 1)(x + 2)(x - 6)$

f.



101. $P(x) = (x - 5i)(x + 5i)$

103. $P(x) = (x - (1 - 2i))(x - (1 + 2i))$

105. $2i, 3 - i$

107. $-i, 2 + I$

109. $-i, 4, -1$; $P(x) = (x - i)(x + i)(x - 4)(x + 1)$

111. $3i, 1 \pm i$; $P(x) = (x - 3i)(x + 3i)(x - (1 + i))(x - (1 - i))$

113. $P(x) = (x - 3)(x + 3)(x - 3i)(x + 3i)$

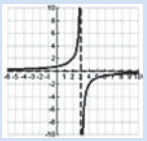
115. $P(x) = (x - 2i)(x + 2i)(x - 1)$

117. HA: $y = -1$ VA: $x = -2$

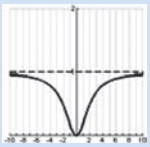
119. HA: none VA: $x = -1$ Slant: $y = 4x - 4$

121. HA: $y = 2$ VA: none

123.

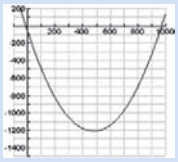


125.

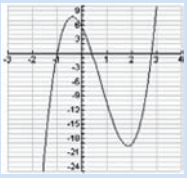


127.

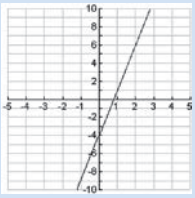
129. **a.** $(480, -1211)$ **b.** $(0, -59)$ **c.** $(-12.14, 0), (972.14, 0)$
d. $x = 480$



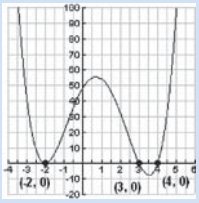
131. x-intercepts: $(-1, 0), (0.4, 0), (2.8, 0)$; zeros: $-1, 0.4, 2.8$, each with multiplicity 1



133. linear function



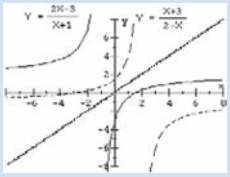
135. **a.** -2 (multiplicity 2), 3, 4
b. $P(x) = (x + 2)^2(x - 3)(x - 4)$



137. $\frac{7}{2}, -2 \pm 3i$; $P(x) = (2x - 7)(x + 2 - 3i)(x + 2 + 3i)$

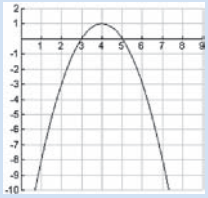
139. **a.** yes, one-to-one **b.** $f^{-1}(x) = \dfrac{-3 - x}{x - 2}$

c.



Practice Test

1.



3. $\left(3, \frac{1}{2}\right)$

5. $f(x) = x(x - 2)^3(x - 1)^2$

7. $Q(x) = -2x^2 - 2x - \frac{11}{2}$, $r(x) = -\frac{19}{2}x + \frac{7}{2}$

9. yes

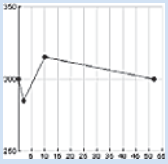
11. $P(x) = (x - 7)(x + 2)(x - 1)$

13. yes, complex zero

15. possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

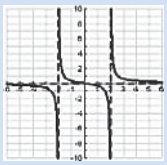
17. $\frac{3}{2}, \pm 2i$

19. degree 3

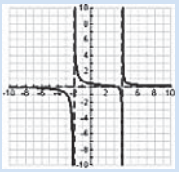


21. degree 3

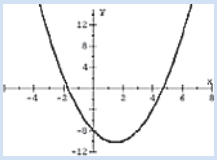
23. **a.** x-intercept: $(0, 0)$, y-intercept: $(0, 0)$
b. $x = \pm 2$ **c.** $y = 0$ **d.** none
e.



25. **a.** x-intercept: $(3, 0)$, y-intercept: $\left(0, \frac{3}{8}\right)$
b. $x = -2, x = 4$ **c.** $y = 0$
d. none **e.**



27. **a.** $y = x^2 - 3x - 7.99$
b. $y = (x - 1.5)^2 - 10.24$ **c.** $(-1.7, 0)$ and $(4.7, 0)$
d. yes



Cumulative Test

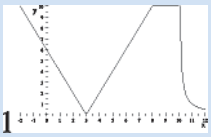
1. $f(2) = \frac{15}{2}, f(-1) = -5, f(1 + h) = 4 + 4h - \dfrac{1}{\sqrt{h + 3}}$,

$f(-x) = -4x - \dfrac{1}{\sqrt{2 - x}}$

3. $f(-3) = \frac{7}{2}, f(0) = -\frac{5}{2}, f(4) = -\frac{7}{18}$, $f(1)$ is undefined.

5. $\dfrac{1}{\sqrt{x + h} + \sqrt{x}} + \dfrac{2x + h}{x^2(x + h)^2}$

7. **a.**



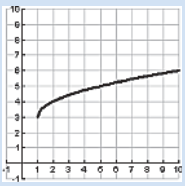
b. domain: $(-\infty, 10) \cup (10, \infty)$, range: $[0, \infty)$

c. increasing: $(3, 8)$, decreasing: $(-\infty, 3) \cup (10, \infty)$, constant: $(8,10)$

9. $-\frac{1}{28}$

11. neither

13. right one unit and then up three units



15. $g(f(-1)) = 0$

17. $f(x) = (x + 2)^2 + 3$

19. $Q(x) = 4x^2 + 4x + 1,\ r(x) = -8$

21. possible rational zeros:
$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}, \pm\frac{2}{3}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{1}{4}$
zeros: $-2, -\frac{3}{4}, \frac{1}{3}$

23. $P(x) = (x + 1)(x - 2)(x - 4)$

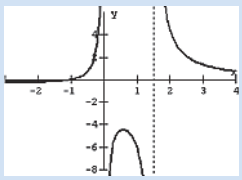
25. HA: $y = 0$ VA: $x = \pm 2$

27. $f(x) = \dfrac{3(x + 1)}{x(2x - 3)}$ x-intercepts: $(-1, 0)$

HA: $y = 0$

VA: $x = 0, x = \frac{3}{2}$

yes



CHAPTER 3

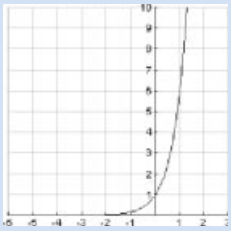
Section 3.1

1. $\frac{1}{25}$ **3.** 4 **5.** 27 **7.** 9.7385 **9.** 7.3891

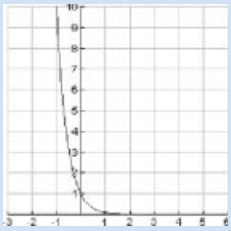
11. 0.0432 **13.** 27 **15.** 16 **17.** 4

19. 19.81 **21.** f **23.** e **25.** b

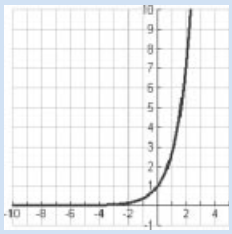
27. y-intercept: $(0, 1)$ HA: $y = 0$
domain: $(-\infty, \infty)$ range: $(0, \infty)$
other points: $\left(-1, \frac{1}{6}\right), (1, 6)$



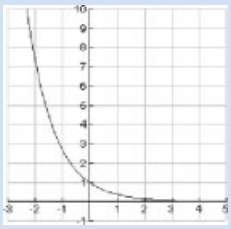
29. y-intercept: $(0, 1)$ HA: $y = 0$
domain: $(-\infty, \infty)$ range: $(0, \infty)$
other points: $(1, 0.1), (-1, 10)$



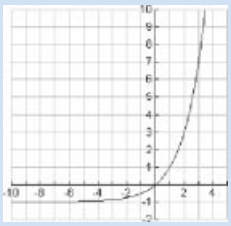
31. y-intercept: $(0, 1)$ HA: $y = 0$
domain: $(-\infty, \infty)$ range: $(0, \infty)$
other points: $(1, e), (2, e^2)$



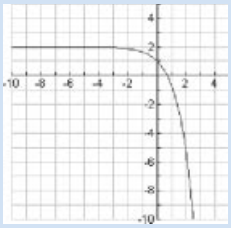
33. y-intercept: $(0, 1)$ HA: $y = 0$
domain: $(-\infty, \infty)$ range: $(0, \infty)$
other points: $\left(1, \frac{1}{e}\right), (-1, e)$



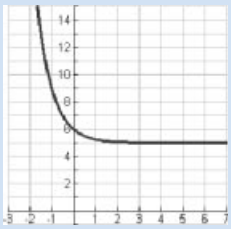
35. y-intercept: $(0, 0)$ HA: $y = -1$
domain: $(-\infty, \infty)$ range: $(-1, \infty)$
other points: $(2, 3), (1, 1)$



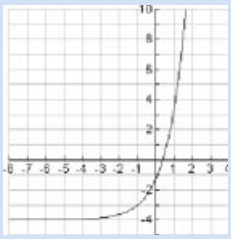
37. y-intercept: $(0, 1)$ HA: $y = 2$
domain: $(-\infty, \infty)$ range: $(-\infty, 2)$
other points: $(1, 2 - e), \left(-1, 2 - \frac{1}{e}\right)$



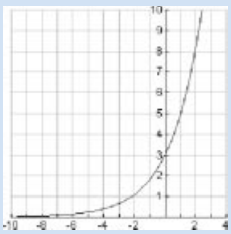
39. y-intercept: $(0, 6)$ HA: $y = 5$
domain: $(-\infty, \infty)$ range: $(5, \infty)$
other points: $(1, 5.25)$, $(-1, 9)$



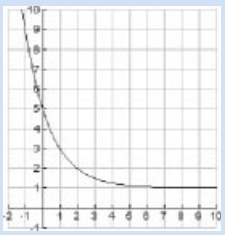
41. y-intercept: $(0, e - 4)$ HA: $y = -4$
domain: $(-\infty, \infty)$ range: $(-4, \infty)$
other points: $(-1, -3)$, $(1, e^2 - 4)$



43. y-intercept: $(0, 3)$ HA: $y = 0$
domain: $(-\infty, \infty)$ range: $(0, \infty)$
other points: $(2, 3e)$, $\left(1, 3\sqrt{e}\right)$



45. y-intercept: $(0, 5)$ HA: $y = 1$
domain: $(-\infty, \infty)$ range: $(1, \infty)$
other points: $(0, 5)$, $(2, 2)$



47. 10.4 million

49. $P(30) = 1500\left(2^{30/5}\right) \cong 96{,}000$

51. 168 mg

53. 2 mg

55. \$3031

57. \$3448.42

59. \$13,011.03

61. \$4319.55

63. \$13,979.42

65. 3.4 mg/L

67.

p(PRICE PER UNIT)	$D(p)$—APPROXIMATE DEMAND FOR PRODUCT IN UNITS
1.00	1,955,000
5.00	1,020,500
10.00	452,810
20.00	89,147
40.00	3455
60.00	134
80.00	5
90.00	1

69. The mistake is that $4^{-1/2} \neq 4^2$. Rather,

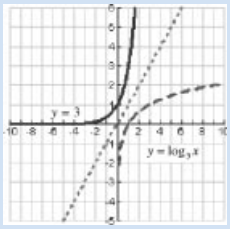
$$4^{-1/2} = \frac{1}{4^{1/2}} = \tfrac{1}{2}.$$

71. $r = 0.025$ rather than 2.5

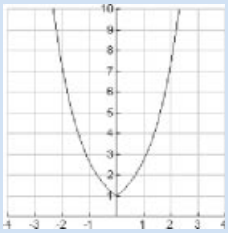
73. false

75. true

77.

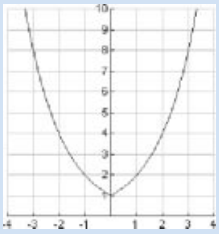


79.

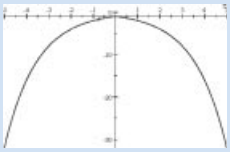


81. y-intercept: $(0, be - a)$ HA: $y = -a$

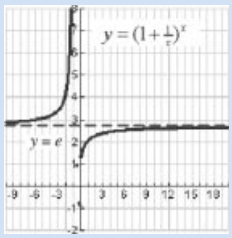
83. Domain: $(-\infty, \infty)$



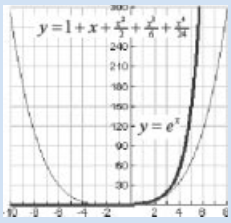
85.



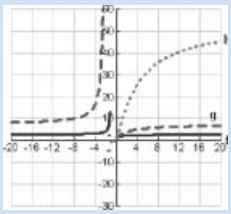
87.



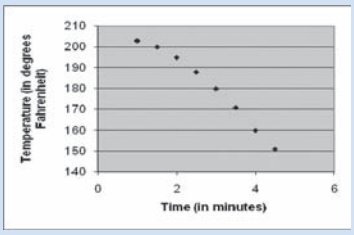
89. close on the interval $(-3, 3)$



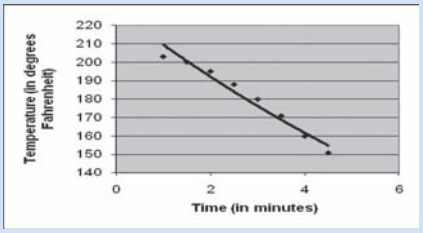
91. as x increases,
$f(x) \to e$, $g(x) \to e^2$, $h(x) \to e^4$



93. a.



b. The best fit exponential curve is $y = 228.34\,(0.9173)^x$ with $r^2 = 0.9628$. This best fit curve is shown below on the scatterplot. The fit is very good, as evidenced by the fact that the square of the correlation coefficient is very close to 1.



c. (i) Compute the y-value when $x = 6$ to obtain about 136°F.
(ii) The temperature of the soup the moment it was taken out of the microwave is the y-value at $x = 0$, namely, about 228°F.

d. The shortcoming of this model for large values of x is that the curve approaches the x-axis, not 72°. As such, it is no longer useful for describing the temperature beyond the x-value at which the temperature is 72°.

95. odd

Section 3.2

1. $81^{1/4} = 3$ **3.** $2^{-5} = \frac{1}{32}$ **5.** $10^{-2} = 0.01$

7. $10^4 = 10{,}000$ **9.** $\left(\frac{1}{4}\right)^{-3} = 64$ **11.** $e^{-1} = \frac{1}{e}$

13. $e^0 = 1$ **15.** $e^x = 5$ **17.** $x^z = y$

19. $y^x = x + y$ **21.** $\log(0.00001) = -5$

23. $\log_5 78{,}125 = 5^7$ **25.** $\log_{225}(15) = \frac{1}{2}$

27. $\log_{2/5}\left(\frac{8}{125}\right) = 3$ **29.** $\log_{1/27}(3) = -\frac{1}{3}$ **31.** $\ln 6 = x$

33. $\log_y x = z$ **35.** 0 **37.** 5

39. 7 **41.** -6 **43.** undefined

45. undefined **47.** 1.46 **49.** 5.94

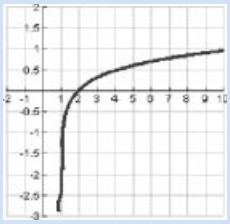
51. undefined **53.** -8.11 **55.** $(-5, \infty)$

57. $\left(-\infty, \frac{5}{2}\right)$ **59.** $\left(-\infty, \frac{7}{2}\right)$ **61.** $(-\infty, 0) \cup (0, \infty)$

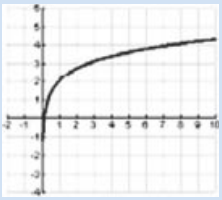
63. $\mathbb{R}$ **65.** $(-2, 5)$ **67.** b

69. c **71.** d

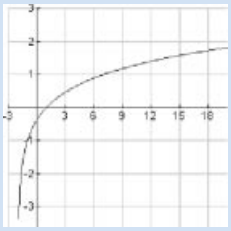
73. domain: $(1, \infty)$
range: $(-\infty, \infty)$



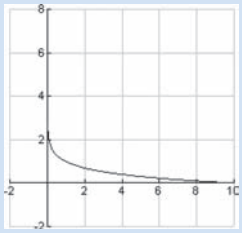
75. domain: $(0, \infty)$
range: $(-\infty, \infty)$



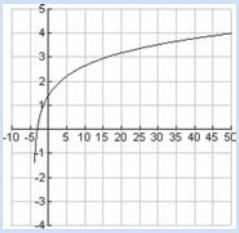
77. domain: $(-2, \infty)$
range: $(-\infty, \infty)$



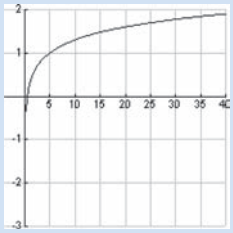
79. domain: $(0, \infty)$
range: $(-\infty, \infty)$



81. domain: $(-4, \infty)$
range: $(-\infty, \infty)$



83. domain: $(0, \infty)$
range: $(-\infty, \infty)$



85. 60 dB **87.** 117 dB **89.** 8.5 **91.** 6.6 **93.** 3.3

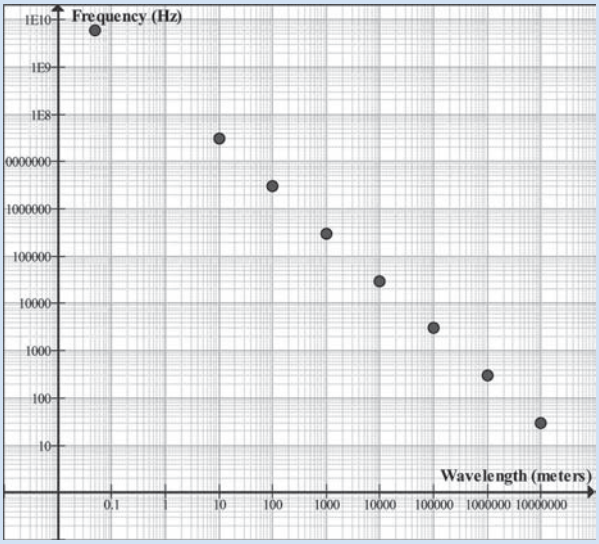
95. Normal rainwater: 5.6
Acid rain/tomato juice: 4

97. 3.6 **99.** 13,236 yr **101.** 25 dB loss

103. a.

Usage	Wavelength	Frequency
Super Low Frequency—Communication with Submarines	10,000,000 m	30 Hz
Ultra Low Frequency—Communication within Mines	1,000,000 m	300 Hz
Very Low Frequency—Avalanche Beacons	100,000 m	3000 Hz
Low Frequency—Navigation, AM Longwave Broadcasting	10,000 m	30,000 Hz
Medium Frequency—AM Bradcasts, Amatuer Radio	1,000 m	300,000 Hz
High Frequency—Shortwave broadcasts, Citizens Band Radio	100 m	3,000,000 Hz
Very High Frequency—FM Radio, Television	10 m	30,000,000 Hz
Ultra High Frequency—Television, Mobile Phones	0.050 m	6,000,000,000 Hz

b.



105. $\log_2 4 = x$ is equivalent to $2^x = 4$ (not $x = 2^4$).

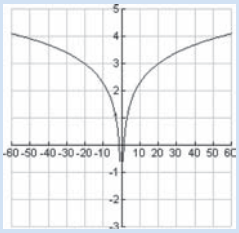
107. The domain is the set of all real numbers such that $x + 5 > 0$, which is written as $(-5, \infty)$.

109. false

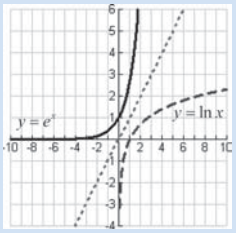
111. true

113. domain: (a, ∞) range: $(-\infty, \infty)$
x-intercept: $(a + e^b, 0)$

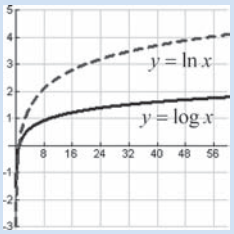
115.



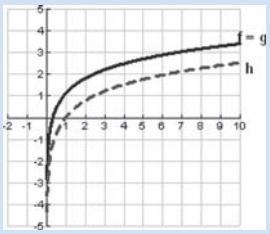
117. $y = x$



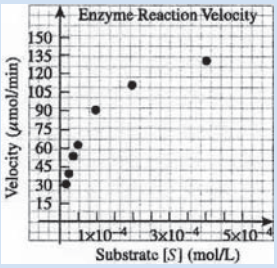
119. x-intercept: (1, 0)
VA: $x = 0$



121. $(0, \infty)$



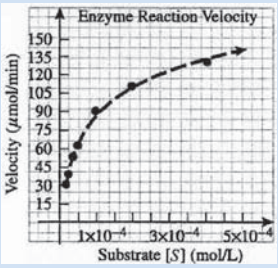
123. a.



b. A reasonable estimate for V_{max} is about 156 μmol/min.

c. K_m is the value of $[S]$ that results in the velocity being half of its maximum value, which by (b) is about 156. So, we need the value of $[S]$ that corresponds to $v = 78$. From the graph, this is very difficult to ascertain because of the very small units. We can simply say that it occurs between 0.0001 and 0.0002. A more accurate estimate can be obtained if a best fit curve is known.

d. (i) $v = 33.70 \ln([S]) + 395.80$ with $r^2 = 0.9984$.
It is shown on the scatterplot below.



(ii) Using the equation, we must solve the following equation for $[S]$:

$$\begin{aligned} 100 &= 33.70\ln([S]) + 395.80 \\ -295.80 &= 33.70\ln[S] \\ -8.77745 &= \ln[S] \\ e^{-8.77745} &= S \\ S &= 0.000154171 \end{aligned}$$

125. $f'(x) = e^x$ **127.** $(f^{-1})'(x) = \dfrac{1}{x}$

Section 3.3

1. 0 **3.** 1 **5.** 8

7. -3 **9.** $\frac{3}{2}$ **11.** 5

13. $x + 5$ **15.** 8 **17.** $\frac{1}{9}$

19. $\dfrac{7}{x^3}$ **21.** $3\log_b(x) + 5\log_b(y)$

23. $\frac{1}{2}\log_b(x) + \frac{1}{3}\log_b(y)$ **25.** $\frac{1}{3}\log_b(r) - \frac{1}{2}\log_b(s)$

27. $\log_b(x) - \log_b(y) - \log_b(z)$ **29.** $2\log x + \frac{1}{2}\log(x + 5)$

31. $3\ln(x) + 2\ln(x - 2) - \frac{1}{2}\ln(x^2 + 5)$

33. $2\log(x - 1) - \log(x - 3) - \log(x + 3)$

35. $\frac{1}{2}\ln(x + 5) - \frac{1}{2}\ln(x - 1)$ **37.** $\log_b(x^3y^5)$

39. $\log_b\left(\dfrac{u^5}{v^2}\right)$ **41.** $\log_b\left(x^{1/2}y^{2/3}\right)$ **43.** $\log\left(\dfrac{u^2}{v^3z^2}\right)$

45. $\ln\left(\dfrac{x^2 - 1}{(x^2 + 3)^2}\right)$ **47.** $\ln\left(\dfrac{(x + 3)^{1/2}}{x(x + 2)^{1/3}}\right)$ **49.** 1.2091

51. -2.3219 **53.** 1.6599 **55.** 2.0115

57. 3.7856 **59.** 110 dB **61.** 5.5

63. 0.0458 **65.** 16 times

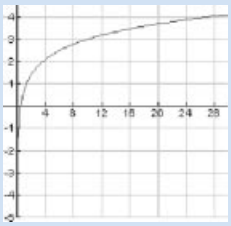
67. $3\log 5 - \log 5^2 = 3\log 5 - 2\log 5 = \log 5$

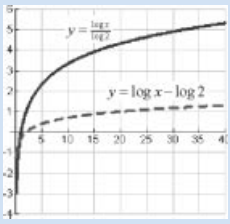
69. Cannot apply the product and quotient properties to logarithms with different bases. Cannot reduce the given expression further without using the change of base formula.

71. true **73.** false **75.** false

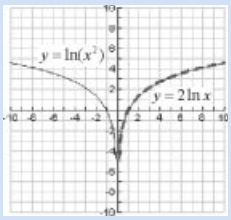
79. $6\log_b x - 9\log_b y + 15\log_b z$

83. yes **85.** no

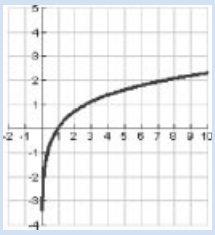




87. no



89. yes



91. $f'(x) = \frac{1}{x} + \frac{1}{x} = \frac{2}{x}$ **93.** $f'(x) = -\frac{2}{x}$

Section 3.4

1. $x = \pm 2$ **3.** $x = -4$ **5.** $x = -\frac{3}{2}$

7. $x = -1$ **9.** $x = 3, 4$ **11.** $x = 0, 6$

13. $x = 1, 4$ **15.** $x = \dfrac{\log_2(27) + 1}{3} \approx 1.918$

17. $x = \ln 5 \approx 1.609$ **19.** $x = 10\ln 4 \approx 13.863$

21. $x = \log_3(10) \approx 2.096$ **23.** $x = \dfrac{\ln(22) - 4}{3} \approx -0.303$

25. $x = \dfrac{\ln 6}{2} \approx 0.896$ **27.** 0.223 **29.** ± 2.282

31. $x = \ln\left(\dfrac{-7 + \sqrt{61}}{2}\right) \approx -0.904$ **33.** $x = 0$

35. $x = \ln 7 \approx 1.946$ **37.** $x = 0$

39. $x = \dfrac{\log_{10}(9)}{2} \approx 0.477$ **41.** $x = 40$

43. $x = \frac{9}{32}$ **45.** $x = \pm 3$ **47.** $x = 5$

49. $x = 6$ **51.** $x = -1$ **53.** no solution

55. $x = \frac{25}{8}$ **57.** $x = -\frac{4}{5}$ **59.** $x = 47.5$

61. $x \approx \pm 7.321$ **63.** $x \approx -1.432$ **65.** $x \cong -1.25$

67. $x \approx 8.456$ **69.** $x = \dfrac{-3 + \sqrt{13}}{2} \cong 0.303$

71. $x \approx 3.646$

73. **a.** 151 beats per min **b.** 7 min **c.** 66 beats per min

75. 31.9 yr **77.** 19.74 yr **79.** 3.16×10^{15} J

81. 1 W/m^2 **83.** 4.61 hr **85.** 15.89 yr

87. 6.2

89. $\ln(4e^x) \neq 4x$. Should first divide both sides by 4, then take the natural log:

$$\begin{aligned} 4e^x &= 9 \\ e^x &= \tfrac{9}{4} \\ \ln(e^x) &= \ln\left(\tfrac{9}{4}\right) \\ x &= \ln\left(\tfrac{9}{4}\right) \end{aligned}$$

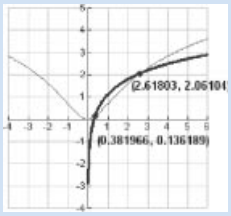
91. $x = -5$ is not a solution since $\log(-5)$ is not defined.

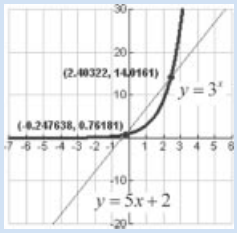
93. true **95.** false **97.** false

99. $x = \dfrac{1 + \sqrt{1 + 4b^2}}{2}$ **101.** $t = -5\ln\left(\frac{3000 - y}{2y}\right)$

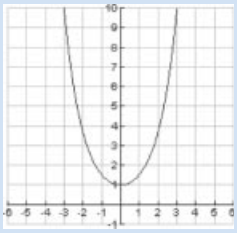
103. $f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

105. $x = \frac{3 \pm \sqrt{5}}{2}$ **107.**





109. domain: $(-\infty, \infty)$ y-axis symmetry



111. $f^{-1}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$ **113.** $\ln y = x \ln 2$

Section 3.5

1. c (iv) **3.** a (iii) **5.** f (i)

7. 94 million **9.** 5.5 yr, 2008

11. 799.6 subscribers **13.** \$455,000 **15.** 332 million

17. 1.45 million **19.** 13.53 ml

21. **a.** $k = -\ln\left(\dfrac{8}{15}\right) \cong 0.6286$

b. 636,000 mp3 players

23. 7575 yr **25.** 131,158,556 years old

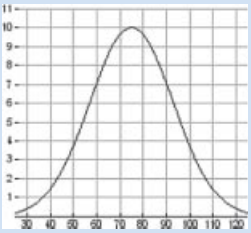
27. 105°F **29.** 3.8 hr before 7 A.M.

31. \$19,100

33. **a.** 84,520 **b.** 100,000 **c.** 100,000

35. 29,551 cases **37.** 1.89 yr **39.** $r = 0$

41. **a.** **b.** 75 **c.** 4 **d.** 4



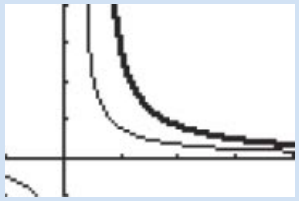
43. **a.** 18 yr **b.** 10 yr **45.** **a.** 30 yr **b.** \$328,120

47. $r = 0.07$, not 7 **49.** true **51.** false

53. less time **55.** about 10.9 days

57. $k_1 = k_2 + \ln\left(\frac{2 + c}{c}\right)$

59. **a.** For the same periodic payment, it will take Wing Shan fewer years to pay off the loan if she can afford to pay biweekly.



b. 11.58 yr **c.** 10.33 yr **d.** 8.54 yr, 7.69 yr, respectively

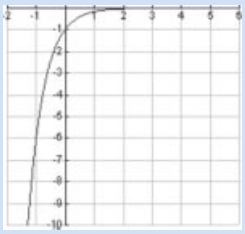
61. $Pe^{kx}\dfrac{(e^{kh} - 1)}{h}$ **63.** $f'(x) = e^x + 1$

Review Exercises

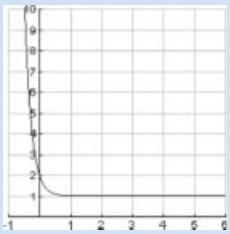
1. 17,559.94 **3.** 5.52 **5.** 24.53 **7.** 5.89

9. 73.52 **11.** 6.25 **13.** b **15.** c

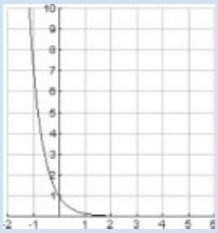
17. y-intercept: $(0, -1)$
HA: $y = 0$



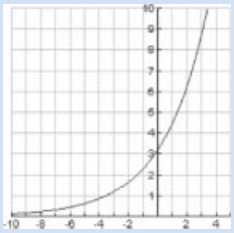
19. y-intercept: $(0, 2)$
HA: $y = 1$



21. y-intercept: $(0, 1)$
HA: $y = 0$



23. y-intercept: $(0, 3.2)$
HA: $y = 0$



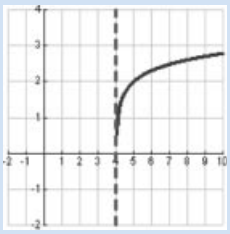
25. \$6144.68 **27.** \$23,080.29 **29.** $4^3 = 64$

31. $10^{-2} = \frac{1}{100}$ **33.** $\log_6 216 = 3$ **35.** $\log_{2/13}\left(\frac{4}{169}\right) = 2$

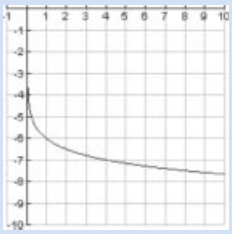
37. 0 **39.** -4 **41.** 1.51 **43.** -2.08

45. $(-2, \infty)$ **47.** $(-\infty, \infty)$ **49.** b **51.** d

53.



55.



57. 6.5 **59.** 50 dB **61.** 1

63. 6 **65.** $a \log_c(x) + b \log_c(y)$

67. $\log_j(r) + \log_j(s) - 3\log_j(t)$

69. $\frac{1}{2}\log(a) - \frac{3}{2}\log(b) - \frac{2}{5}\log(c)$

71. 0.5283 **73.** 0.2939 **75.** $x = -4$

77. $x = \frac{4}{3}$ **79.** $x = -6$ **81.** $x \approx -0.218$

83. no solution **85.** $x = 0$ **87.** $x = \frac{100}{3}$

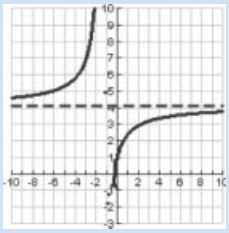
89. $x = 128\sqrt{2}$ **91.** $x \approx \pm 3.004$ **93.** $x \approx 0.449$

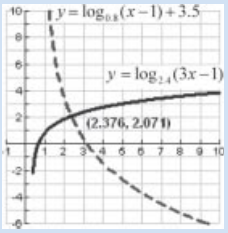
95. \$28,536.88 **97.** 16.6 yr **99.** 3.72 million

101. 6250 bacteria **103.** 56 yr **105.** 16 fish

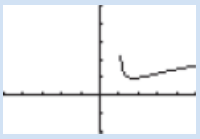
107. 343 mice

109. HA: $y = e^{\sqrt{2}} \approx 4.11$ **111.** (2.376, 2.071)

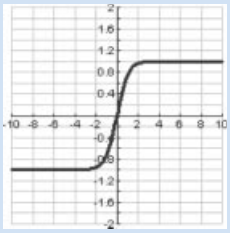




113. $(0, \infty)$



115. domain: $(-\infty, \infty)$ symmetric origin
HA: $y = -1$ (as $x \to -\infty$), $y = 1$ (as $x \to \infty$)



117. **a.** $N = 4e^{-0.038508t} \approx 4(0.9622)^t$ **b.** $N = 4(0.9622)^t$
c. yes

Practice Test

1. x^3 **3.** -4

5. $x = \pm\sqrt{1 + \ln 42} \approx \pm 2.177$

7. $x = \dfrac{-1 + \ln\left(\frac{300}{27}\right)}{0.2} \cong 7.04$

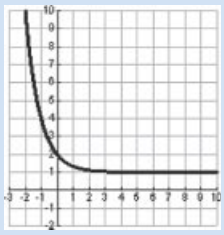
9. $x = 4 + e^2 \approx 11.389$

11. $x = e^e \approx 15.154$ **13.** $x = 9$

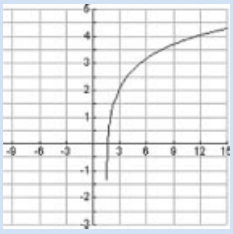
15. $x = \dfrac{-3 \pm \sqrt{9 - 4(-e)}}{2} \approx 0.729$

17. $x = \ln\left(\frac{1}{2}\right) \approx -0.693$ **19.** $(-1, 0) \cup (1, \infty)$

21. x-intercept: none
y-intercept: (0, 2)
HA: $y = 1$



23. x-intercept: $\left(\dfrac{3 + \frac{1}{e}}{2}, 0\right)$
y-intercept: none
VA: $x = \frac{3}{2}$

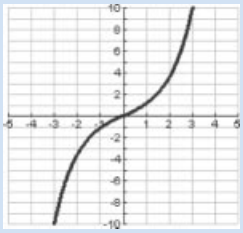


25. \$8051.62 **27.** 90 dB

29. $7.9 \times 10^{11} < E < 2.5 \times 10^{13}$ J

31. 7800 bacteria **33.** 3 days

35. domain $(-\infty, \infty)$
symmetric origin



Cumulative Test

1. domain: $(-\infty, -3) \cup (3, \infty)$; range: $(0, \infty)$

3. $g(x) = e^{2x}$ and $f(x) = \dfrac{1 - x}{1 + x}$ or

$g(x) = e^x$ and $f(x) = \dfrac{1 - x^2}{1 + x^2}$

5. $f(x) = -\frac{4}{9}x^2 - \frac{16}{9}x + \frac{11}{9}$ **7.** 2.197

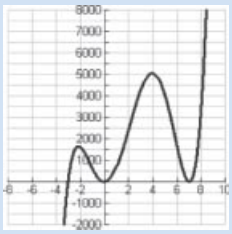
9. **a.** 1 **b.** 5 **c.** 1 **d.** undefined
e. domain: $(-2, \infty)$ range: $(0, \infty)$
f. Increasing: $(4, \infty)$, decreasing: $(0, 4)$, constant: $(-2, 0)$

11. yes **13.** $(1, -1)$

15. $Q(x) = -x^3 + 3x - 5$, $r(x) = 0$

17. HA: none VA: $x = 3$ Slant: $y = x + 3$

19.



21. 5 **23.** $x = 0.5$ **25.** 8.62 yr

27. **a.** $N = 6e^{-0.247553t} \approx 6(0.9755486421)^t$
b. 2.72 g

CHAPTER 4

Section 4.1

1. **a.** 72° **b.** 162° **3.** **a.** 48° **b.** 138° **5.** **a.** 1° **b.** 91°

7. 0.18 **9.** 0.02 **11.** 0.125

13. $\frac{\pi}{6}$ **15.** $\frac{\pi}{4}$ **17.** $\frac{7\pi}{4}$

19. $\frac{5\pi}{12}$ **21.** $\frac{17\pi}{18}$ **23.** $\frac{13\pi}{3}$

25. $-\frac{7\pi}{6}$ **27.** -20π **29.** 30°

31. 135° **33.** 67.5° **35.** 75°

37. 1620° **39.** 171° **41.** −84°

43. 229.18° **45.** 48.70° **47.** −160.37°

49. 198.48° **51.** 0.820 **53.** 1.95

55. 0.986 **57.** QII **59.** negative y-axis

61. negative x-axis **63.** I **65.** IV

67. II **69.** 52° **71.** 268°

73. 330° **75.** $\frac{5\pi}{3}$ **77.** $\frac{11\pi}{9}$

79. 1.42 **81.** $\frac{2\pi}{3}$ ft **83.** $\frac{5}{2}$ in.

85. $\frac{11\pi}{5}$ μm **87.** $\frac{200\pi}{3}$ km **89.** 2.85 km^2

91. 8.62 cm^2 **93.** 0.0236 ft^2 **95.** $\frac{2}{5}$ m/sec

97. 272 km/hr **99.** 9.8 m **101.** 1.5 mi

103. $\frac{5\pi}{2}$ rad/sec **105.** $\frac{2\pi}{9}$ rad/sec **107.** 6π in./sec

109. $\frac{\pi}{4}$ mm/sec **111.** 26.2 cm

113. 653 in. or 54.5 ft **115.** 1,440°

117. 69.82 mph **119.** 10.11 rad/sec = 1.6 rotations/sec

121. 7.8 cm or 78 mm **123.** 4.0 cm or 40 mm

125. $\frac{189\pi}{8} \cong 74$ sq mi

127. $\frac{(4.5 \times 10^{-9})}{2}\frac{\pi}{5} \cong 1.4 \times 10^{-9}$ m or 1.4 nm

129. Angular velocity must be expressed in radians (not degrees) per second. Use $\pi \frac{\text{rad}}{\text{sec}}$ in place of $\frac{180°}{\text{sec}}$.

131. true **133.** true **135.** 110°

137. $\frac{15\pi}{2}$ units^2 **139.** 68.8° **141.** $\theta = \frac{\pi}{5}$

143. $\frac{\pi}{3}$

Section 4.2

1. $\frac{\sqrt{5}}{5}$ **3.** $\sqrt{5}$ **5.** 2

7. $\frac{2\sqrt{10}}{7}$ **9.** $\frac{7}{3}$ **11.** $\frac{3\sqrt{10}}{20}$

13. a **15.** b **17.** c

19. $\frac{\sqrt{3}}{3}$ **21.** $\sqrt{3}$ **23.** $\frac{2\sqrt{3}}{3}$

25. $\frac{2\sqrt{3}}{3}$ **27.** $\frac{\sqrt{3}}{3}$ **29.** $\sqrt{2}$

31. 0.6018 **33.** 0.1392 **35.** 0.2588

37. −0.8090 **39.** 1.3764 **41.** 0.4142

43. 1.0034 **45.** 0.7002 **47.** 18 ft

49. 5.50 mi **51.** 12 km **53.** 62°

55. $\beta = 58°$, $a \approx 6.4$ ft, $b \approx 10$ ft

57. $\alpha = 18°$, $b \approx 9.2$ mm, $a \approx 3.0$ mm

59. $\beta = 35.8°$, $c \approx 137$ mi, $b \approx 80.1$ mi

61. $\alpha \approx 56.0°$; $\beta \approx 34.0°$; $c \approx 51.3$ ft

63. $\alpha \approx 55.480°$; $\beta \approx 34.520°$; $b \approx 24{,}235$ km

65. $c \approx 27.0$ in.; $a \approx 24.4$ in.; $\alpha \approx 64.6°$

67. 88 ft **69.** 260 ft **71.** 11.0°; too low

73. 80 ft **75.** 170 m **77.** 0.000016°

79. 4,414 ft **81.** 136.69° **83.** 120.93 ft

85. 24 ft **87.** 3.5 ft **89.** 4.7 in.

91. Opposite side has length 3, not 4.

93. $\sec x = \frac{1}{\cos x}$, not $\frac{1}{\sin x}$. **95.** true

97. false **99.** 0 **101.** 0

103. $\frac{1}{2}$ **105.** **a.** 0.342; 2.92398 **b.** 2.92380

107. **a.** 1.423; 0.70281 **b.** 0.70281

109. $\frac{6-2\sqrt{3}}{3}$ **111.** $\sqrt{3}-1$

Section 4.3

1. $\sin\theta = \frac{2\sqrt{5}}{5}$, $\cos\theta = \frac{\sqrt{5}}{5}$, $\tan\theta = 2$,
$\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = \sqrt{5}$, $\cot\theta = \frac{1}{2}$

3. $\sin\theta = \frac{4\sqrt{41}}{41}$, $\cos\theta = \frac{5\sqrt{41}}{41}$, $\tan\theta = \frac{4}{5}$,
$\csc\theta = \frac{\sqrt{41}}{4}$, $\sec\theta = \frac{\sqrt{41}}{5}$, $\cot\theta = \frac{5}{4}$

5. $\sin\theta = \frac{2\sqrt{5}}{5}$, $\cos\theta = -\frac{\sqrt{5}}{5}$, $\tan\theta = -2$,
$\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$, $\cot\theta = -\frac{1}{2}$

7. $\sin\theta = -\frac{7\sqrt{65}}{65}$, $\cos\theta = -\frac{4\sqrt{65}}{65}$, $\tan\theta = \frac{7}{4}$,
$\csc\theta = -\frac{\sqrt{65}}{7}$, $\sec\theta = -\frac{\sqrt{65}}{4}$, $\cot\theta = \frac{4}{7}$

9. $\sin\theta = \frac{\sqrt{15}}{5}$, $\cos\theta = -\frac{\sqrt{10}}{5}$, $\tan\theta = -\frac{\sqrt{6}}{2}$,
$\csc\theta = \frac{\sqrt{15}}{3}$, $\sec\theta = -\frac{\sqrt{10}}{2}$, $\cot\theta = -\frac{\sqrt{6}}{3}$

11. $\sin\theta = -\frac{\sqrt{6}}{4}$, $\cos\theta = -\frac{\sqrt{10}}{4}$, $\tan\theta = \frac{\sqrt{15}}{5}$,
$\csc\theta = -\frac{2\sqrt{6}}{3}$, $\sec\theta = -\frac{2\sqrt{10}}{5}$, $\cot\theta = \frac{\sqrt{15}}{3}$

13. $\sin\theta = -\frac{2\sqrt{29}}{29}$, $\cos\theta = -\frac{5\sqrt{29}}{29}$, $\tan\theta = \frac{2}{5}$,
$\csc\theta = -\frac{\sqrt{29}}{2}$, $\sec\theta = -\frac{\sqrt{29}}{5}$, $\cot\theta = \frac{5}{2}$

15. QIV **17.** QII **19.** QI

21. QI **23.** QIII **25.** $-\frac{4}{5}$

27. $-\frac{60}{11}$ **29.** $-\frac{84}{85}$ **31.** $-\sqrt{3}$

33. $-\frac{\sqrt{3}}{3}$ **35.** $\frac{2\sqrt{3}}{3}$ **37.** 1

39. -1 **41.** 0 **43.** 1

45. 1 **47.** possible **49.** not possible

51. possible **53.** possible **55.** possible

57. $-\frac{1}{2}$ **59.** $-\frac{\sqrt{3}}{2}$ **61.** $\frac{\sqrt{3}}{3}$

63. 1 **65.** -2 **67.** 1

69. $\theta = 30°$ or $330°$ **71.** $\theta = 210°$ or $330°$

73. $\theta = 90°$ or $270°$ **75.** $\theta = 270°$ **77.** 110°

79. 143° **81.** 322° **83.** 140°

85. 340° **87.** 1° **89.** 335°

91. 1.3 **93.** 12°

95. 15°; The lower leg is bent at the knee in a backward direction at an angle of 15°.

97. 75.5°

99. Reference angle is measured between the terminal side and the x-axis, not the y-axis. The reference angle is 60°, $\sec 120° = -2$

101. true **103.** false **105.** false

107. true **109.** $-\frac{3}{5}$

111. $y = (\tan\theta)(a-x)$ **113.** $-\frac{a}{\sqrt{a^2+b^2}}$

115. $-\frac{\sqrt{a^2-b^2}}{b}$ **117.** 0 **119.** 0

121. 0 **123.** does not exist

125. $\frac{12+3\sqrt{2}+\sqrt{3}}{6}$ **127.** $\frac{8}{3}$

Section 4.4

1. SSA **3.** SSS **5.** ASA

7. $\gamma = 75°$, $b \approx 12.2$ m, $c \approx 13.66$ m

9. $\beta = 62°$, $a \approx 163$ cm, $c \approx 215$ cm

11. $\beta = 116.1°$, $a \approx 80.2$ yd, $b \approx 256.6$ yd

13. $\gamma = 120°$, $a \approx 7$ m, $b \approx 7$ m

15. $\alpha = 97°$, $a \approx 118$ yd, $b \approx 52$ yd

17. $\beta_1 \approx 20°$, $\gamma_1 \approx 144°$, $c_1 \approx 9$; $\beta_2 \approx 160°$, $\gamma_2 \approx 4°$, $c_2 \approx 1$

19. $\alpha \approx 40°$, $\beta \approx 100°$, $b \approx 18$

21. no triangle

23. $\beta = 90°$, $\gamma \approx 60°$, $c \approx 16$

25. $\beta \approx 23°$, $\gamma \approx 123°$, $c \approx 15$

27. $\beta_1 \approx 21.9°$, $\gamma_1 \approx 136.8°$, $c_1 \approx 11.36$;
$\beta_2 \approx 158.1°$, $\gamma_2 \approx 0.6°$, $c_2 \approx 0.17$

29. $\beta \approx 62°$, $\gamma \approx 2°$, $c \approx 0.275$

31. $\beta_1 \approx 77°$, $\alpha_1 \approx 63°$, $a_1 \approx 457$;
$\beta_2 \approx 103°$, $\alpha_2 \approx 37°$, $a_2 \approx 309$

33. $\alpha \approx 31°$, $\gamma \approx 43°$, $c \approx 2$

35. 1,246 ft **37.** 1.7 mi **39.** 1.3 mi

41. 26 ft **43.** 270 ft **45.** 63.83°

47. each 76.5 ft **49.** 60.14 ft **51.** 1.2 cm

53. The value of β is incorrect. Should be $\sin\beta = \frac{9\sin 120°}{7}$.

55. false **57.** true **59.** true

71. 27 in. **73.** 22 m

Section 4.5

1. $b \approx 5, \gamma \approx 33°, \alpha \approx 47°$ **3.** $a \approx 5, \gamma \approx 6°, \beta \approx 158°$

5. $a \approx 2, \beta \approx 80°, \gamma \approx 80°$ **7.** $b \approx 5, \alpha \approx 43°, \gamma \approx 114°$

9. $b \approx 7, \alpha \approx 30°, \gamma \approx 90°$ **11.** $\alpha \approx 93°, \beta \approx 39°, \gamma \approx 48°$

13. $\gamma \approx 77°, \beta \approx 51.32°, \alpha \approx 51°$

15. $\alpha \approx 75°, \beta \approx 57°, \gamma \approx 48°$

17. no triangle

19. $\gamma \approx 90°, \beta \approx 23°, \alpha \approx 67°$

21. $\gamma = 105°, b \approx 5, c \approx 9$

23. $\beta \approx 12°, \gamma \approx 137°, c \approx 16$

25. $\beta \approx 77°, \alpha \approx 66°, \gamma \approx 37°$

27. $\gamma \approx 2°, \alpha \approx 168°, a \approx 13$

29. 55.4 **31.** 0.5 **33.** 23.6

35. 6.4 **37.** 4,408.4 **39.** 97.4

41. 25.0 **43.** 26.7 **45.** 111.64

47. 111,632,076 **49.** no triangle **51.** 2710 mi

53. 1280 mi **55.** 16 ft **57.** 26.0 cm

59. 21.67° **61.** 83.07° **63.** 47,128 sq ft

65. 23.38 sq ft **67.** 8.73

69. Should have used the smaller angle β in Step 2

71. false **73.** true **75.** true

79. $\cos\left(\frac{X}{2}\right) = \sqrt{\dfrac{1 - \cos\left(2\cos^{-1}\left(\frac{1}{4}\right)\right)}{2}}$ **83.** 0.69 sq units

91. 333 mi **93.** 28 m

Review Exercises

1. a. 62° **b.** 152° **3. a.** 55° **b.** 145°

5. a. 0.99° **b.** 90.99° **7.** $\frac{3\pi}{4}$

9. $\frac{11\pi}{6}$ **11.** $\frac{6\pi}{5}$

13. 9π **15.** 60° **17.** 225°

19. 100° **21.** 1800° **23.** 150°

25. 754 in./min **27.** $\frac{2\sqrt{13}}{13}$ **29.** $\frac{\sqrt{13}}{2}$

31. $\frac{3}{2}$ **33.** b **35.** b

37. c **39.** 0.6691 **41.** 0.9548

43. 1.5399 **45.** 1.5477 **47.** 75 ft

49. $\sin\theta = -\frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = -\frac{4}{3},$
$\cot\theta = -\frac{3}{4}, \sec\theta = \frac{5}{3}, \csc\theta = -\frac{5}{4}$

51. $\sin\theta = \frac{\sqrt{10}}{10}, \cos\theta = -\frac{3\sqrt{10}}{10}, \tan\theta = -\frac{1}{3},$
$\cot\theta = -3, \sec\theta = -\frac{\sqrt{10}}{3}, \csc\theta = \sqrt{10}$

53. $\sin\theta = \frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = \frac{\sqrt{3}}{3},$
$\cot\theta = \sqrt{3}, \sec\theta = \frac{2\sqrt{3}}{3}, \csc\theta = 2$

55. $\sin\theta = -\frac{\sqrt{5}}{5}, \cos\theta = \frac{2\sqrt{5}}{5}, \tan\theta = -\frac{1}{2},$
$\cot\theta = -2, \sec\theta = \frac{\sqrt{5}}{2}, \csc\theta = -\sqrt{5}$

57. $\sin\theta = -\frac{\sqrt{7.2}}{3}, \cos\theta = -\frac{\sqrt{7.2}}{6}, \tan\theta = 2,$
$\cot\theta = \frac{1}{2}, \sec\theta = -\frac{\sqrt{7.2}}{1.2}, \csc\theta = -\frac{\sqrt{7.2}}{2.4}$

59. $-\frac{1}{2}$ **61.** $-\frac{\sqrt{3}}{3}$ **63.** $-\frac{2\sqrt{3}}{3}$

65. $-\frac{\sqrt{2}}{2}$ **67.** $\sqrt{3}$ **69.** $-\sqrt{2}$

71. $-\frac{2\sqrt{3}}{3}$ **73.** $\gamma = 150°, b \approx 8, c \approx 12$

75. $\gamma = 130°, a \approx 1, b \approx 9$

77. $\beta = 158°, a \approx 11, b \approx 22$

79. $\beta = 90°, a \approx \sqrt{2}, c \approx \sqrt{2}$

81. $\beta = 146°, b \approx 266, c \approx 178$

83. $\beta_1 \approx 26°, \gamma_1 \approx 134°, c_1 \approx 15;$
$\beta_2 \approx 154°, \gamma_2 \approx 6°, c_2 \approx 2$

85. $\gamma_1 \approx 29°, \alpha_1 \approx 127°, b_1 \approx 20;$
$\gamma_2 \approx 151°, \beta_2 \approx 5°, b_2 \approx 2$

87. no triangle

89. $\beta_1 \approx 15°, \gamma_1 \approx 155°, c_1 \approx 10;$
$\beta_2 \approx 165°, \gamma_2 \approx 5°, c_2 \approx 2$

91. $c \approx 46, \alpha \approx 42°, \beta \approx 88°$

93. $\gamma \approx 75°, \beta \approx 54°, \alpha \approx 51°$

95. $\gamma = 90°, \beta \approx 48°, \alpha \approx 42°$

97. $a \approx 4, \beta \approx 28°, \gamma \approx 138°$

99. $a \approx 11, \beta \approx 68°, \gamma \approx 22°$

101. $\gamma \approx 70°, \beta \approx 59°, \alpha \approx 51°$

103. $a \approx 26, \beta \approx 37°, \gamma \approx 43°$

105. $a \approx 28, \beta \approx 4°, \gamma \approx 166°$

107. no triangle **109.** $\beta \approx 10°, \gamma \approx 155°, c \approx 10.3$

111. 141.8 **113.** 51.5 **115.** 89.8

117. 41.7 **119.** 5.2 in.

Practice Test

1. 6000 ft

3. exact value versus approximate value

5. QIV **7.** 585° **9.** $\frac{15\pi}{4}$ in^2

11. $\gamma = 110°$, $a \approx 7.8$, $c \approx 14.6$

13. $\gamma \approx 96.4°$, $\beta \approx 48.2°$, $\alpha \approx 35.4°$

15. no triangle

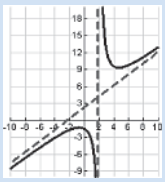
17. $\gamma = 50°$, $b \approx 1.82$, $c \approx 4.08$ **19.** 57

Cumulative Test

1. $-\frac{5}{8}$ **3.** $-2x - h$ **5.** 1

7. $y = -2x^2 + 7$

9. VA: $x = 2$, HA: none,
SA: $y = x + 2$



11. \$37,250 **13.** $x \approx 0.440$ **15.** $15\sqrt{2}$ ft

17. $\frac{12\pi}{5}$ **19.** $\sqrt{3}$ **21.** 1.6616

23. $\alpha = 138°$, $c \approx 8$ cm, $c \approx 9$ cm

CHAPTER 5

Section 5.1

1. $-\frac{\sqrt{3}}{2}$ **3.** $-\frac{\sqrt{3}}{2}$ **5.** $\frac{\sqrt{2}}{2}$ **7.** -1

9. $-\sqrt{2}$ **11.** $\sqrt{3}$ **13.** 2 **15.** $-\frac{\sqrt{3}}{2}$

17. $-\frac{\sqrt{3}}{2}$ **19.** $-\frac{\sqrt{2}}{2}$ **21.** $-\frac{\sqrt{3}}{2}$ **23.** $\frac{\sqrt{2}}{2}$

25. 1 **27.** $\frac{\sqrt{2}}{2}$ **29.** 0 **31.** -2

33. $\frac{\sqrt{3}}{3}$ **35.** $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$ **37.** $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$

39. $\theta = 0, \pi, 2\pi, 3\pi, 4\pi$ **41.** $\theta = \pi, 3\pi$

43. $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ **45.** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ **47.** $\theta = 0, \pi, 2\pi$

49. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ **51.** $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ **53.** $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$

55. 22.9°F **57.** 99.1°F **59.** 2.6 ft

61. 135 lb **63.** 10,000 guests

65. 10.7 μg/μL **67.** 35°C

69. Should have used $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

71. true **73.** false **75.** true

77. odd **79.** $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ **81.** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

83. $\theta = \frac{\pi}{3} + n\pi, \frac{2\pi}{3} + n\pi$ **85.** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

87. 0.891; -0.891 **89.** 6.314; -6.314

91. 0.5 **93.** 0.866 **95.** 2

97. $\frac{3 - \sqrt{3}}{3}$

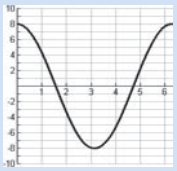
Section 5.2

1. c **3.** a **5.** h **7.** b

9. e **11.** $\frac{3}{2}$; $p = \frac{2\pi}{3}$ **13.** 1; $p = \frac{2\pi}{5}$ **15.** $\frac{2}{3}$; $p = \frac{4\pi}{3}$

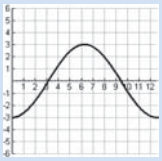
17. 3; $p = 2$ **19.** 5; $p = 6$

21.



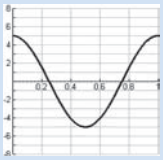
23.

25.

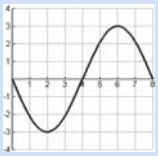


27.

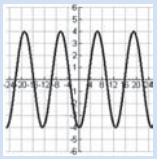
29.



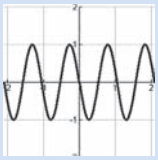
31.



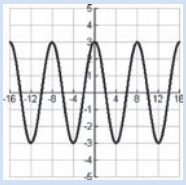
33.



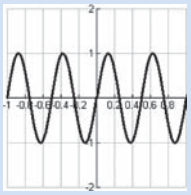
35.



37.



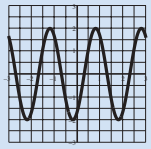
39.



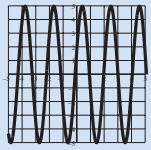
41. $y = -\sin(2x)$ **43.** $y = \cos(\pi x)$

45. $y = -2\sin\left(\frac{\pi}{2}x\right)$ **47.** $y = \sin(8\pi x)$

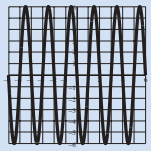
49. 2; 2; $\frac{1}{\pi}$



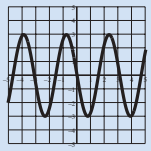
51. 5; $\frac{\pi}{2}$; $-\frac{2}{3}$



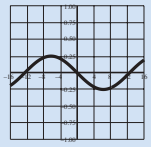
53. 6; 2; -2



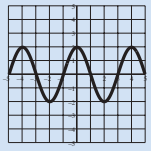
55. 3; π; $\frac{\pi}{2}$ left



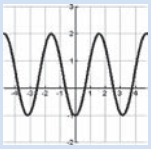
57. $\frac{1}{4}$; 8π; 2π right



59. 2; 4; -4 right

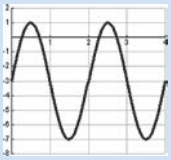


61.

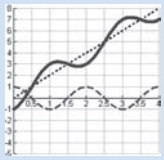


63.

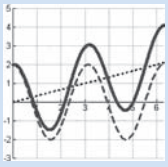
65.



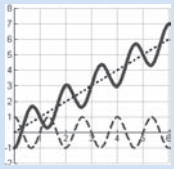
67.



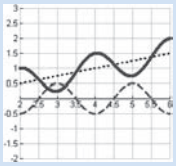
69.



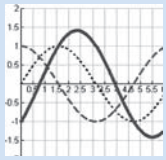
71.



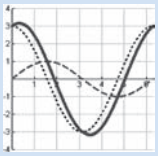
73.



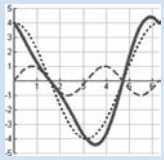
75.



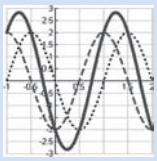
77.



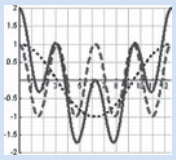
79.



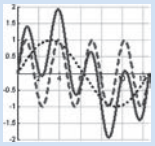
81.



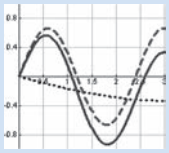
83.



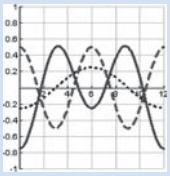
85.



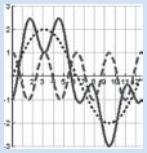
87.



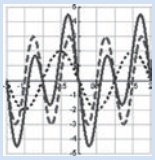
89.



91.



93.



95. 3.5 or 3500 widgets

97. 1 mg/L

99. 4 cm; 4 g

101. $\frac{1}{4\pi}$ cycles/sec

103. 0.005 cm; 256 Hz

105. 0.008 cm; 375 Hz

107. 660 m/sec

109. 660 m/sec

111. $y = 25 - 25\cos\left(\frac{\pi t}{2}\right)$; 0 candelas/m^2

113. The correct graph is reflected over the x-axis.

115. true

117. false

119. $(0, A)$

121. $x = \frac{n\pi}{B}$

123. $\left(0, -\frac{A}{2}\right)$

125. $\left(\frac{3\pi}{2B} + \frac{2n\pi}{B}, 0\right)$

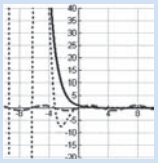
127. $\left[-\frac{5A}{2}, \frac{3A}{2}\right]$

129. no

131. graphs same

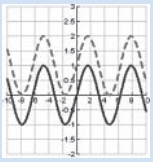
133. **a.** left $\frac{\pi}{3}$ units **b.** right $\frac{\pi}{3}$ units

135. $y_1 = e^{-t}$, $y_2 = \sin t$, $y_3 = e^{-t}\sin t$



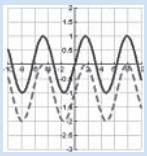
As t increases, the graph of y_1 approaches the t-axis from above, the graph of y_2 oscillates, keeping its same form, the graph of y_3 oscillates, but dampens so that it approaches the t-axis from below and above.

137. **a.** $y_1 = \sin x$(solid), $y_2 = \sin x + 1$ (dashed)



The graph of y_2 is that of y_1 shifted up one unit.

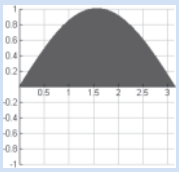
b. $y_1 = \sin x$ (solid)
$y_2 = \sin x - 1$ (dashed)



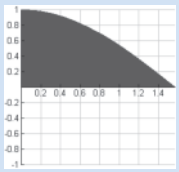
The graph of y_2 is that of y_1 shifted down 1 unit.

139. 5

141. 2



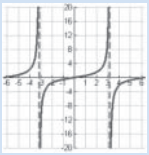
143. 1



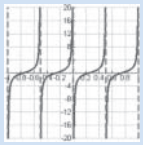
Section 5.3

1. b **3.** h **5.** c **7.** d

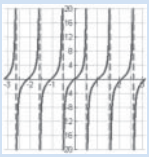
9.



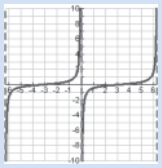
11.



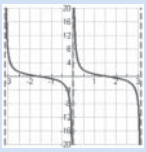
13.



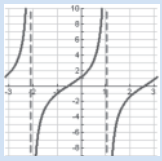
15.



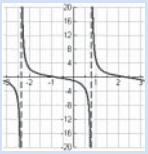
17.



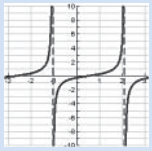
19.



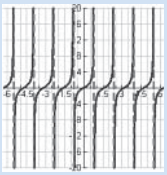
21.



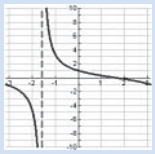
23.



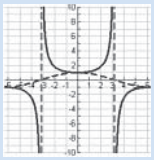
25.



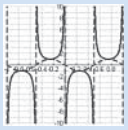
27.



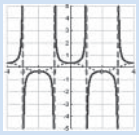
29.



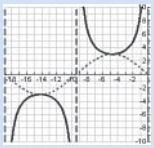
31.



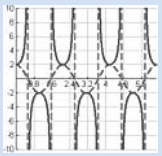
33.



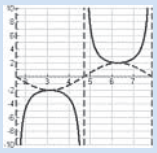
35.



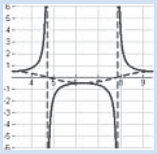
37.



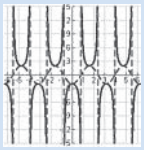
39.



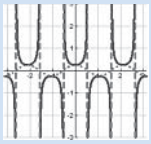
41.



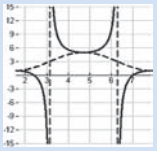
43.



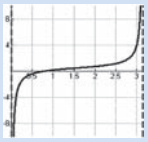
45.



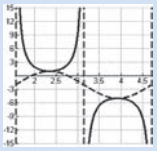
47.



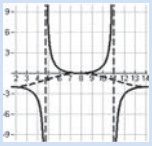
49.



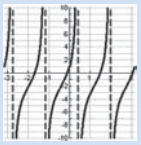
51.



53.



55.



57. domain: $x \neq n$, n an integer range: $\mathbb{R}$

59. domain: $x \neq \frac{2n+1}{10}\pi$, n an integer
range: $(-\infty, -2] \cup [2, \infty)$

61. domain: $x \neq 2n\pi$, n an integer range: $(-\infty, 1] \cup [3, \infty)$

63. domain: $x \neq 4n + 6$, n an integer range: $\mathbb{R}$

65. domain: $x \neq n$, n an integer range: $\left(-\infty, -\frac{5}{2}\right] \cup \left]\left[-\frac{3}{2}, \infty\right)\right.$

67. 48 m

69. **a.** -5.2 mi **b.** -3 mi **c.** 0 **d.** 3 mi **e.** 5.2 mi

71. Forgot that the amplitude is 3, not 1. Guide function should have been $y = 3\sin(2x)$.

73. true

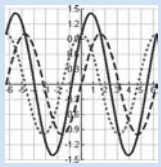
75. n, n an integer

77. $x = 0, \pm\frac{\pi}{2}, \pm\pi$

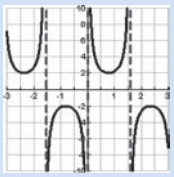
79. $x = \frac{n\pi - C}{B}$, n an integer

81. infinitely many solutions

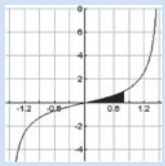
83. $\sqrt{2}$



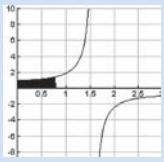
85. π



87. $-\ln\frac{\sqrt{2}}{2}$



89. $\ln\left(\sqrt{2} + 1\right)$



Review Exercises

1. $-\frac{\sqrt{3}}{3}$ **3.** $-\frac{1}{2}$ **5.** 1

7. -1 **9.** -1 **11.** $\frac{1}{2}$

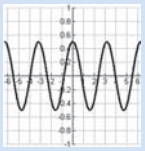
13. $\frac{\sqrt{2}}{2}$ **15.** 1 **17.** $-\frac{\sqrt{3}}{2}$

19. $\frac{-\sqrt{3}}{3}$ **21.** 2π **23.** $y = 4\cos x$

25. 5 **27.** 2; $p = 1$ **29.** $\frac{1}{5}$; $p = \frac{2\pi}{3}$

31.

33.

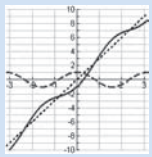


35. 3; 2π; $\frac{\pi}{2}$; 2 (up)

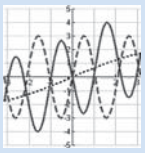
37. 4; $\frac{2\pi}{3}$; $-\frac{\pi}{4}$; -2 (down)

39. $\frac{1}{3}$; 2; $\frac{1}{2\pi}$; $\frac{1}{2}$ (down)

41.



43.

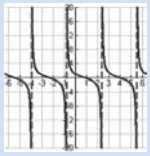


45. domain: $x \neq n\pi$, n an integer range: $\mathbb{R}$

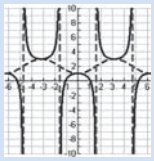
47. domain: $x \neq \frac{2n+1}{4}\pi$, n an integer
range: $(-\infty, -3] \cup [3, \infty)$

49. domain: $x \neq \frac{6n+7}{6}$, n an integer
range: $\left(-\infty, -\frac{3}{4}\right] \cup \left[-\frac{1}{4}, \infty\right)$

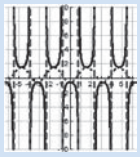
51.



53.

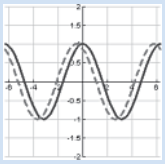


55.

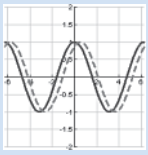


57. -0.9659

59. **a.** $y_1 = \cos x$ (solid)
$y_2 = \cos\left(x + \frac{\pi}{6}\right)$ (dashed)
The graph of y_2 is that of y_1 shifted left $\frac{\pi}{6}$ units.



b. $y_1 = \cos x$ (solid)
$y_2 = \cos\left(x - \frac{\pi}{6}\right)$ (dashed)
The graph of y_2 is that of y_1 shifted right $\frac{\pi}{6}$ units.

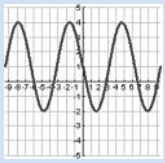


61. approximately 5

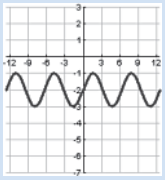
Practice Test

1. 5, $p = \frac{2\pi}{3}$

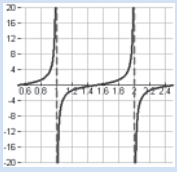
3.



5.



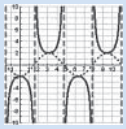
7.



9. $x = \frac{n\pi}{2}$, n an integer

11. $(-\infty, -4] \cup [2, \infty)$

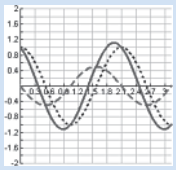
13.



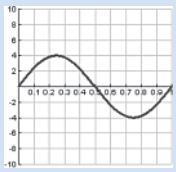
15. true

17. $y = 4\sin\left[\pi\left(x + \frac{3}{2}\right)\right] - \frac{1}{2}$

19.

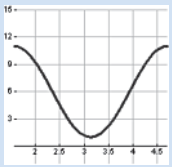


21. **a.**



b. 4 **c.** 1

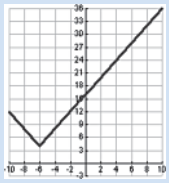
23. **a.** $y = 6 + 5\cos\left[2\left(x - \frac{\pi}{2}\right)\right]$
b. amplitude: 5, period: π, phase shift: $\frac{\pi}{2}$
c.



Cumulative Test

1. $(-5, \infty)$

3. $y = 2|x + 6| + 4$



5. $f^{-1}(x) = \frac{5x + 2}{1 - 3x}$
domain f = range f^{-1}: $\left(-\infty, -\frac{5}{3}\right) \cup \left(-\frac{5}{3}, \infty\right)$
domain f^{-1} = range f: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

7. $Q(x) = 3x^2 - \frac{5}{2}x + \frac{9}{2}$, $r(x) = \frac{9}{2}x + \frac{1}{2}$

9. $P(x) = \left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right)(x - i)(x + i)$

11. \$3382

13. $3\ln a - 2\ln b - 5\ln c$

15. $x = 1$

17. $\alpha = 63°$, $b \approx 6.36$ in., $a \approx 12.47$ in.

19. $\beta \approx 71.17°$, $\gamma = 40.83°$, $c \approx 16.92$ m or
$\beta = 108.83°$, $\gamma = 3.17°$, $c \approx 1.43$ m

21. $\cos\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = -\frac{\sqrt{3}}{3}$, $\cot\theta = -\sqrt{3}$,
$\sec\theta = -\frac{2\sqrt{3}}{3}$, $\csc\theta = 2$

23. $f = \frac{2}{\pi}$

CHAPTER 6

Section 6.1

1. 30° **3.** $90° - x$ **5.** 60°

7. $\cos(90° - x - y)$ **9.** $\sin(70° - A)$ **11.** $\tan(45° + x)$

13. $\sec(30° + \theta)$ **15.** 1 **17.** $\csc x$

19. -1 **21.** $\sec^2 x$ **23.** 1

25. $\sin^2 x - \cos^2 x$ **27.** $\sec x$ **29.** 1

31. $\sin^2 x$ **33.** $\csc^2 x$ **35.** $-\cos x$

37. 1 **65.** conditional **67.** identity

69. conditional **71.** conditional **73.** conditional

75. identity **77.** conditional **81.** $|\sec\theta|$

83. Simplified the two fractions in Step 2 incorrectly

85. $\tan^2 x = 1$ is a conditional, which does not hold for all values of x.

87. false **89.** QI or QIV **91.** QI or QII

93. no **95.** no **97.** $a^2 + b^2$

101. $\sec\theta$ **103.** $\cos A\cos B - \sin A\sin B$

105. $\sin A\cos B + \cos A\sin B$

107. $|a|\cos\theta$ **109.** $|a|\tan\theta$

Section 6.2

1. $\frac{\sqrt{6} - \sqrt{2}}{4}$ **3.** $\frac{\sqrt{6} - \sqrt{2}}{4}$ **5.** $-2 + \sqrt{3}$

7. $\frac{\sqrt{2} + \sqrt{6}}{4}$ **9.** $2 + \sqrt{3}$ **11.** $2 + \sqrt{3}$

13. $\sqrt{2} - \sqrt{6}$ **15.** $\sqrt{6} - \sqrt{2}$ **17.** $\cos x$

19. $-\sin x$ **21.** 0 **23.** $-2\cos(A - B)$

25. $-2\sin(A + B)$ **27.** $\tan(26°)$ **29.** $\frac{1 + 2\sqrt{30}}{12}$

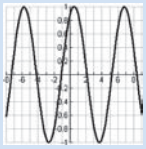
31. $\frac{-6\sqrt{6}+4}{25}$ **33.** $\frac{192-25\sqrt{15}}{-119}$ **35.** identity

37. conditional **39.** identity **41.** identity

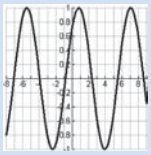
43. identity **45.** conditional **47.** identity

49. identity **51.** conditional

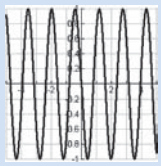
53. $y = \sin\left(x + \frac{\pi}{3}\right)$



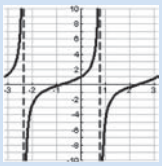
55. $y = \cos\left(x - \frac{\pi}{4}\right)$



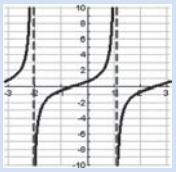
57. $y = -\sin 4x$



59. $y = \tan\left(x - \frac{\pi}{4}\right)$



61. $y = \tan\left(x + \frac{\pi}{6}\right)$



63. $\frac{\sqrt{2}}{2}\left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - - + + \cdots\right)$

67. $E = A\cos(kz)\cos(ct)$ **69.** $T(t) = 38 - 2.5\sin\left(\frac{\pi}{6}t\right)$

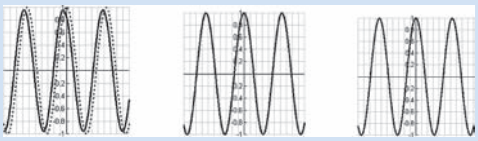
71. $\tan(A + B) \neq \tan A + \tan B$. Should have used

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

73. false **75.** false

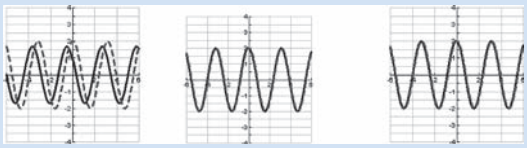
79. $B = 2m\pi, \quad A = 2n\pi$; n, m integers

81. **a.** **b.** **c.**



$y = \cos x$ as $h \to 0$

83. **a.** **b.** **c.**



$y = 2\cos(2x)$ as $h \to 0$

85. $\tan x = -\tan y$ **87.** $\tan x = \dfrac{2 - \tan y}{1 + 2\tan y}$

Section 6.3

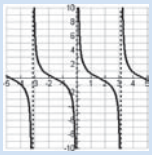
1. $-\frac{4}{5}$ **3.** $\frac{120}{119}$ **5.** $\frac{120}{169}$

7. $-\frac{4}{3}$ **9.** $\frac{\sqrt{19}}{10}$ **11.** $\frac{119}{120}$

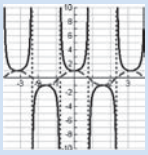
13. $\frac{\sqrt{3}}{3}$ **15.** $\frac{\sqrt{2}}{4}$ **17.** $\cos(4x)$

19. $-\frac{\sqrt{3}}{3}$ **21.** $-\frac{\sqrt{3}}{2}$ **23.** $-\frac{\sqrt{3}}{2}$

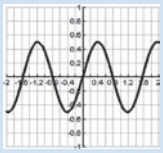
41.



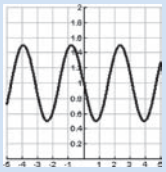
43.



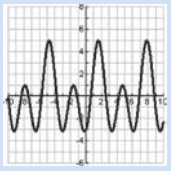
45.



47.



49.



51. $\frac{\sqrt{2} - \sqrt{3}}{2}$ **53.** $-\frac{\sqrt{2} + \sqrt{3}}{2}$ **55.** $\frac{\sqrt{2} - \sqrt{3}}{2}$

57. $\sqrt{3 + 2\sqrt{2}}$ **59.** $-\frac{2}{\sqrt{2 + \sqrt{2}}}$

61. $-\frac{1}{\sqrt{3 + 2\sqrt{2}}}$ or $1 - \sqrt{2}$

63. $-\frac{2}{\sqrt{2 - \sqrt{2}}}$ **65.** 1 **67.** $\frac{2\sqrt{13}}{13}$ **69.** $\frac{3\sqrt{13}}{13}$

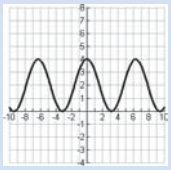
71. $\frac{\sqrt{5} - 1}{2}$ or $\sqrt{\frac{3 - \sqrt{5}}{2}}$ **73.** $\sqrt{\frac{3 + 2\sqrt{2}}{6}}$

75. $-\frac{\sqrt{15}}{5}$ **77.** $\sqrt{\dfrac{1 - \frac{24}{\sqrt{601}}}{2}}$

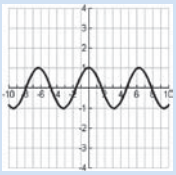
79. $-\sqrt{\frac{1 - \sqrt{0.91}}{1 - \sqrt{0.91}}}$ **81.** $\sqrt{\frac{7}{3}}$ **83.** $\cos\left(\frac{5\pi}{12}\right)$

85. $\tan(75°)$ **87.** $-\tan\left(\frac{5\pi}{8}\right)$

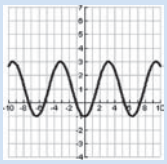
101.



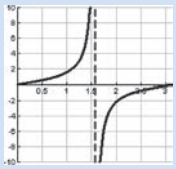
103.



105.



107.



109. $C(t) = 2 + 10\cos(2t)$

111. 22,565,385 lb

115. $\sqrt{2}$ ft

117. $\frac{1}{3}$

119. Should use $\sin x = -\frac{2\sqrt{2}}{3}$ since we are assuming that $\sin x < 0$.

121. If $\pi < x < \frac{3\pi}{2}$, then $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$, so that $\sin\left(\frac{x}{2}\right)$ is positive, not negative.

123. false

125. false

127. false

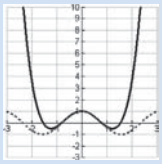
129. false

131. cannot evaluate the identity at $A = \pi$

133. no

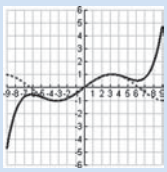
137. $0 < x < \pi$

139.



y_1 (dotted graph) is a good approximation of y_2 on $[-1, 1]$.

141.



y_1 (dotted graph) is a good approximation of y_2 on $[-1, 1]$.

143. $\tan(2x)$

145. $\sin(3x)$

Section 6.4

1. $\frac{1}{2}[\sin(3x) + \sin x]$

3. $\frac{5}{2}[\cos(2x) - \cos(10x)]$

5. $2[\cos x + \cos(3x)]$

7. $\frac{1}{2}[\cos x - \cos(4x)]$

9. $\frac{1}{2}[\cos(2\pi x) - \cos(3\pi x)]\left[\cos(2x) + \cos\left(\frac{2x}{3}\right)\right]$

11. $-\frac{3}{2}[\cos(1.9x) + \cos(1.1x)]$

13. $2\left[\sin\left(2\sqrt{3}x\right) - \sin\left(4\sqrt{3}x\right)\right]$

15. $2\cos(4x)\cos x$

17. $2\sin x\cos(2x)$

19. $-2\sin x\cos\left(\frac{3x}{2}\right)$

21. $2\cos\left(\frac{3}{2}x\right)\cos\left(\frac{5}{6}x\right)$

23. $2\sin(0.5x)\cos(0.1x)$

25. $-2\sin\left(\sqrt{5}x\right)\cos\left(2\sqrt{5}x\right)$

27. $2\cos\left(\frac{\pi}{24}x\right)\cos\left(\frac{5\pi}{24}x\right)$

29. $-\tan x$

31. $\tan(2x)$

33. $\cot\left(\frac{3x}{2}\right)$

43. $P(t) = \sqrt{3}\cos\left(\frac{\pi}{6}t + \frac{4}{3}\pi\right)$

45. $2\cos(886\pi t)\cos(102\pi t)$; 102 Hz; 443 Hz

47. $2\sin\left[\frac{2\pi tc}{2}\left(\frac{1}{1.55} + \frac{1}{0.63}\right)10^6\right] \cdot \cos\left[\frac{2\pi tc}{2}\left(\frac{1}{1.55} - \frac{1}{0.63}\right)10^6\right]$

49. $2\sin(1979\pi t)\cos(439\pi t)$

51. 5.98 ft^2

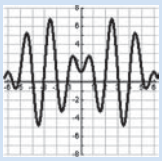
53. $\cos A\cos B \neq \cos(AB)$ and $\sin A\sin B \neq \sin(AB)$, should have used the product-to-sum identities.

55. false

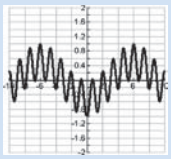
57. true

59. $\frac{1}{4}[\sin(A - B + C) + \sin(C - A + B) - \sin(A + B + C) - \sin(A + B - C)]$

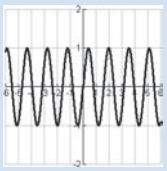
63.



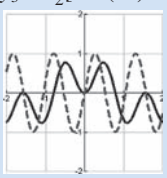
65.



67. $\sin(4x)$



69. $y_1 = \sin(4x)\sin(2x)$ (solid), $y_2 = \sin(6x)$ (dashed), $y_3 = \frac{1}{2}[\cos(2x) - \cos(6x)]$



Y_1 and Y_3 are the same.

71. $\cos x = \cos y - \frac{2}{5}$

73. $2\sin y = \sec x$

Section 6.5

1. $\frac{\pi}{4}$

3. $-\frac{\pi}{3}$

5. $\frac{3\pi}{4}$

7. $\frac{\pi}{6}$

9. $\frac{\pi}{6}$

11. $-\frac{\pi}{3}$

13. 0

15. π

17. 60°

19. 45°

21. 120°

23. 30°

25. −30°

27. 135°

29. −90°

31. 90°

33. 57.10°

35. 62.18°

37. 48.10°

39. −15.30°

41. 166.70°

43. −0.63

45. 1.43

47. 0.92

49. 2.09 **51.** 0.31 **53.** $\frac{5\pi}{12}$

55. undefined **57.** $\frac{\pi}{6}$ **59.** $\frac{2\pi}{3}$

61. $\sqrt{3}$ **63.** $\frac{\pi}{3}$ **65.** undefined

67. 0 **69.** $-\frac{\pi}{4}$ **71.** not possible

73. $\frac{2\pi}{3}$ **75.** $-\frac{\pi}{4}$ **77.** $\frac{\sqrt{7}}{4}$ **79.** $\frac{12}{13}$

81. $\frac{3}{4}$ **83.** $\frac{5\sqrt{23}}{23}$ **85.** $\frac{4\sqrt{15}}{15}$ **87.** $\frac{11}{60}$

89. $\frac{24}{25}$ **91.** $\frac{56}{65}$ **93.** $\frac{24}{25}$ **95.** $\frac{120}{119}$

97. $\sqrt{1 - u^2}$ **99.** $\frac{\sqrt{1 - u^2}}{u}$ **101.** April and October

103. 3rd mo **105.** 0.026476 sec = 26 ms

107. 173.4; June 22–23 **109.** 11 yr

111. $\tan\theta = \frac{8x}{x^2 - 7}$ **113.** 0.70 m; 0.24 m

115. $\theta = \pi - \tan^{-1}\left(\frac{300}{200 - x}\right) - \tan^{-1}\left(\frac{150}{x}\right)$

117. The identity $\sin^{-1}(\sin x) = x$ is valid only for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, not $[0, \pi]$.

119. In general, $\cot^{-1} x \neq \frac{1}{\tan^{-1} x}$.

121. false **123.** false

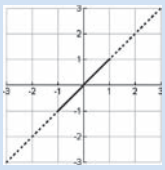
125. $\frac{1}{2}$ is not in the domain of the inverse secant function.

127. $\frac{\sqrt{6} - \sqrt{2}}{4}$ **129.** 0

131. **a.** $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ **b.** $f^{-1}(x) = \frac{\pi}{4} + \cos^{-1}(x - 3)$, $[2, 4]$

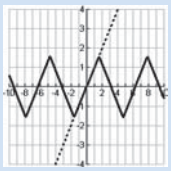
133. **a.** $\left(-\frac{\pi}{12}, \frac{5\pi}{12}\right)$ **b.** $f^{-1}(x) = \frac{\pi}{12} + \frac{1}{2}\cot^{-1}(4x - 8)$, $\mathbb{R}$

135. $Y_1 = \sin\left(\sin^{-1} x\right)$, $Y_2 = x$



Identity $\sin\left(\sin^{-1} x\right) = x$ only holds for $-1 \le x \le 1$.

137. $Y_1 = \csc^{-1}(\csc x)$, $Y_2 = x$



$x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

139. **a.** $\frac{720}{1681}$ **b.** 0.42832 **c.** yes

141. $\sec^2 y = 1 + x^2$ **143.** $\sec y \tan y = x\sqrt{x^2 - 1}$

Section 6.6

1. $\frac{3\pi}{4}, \frac{5\pi}{4}$ **3.** $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$

5. $n\pi$, n an integer **7.** $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

9. $\frac{7\pi}{3} + 4n\pi, \frac{11\pi}{3} + 4n\pi$, n an integer

11. $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}, -\frac{\pi}{3}, -\frac{5\pi}{6}, -\frac{4\pi}{3}, -\frac{11\pi}{6}$

13. $-\frac{2\pi}{3}, -\frac{4\pi}{3}$ **15.** $\frac{\pi(2+3n)}{12}$, n an integer

17. $\frac{(2n+1)\pi}{3}$, n an integer **19.** $-\frac{\pi}{2}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$

21. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ **23.** $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

25. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **27.** $\frac{2\pi}{3}$

29. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **31.** $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{3}$

33. $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{3\pi}{2}$ **35.** $\frac{\pi}{2}$

37. $0, \pi$

39. $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

41. 115.83°, 295.83°, 154.17°, 334.17°

43. 333.63° **45.** 29.05°, 209.05°

47. 200.70°, 339.30° **49.** 41.41°, 318.59°

51. 56.31°, 126.87°, 236.31°, 306.87°

53. 101.79°, 281.79°, 168.21°, 348.21°, 9.74°, 189.74°, 80.26°, 260.26°

55. 80.12°, 279.88°

57. 64.93°, 121.41°, 244.93°, 301.41°

59. 15°, 45°, 75°, 105°, 135°, 165°, 195°, 225°, 255°, 285°, 315°, 345°

61. $\frac{\pi}{4}, \frac{5\pi}{4}$ **63.** π **65.** $\frac{\pi}{6}$ **67.** $\frac{\pi}{3}$

69. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **71.** $\frac{\pi}{2}, \frac{3\pi}{2}$

73. $0, \pi, 2\pi, \frac{\pi}{4}, \frac{7\pi}{4}$ **75.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

77. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ **79.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

81. $\frac{3\pi}{2}$ **83.** $\frac{2\pi}{3}, \frac{4\pi}{3}$ **85.** $\frac{\pi}{3}, \frac{5\pi}{3}, \pi$

87. $\frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24}, \frac{41\pi}{24}$

89. 57.47°, 122.53°, 216.38°, 323.62°

91. 30°, 150°, 199.47°, 340.53°

93. 14.48°, 165.52°, 270°

95. 111.47°, 248.53° **97.** $\frac{\pi}{3}, \frac{5\pi}{3}$

99. 4th quarter of 2008, 2nd quarter of 2009, and 4th quarter of 2010

101. 9 P.M. **103.** March

105. $A = x^2\left[\sin\theta + \frac{\sin(2\theta)}{2}\right]$

107. 2001 **109.** 24° **111.** $\frac{3}{4}$ sec

113. $(0, 1), \left(\frac{\pi}{3}, \frac{3}{2}\right), \left(\frac{5\pi}{3}, \frac{3}{2}\right), (\pi, -3), (2\pi, 1)$

115. March and September

117. 1 A.M. and 11 A.M.

119. The value $\theta = \frac{3\pi}{2}$ does not satisfy the original equation. $\sqrt{2 + \sin\left(\frac{3\pi}{2}\right)} = \sqrt{2 - 1} = 1$, while $\sin\left(\frac{3\pi}{2}\right) = -1$. So, this value of θ is an extraneous solution.

121. Cannot divide by $\cos x$ since it could be zero. Factor as $2\cos x(3\sin x - 1) = 0$.

123. false **125.** true **127.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

129. $\frac{\pi}{6}$ or 30° **131.** no solution

133. $5 + 2n$, n an integer **135.** $\frac{\pi}{6}, \frac{5\pi}{6}$

137. no solutions

139. infinitely many negative solutions

141. 7.39° **143.** 79.07° **145.** 127.1°

147. $x = 0, 2\pi$ **149.** $x = 0, \frac{\pi}{3}, -\frac{\pi}{3}$

Review Exercises

1. 60° **3.** 45° **5.** 60°

7. $\sec^2 x$ **9.** $\sec^2 x$ **11.** $\cos^2 x$

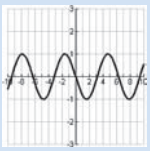
13. $-\left(4 + 2\csc x + \csc^2 x\right)$ **21.** identity

23. conditional **25.** identity **27.** $\frac{\sqrt{2} - \sqrt{6}}{4}$

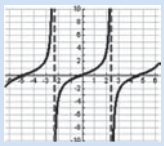
29. $\sqrt{3} - 2$ **31.** $\sin x$ **33.** $\tan x$

35. $\frac{117}{44}$ **37.** $-\frac{897}{1025}$ **39.** identity

41.



43.



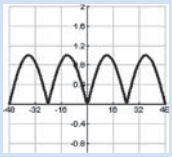
45. $\frac{7}{25}$ **47.** $\frac{671}{1800}$ **49.** $\frac{336}{625}$

51. $\frac{\sqrt{3}}{2}$ **53.** $\frac{3}{2}$ **61.** $-\frac{\sqrt{2 - \sqrt{2}}}{2}$

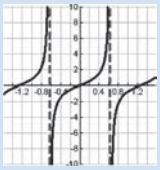
63. $\frac{1}{\sqrt{3 + 2\sqrt{2}}}$ **65.** $\frac{2}{\sqrt{2 + \sqrt{3}}}$ **67.** $\frac{7\sqrt{2}}{10}$

69. $-\frac{5}{4}$ **71.** $\sin\left(\frac{\pi}{12}\right)$

77.



79.



81. $3[\sin(7x) + \sin(3x)]$ **83.** $-2\sin(4x)\sin(x)$

85. $2\sin\left(\frac{x}{3}\right)\cos x$ **87.** $\cot(3x)$ **93.** $\frac{\pi}{4}$

95. $\frac{\pi}{2}$ **97.** $\frac{\pi}{6}$ **99.** $-90°$

101. 60° **103.** $-60°$ **105.** $-37.50°$

107. 22.50° **109.** 1.75 **111.** -0.10

113. $-\frac{\pi}{4}$ **115.** $\sqrt{3}$ **117.** $\frac{\pi}{3}$

119. $\frac{60}{61}$ **121.** $\frac{7}{6}$ **123.** $\frac{6\sqrt{35}}{35}$

125. $\frac{2\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{6}, \frac{11\pi}{6}$ **127.** $-\frac{\pi}{2}, -\frac{3\pi}{2}$ **129.** $\frac{9\pi}{4}, \frac{21\pi}{4}$

131. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **133.** $\frac{3\pi}{8}, \frac{11\pi}{8}, \frac{7\pi}{8}, \frac{15\pi}{8}$

135. $0, \pi, 2\pi, \frac{3\pi}{4}, \frac{7\pi}{4}$

137. 80.46°, 260.46°, 170.46°, 350.46°

139. 90°, 270°, 138.59°, 221.41°

141. 17.62°, 162.38° **143.** $\frac{\pi}{4}, \frac{5\pi}{4}$ **145.** $\pi, \frac{\pi}{3}$

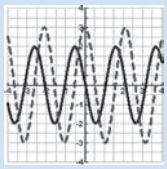
147. $0, \pi, 2\pi, \frac{\pi}{6}, \frac{11\pi}{6}$ **149.** $\frac{3\pi}{2}$ **151.** π

153. $\frac{\pi}{2}, \frac{3\pi}{2}$ **155.** 90°, 270°, 135°, 315°

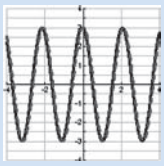
157. 0°, 360° **159.** 90°, 270°, 60°, 300°

161. **a.** 0.2924 **b.** 0.0437 **c.** 0.2924

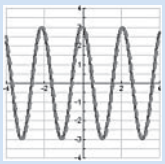
163. **a.**



b.



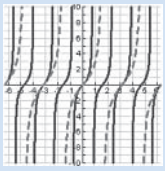
c.



$y = 3\cos(3x)$ as $h \to 0$

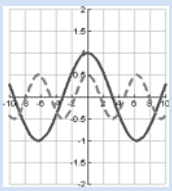
165. $y_1 = \tan(2x)$ (solid), $y_2 = 2\tan x$ (dashed),

$$y_3 = \frac{2\tan x}{1 - \tan^2 x}$$



Y_1 and Y_3 are the same.

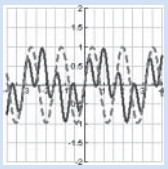
167. $y_1 = \cos\left(\frac{x}{2}\right)$ (solid), $y_2 = \frac{1}{2}\cos x$ (dashed),



$$y_3 = \sqrt{\frac{1 + \cos x}{2}}$$

Y_1 and Y_3 are the same.

169. $y_1 = \sin(5x)\cos(3x)$ (solid), $y_2 = \sin(4x)$ (dashed),
$y_3 = \frac{1}{2}[\sin(8x) + \sin(2x)]$



Y_1 and Y_3 are the same.

171. **a.** $-\frac{3}{5}$ **b.** -0.6 **c.** yes

173. 0.579 rad

Practice Test

1. $x = \frac{(2n+1)\pi}{2}$, n an integer **3.** $-\frac{\sqrt{2-\sqrt{2}}}{2}$

5. $\frac{\sqrt{30}}{10}$ **7.** $\cos(10x)$ **9.** $\cos\left(\frac{a+b}{2}\right)$

11. $20\cos x\cos 3$

13. $\theta = \frac{4\pi}{3} + 2n\pi, \frac{5\pi}{3} + 2n\pi$, n an integer

15. 14.48°, 165.52°, 90°, 270° **17.** conditional

19. $-\frac{\sqrt{26}}{26}$ **21.** $\cot\left(\frac{\pi}{6}x + \frac{\pi}{8}\right)$

23. $\left[c - \frac{1}{2}, c\right) \cup \left(c, c + \frac{1}{2}\right]$; $f^{-1}(x) = \frac{1}{\pi}\csc^{-1}\left(\frac{x-a}{b}\right) - \frac{c}{\pi}$

25. $\frac{8}{3} + 8n, \frac{16}{3} + 8n$, n an integer

27. $\frac{\pi}{2} + 6n\pi, \frac{7\pi}{2} + 6n\pi$, n an integer

29. **a.** $\sqrt{\frac{9}{10}}$ **b.** 0.3163 **c.** yes

Cumulative Test

1. **a.** $\frac{\sqrt{3}}{2}$ **b.** $\sqrt{3}$ **c.** -2

3. **a.** $\frac{2\pi}{3}$ **b.** $-\frac{\pi}{6}$ **c.** $\frac{5\pi}{6}$

5. even **7.** $\frac{1}{x^3} - 1$; $(-\infty, 0) \cup (0, \infty)$

9. $\left(-\frac{6}{5}, -\frac{3}{5}\right)$ **11.** $Q(x) = 5x - 4$, $r(x) = -5x + 7$

13. HA: $y = 0.7$; VA: $x = -2$, $x = 3$

15. $(-3, \infty)$ **17.** 4 **19.** 0.4695

21. $-\frac{7\pi}{12}$ **23.** conditional **25.** $\frac{5}{12}$

27. 1.3994

CHAPTER 7

Section 7.1

1. $\sqrt{13}$ **3.** $5\sqrt{2}$ **5.** 25

7. $\sqrt{73}$, 69.4° **9.** $\sqrt{26}$, 348.7° **11.** $\sqrt{17}$, 166.0°

13. 8, 180° **15.** $2\sqrt{3}$, 60° **17.** $\langle -2, -2\rangle$

19. $\langle -12, 9\rangle$ **21.** $\langle 0, -14\rangle$ **23.** $\langle -36, 48\rangle$

25. $\langle 6.3, 3.0\rangle$ **27.** $\langle -2.8, 15.8\rangle$ **29.** $\langle 2.6, -3.1\rangle$

31. $\langle 8.2, -3.8\rangle$ **33.** $\langle -1, 1.7\rangle$ **35.** $\left\langle -\frac{5}{13}, -\frac{12}{13}\right\rangle$

37. $\left\langle \frac{60}{61}, \frac{11}{61}\right\rangle$ **39.** $\left\langle \frac{24}{25}, -\frac{7}{25}\right\rangle$ **41.** $\left\langle -\frac{3}{5}, -\frac{4}{5}\right\rangle$

43. $\left\langle \frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}\right\rangle$ **45.** $7\vec{i} + 3\vec{j}$ **47.** $5\vec{i} - 3\vec{j}$

49. $-\vec{i} + 0\vec{j}$ **51.** $2\vec{i} + 0\vec{j}$ **53.** $-5\vec{i} + 5\vec{j}$

55. $7\vec{i} + 0\vec{j}$ **57.** H: 1905 ft/sec; V: 1100 ft/sec

59. 2801 lb **61.** 11.7 mph, 31° west of due north

63. 52.41° east of due north, 303 mph **65.** 250 lb

67. V: 51.4 ft/sec; H: 61.3 ft/sec **69.** 29.93 yd

71. 10.9° **73.** 1156 lb

75. 23.75°, 351.16 **77.** $5\sqrt{2}$, 5

79. **a.** $|\overrightarrow{OA}| = 4$, $|\overrightarrow{OA}| = 5$,
$|\overrightarrow{OA}| = 3\sqrt{2}$, $|\overrightarrow{OA}| = \sqrt{10}$, $|\overrightarrow{OA}| = 1$
b. $\left(10 + \sqrt{10} + 3\sqrt{2}\right)$ units

81. 8.97 Nm **83.** 31.95 Nm

85. 8.67, 18.05° counterclockwise of south

87. 1611 N; 19°

89. Magnitude cannot be negative. Oserve that
$|\langle -2, -8\rangle| = \sqrt{(-2)^2 + (-8)^2} = \sqrt{68} = 2\sqrt{17}$.

91. false **93.** true **95.** vector

97. $\sqrt{a^2 + b^2}$ **105.** [[−13][28]] **107.** [[5/13][−12/13]]

109. 183, −79.61114 **111.** 1 **113.** $\langle 1, 2t + h\rangle$

Section 7.2

1. 2 **3.** -3 **5.** 42

7. 11 **9.** $-13a$ **11.** -1.4

13. 98° **15.** 109° **17.** 3°

19. 30° **21.** 105° **23.** 180°

25. not orthogonal **27.** orthogonal **29.** not orthogonal

31. orthogonal **33.** orthogonal **35.** orthogonal

37. 400 ft-lb **39.** 80,000 ft-lb **41.** 1299 ft-lb

43. 148 ft-lb **45.** 1607 lb **47.** 694,593 ft-lb

49. $49,300; total cost

53. **a.** $\cos\beta = \dfrac{v_1}{\sqrt{v_1^2 + v_2^2}}$, $\sin\beta = \dfrac{v_2}{\sqrt{v_1^2 + v_2^2}}$,
$\cos\alpha = \dfrac{w_1}{\sqrt{w_1^2 + w_2^2}}$, $\sin\alpha = \dfrac{w_2}{\sqrt{w_1^2 + w_2^2}}$

55. **a.** -14.72 **b.** 64.57°

57. $\vec{n} = \langle r, 0\rangle$; $\vec{u} \cdot \vec{n} = |\vec{u}|r$ **59.** -2

61. The dot product of two vectors is a scalar, not a vector.

63. false **65.** true **67.** 17

75. **a.** $\text{proj}_{-\vec{u}}\, 2\vec{u} = -2\vec{u}$ **b.** $\text{proj}_{-\vec{u}}\, c\vec{u} = -c\vec{u}$

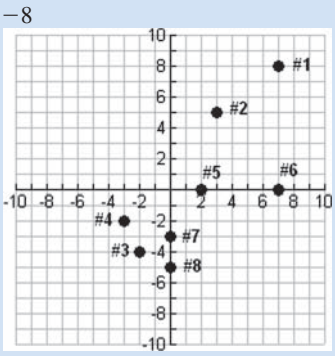
77. $\vec{u}$ is perpendicular to $\vec{v}$, $\theta = 90°$

79. $-6, 6$ **81.** -1083 **83.** 31.4°

85. 1220 **87.** $2t^2 - 3t^3$ **89.** 1.518 rad

Section 7.3

1. -8



9. $\sqrt{2}\left[\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right] = \sqrt{2}[\cos(315°) + i\sin(315°)]$

11. $2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = 2[\cos(60°) + i\sin(60°)]$

13. $4\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] = 4\sqrt{2}[\cos(135°) + i\sin(135°)]$

15. $2\sqrt{3}\left[\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right] = 2\sqrt{3}[\cos(300°) + i\sin(300°)]$

17. $3[\cos(0) + i\sin(0)] = 3[\cos(0°) + i\sin(0°)]$

19. $\frac{\sqrt{2}}{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] = \frac{\sqrt{2}}{2}[\cos(225°) + i\sin(225°)]$

21. $2\sqrt{3}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] =$
$2\sqrt{3}[\cos(225°) + i\sin(225°)]$

23. $5\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right] =$
$5\sqrt{2}[\cos(135°) + i\sin(135°)]$

25. $\sqrt{58}[\cos(293.2°) + i\sin(293.2°)]$

27. $\sqrt{61}[\cos(140.2°) + i\sin(140.2°)]$

29. $13[\cos(112.6°) + i\sin(112.6°)]$

31. $10[\cos(323.1°) + i\sin(323.1°)]$

33. $\frac{\sqrt{13}}{4}[\cos(123.7°) + i\sin(123.7°)]$

35. $5.59[\cos(24.27°) + i\sin(24.27°)]$

37. $\sqrt{17}[\cos(212.84°) + i\sin(212.84°)]$

39. $4.54[\cos(332.31°) + i\sin(332.31°)]$

41. -5 **43.** $\sqrt{2} - \sqrt{2}i$ **45.** $-2 - 2\sqrt{3}i$

47. $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$ **49.** $1 + i$ **51.** $-3\sqrt{2} + 3\sqrt{2}i$

53. $2.1131 - 4.5315i$ **55.** $-0.5209 + 2.9544i$

57. $5.3623 - 4.4995i$ **59.** $-2.8978 + 0.7765i$

61. $0.6180 - 1.9021i$ **63.** $-0.87 - 4.92i$

65. **a.** 59.7 mi **b.** $59.7[\cos(68.8°) + i\sin(68.8°)]$ **c.** 6.8°

67. **a.** $\overrightarrow{AB} = B - A = -2i$, $\overrightarrow{BC} = C - B = 3 - 3i$,
$\overrightarrow{CD} = D - C = -i$
b. $3\sqrt{5}[\cos(297°) + i\sin(297°)]$

69. $z = 2\sqrt{13}[\cos(56.31°) + i\sin(56.31°)]$

71. The point is in QIII not QI. Add 180° to $\tan^{-1}\left(\frac{8}{3}\right)$.

73. true **75.** true **77.** 0°

79. $\sqrt{b^2} = |b|$ **81.** $a\sqrt{5}[\cos(296.6°) + i\sin(296.6°)]$

83. $-\frac{\pi}{2} - \frac{\pi\sqrt{3}}{2}i$ **85.** $8.79[\cos(28°) + i\sin(28°)]$

87. $-z = r[\cos(\theta + \pi) + i\sin(\theta + \pi)]$

89. 1.41421, 45°, $1.4142[\cos(45°) + i\sin(45°)]$

91. $\sqrt{5}[\cos(26.57°) + i\sin(26.57°)]$

93. $35[\cos(323°) + i\sin(323°)]$

95. $r = 5$ **97.** $r = 2\sin\theta$

Section 7.4

1. $-6 + 6\sqrt{3}i$ **3.** $-4\sqrt{2} - 4\sqrt{2}i$ **5.** $0 + 8i$

7. $\frac{9\sqrt{2}}{2} + \frac{9\sqrt{2}}{2}i$ **9.** $0 + 12i$ **11.** $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

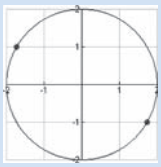
13. $-\sqrt{2} + \sqrt{2}i$ **15.** $0 - 2i$ **17.** $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

19. $-\frac{5}{2} - \frac{5\sqrt{3}}{2}i$ **21.** $4 - 4i$ **23.** $-64 + 0i$

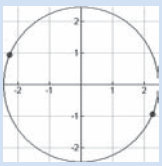
25. $-8 + 8\sqrt{3}i$ **27.** $1{,}048{,}576 + 0i$

29. $-1{,}048{,}576\sqrt{3} - 1{,}048{,}576i$

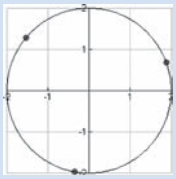
31. $2[\cos(150°) + i\sin(150°)]$,
$2[\cos(330°) + i\sin(330°)]$



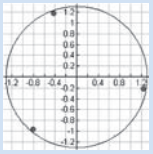
33. $\sqrt{6}[\cos(157.5°) + i\sin(157.5°)]$,
$\sqrt{6}[\cos(337.5°) + i\sin(337.5°)]$



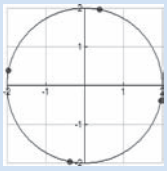
35. $2[\cos(20°) + i\sin(20°)]$,
$2[\cos(140°) + i\sin(140°)]$,
$2[\cos(260°) + i\sin(260°)]$



37. $\sqrt[3]{2}[\cos(110°) + i\sin(110°)]$, $\sqrt[3]{2}[\cos(230°) + i\sin(230°)]$,
$\sqrt[3]{2}[\cos(350°) + i\sin(350°)]$



39. $2[\cos(78.75°) + i\sin(78.75°)]$,
$2[\cos(168.75°) + i\sin(168.75°)]$,
$2[\cos(258.75°) + i\sin(258.75°)]$,
$2[\cos(348.75°) + i\sin(348.75°)]$



41. $\pm 2, \pm 2i$ **43.** $-2, 1 - \sqrt{3}i, 1 + \sqrt{3}i$

45. $\sqrt{2} + \sqrt{2}i, -\sqrt{2} + \sqrt{2}i, -\sqrt{2} - \sqrt{2}i, \sqrt{2} - \sqrt{2}i$

47. $1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$

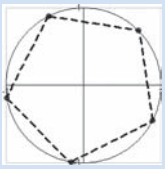
49. $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

51. $\sqrt[4]{2}\left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right]$,
$\sqrt[4]{2}\left[\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right]$,
$\sqrt[4]{2}\left[\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right]$,
$\sqrt[4]{2}\left[\cos\left(\frac{13\pi}{8}\right) + i\sin\left(\frac{13\pi}{8}\right)\right]$

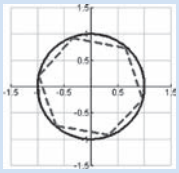
53. $2\left[\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right]$,
$2\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right]$,
$2[\cos(\pi) + i\sin(\pi)]$
$2\left[\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right]$,
$2\left[\cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)\right]$

55. $\pi^2\left[\cos\left(\frac{\pi}{14}\right) + i\sin\left(\frac{\pi}{14}\right)\right]$,
$\pi^2\left[\cos\left(\frac{5\pi}{14}\right) + i\sin\left(\frac{5\pi}{14}\right)\right]$,
$\pi^2\left[\cos\left(\frac{9\pi}{14}\right) + i\sin\left(\frac{9\pi}{14}\right)\right]$
$\pi^2\left[\cos\left(\frac{13\pi}{14}\right) + i\sin\left(\frac{13\pi}{14}\right)\right]$,
$\pi^2\left[\cos\left(\frac{17\pi}{14}\right) + i\sin\left(\frac{17\pi}{14}\right)\right]$,
$\pi^2\left[\cos\left(\frac{21\pi}{14}\right) + i\sin\left(\frac{21\pi}{14}\right)\right]$
$\pi^2\left[\cos\left(\frac{25\pi}{14}\right) + i\sin\left(\frac{25\pi}{14}\right)\right]$

57. $[\cos(45°) + i\sin(45°)]$,
$[\cos(117°) + i\sin(117°)]$,
$[\cos(189°) + i\sin(189°)]$,
$[\cos(261°) + i\sin(261°)]$,
$[\cos(333°) + i\sin(333°)]$



59. $[\cos(50° + 60°k) + i\sin(50° + 60°k)]$,
$k = 0, 1, 2, 3, 4, 5$



61. Reversed order of angles being subtracted.

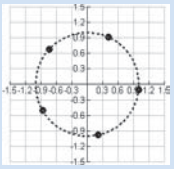
63. Should use DeMoivre's formula. In general, $(a + b)^6 \neq a^6 + b^6$.

65. true **67.** false **69.** true

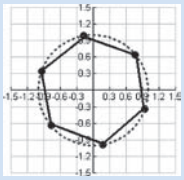
71. true **79.** $\cos(3x) = 4\cos^3 x - 3\cos x$

81. $2^{\frac{n+m}{2}} e^{\frac{\pi}{4}(m-n)i}$

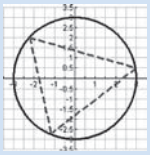
83. $\left[\cos\left(\frac{11\pi}{6(5)} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{11\pi}{6(5)} + \frac{2k\pi}{5}\right)\right]$,
$k = 0, 1, 2, 3, 4$



85. $\left[\cos\left(\frac{4\pi}{3(6)} + \frac{2k\pi}{6}\right) + i\sin\left(\frac{4\pi}{3(6)} + \frac{2k\pi}{6}\right)\right]$,
$k = 0, 1, 2, 3, 4, 5$

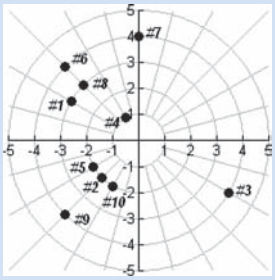


87. $(27)^{1/3}\left[\cos\left(\frac{\frac{\pi}{4}}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4}}{3} + \frac{2k\pi}{3}\right)\right]$,
$k = 0, 1, 2$



Section 7.5

1. -10



11. $\left(4, \frac{\pi}{3}\right)$ **13.** $\left(2, \frac{4\pi}{3}\right)$ **15.** $\left(4\sqrt{2}, \frac{3\pi}{4}\right)$

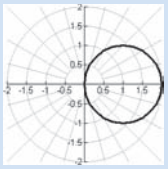
17. $(3, 0)$ **19.** $\left(2, \frac{7\pi}{6}\right)$ **21.** $\left(2, -2\sqrt{3}\right)$

23. $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **25.** $(0, 0)$ **27.** $\left(-1, -\sqrt{3}\right)$

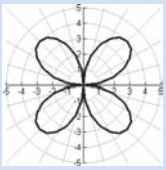
29. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ **31.** d **33.** a

35.

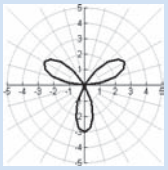
37.



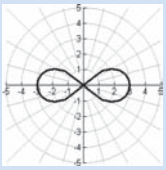
39.



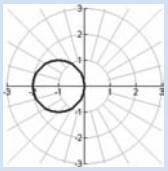
41.



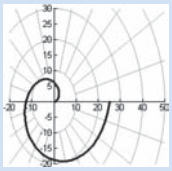
43.



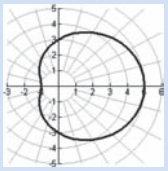
45.



47.

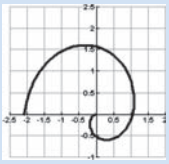


49.

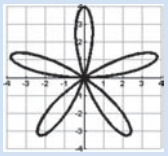


51. $y = -2x + 1$, line **53.** $(x-1)^2 + y^2 = 9$, circle

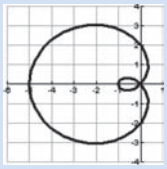
55.



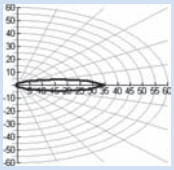
57.



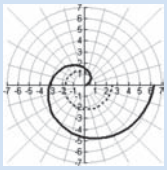
59.



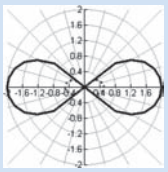
61.



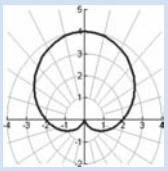
63. $r = \sqrt{\theta}$ more tightly wound than graph of $r = \theta$



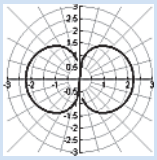
65. $r^2 = \frac{1}{4}\cos(2\theta)$ much closer to the origin than graph of $r^2 = 4\cos(2\theta)$



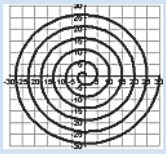
67.



69. a.–c. All three graphs are figure eights. Extending the domain in **b** results in twice as fast movement, while doing so in **c** results in movement that is four times as fast.



71. 6 times, $r = \frac{5}{2\pi}\theta, 0 \le \theta \le 12\pi$

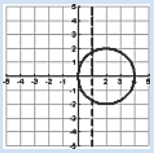


73. a. $r = 8\sin(3\theta), 0 \le \theta \le 2\pi$ **b.** 50 times

75. The point is in QIII; so needed to add π to the angle.

77. true **79.** $r = \frac{a}{\cos\theta}$ **81.** $(-a, \theta \pm 180°)$

83. $\left(2, \frac{\pi}{3}\right), \left(2, \frac{5\pi}{3}\right)$



85. $x^3 - y^3 - 2axy = 0$

87. circle with radius a centered (a, b)

89. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ **91.** $\theta = \frac{\pi}{2}, \theta = \cos^{-1}\left(-\frac{1}{3}\right), \theta = \frac{3\pi}{2}$

93. $\theta = \frac{\pi}{8} + \frac{n\pi}{2}$, n an integer

95. $\left(2\sqrt{2}, \frac{\pi}{4}\right), \left(-2\sqrt{2}, \frac{5\pi}{4}\right)$

97. $\left(\frac{2-\sqrt{2}}{2}, \frac{3\pi}{4}\right), \left(\frac{2+\sqrt{2}}{2}, \frac{7\pi}{4}\right)$

Review Exercises

1. 13 **3.** 13 **5.** 26; 112.6°

7. 20; 323.1° **9.** $\langle 2, 11\rangle$ **11.** $\langle 38, -7\rangle$ **13.** $\langle 2.6, 9.7\rangle$

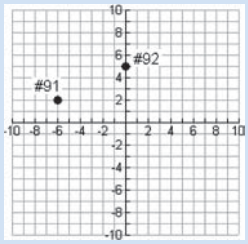
15. $\langle -3.1, 11.6\rangle$ **17.** $\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$ **19.** $5\vec{i} + 2\vec{j}$

21. -6 **23.** -9 **25.** 16 **27.** 59° **29.** 49°

31. 166° **33.** not orthogonal **35.** orthogonal

37. not orthogonal **39.** not orthogonal

41.



43. $2[\cos(315°) + i\sin(315°)]$

45. $8[\cos(270°) + i\sin(270°)]$

47. $61[\cos(169.6°) + i\sin(169.6°)]$

49. $17[\cos(28.1°) + i\sin(28.1°)]$

51. $3 - 3\sqrt{3}i$ **53.** $-1 + i$

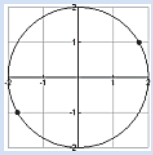
55. $-3.7588 - 1.3681i$

57. $-12i$ **59.** $-\frac{21}{2} - \frac{21\sqrt{3}}{2}i$

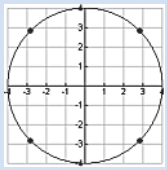
61. $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **63.** -6

65. -324 **67.** $16 - 16\sqrt{3}i$

69. $2[\cos(30°) + i\sin(30°)]$,
$2[\cos(210°) + i\sin(210°)]$



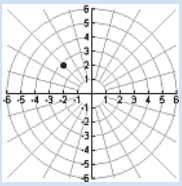
71. $4[\cos(45°) + i\sin(45°)]$,
$4[\cos(135°) + i\sin(135°)]$,
$4[\cos(225°) + i\sin(225°)]$,
$4[\cos(315°) + i\sin(315°)]$



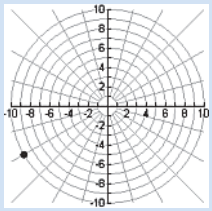
73. $3 + 3\sqrt{3}i, -6, 3 - 3\sqrt{3}i$

75. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

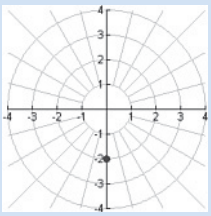
77. $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$



79. $\left(10, \frac{7\pi}{6}\right)$

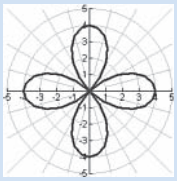


81. $\left(2, \frac{3\pi}{2}\right)$

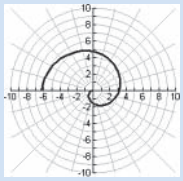


83. $\left(-\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ **85.** $\left(1, \sqrt{3}\right)$ **87.** $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

89.



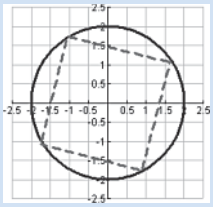
91.



93. 65, $-67.38014°$ **95.** $40°$

97. $12[\cos(246°) + i\sin(246°)]$

99. $\sqrt[4]{16}\left[\cos\left(\frac{120°}{4} + \frac{360°k}{4}\right) + i\sin\left(\frac{120°}{4} + \frac{360°k}{4}\right)\right]$, $k = 0, 1, 2, 3$



101. $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

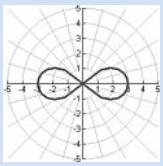
Practice Test

1. 13, $112.6°$ **3. a.** $\langle -14, 5\rangle$ **b.** -16

5. 9 **7.** $32{,}768\left[-1 + \sqrt{3}i\right]$

9. $\frac{-3\sqrt{3}}{2} - \frac{3}{2}i$ **11.** $\left(15\sqrt{5}, 333.4°\right)$

13.



17. -20 **19.** $\sqrt{2}$ **21.** 50,769 lb, $\alpha \approx 12°$

23. $4[\cos(67.5° + 90°k) + i\sin(67.5° + 90°k)]$, $k = 0, 1, 2, 3$

25. $112°$

Cumulative Test

1. $(f \circ g)(x) = \dfrac{-12x - 19}{3x + 5}$,
domains: $f = (-\infty, \infty)$, $g = f \circ g = \left(-\frac{5}{3}, \infty\right)$

3. $y = -\frac{1}{3}(x + 1)^2 + 2 = -\frac{1}{3}x^2 - \frac{2}{3}x + \frac{5}{3}$

5. no HA, VA: $x = 1$, slant: $y = x - 1$

7. $\log_{625} 5 = \frac{1}{4}$

9. $\sin\theta = \frac{3}{8}, \cos\theta = \frac{\sqrt{55}}{8}, \tan\theta = \frac{3\sqrt{55}}{55}$
$\cot\theta = \frac{\sqrt{55}}{3}, \sec\theta = \frac{8\sqrt{55}}{55}, \csc\theta = \frac{8}{3}$

11. $c \approx 13.1$ m, $\alpha \approx 56.1°$, $\beta = 73.9°$

13. Amplitude $= \frac{1}{3}$, Period $= \frac{\pi}{2}$, Phase Shift $= \frac{\pi}{4}$ (right), Vertical Shift $= 4$ (up)

15. $\frac{25\sqrt{3} - 48}{11}$

CHAPTER 8

Section 8.1

1. $(8, -1)$ **3.** $(1, -1)$ **5.** $(1, 2)$

7. $u = \frac{32}{17}, v = \frac{11}{17}$ **9.** no solution

11. infinitely many solutions **13.** infinitely many solutions

15. $(1, 3)$ **17.** $(6, 8)$ **19.** $(-6.24, -2.15)$

21. $(2, 5)$ **23.** $(-3, 4)$ **25.** $\left(1, -\frac{2}{7}\right)$

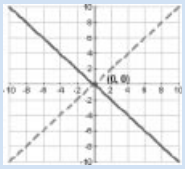
27. $\left(\frac{19}{7}, \frac{11}{35}\right)$ **29.** infinitely many solutions

31. $(4, 0)$ **33.** $(-2, 1)$ **35.** $(3, -2)$

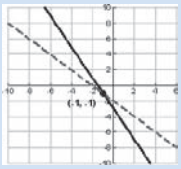
37. $\left(\frac{75}{32}, \frac{7}{16}\right)$ **39.** $(4.2, -3.5)$ **41.** c

43. d

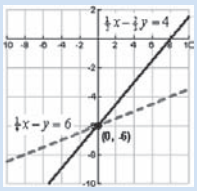
45. (0, 0)



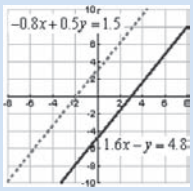
47. $(-1, -1)$



49. $(0, -6)$



51. no solutions



53. (3, 1)

55. (1, 0)

57. Infinitely many solutions: $\left(a, \frac{1.25 - 0.02a}{0.05}\right)$

59. 6 AusPens per kit

61. 15.86 ml of 8% HCl, 21.14 ml of 15% HCl

63. \$300,000 of sales

65. 169 highway miles, 180.5 city miles

67. plane speed: 450 mph, wind speed: 50 mph

69. 10% stock: \$3500, 14% stock: \$6500

71. 8 CD players

73. Type I: \$8; Type II: \$9

75. males: 8,999,215; females: 9,329,125

77. Every term is the first equation is not multiplied by -1 correctly. The equation should be $-2x - y = 3$, and the resulting solution should be $x = 11$, $y = -25$.

79. Did not distribute -1 correctly. In Step 3, the calculation should be $-(-3y - 4) = 3y + 4$.

81. false

83. false

85. $A = -4, B = 7$

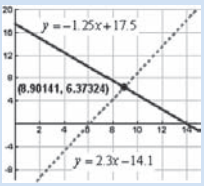
87. 2% drink: 8 cups, 4% drink: 96 cups

89. $(-1, 2)$

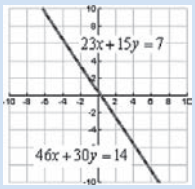
91. $(2, 6)$

93. $(\sqrt{3}, -2), (\sqrt{3}, 2), (-\sqrt{3}, -2), (-\sqrt{3}, 2)$

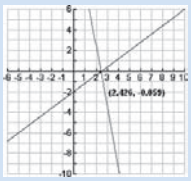
95. (8.9, 6.4)



97. Infinitely many solutions



99. (2.426, -0.059)



101. $A = \frac{3}{2}, B = -\frac{1}{2}$

103. $A = \frac{4}{5}, B = \frac{1}{5}$

Section 8.2

1. $x = -\frac{3}{2}, y = -3, z = \frac{9}{2}$

3. $x = -2, y = \frac{9}{2}, z = \frac{1}{2}$

5. $x = 5, y = 3, z = -1$

7. $x = \frac{90}{31}, y = \frac{103}{31}, z = \frac{9}{31}$

9. $x = -\frac{13}{4}, y = \frac{1}{2}, z = -\frac{5}{2}$

11. $x = -2, y = -1, z = 0$

13. $x = 2, y = 5, z = -1$

15. no solution

17. no solution

19. $x = 1 - a, y = -(a + \frac{1}{2}), z = a$

21. $x = 41 + 4a, y = 31 + 3a, z = a$

23. $x_1 = -\frac{1}{2}, x_2 = \frac{7}{4}, x_3 = -\frac{3}{4}$

25. no solution

27. $x_1 = 1, x_2 = -1 + a, x_3 = a$

29. $x = \frac{2}{3}a + \frac{8}{3}, y = -\frac{1}{3}a - \frac{10}{3}, z = a$

31. $x = a, y = \frac{20}{3} - \frac{13}{3}a, z = 5 - 3a$

33. 100 basic widgets, 100 midprice widgets, and 100 top-of-the-line widgets produced

35. 6 Mediterranean chicken sandwiches
3 six-in. tuna sandwiches
5 six-in. roast beef sandwiches

37. $h_0 = 0, v_0 = 52, a = -32$

39. $y = -0.0625x^2 + 5.25x - 50$

41. money market: \$10,000,
mutual fund: \$4000, stock: \$6000

43. 33 regular model skis, 72 trick skis,
5 slalom skis

45. game 1: 885 points, game 2: 823 points,
game 3: 883 points

47. eagle balls: 1250, birdie balls: 3750,
bogey balls: 5000

49. 10 sec: 2, 20 sec: 3, 40 sec: 1

51. Equation (2) and Equation (3) must be added correctly – should be $2x - y + z = 2$. Also, should begin by eliminating one variable from Equation (1).

53. true

55. $a = 4, b = -2, c = -4$

57. $x^2 + y^2 + 4x - 2y - 4 = 0$

59. $a = -\frac{55}{24}, b = -\frac{1}{4}, c = \frac{223}{24}, d = \frac{1}{4}, e = 44$

61. no solution

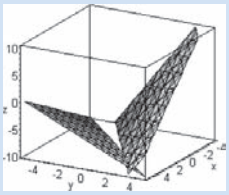
63. $x_1 = -2, x_2 = 1, x_3 = -4, x_4 = 5$

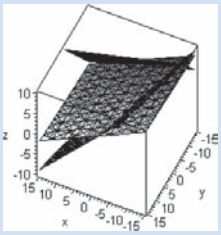
65. $A = 0, B = 1, C = 1, D = 0$

67. $x = 41 + 4a, y = 31 + 3a, z = a$

69. same as answer in Example 57

71. $\left(-\frac{80}{7}, -\frac{80}{7}, \frac{48}{7}\right)$





73. $A = 2, B = 3, C = -4$

75. $A = \frac{4}{3}, B = -1, C = -\frac{1}{3}$

Section 8.3

1. 2×3

3. 1×4

5. 1×1

7. $\left[\begin{array}{rr|r} 3 & -2 & 7 \\ -4 & 6 & -3 \end{array}\right]$

9. $\left[\begin{array}{rrr|r} 2 & -3 & 4 & -3 \\ -1 & 1 & 2 & 1 \\ 5 & -2 & -3 & 7 \end{array}\right]$

11. $\left[\begin{array}{rrr|r} 1 & 1 & 0 & 3 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 5 \end{array}\right]$

13. $\left[\begin{array}{rrr|r} -4 & 3 & 5 & 2 \\ 2 & -3 & -2 & -3 \\ -2 & 4 & 3 & 1 \end{array}\right]$

15. $\begin{cases} -3x + 7y = 2 \\ x + 5y = 8 \end{cases}$

17. $\begin{cases} -x = 4 \\ 7x + 9y + 3z = -3 \\ 4x + 6y - 5z = 8 \end{cases}$

19. $\begin{cases} x = a \\ y = b \end{cases}$

21. not reduced form

23. reduced form

25. not reduced form

27. reduced form

29. reduced form

31. $\left[\begin{array}{rr|r} 1 & -2 & -3 \\ 0 & 7 & 5 \end{array}\right]$

33. $\left[\begin{array}{rrr|r} 1 & -2 & -1 & 3 \\ 0 & 5 & -1 & 0 \\ 3 & -2 & 5 & 8 \end{array}\right]$

35. $\left[\begin{array}{rrrr|r} 1 & -2 & 5 & -1 & 2 \\ 0 & 1 & 1 & -3 & 3 \\ 0 & -2 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 & -6 \end{array}\right]$

37. $\left[\begin{array}{rrrr|r} 1 & 0 & 5 & -10 & -5 \\ 0 & 1 & 2 & -3 & -2 \\ 0 & 0 & -7 & 6 & 3 \\ 0 & 0 & 8 & -10 & -9 \end{array}\right]$

39. $\left[\begin{array}{rrrr|r} 1 & 0 & 4 & 0 & 27 \\ 0 & 1 & 2 & 0 & -11 \\ 0 & 0 & 1 & 0 & 21 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right]$

41. $\left[\begin{array}{rr|r} 1 & 0 & -8 \\ 0 & 1 & 6 \end{array}\right]$

43. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

45. $\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$

47. $\left[\begin{array}{rrr|r} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 2 \end{array}\right]$

49. $\left[\begin{array}{rrr|r} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$

51. $x = -7, y = 5$

53. $x - 2y = -3$ or $x = 2a - 3, y = a$

55. no solution

57. $x = 4a + 41, y = 31 + 3a, z = a$

59. $x_1 = -\frac{1}{2}, x_2 = \frac{7}{4}, x_3 = -\frac{3}{4}$

61. no solution

63. $x_1 = 1, x_2 = a - 1, x_3 = a$

65. $x = \frac{2}{3}(a + 4), y = -\frac{1}{3}(a + 10), z = a$

67. no solution

69. $x_1 = -2, x_2 = 1, x_3 = -4, x_4 = 5$

71. $(1, -2)$

73. no solution

75. $(-2, 1, 3)$

77. $(3, -2, 2)$

79. no solution

81. $x = \frac{a}{4} + 3, y = \frac{7a}{4} - \frac{53}{3}, z = a$

83. $x = \frac{72 - 11a}{14}, y = \frac{13a + 4}{14}, z = a$

85. $x = 1, y = 2, z = -3, w = 1$

87. 960 red dwarf, 8 blue stars, 2,880,000 yellow stars

89. 2 chicken, 2 tuna, 8 roast beef, 2 turkey bacon

91. initial height: 0 ft, initial velocity: 50 ft/sec, acceleration: -32 ft/sec^2

93. $y = -0.053x^2 + 4.58x - 34.76$

95. about 88 ml of the 1.5% solution and 12 ml of the 30% solution

97. 200 basic widgets, 100 midprice widgets, and 75 top-of-the-line widgets produced

99. money market: \$5500, mutual fund: \$2500, stock: \$2000

101. product x: 25 units, product y: 40 units, product z: 6 units

103. general admission: 25, reserved: 30, end zone: 45

105. $a = -\frac{22}{17}, b = -\frac{44}{17}, c = -\frac{280}{17}$

107. Need to line up a single variable in a given column before forming the augmented matrix. The correct martrix is

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 2 \\ 1 & 1 & -2 & -3 \\ 1 & 1 & 1 & 6 \end{array}\right], \text{ after reducing, } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

109. Row 3 is not inconsistent. It implies $z = 0$.

111. false **113.** true **115.** false **117.** false

119. $f(x) = -\frac{11}{6}x^4 + \frac{44}{3}x^3 - \frac{223}{6}x^2 + \frac{94}{3}x + 44$

121. 35 hr **123.** $(-1, 2)$, $(2, -1)$, and $(3, -1)$

125.
```
rref([A])
[[1 0-4 41]
 [0 1-3 31]
 [0 0 0 0 ]]
```

127. a.
```
rref([A])
[[1 0 0 -.23653…
 [0 1 0 .928846…
 [0 0 1 6.08846…
```
$y = -0.24x^2 + 0.93x + 6.09$

b.
```
QuadReg
 y=ax²+bx+c
 a=-.2365384615
 b=.9288461538
 c=6.088461538
```
$y = -0.24x^2 + 0.93x + 6.09$

129. $c_1 = \frac{3}{4}, c_2 = -\frac{3}{4}$ **131.** $c_1 = 2, c_2 = -3, c_3 = 1$

Section 8.4

1. 2×3 **3.** 2×2 **5.** 3×3

7. 4×4 **9.** $x = -5, y = 1$

11. $x = -3, y = -2, z = 3$ **13.** $x = 6, y = 3$

15. $\begin{bmatrix} -1 & 5 & 1 \\ 5 & 2 & 5 \end{bmatrix}$ **17.** $\begin{bmatrix} -2 & 4 \\ 2 & -2 \\ -1 & 3 \end{bmatrix}$

19. not defined **21.** not defined

23. $\begin{bmatrix} -2 & 12 & 3 \\ 13 & 2 & 14 \end{bmatrix}$ **25.** $\begin{bmatrix} 8 & 3 \\ 11 & 5 \end{bmatrix}$

27. $\begin{bmatrix} -3 & 21 & 6 \\ -4 & 7 & 1 \\ 13 & 14 & 9 \end{bmatrix}$ **29.** $\begin{bmatrix} 3 & 6 \\ -2 & -2 \\ 17 & 24 \end{bmatrix}$

31. not defined **33.** $[0 \quad 60]$

35. $[-6 \quad 1 \quad -9]$ **37.** $\begin{bmatrix} 7 & 10 & -8 \\ 0 & 15 & 5 \\ 23 & 0 & -7 \end{bmatrix}$

39. $\begin{bmatrix} 12 & 20 \\ 30 & 42 \end{bmatrix}$ **41.** $\begin{bmatrix} -4 \\ -4 \\ -16 \end{bmatrix}$

43. not defined **45.** yes

47. yes **49.** yes

51. $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ **53.** $\begin{bmatrix} -\frac{1}{13} & \frac{8}{39} \\ \frac{20}{39} & -\frac{4}{117} \end{bmatrix}$

55. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ **57.** A^{-1} does not exist.

59. $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & -1 & 1 \end{bmatrix}$ **61.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{4} & \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{2} \end{bmatrix}$

63. $x = 2, y = -1$ **65.** $x = \frac{1}{2}, y = \frac{1}{3}$

67. $x = 0, y = 0, z = 1$

69. A^{-1} does not exist

71. $x = -1, y = 1, z = -7$ **73.** $x = 3, y = 5, z = 4$

75. $A = \begin{bmatrix} 0.70 \\ 0.30 \end{bmatrix}$, $B = \begin{bmatrix} 0.89 \\ 0.84 \end{bmatrix}$

a. $46A = \begin{bmatrix} 32.2 \\ 13.8 \end{bmatrix}$, out of 46 million people, 32.2 million said that they had tried to quit smoking, while 13.8 million said that they had not.

b. $46B = \begin{bmatrix} 40.94 \\ 38.64 \end{bmatrix}$, out of 46 million people, 40.94 million believed that smoking would increase the chance of getting lung cancer, and that 38.64 million believed that smoking would shorten their lives.

77. $A = \begin{bmatrix} 0.589 & 0.628 \\ 0.414 & 0.430 \end{bmatrix}$, $B = \begin{bmatrix} 100M \\ 110M \end{bmatrix}$

$AB = \begin{bmatrix} 127.98M \\ 88.7M \end{bmatrix}$ 127.98 million registered voters, of those 88.7 million actually vote

79. $A = [0.45 \quad 0.50 \quad 1.00]$

$$B = \begin{bmatrix} 7{,}523 \\ 2{,}700 \\ 15{,}200 \end{bmatrix} \quad AB = [19{,}935.35]$$

81. $AB = \begin{bmatrix} 0.228 \\ 0.081 \\ 0.015 \end{bmatrix}$, total cost per mile to run each type of automobile

83. $N = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad XN = \begin{bmatrix} 10 \\ 16 \\ 20 \end{bmatrix}$

The nutritional content of the meal is 10 g of carbohydrates, 16 g of protein, and 20 g of fat.

85. $N = \begin{bmatrix} 200 \\ 25 \\ 0 \end{bmatrix} \qquad XN = \begin{bmatrix} 9.25 \\ 13.25 \\ 15.75 \end{bmatrix}$

Company 1 would charge \$9.25, Company 2 would charge \$13.25, and Company 3 would charge \$15.75, respectively, for 200 minutes of talking and 25 text message. The better cell phone provider for this employee would be Company 1.

87. JAW **89.** LEG **91.** EYE

93. $X = \begin{bmatrix} 8 & 4 & 6 \\ 6 & 10 & 5 \\ 10 & 4 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ 21 \\ 22 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The combination of one serving each of food A, B, and C will create a meal of 18 g carbohydrates, 21 g of protein, and 22 g of fat.

95. $X = \begin{bmatrix} 0.03 & 0.06 & 0.15 \\ 0.04 & 0.05 & 0.18 \\ 0.05 & 0.07 & 0.13 \end{bmatrix}^{-1} \begin{bmatrix} 49.50 \\ 52.00 \\ 58.50 \end{bmatrix} = \begin{bmatrix} 350 \\ 450 \\ 100 \end{bmatrix}$

The employee's normal monthly usage is 350 minutes talking, 400 text messages, and 100 MB of data usage.

97. Not multiplying correctly. It should be:
$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -7 & 19 \\ -9 & 23 \end{bmatrix}$

99. A is not invertible because the identity matrix was not reached.

101. false **103.** true **105.** false

107. $\begin{bmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{22}^2 + a_{21}a_{12} \end{bmatrix}$

109. $x = 9$

111. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $A^3 = 2^{n-1}A$, $n \geq 1$

113. must have $m = p$

115. $A \cdot A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \left(\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$

$$= \frac{1}{ad - bc} \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= \begin{bmatrix} \frac{ad - bc}{ad - bc} & 0 \\ 0 & \frac{ad - bc}{ad - bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

117. $ad - bc = 0$

119. $\begin{bmatrix} 33 & 35 \\ -96 & -82 \\ 31 & 19 \\ 146 & 138 \end{bmatrix}$ **121.** not defined

123. $\begin{bmatrix} 5 & -4 & 4 \\ 2 & -15 & -3 \\ 26 & 4 & -8 \end{bmatrix}$

125. $\begin{bmatrix} -\frac{115}{6008} & \frac{431}{6008} & \frac{-1067}{6008} & \frac{103}{751} \\ \frac{411}{6008} & \frac{-391}{6008} & \frac{731}{6008} & \frac{-22}{751} \\ \frac{57}{751} & \frac{28}{751} & \frac{-85}{751} & \frac{3}{751} \\ \frac{-429}{6008} & \frac{145}{6008} & \frac{1035}{6008} & \frac{12}{751} \end{bmatrix}$

127. $\begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4y} & -\frac{1}{4y} \end{bmatrix}$ **129** $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Section 8.5

1. -2 **3.** 31 **5.** -28

7. -0.6 **9.** 0 **11.** $x = 5, y = -6$

13. $x = -2, y = 1$ **15.** $x = -3, y = -4$

17. $x = -2, y = 5$ **19.** $x = 2, y = 2$

21. $D = 0$, inconsistent or dependent system

23. $D = 0$, inconsistent or dependent system

25. $x = \frac{1}{2}, y = -1$ **27.** $x = 1.5, y = 2.1$

29. $x = 0, y = 7$ **31.** 7

33. -25 **35.** -180

37. 0 **39.** 238

41. 0 **43.** $x = 2, y = 3, z = 5$

45. $x = -2, y = 3, z = 5$ **47.** $x = 2, y = -3, z = 1$

49. $D = 0$, inconsistent or dependent system

51. $D = 0$, inconsistent or dependent system

53. $x = -3, y = 1, z = 4$ **55.** $x = 2, y = -3, z = 5$

57. $x = -2, y = \frac{3}{2}, z = 3$ **59.** yes

61. 6 units^2 **63.** 6 units^2

65. $y = 2x$ **67.** $I_1 = \frac{7}{2}, I_2 = \frac{5}{2}, I_3 = 1$

69. The second determinant should be subtracted; that is, it should be $-1\begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$.

71. In D_x and D_y, the column $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ should replace the column corresponding to the variable that is being solved for in each case. Precisely, D_x should be $\begin{bmatrix} 6 & 3 \\ -3 & -1 \end{bmatrix}$ and D_y should be $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$.

73. true **75.** false

77. abc **79.** -419

81. $-b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$= -b_1[(a_2)(c_3) - (a_3)(c_2)] + b_2[(a_1)(c_3) - (a_3)(c_1)]$
$\quad -b_3[(a_1)(c_2) - (a_2)(c_1)]$

$= -a_2b_1c_3 + a_3b_1c_2 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_2b_3c_1$

85. -180 **87.** -1019

89. $x = -6.4, y = 1.5, z = 3.4$ **91.** r

93. $-\rho^2\sin\varphi$

Section 8.6

1. d **3.** a **5.** b

7. $\dfrac{A}{x-5} + \dfrac{B}{x+4}$ **9.** $\dfrac{A}{x-4} + \dfrac{B}{x} + \dfrac{C}{x^2}$

11. $2x - 6 + \dfrac{3x+33}{x^2+x+5}$ **13.** $\dfrac{Ax+B}{x^2+10} + \dfrac{Cx+D}{(x^2+10)^2}$

15. $\dfrac{1}{x} - \dfrac{1}{x+1}$ **17.** $\dfrac{1}{x-1}$

19. $\dfrac{2}{x-3} + \dfrac{7}{x+5}$ **21.** $\dfrac{3}{x-1} + \dfrac{4}{(x-1)^2}$

23. $\dfrac{4}{x+3} - \dfrac{15}{(x+3)^2}$ **25.** $\dfrac{3}{x+1} + \dfrac{1}{x-5} + \dfrac{2}{(x-5)^2}$

27. $\dfrac{-2}{x+4} + \dfrac{7x}{x^2+3}$ **29.** $\dfrac{-2}{x-7} + \dfrac{4x-3}{3x^2-7x+5}$

31. $\dfrac{x}{x^2+9} - \dfrac{9x}{(x^2+9)^2}$ **33.** $\dfrac{2x-3}{x^2+1} + \dfrac{5x+1}{(x^2+1)^2}$

35. $\dfrac{1}{x-1} + \dfrac{1}{2(x+1)} + \dfrac{-3x-1}{2(x^2+1)}$

37. $\dfrac{3}{x-1} + \dfrac{2x+5}{x^2+2x-1}$ **39.** $\dfrac{1}{x-1} + \dfrac{1-x}{x^2+x+1}$

41. $\dfrac{1}{d_o} + \dfrac{1}{d_i} = \dfrac{1}{f}$ **43.** $-\frac{11}{4}e^{2t} + \frac{7}{4}e^{-2t}$

45. The form of the decomposition is incorrect. It should be $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$. Once this correction is made, the correct decomposition is $\dfrac{1}{x} + \dfrac{2x+3}{x^2+1}$.

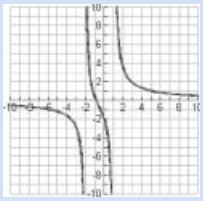
47. false **49.** true **51.** false

53. $\dfrac{1}{x-1} - \dfrac{1}{x+2} + \dfrac{1}{x-2}$

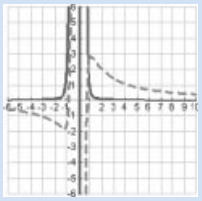
55. $\dfrac{1}{x} + \dfrac{1}{x+1} - \dfrac{1}{x^3}$

57. $\dfrac{x}{x^2+1} - \dfrac{2x}{(x^2+1)^2} + \dfrac{x+2}{(x^2+1)^3}$

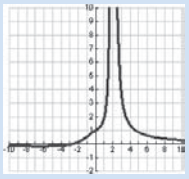
59. yes



61. no



63. yes

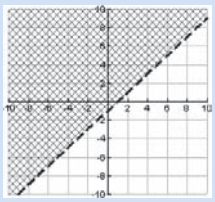


65. $\dfrac{3}{k} - \dfrac{3}{k+3}$ **67.** $\dfrac{1}{k^2} - \dfrac{1}{(k+1)^2}$

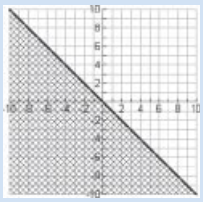
Section 8.7

1. d **3.** b

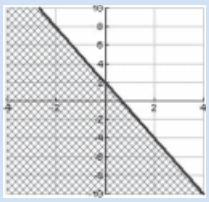
5.



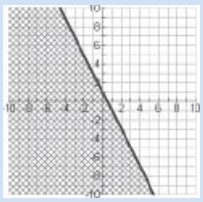
7.



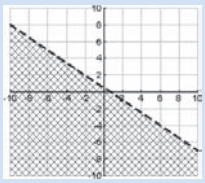
9.



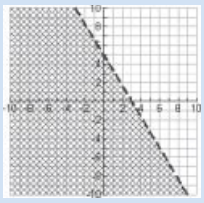
11.



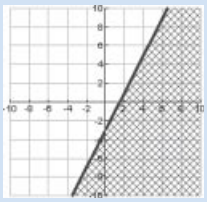
13.



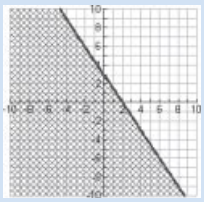
15.



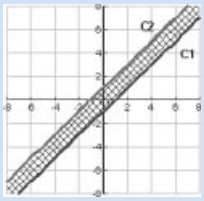
17.



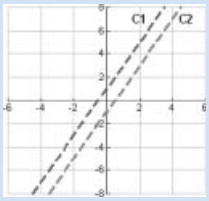
19.



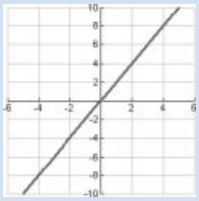
21.



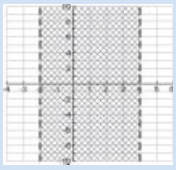
23.



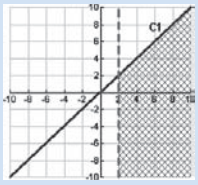
25.



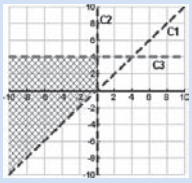
27.



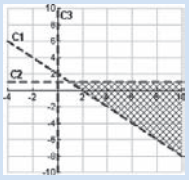
29.



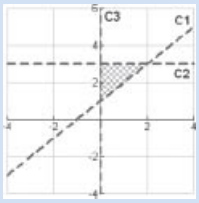
31.



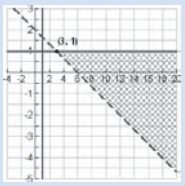
33.



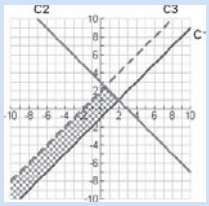
35.



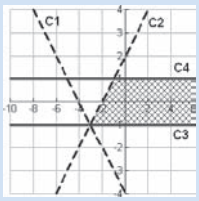
37.



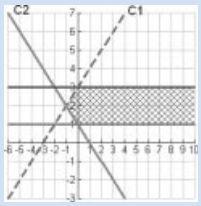
39.



41.

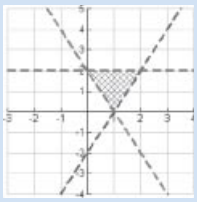


43.

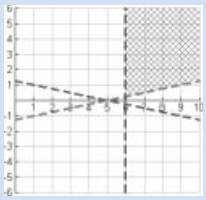


45. no solution

47.



49.



51. $f(x, y) = z = 2x + 3y$
$f(-1, 4) = 10$
$f(2, 4) = 16$ (MAX),
$f(-2, -1) = -7$ (MIN),
$f(1, -1) = -1$

53. $f(x, y) = z = 1.5x + 4.5y$
$f(-1, 4) = 16.5$
$f(2, 4) = 21$ (MAX)
$f(-2, -1) = -7.5$ (MIN)
$f(1, -1) = -3$

55. minimize at $f(0, 0) = 0$

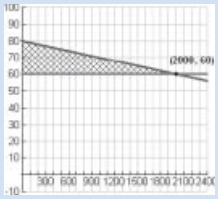
57. no maximum

59. minimize at $f(0, 0) = 0$

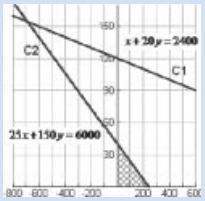
61. maximize at $f(1, 6) = \frac{53}{20} = 2.65$

63. $\begin{cases} P \le 80 - 0.01x \\ P \ge 60 \\ x \ge 0 \end{cases}$

65. 20,000 units^2



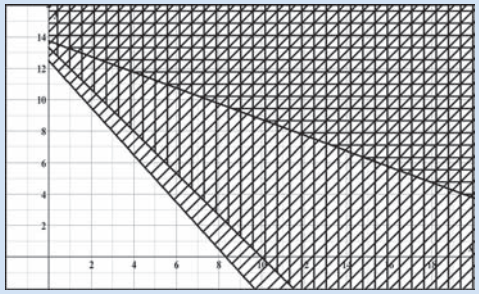
67. $\begin{cases} x \ge 0, y \ge 0 \\ x + 20y \le 2400 \\ 25x + 150y \le 6000 \end{cases}$



69. Frances T-shirts: 130
Charley T-shirts: 50 (profit \$950)

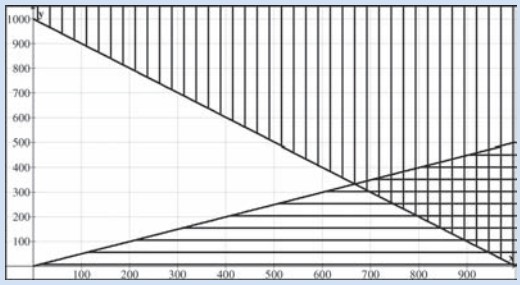
71. **a.** $275 \le 10x + 20y$
$125 \le 15x + 10y$
$200 \le 20x + 15y$
$x \ge 0, y \ge 0$

b.



73. **a.** $x \ge 2y$
$x + y \ge 1000$
$x \ge 0, y \ge 0$

b.



c. Two possible solutions would be for the manufacturer to produce 700 USB wireless mice and 300 Bluetooth mice or 800 USB wireless mice and 300 Bluetooth mice.

75. laptops: 25, desktops: 0 (profit \$7500)

77. first-class cars: 3, second-class cars: 27

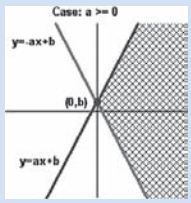
79. 200 of each type of ski

81. The shading should be above the line.

83. true **85.** false **87.** false

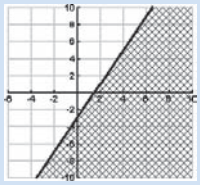
89. shaded rectangle

91.

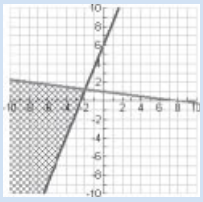


93. maximum at $(0, a)$ and is a

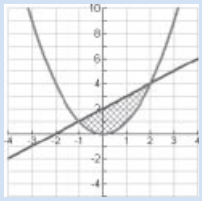
95.



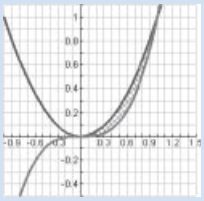
97.



99.



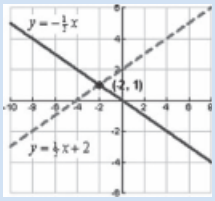
101.

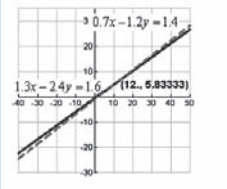


Review Exercises

1. $(3, 0)$ **3.** $\left(\frac{13}{4}, 8\right)$ **5.** $(2, 1)$ **7.** $\left(\frac{19}{8}, \frac{13}{8}\right)$

9. $(-2, 1)$ **11.** $\left(12, 5.8\overline{3}\right)$





13. $(3, -2)$ **15.** $(-1, 2)$

17. c **19.** d

21. 6% NaCl: 10.5 ml, 18% NaCl: 31.5 ml

23. $x = -1, y = -a + 2, z = a$

25. no solution

27. $y = -0.0050x^2 + 0.4486x - 3.8884$

29. $\left[\begin{array}{cc|c} 5 & 7 & 2 \\ 3 & -4 & -2 \end{array}\right]$ **31.** $\left[\begin{array}{ccc|c} 2 & 0 & -1 & 3 \\ 0 & 1 & -3 & -2 \\ 1 & 0 & 4 & -3 \end{array}\right]$

33. no **35.** no

37. $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -1 \end{array}\right]$ **39.** $\left[\begin{array}{ccc|c} 1 & -4 & 3 & -1 \\ 0 & -2 & 3 & -2 \\ 0 & 1 & -4 & 8 \end{array}\right]$

41. $\left[\begin{array}{cc|c} 1 & 0 & \frac{3}{5} \\ 0 & 1 & -\frac{1}{5} \end{array}\right]$ **43.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -4 \end{array}\right]$

45. $x = \frac{5}{4}, y = \frac{7}{8}$

47. $x = 2, y = 1$

49. $x = -\frac{74}{21}, y = -\frac{73}{21}, z = -\frac{3}{7}$

51. $x = 1, y = 3, z = -5$

53. $x = -\frac{3}{7}a - 2, y = \frac{2}{7}a + 2, z = a$

55. $y = -0.005x^2 + 0.45x - 3.89$

57. not defined

59. $\begin{bmatrix} 3 & 5 & 2 \\ 7 & 8 & 1 \end{bmatrix}$

61. $\begin{bmatrix} 9 & -4 \\ 9 & 9 \end{bmatrix}$

63. $\begin{bmatrix} 4 & 13 \\ 18 & 11 \end{bmatrix}$

65. $\begin{bmatrix} 0 & -19 \\ -18 & -9 \end{bmatrix}$

67. $\begin{bmatrix} -7 & -11 & -8 \\ 3 & 7 & 2 \end{bmatrix}$

69. $\begin{bmatrix} 10 & -13 \\ 18 & -20 \end{bmatrix}$

71. $\begin{bmatrix} 17 & -8 & 18 \\ 33 & 0 & 42 \end{bmatrix}$

73. $\begin{bmatrix} 10 & 9 & 20 \\ 22 & -4 & 2 \end{bmatrix}$

75. yes

77. yes

79. $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$

81. $\begin{bmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}$

83. $\begin{bmatrix} -\frac{1}{6} & \frac{7}{12} & -\frac{1}{12} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & -\frac{1}{12} & -\frac{5}{12} \end{bmatrix}$

85. $\begin{bmatrix} 0 & -\frac{2}{5} & \frac{1}{5} \\ 1 & -\frac{2}{5} & \frac{1}{5} \\ -\frac{1}{2} & \frac{3}{10} & \frac{1}{10} \end{bmatrix}$

87. $x = 5, y = 4$

89. $x = 8, y = 12$

91. $x = 1, y = 2, z = 3$

93. -8

95. 5.4

97. $x = 3, y = 1$

99. $x = 6, y = 0$

101. $x = 90, y = 155$

103. 11

105. $-abd$

107. $x = 1, y = 1, z = 2$

109. $x = -\frac{15}{7}, y = -\frac{25}{7}, z = \frac{19}{14}$

111. $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+3} + \dfrac{D}{x-5}$

113. $\dfrac{A}{x} + \dfrac{B}{(2x+1)^2} + \dfrac{C}{4x+5} + \dfrac{D}{2x+1}$

115. $\dfrac{A}{x-3} + \dfrac{B}{x+4}$

117. $\dfrac{Ax+B}{x^2+17} + \dfrac{Cx+D}{(x^2+17)^2}$

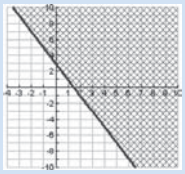
119. $\dfrac{4}{x-1} + \dfrac{5}{x+7}$

121. $\dfrac{1}{2x} + \dfrac{15}{2(x-5)} - \dfrac{3}{2(x+5)}$

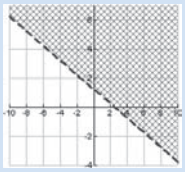
123. $\dfrac{-2}{x+1} + \dfrac{2}{x}$

125. $\dfrac{5}{x+2} - \dfrac{27}{(x+2)^2}$

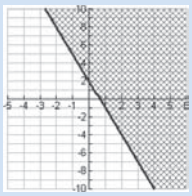
127.



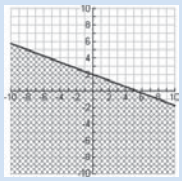
129.



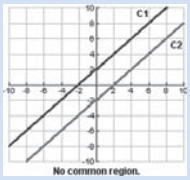
131.



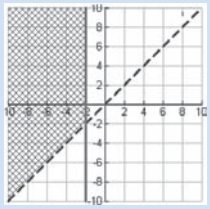
133.



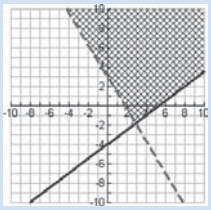
135. no solution



137.



139.



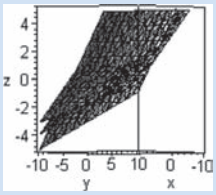
141. minimum value of z: 0, occurs at (0, 0)

143. minimum value of z: -30, occurs at (0, 6)

145. ocean watercolor: 10
geometric shape: 30 (profit \$390)

147. $(2, -3)$

149. (3.6, 3, 0.8)



151. a.

```
rref([A])
[[1  0  0  .161066…
 [0  1  0  -.04946…
 [0  0  1  -4.1012…
```

$y = 0.16x^2 - 0.05x - 4.10$

b.

```
QuadReg
 y=ax²+bx+c
 a=.1610661269
 b=-.0494601889
 c=-4.101214575
```

$y = 0.16x^2 - 0.05x - 4.10$

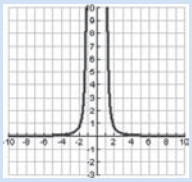
153. $\begin{bmatrix} -238 & 206 & 50 \\ -113 & 159 & 135 \\ 40 & -30 & 0 \end{bmatrix}$

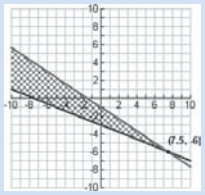
155. $x = 2.25, y = -4.35$

157. $x = -9.5, y = 3.4$

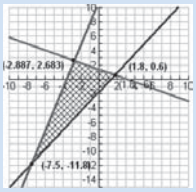
159. yes

161.





163. maximum is 12.06, occurs at (1.8, 0.6)



Practice Test

1. (7, 3)

3. $x = a, y = a - 2$

5. $x = 1, y = -5, z = 3$

7. $\left[\begin{array}{ccc|c} 6 & 9 & 1 & 5 \\ 2 & -3 & 1 & 3 \\ 10 & 12 & 2 & 9 \end{array}\right]$

9. $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & -11 \\ 0 & 7 & 15 \end{bmatrix}$

11. $x = -\frac{1}{3}a + \frac{7}{6}, y = \frac{1}{9}a - \frac{2}{9}, z = a$

13. $\begin{bmatrix} -11 & 19 \\ -6 & 8 \end{bmatrix}$

15. $\begin{bmatrix} \frac{1}{19} & \frac{3}{19} \\ \frac{5}{19} & -\frac{4}{19} \end{bmatrix}$

17. $x = -3, y = 1, z = 7$

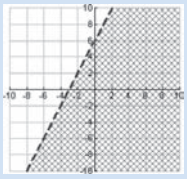
19. -31

21. $x = 1, y = -1, z = 2$

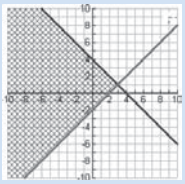
23. $\dfrac{5}{x} - \dfrac{3}{x + 1}$

25. $\dfrac{1}{3x} + \dfrac{2}{3(x - 3)} - \dfrac{1}{x + 3}$

27.



29.



31. minimum value of z: 7, occurs at (0, 1)

33. (12.5, −6.4)

Cumulative Test

1. 27

3. $2x + h - 3$

5. (15, 6)

7. VA: $x = 3$; HA: $y = -5$

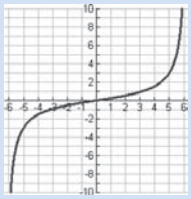
9. -1

11. 2.585

13. $\frac{\sqrt{3}}{2}$

15. $\sin\theta = \frac{2\sqrt{29}}{29}, \cos\theta = \frac{-5\sqrt{29}}{29}, \tan\theta = \frac{-2}{5}, \cot\theta = -\frac{5}{2},$
$\sec\theta = -\frac{\sqrt{29}}{5}, \csc\theta = \frac{\sqrt{29}}{2}$

17.



19. domain: $x \neq \frac{(2n + 1)\pi}{2} + \frac{\pi}{2} = \frac{(2n + 2)\pi}{2} = (n + 1)\pi$, n any integer, range: all reals

21. 1

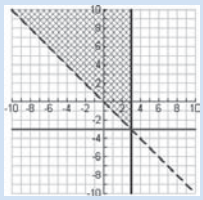
23. $\alpha = 36.3°, b \approx 123.4$ m, $c \approx 78.1$ m

25. $-2, 1 - \sqrt{3}i, 1 + \sqrt{3}i$

27. $\begin{bmatrix} 72 & -18 & 54 \\ 26 & -2 & 4 \end{bmatrix}$

29. $x = \frac{3}{11}, y = -\frac{2}{11}$

31.



CHAPTER 9

Section 9.1

1. hyperbola

3. circle

5. hyperbola

7. ellipse

9. parabola

11. circle

Section 9.2

1. c

3. d

5. c

7. a

9. $x^2 = 12y$

11. $y^2 = -20x$

13. $(x - 3)^2 = 8(y - 5)$

15. $(y - 4)^2 = -8(x - 2)$

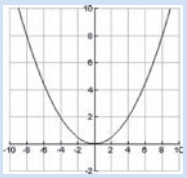
17. $(x - 2)^2 = 4(3)(y - 1) = 12(y - 1)$

19. $(y + 1)^2 = 4(1)(x - 2) = 4(x - 2)$

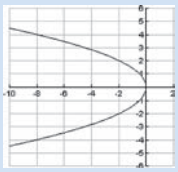
21. $(y - 2)^2 = 8(x + 1)$

23. $(x - 2)^2 = -8(y + 1)$

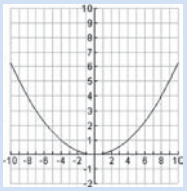
25. vertex: (0, 0)
focus: (0, 2)
directrix: $y = -2$
length of latus rectum: 8



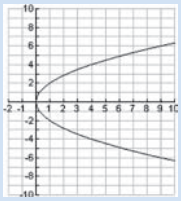
27. vertex: (0, 0)
focus: $\left(-\frac{1}{2}, 0\right)$
directrix: $x = \frac{1}{2}$
length of latus rectum: 2



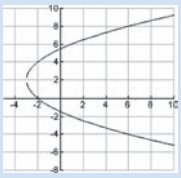
29. vertex: (0, 0)
focus: (0, 4)
directrix: $y = -4$
length of latus rectum: 16



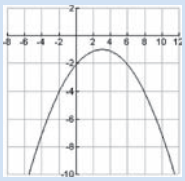
31. vertex: (0, 0)
focus: (1, 0)
directrix: $x = -1$
length of latus rectum: 4



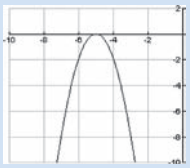
33. vertex: $(-3, 2)$



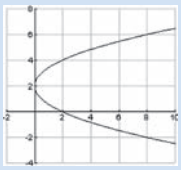
35. vertex: $(3, -1)$



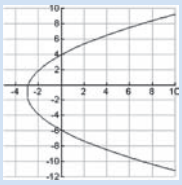
37. vertex: $(-5, 0)$



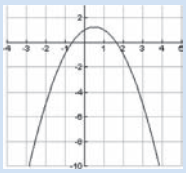
39. vertex: (0, 2)



41. vertex: $(-3, -1)$



43. vertex: $\left(\frac{1}{2}, \frac{5}{4}\right)$



45. (0, 2), receiver placed 2 ft from vertex

47. opens up: $y = \frac{1}{8}x^2$, for any x in $[-2.5, 2.5]$
opens right: $x = \frac{1}{8}y^2$, for any y in $[-2.5, 2.5]$

49. $x^2 = 4(40)y = 160y$

51. yes, opening height 18.75 ft, mast 17 ft

53. 374.25 ft, $x^2 = 1497y$

55. 55 pulses per min

57. The maximum profit of $400,000 is achieved when 3000 units are produced.

59. If the vertex is at the origin and the focus is at (3, 0), then the parabola must open to the right. So, the general equation is $y^2 = 4px$, for some $p > 0$.

61. true

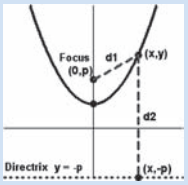
63. false

65. $x = h - 1$

67. $\left(6, \frac{13}{2}\right)$

69. Equate d_1 and d_2 and simplify:

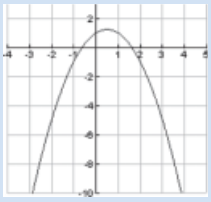
$$\sqrt{(x-0)^2 + (y-p)^2} = |y + p|$$
$$x^2 = 4py$$



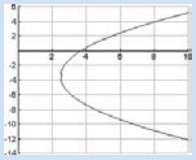
71. $(y - 6)^2 = 8(x - 2)$, $(y - 6)^2 = -4(x - 6)$

73. $(2, 3)$, $(-2, 3)$

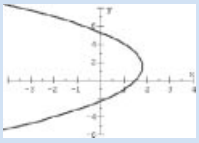
75.



77. vertex: $(2.5, -3.5)$
opens right



79. vertex: (1.8, 1.5)
opens left



81. $(2, 3)$, $(-2, 3)$

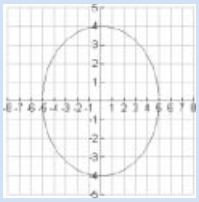
83. $(1, 7)$, $(10, 10)$

Section 9.3

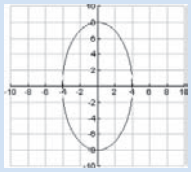
1. d

3. a

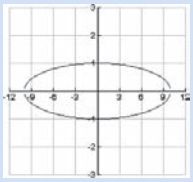
5. center: (0, 0)
vertices: $(\pm 5, 0)$, $(0, \pm 4)$



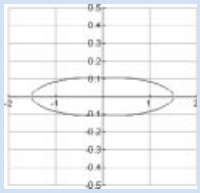
7. center: (0, 0)
vertices: $(\pm 4, 0)$, $(0, \pm 8)$



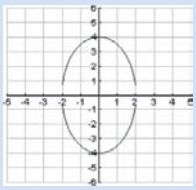
9. center: (0, 0)
vertices: $(\pm 10, 0), (0, \pm 1)$



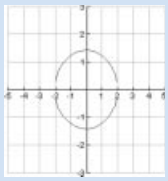
11. center: (0, 0)
vertices: $\left(\pm\frac{3}{2}, 0\right), \left(0, \pm\frac{1}{9}\right)$



13. center: (0, 0)
vertices: $(\pm 2, 0), (0, \pm 4)$



15. center: (0, 0)
vertices: $(\pm 2, 0), \left(0, \pm\sqrt{2}\right)$



17. $\frac{x^2}{36} + \frac{y^2}{20} = 1$

19. $\frac{x^2}{7} + \frac{y^2}{16} = 1$

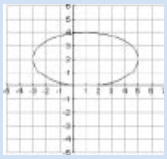
21. $\frac{x^2}{4} + \frac{y^2}{16} = 1$

23. $\frac{x^2}{9} + \frac{y^2}{49} = 1$

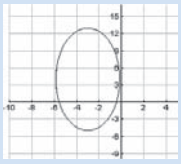
25. c

27. b

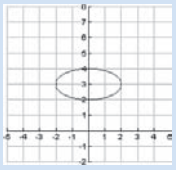
29. center: (1, 2)
vertices: $(-3, 2), (5, 2), (1, 0), (1, 4)$



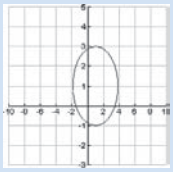
31. center: $(-3, 4)$
vertices: $\left(-2\sqrt{2} - 3, 4\right), \left(2\sqrt{2} - 3, 4\right),$
$\left(-3, 4 + 4\sqrt{5}\right), \left(-3, 4 - 4\sqrt{5}\right)$



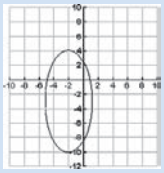
33. center: (0, 3)
vertices: $(-2, 3), (2, 3), (0, 3), (0, 4)$



35. center: (1, 1)
vertices: $\left(1 \pm 2\sqrt{2}, 1\right), (1, 3), (1, -1)$



37. center: $(-2, -3)$
vertices: $\left(-2 \pm \sqrt{10}, -3\right), \left(-2, -3 \pm 5\sqrt{2}\right)$



39. $\frac{(x-2)^2}{25} + \frac{(y-5)^2}{9} = 1$

41. $\frac{(x-4)^2}{7} + \frac{(y+4)^2}{16} = 1$

43. $\frac{(x-3)^2}{4} + \frac{(y-2)^2}{16} = 1$

45. $\frac{(x+1)^2}{9} + \frac{(y+4)^2}{25} = 1$

47. $\frac{x^2}{225} + \frac{y^2}{5625} = 1$

49. **a.** $\frac{x^2}{5625} + \frac{y^2}{400} = 1$

b. width at end of field is 24 yd; no, since width of football field is 30 yd wide

51. $\frac{x^2}{5{,}914{,}000{,}000^2} + \frac{y^2}{5{,}729{,}000{,}000^2} = 1$

53. $\frac{x^2}{150{,}000{,}000^2} + \frac{y^2}{146{,}000{,}000^2} = 1$

55. straight line

57. 2 mg/cm^3

59. **a.** $\frac{x^2}{64} + \frac{y^2}{25} = 1$ **b.** 42 in. **c.** 1509 steps

61. It should be $a^2 = 6$, $b^2 = 4$, so that $a = \pm\sqrt{6}$, $b = \pm 2$.

63. false

65. true

67. three ellipses

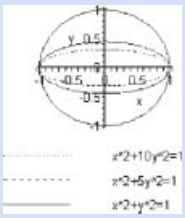
69. one point

71. Pluto: $e \cong 0.25$ Earth: $e \cong 0.02$

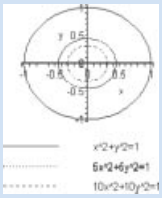
73. $x^2 + 3y^2 = 28$

75. $8x^2 + 9y^2 - 32x + 54y + 41 = 0$

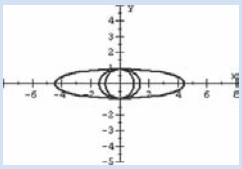
77. as c increases, ellipse more elongated



79. as c increases, circle gets smaller



81. as c decreases, the major axis becomes longer



83. maximum: -3, minimum: -7

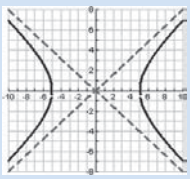
85. maximum: 1, minimum: -17

Section 9.4

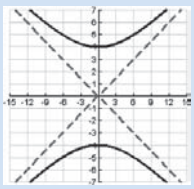
1. b

3. d

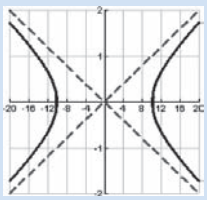
5.



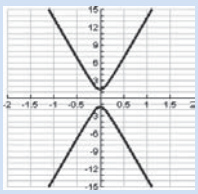
7.



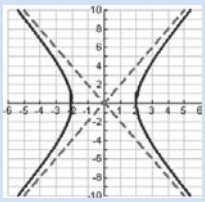
9.



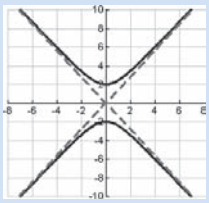
11.



13.



15.



17. $\frac{x^2}{16} - \frac{y^2}{20} = 1$

19. $\frac{y^2}{9} - \frac{x^2}{7} = 1$

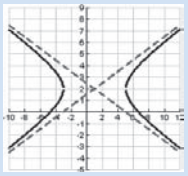
21. $x^2 - y^2 = a^2$

23. $\frac{y^2}{4} - x^2 = b^2$

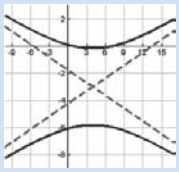
25. c

27. b

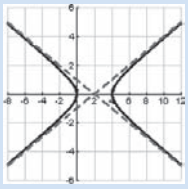
29.



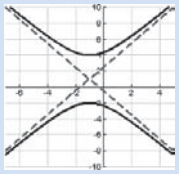
31.



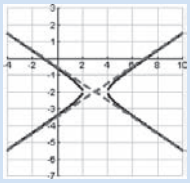
33.



35.



37.



39. $\frac{(x-2)^2}{16} - \frac{(y-5)^2}{9} = 1$

41. $\frac{(y+4)^2}{9} - \frac{(x-4)^2}{7} = 1$

43. Ship will come ashore between the two stations 28.5 mi from one and 121.5 mi from the other.

45. 0.000484 sec

47. $y^2 - \frac{4}{5}x^2 = 1$

49. 275 ft

51. (76, 50)

53. (109.4, 60)

55. The transverse axis should be vertical. The points are (3, 0), $(-3, 0)$ and the vertices are (0, 2), $(0, -2)$.

57. false

59. true

61. $(p, -q)$, $(-p, q)$ and $(-p, -q)$

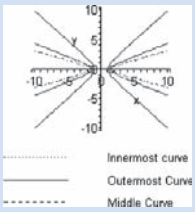
63. $r > q$

65. $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$, which is equivalent to $x^2 - y^2 = a^2$

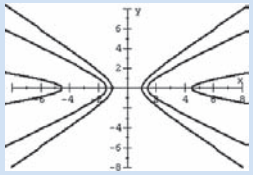
67. $y = -\frac{4}{3}x + \frac{2}{3}$, $y = \frac{4}{3}x + \frac{10}{3}$

69. $(\pm\sqrt{34}, 0)$

71. As c increases, the graphs become more squeezed down toward the x-axis.



73. As c decreases, the vertices are located at $\left(\pm\frac{1}{c}, 0\right)$ are moving away from the origin.



77. $\dfrac{-(2x + h)}{\sqrt{1 + (x + h)^2} + \sqrt{1 + x^2}}$

Section 9.5

1. (2, 6), (−1, 3) **3.** (1, 0)

5. no solution **7.** (0, 1)

9. (0.63, −1.61), (−0.63, −1.61)

11. no solution **13.** (1, 1)

15. $\left(2\sqrt{2}, \sqrt{2}\right), \left(-2\sqrt{2}, -\sqrt{2}\right),$ $\left(\sqrt{2}, 2\sqrt{2}\right), \left(-\sqrt{2}, -2\sqrt{2}\right)$

17. (−6, 33), (2, 1) **19.** (3, 4) (−2, −1)

21. (0, −3), $\left(\frac{2}{5}, -\frac{11}{5}\right)$ **23.** (−1, −1), $\left(\frac{1}{4}, \frac{3}{2}\right)$

25. (−1, −4), (4, 1) **27.** (1, 3), (−1, −3)

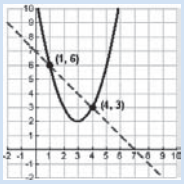
29. (−4, −1), (4, 1)

31. (−2, −1), (−2, 1), (2, −1), (2, 1)

33. (2, 4) **35.** $\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{2}, -\frac{1}{3}\right)$

37. no solution **39.** no solution

41.



43. 8 cm × 10 cm **45.** 3 and 7

47. 8 and 9, −8 and −9 **49.** 8 cm × 10 cm

51. 400 ft × 500 ft or $\frac{1000}{3}$ ft × 600 ft

53. professor: 2 m/sec, Jeremy: 10 m/sec

55. 60 mph **57.** 4763

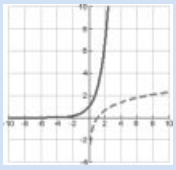
59. In general, $y^2 - y \neq 0$. Must solve this system using substitution.

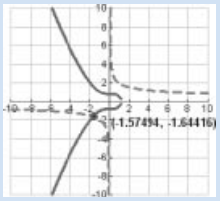
61. false **63.** false **65.** $2n$

67. Consider $\begin{cases} y = x^2 + 1 \\ y = 1 \end{cases}$. Any system in which the linear equation is the tangent line to the parabola at its vertex will have only one solution.

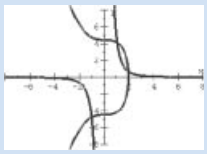
69. (1, 2), (−1, 2), (1, −2), (−1, −2) **71.** no solution

73. no solution **75.** (−1.57, −1.64)





77. (1.067, 4.119), (1.986, 0.638), (−1.017, −4.757)

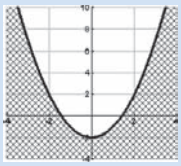


79. $y = \sqrt{\frac{8 - x^2}{4}}$ **81.** $y = \dfrac{3}{x^{3/2}}$

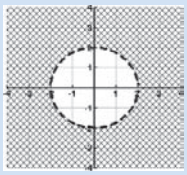
Section 9.6

1. b **3.** j **5.** h **7.** c **9.** d **11.** k

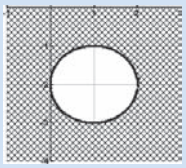
13.



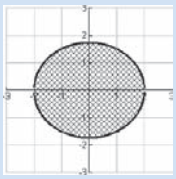
15.



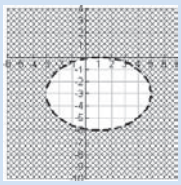
17.



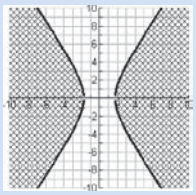
19.



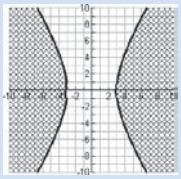
21.



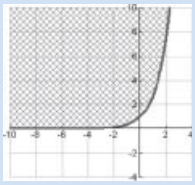
23.



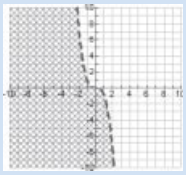
25.



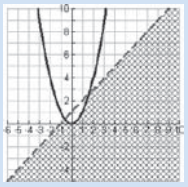
27.



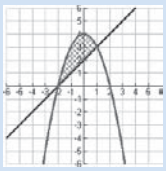
29.



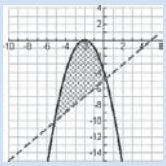
31.



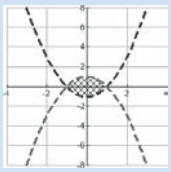
33.



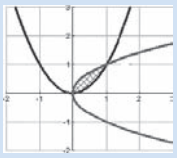
35.



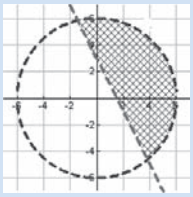
37.



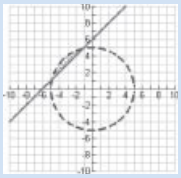
39.



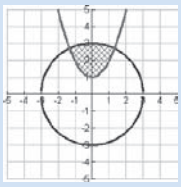
41.



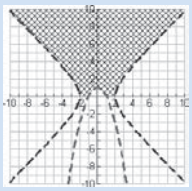
43.



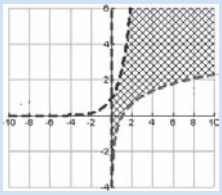
45.



47.



49.



51. $\frac{9}{2}\pi$ units^2 **53.** 2π units^2 **55.** 90 units^2

57. There is no common region here—it is empty, as is seen in the graph below:

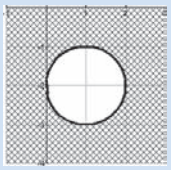
59. false **61.** true

63. true **65.** false

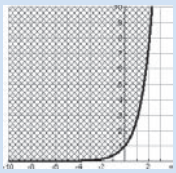
67. $0 \le a \le b$ **69.** $a = 36$

71. $h = 0$

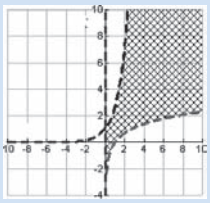
73.



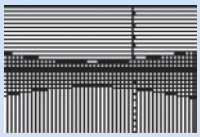
75.



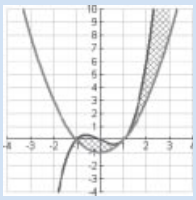
77.



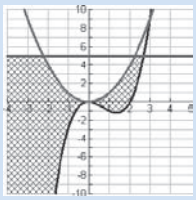
79.



81.



83.

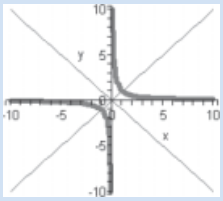


Section 9.7

1. $(3\sqrt{2}, \sqrt{2})$ **3.** $\left(-\frac{3\sqrt{3}}{2} + 1, \frac{3}{2} + \sqrt{3}\right)$

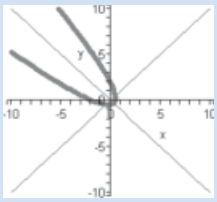
5. $\left(-\frac{1 + 3\sqrt{3}}{2}, \frac{\sqrt{3} - 3}{2}\right)$ **7.** $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

9. **a.** hyperbola **b.** $\frac{X^2}{2} - \frac{Y^2}{2} = 1$
c.



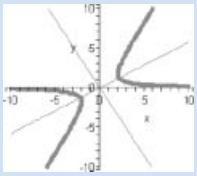
11. a. parabola **b.** $2X^2 - 2Y - 1 = 0$

c.



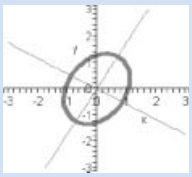
13. a. hyperbola **b.** $\frac{X^2}{6} - \frac{Y^2}{2} = 1$

c.



15. a. ellipse **b.** $\frac{X^2}{2} + \frac{Y^2}{1} = 1$

c.

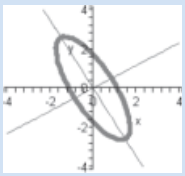


17. a. parabola **b.** $2X^2 - 2Y - 1 = 0$

c.

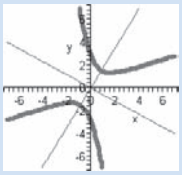
19. a. ellipse **b.** $\frac{X^2}{1} + \frac{Y^2}{9} = 1$

c.



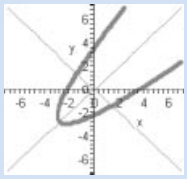
21. a. hyperbola **b.** $\frac{X^2}{3} - \frac{Y^2}{2} = 1$

c.



23. a. parabola **b.** $Y^2 - X - 4 = 0$

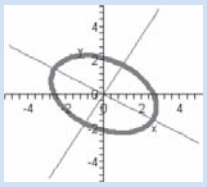
c.



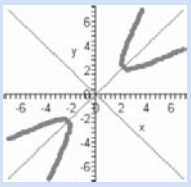
25. 45° **27.** 60° **29.** 30° **31.** 45°

33. 15° **35.** 40.3° **37.** 50.7°

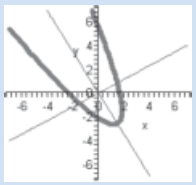
39.



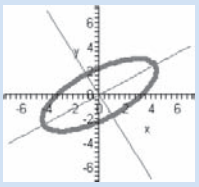
41.



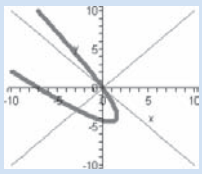
43.



45.



47.



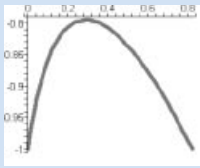
49. true **51.** true

53. a. $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ **b.** The original equation

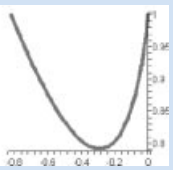
55. $a < 0$: hyperbola; $a = 0$: parabola; $a > 0$: ellipse; $a = 1$: circle

57.

a.



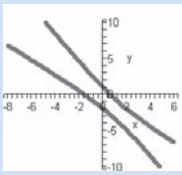
b.



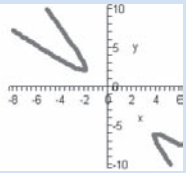
rotation

59.

a.

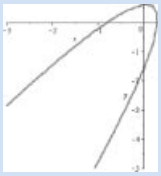


b.

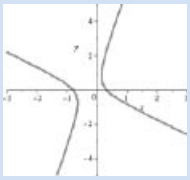


vertices are separating

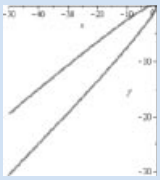
61. **a.** $-26.57°$



b. $19.33°$



c. $26.57°$



63. $(-\sqrt{2}, -2\sqrt{2}), (\sqrt{2}, 2\sqrt{2})$

65. $(-1, 0), (1, 0), (3, 1), (-3, -1)$

Section 9.8

1. $r = \dfrac{5}{2 - \sin\theta}$ **3.** $r = \dfrac{8}{1 + 2\sin\theta}$ **5.** $r = \dfrac{1}{1 + \cos\theta}$

7. $r = \dfrac{6}{4 + 3\cos\theta}$ **9.** $r = \dfrac{12}{3 - 4\cos\theta}$ **11.** $r = \dfrac{3}{1 - \sin\theta}$

13. $r = \dfrac{18}{5 + 3\sin\theta}$ **15.** parabola **17.** ellipse

19. hyperbola **21.** ellipse **23.** parabola

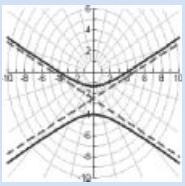
25. hyperbola

27. **a.** parabola **b.** $e = 1$, $(0, 1)$

c.

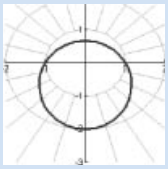
29. **a.** hyperbola **b.** $e = 2$, $(0, -4)$, $\left(0, -\frac{4}{3}\right)$

c.



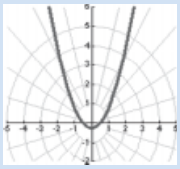
31. **a.** ellipse **b.** $e = \frac{1}{2}$, $\left(0, \frac{1}{2}\right)$, $(0, -2)$

c.



33. **a.** parabola **b.** $e = 1$, $\left(0, -\frac{1}{4}\right)$

c.

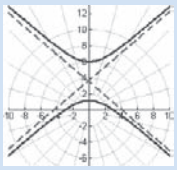


35. **a.** ellipse **b.** $e = \frac{1}{3}$, $(1, 0)$, $(-2, 0)$

c.

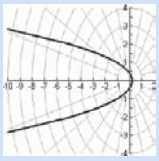
37. **a.** hyperbola **b.** $e = \frac{3}{2}$, $\left(0, \frac{6}{5}\right)$, $(0, 6)$

c.



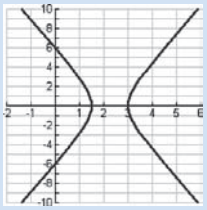
39. **a.** parabola **b.** $e = 1$, $\left(0, \frac{1}{5}\right)$

c.



41. **a.** hyperbola **b.** $e = \frac{3}{5}$, $\left(\frac{15}{8}, \frac{\pi}{2}\right)$, $\left(\frac{15}{2}, \frac{3\pi}{2}\right)$

c.



43. 0.248, $r = \dfrac{5{,}913{,}500{,}000\left(1 - 0.248^2\right)}{1 - 0.248\cos\theta}$

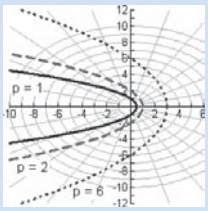
45. $r = \dfrac{150{,}000{,}000\left(1 - 0.223^2\right)}{1 - 0.223\cos\theta}$

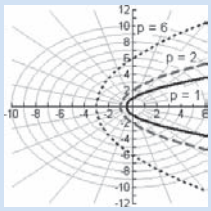
47. **a.** $(-0.0167, 0)$ **b.** $(0.0167, \pi)$ **49.** $(0, -15.406)$

51. $e \to 1$: elongated (more elliptical), $e \to 0$: elliptical (circular)

55. $\frac{2ep}{1 - e^2}$ **57.** $\left(-\dfrac{\frac{ep}{1+e} - \frac{ep}{1-e}}{2}, \pi\right)$

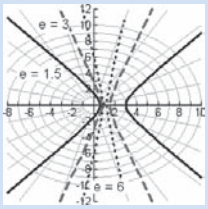
59. $r = \dfrac{p}{1 + \cos\theta}$ $r = \dfrac{p}{1 - \cos\theta}$

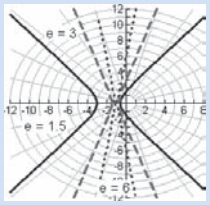




As $p \to \infty$, the graphs get wider, open in opposite directions.

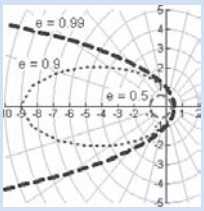
61. $r = \dfrac{e}{1 + e\cos\theta}$ $r = \dfrac{e}{1 - e\cos\theta}$

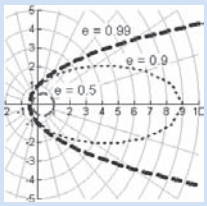




As $e \to \infty$, the graphs get wider, latter family of curves is shifted to the left.

63. $r = \dfrac{e}{1 + e\cos\theta}$ $r = \dfrac{e}{1 - e\cos\theta}$



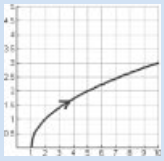


As $e \to 1$, graphs get larger, and centers move accordingly, centers for the latter family of curves move to the right, while the centers for the first family move left.

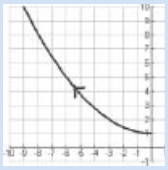
69. $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ **71.** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Section 9.9

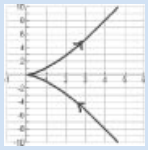
1.



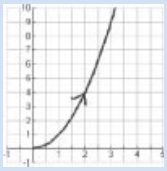
3.



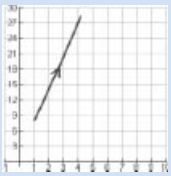
5.



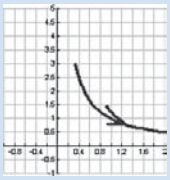
7.



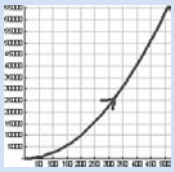
9.



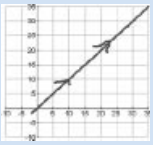
11.



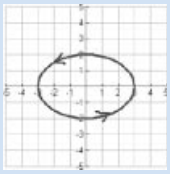
13.



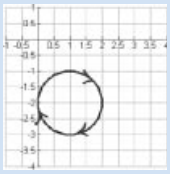
15.



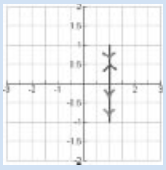
17.



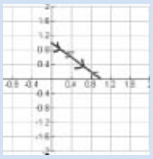
19.



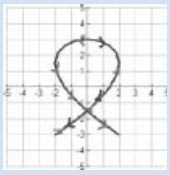
21.



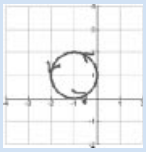
23.



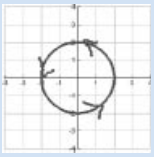
25.



27.



29.



31. $y = \frac{1}{x^2}$ **33.** $y = x - 2$ **35.** $y = \sqrt{x^2 + 1}$

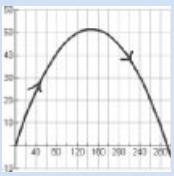
37. $x + y = 2$ **39.** $x + 4y = 8$ **41.** 17.7 sec

43. yes

45. distance: 13,261 ft; max height: 5742 ft

47. 125 sec

49.



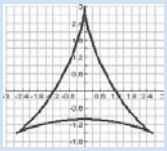
51. $t = 0$: $(A + B, 0)$, $t = \frac{\pi}{2}$: $(0, A + B)$, $t = \pi$: $(-A - B, 0)$, $t = \frac{3\pi}{2}$: $(0, -A - B)$, $t = 2\pi$: $(A + B, 0)$

53. The original domain must be $t \geq 0$. Only the portion of the parabola where $y \geq 0$ is part of the plane curve.

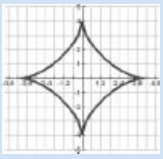
55. false **57.** quarter circle in QI

61. $y = \frac{a-b}{a+b}x + \frac{2ab}{a+b}$ **63.** $y = bx^{1/a}$

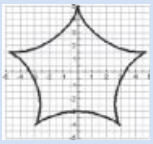
65. $a = 2$



$a = 3$

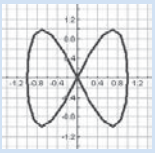


$a = 4$

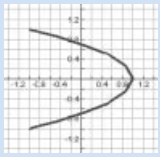


As a increases, distance between vertices and origin gets larger, and number of distinct vertices increases.

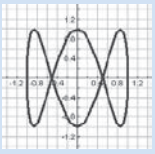
67. $a = 2, b = 4$



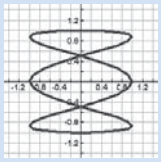
$a = 4, b = 2$



$a = 1, b = 3$

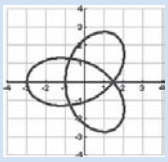


$a = 3, b = 1$

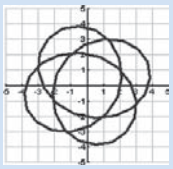


2π units of time

69. $a = 2$



$a = 3$



2π units of time

71. $(-3, 8), (2, 3)$ **73.** $(-100, -96), (1300, -1664)$

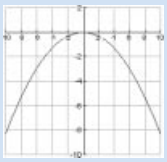
Review Exercises

1. false **3.** true

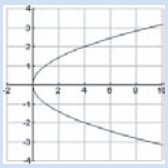
5. $y^2 = 12x$ **7.** $y^2 = -20x$

9. $(x - 2)^2 = 8(y - 3)$ **11.** $(x - 1)^2 = -4(y - 6)$

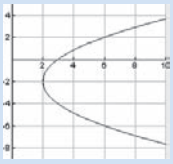
13. F: $(0, -3)$ V: $(0, 0)$, D: $y = 3$, LR: 12



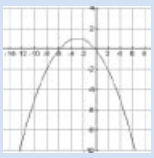
15. F: $(\frac{1}{4}, 0)$, V: $(0, 0)$, D: $x = -\frac{1}{4}$, LR: 1



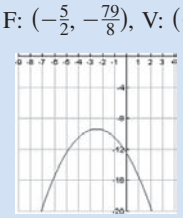
17. F: $(3, -2)$, V: $(0, 0)$, D: $x = 1$, LR: 4



19. F: $(-3, -1)$, V: $(-3, 1)$, D: $y = 3$, LR: 8

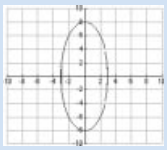


21. F: $(-\frac{5}{2}, -\frac{79}{8})$, V: $(-\frac{5}{2}, -\frac{75}{8})$, D: $y = -\frac{71}{8}$, LR: 2

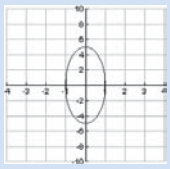


23. 3.125 ft from center

25.



27.

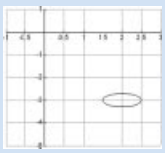


29. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

31. $\frac{x^2}{9} + \frac{y^2}{64} = 1$

33.

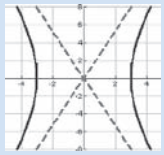
35.



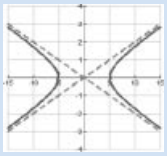
37. $\frac{(x-3)^2}{25} + \frac{(y-3)^2}{9} = 1$

39. $\frac{(x-(3.74\times 10^7))^2}{6.058\times 10^{17}} + \frac{(y-0)^2}{6.044\times 10^{17}} = 1$

41.



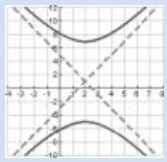
43.



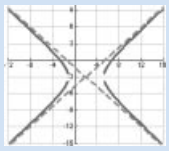
45. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

47. $\frac{y^2}{9} - x^2 = 1$

49.



51.



53. $\frac{(x-4)^2}{16} - \frac{(y-3)^2}{9} = 1$

55. between the stations: 65.36 mi from one, 154.64 mi from the other

57. $(-2, -7), (1, -4)$

59. $(1, 2), (-1, 2)$

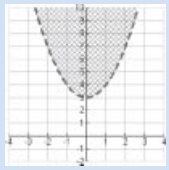
61. no solution

63. no solution

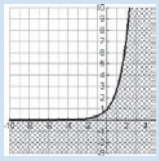
65. $(2, 3), (-3, -2)$

67. $\left(\frac{1}{2}, \frac{1}{\sqrt{7}}\right), \left(-\frac{1}{2}, \frac{1}{\sqrt{7}}\right), \left(\frac{1}{2}, -\frac{1}{\sqrt{7}}\right), \left(-\frac{1}{2}, -\frac{1}{\sqrt{7}}\right)$

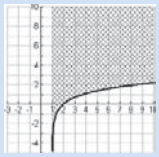
69.



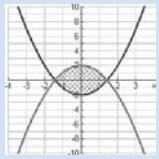
71.



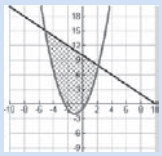
73.



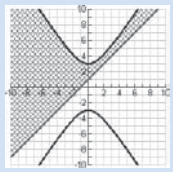
75.



77.

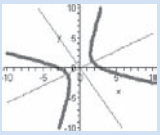


79.



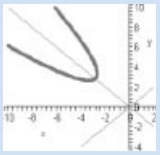
81. $\left(-\frac{3}{2} + \sqrt{3}, \frac{3\sqrt{3}}{2} + 1\right)$

83. $\frac{x^2}{4} - \frac{y^2}{4} = 1$



85. 60°

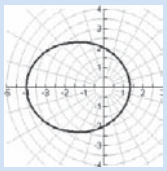
87.



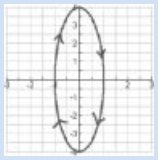
89. $r = \dfrac{21}{7 - 3\sin\theta}$

91. hyperbola

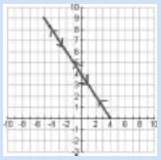
93. $e = \frac{1}{2}, \left(\frac{4}{3}, 0\right), (-4, 0)$



95.



97.

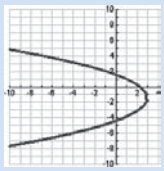


99. $x = 4 - y^2$

101. $y = 2x + 4$

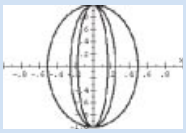
103. $(0.6, -1.2)$, down

105. **a.** $y = -1.4 \pm \sqrt{-3x + 8.81}$

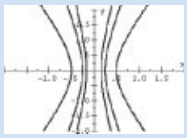


b. $(y - (-1.4))^2 = 4\left(-\frac{3}{4}\right)(x - 2.937)$, $(2.937, -1.4)$, left
c. yes

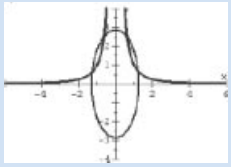
107. increases minor axis (x-axis)



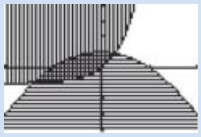
109. move toward origin



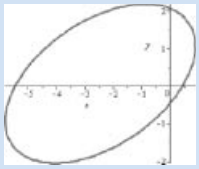
111. $(0.635, 2.480)$, $(-0.635, 2.480)$, $(-1.245, 0.645)$, $(1.245, 0.645)$



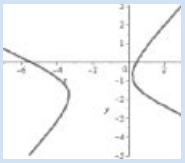
113.



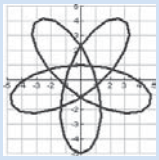
115. **a.** 1.2 rad



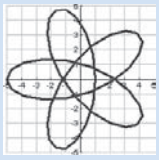
b. 0.2 rad



119. $a = 2, b = 3$



$a = 3, b = 2$



2π units of time

Practice Test

1. c **3.** d **5.** f

7. $y^2 = -16x$

9. $(x + 1)^2 = -12(y - 5)$

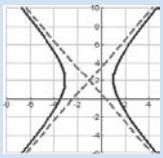
11. $\frac{x^2}{7} + \frac{y^2}{16} = 1$

13. $\frac{(x - 2)^2}{20} + \frac{y^2}{36} = 1$

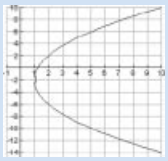
15. $x^2 - \frac{y^2}{4} = 1$

17. $\frac{y^2}{16} - \frac{(x - 2)^2}{20} = 1$

19.

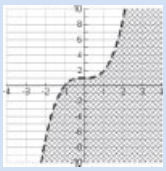


21.

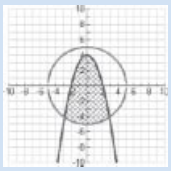


23. $x^2 = 6y$

25.



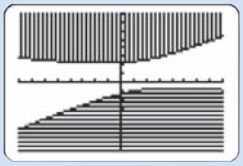
27.



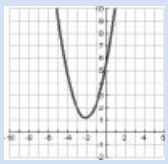
29. ellipse, $e = \frac{2}{3}$

31. 5.3 sec, 450 ft

33.



35. **a.** $y = x^2 + 4.2x + 5.61$



b. $(x - (-1.2))^2 = 4\left(\frac{1}{4}\right)(y - 1.2)$, $(-1.2, 1.2)$, up

c. yes

Cumulative Test

1. $-6, 3$

3. -7

5. $y = \frac{1}{3}(x - 7)^2 + 7$

7. 4

9. 1.9626

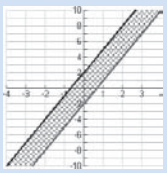
11. $\frac{1}{\cos\theta} + \sin\theta$

13. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

15. $\langle -5.13, 14.10 \rangle$

17. soda: \$1.29; soft pretzel: \$1.45

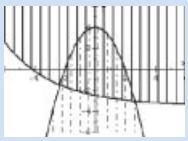
19.



21. $\begin{bmatrix} 7 & -16 & 33 \\ 18 & -3 & -17 \end{bmatrix}$ **23.** $\frac{(x-6)^2}{9} + \frac{(y+2)^2}{25} = 1$

25. (2, 4), (4, 2)

27.



CHAPTER 10

Section 10.1

1. 1, 2, 3, 4 **3.** 1, 3, 5, 7 **5.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

7. $2, 2, \frac{4}{3}, \frac{2}{3}$ **9.** $-x^2, x^3, -x^4, x^5$

11. $-\frac{1}{6}, \frac{1}{12}, -\frac{1}{20}, \frac{1}{30}$ **13.** $\frac{1}{512}$ **15.** $-\frac{1}{420}$

17. $\frac{10201}{10000} = 1.0201$ **19.** 23 **21.** $a_n = 2n$

23. $a_n = \dfrac{1}{(n+1)n}$ **25.** $a_n = \dfrac{(-1)^n 2^n}{3^n}$ **27.** $a_n = (-1)^{n+1}$

29. 72 **31.** 812 **33.** $\frac{1}{5852}$

35. 83, 156, 160 **37.** $\dfrac{1}{(n+1)n}$ **39.** $(2n+3)(2n+2)$

41. 7, 10, 13, 16 **43.** 1, 2, 6, 24 **45.** $100, 50, \frac{25}{3}, \frac{25}{72}$

47. 1, 2, 2, 4 **49.** $1, -1, -2, 5$ **51.** 10

53. 30 **55.** 36 **57.** 5

59. $1 - x + x^2 - x^3$ **61.** $\frac{109}{15}$

63. $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}$ **65.** $\frac{20}{9}$

67. not possible **69.** $\displaystyle\sum_{n=0}^{6} (-1)^n \frac{1}{2^n}$

71. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n$ **73.** $\displaystyle\sum_{n=1}^{6} \frac{(n+1)!}{(n-1)!} = \sum_{n=1}^{5} n(n+1)$

75. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$

77. \$28,640.89; total balance in account after 6 yr (or 72 mo)

79. $s_n = 20 + 2n$; a paralegal with 20 years experience would make \$60 per hour.

81. $a_n = 1.03a_{n-1}$; $a_0 = 30{,}000$

83. $a_{n+1} = 1000 - 75a_n$; approximately 10.7 yr

85. $A_1 = 100, A_2 = 200.10, A_3 = 300.30,$
$A_4 = 400.60, A_{36} = 3663.72$

87. 7; 7.38906 **89.** 0.0953103; 0.0953102

91. The mistake is that $6! \neq 3!2!$, but rather $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

93. $(-1)^{n+1} = \begin{cases} 1, & n = 1, 3, 5, \ldots \\ -1, & n = 2, 4, 6, \ldots \end{cases}$ So, the terms should all be the opposite sign.

95. true **97.** false **99.** false

101. $C, C + D, C + 2D, C + 3D$

103. 1 and 1 **105.** ≈ 2.705; ≈ 2.717; ≈ 2.718;

107. $\frac{109}{15}$ **109.** monotonic, increasing

111. not monotonic

Section 10.2

1. arithmetic, $d = 3$ **3.** not arithmetic

5. arithmetic, $d = -0.03$

7. arithmetic, $d = \frac{2}{3}$ **9.** not arithmetic

11. $3, 1, -1, -3$; arithmetic; $d = -2$

13. 1, 4, 9, 16; not arithmetic

15. 2, 7, 12, 17; arithmetic; $d = 5$

17. 0, 10, 20, 30; arithmetic; $d = 10$

19. $-1, 2, -3, 4$; not arithmetic

21. $a_n = 11 + (n-1)5 = 5n + 6$

23. $a_n = -4 + (n-1)(2) = -6 + 2n$

25. $a_n = 0 + (n-1)\frac{2}{3} = \frac{2}{3}n - \frac{2}{3}$

27. $a_n = 0 + (n-1)e = en - e$ **29.** 124

31. -684 **33.** $\frac{16}{3}$

35. $a_5 = 44, a_{17} = 152; a_n = 8 + (n-1)9 = 9n - 1$

37. $a_7 = -1, a_{17} = -41; a_n = 23 + (n-1)(-4) = -4n + 27$

39. $a_4 = 3, a_{22} = 15; a_n = 1 + (n-1)\frac{2}{3} = \frac{2}{3}n + \frac{1}{3}$

41. 552 **43.** -780 **45.** 51

47. 416 **49.** 3875

51. $\frac{21}{2}\left[\frac{1}{6} - \frac{13}{2}\right] = \frac{21}{2}\left(\frac{1-39}{6}\right) = -\frac{133}{2}$ **53.** 630

55. $S_{43} = \frac{43}{4}(5 - 43) = -\frac{817}{2}$ **57.** 1368

59. Colin: \$347,500; Camden: \$340,000

61. 850 seats

63. 1101 glasses on the bottom row, each row had 20 fewer glasses than the one before.

65. 1600 ft **67.** 210 oranges

69. **a.** 23 seats in the first row **b.** 1125 seats

71. $a_n = a_1 + (n-1)d$, not $a_1 + nd$

73. There are 11 terms, not 10. So, $n = 11$, and thus, $S_{11} = \frac{11}{2}(1 + 21) = 121.$

75. false **77.** true **79.** $\frac{(n+1)[2a+bn]}{2}$

81. 27,420 **83.** 5050 **85.** 2500

87. 18,850 **89.** 1010 **91.** 1.204

Section 10.3

1. yes, $r = 3$ **3.** no

5. yes, $r = \frac{1}{2}$ **7.** yes, $r = 1.7$

9. 6, 18, 54, 162, 486 **11.** 1, −4, 16, −64, 256

13. 10,000, 10,600, 11,236, 11,910.16, 12,624.77

15. $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}$ **17.** $a_n = 5(2)^{n-1}$

19. $a_n = 1(-3)^{n-1}$ **21.** $a_n = 1000(1.07)^{n-1}$

23. $a_n = \frac{16}{3}\left(-\frac{1}{4}\right)^{n-1}$ **25.** $a_7 = -128$

27. $a_{13} = \frac{4096}{3}$ **29.** $a_{15} = 6.10 \times 10^{-16}$

31. $\frac{8191}{3}$ **33.** 59,048

35. $2.\overline{2}$ **37.** 6560

39. 16,383 **41.** 2

43. $-\frac{1}{4}$ **45.** not possible, diverges

47. $-\frac{27}{2}$ **49.** 10,526

51. $\frac{2}{3}$ **53.** 100

55. \$44,610.95

57. $a_n = 2000(0.5)^n$; $a_4 = 125$, $a_7 = 16$

59. 17 ft **61.** 58,640 students

63. 66 days; \$9618 **65.** \$3877.64

67. 26 weeks: \$13,196.88 52 weeks: \$26,811.75

69. \$501,509 **71.** $\frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

73. Should be $r = -\frac{1}{3}$.

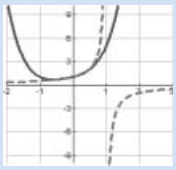
75. Should use $r = -3$ all the way through the calculation. Also, $a_1 - 12$ (not 4).

77. false **79.** true

81. $|b|<1, \frac{a}{1-b}$ **83.** $\frac{47}{99}$

85. −37,529,996,894,754

87. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$



89. converges, 3 **91.** converges, $\frac{1}{2}$

Section 10.4

25. 7 steps **27.** 31 steps

31. false **39.** $\frac{255}{256}$, yes

Section 10.5

1. 35 **3.** 45 **5.** 1 **7.** 1 **9.** 17,296

11. $x^4 + 8x^3 + 24x^2 + 32x + 16$

13. $y^5 - 15y^4 + 90y^3 - 270y^2 + 405y - 243$

15. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

17. $x^3 + 9x^2y + 27xy^2 + 27y^3$

19. $125x^3 - 150x^2 + 60x - 8$

21. $\frac{1}{x^4} + 20\frac{y}{x^3} + 150\frac{y^2}{x^2} + 500\frac{y^3}{x} + 625y^4$

23. $x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8$

25. $a^5x^5 + 5a^4bx^4y + 10a^3b^2x^3y^2 + 10a^2b^2x^2y^3 + 5ab^4xy^4 + b^5y^5$

27. $x^3 + 12x^{5/2} + 60x^2 + 160x^{3/2} + 240x + 192x^{1/2} + 64$

29. $a^3 + 4a^{9/4}b^{1/4} + 6a^{3/2}b^{1/2} + 4a^{3/4}b^{3/4} + b$

31. $x + 8x^{3/4}y^{1/2} + 24x^{1/2}y + 32x^{1/4}y^{3/2} + 16y^2$

33. $r^4 - 4r^3s + 6r^2s^2 - 4rs^3 + s^4$

35. $a^6x^6 + 6a^5bx^5y + 15a^4b^2x^4y^2 + 20a^3b^3x^3y^3 + 15a^2b^4x^2y^4 + 6ab^5xy^5 + b^6y^6$

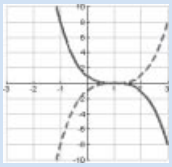
37. 3360 **39.** 5670 **41.** 22,680

43. 70 **45.** 3,838,380 **47.** 2,598,960

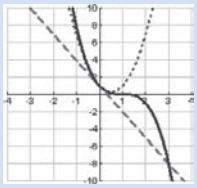
49. $\binom{7}{5} \neq \frac{7!}{5!}$, but rather $\frac{7!}{5!2!} = \frac{7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}(2 \cdot 1)} = 21.$

51. false **53.** true **55.** true

61. $1 - 3x + 3x^2 - x^3$



63. graphs of the respective functions get closer to the graph of $y_4 = (1 - x)^3$ when $1 < x < 2$, when $x > 1$, no longer true



65. graph of the curve better approximation to the graph of $y = \left(1 + \frac{1}{x}\right)^3$, for $1 < x < 2$; no, does not get closer to this graph if $0 < x < 1$

67. $\sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^{k-1}$ **69.** $\sum_{k=0}^{n-1} \frac{n!}{(n-1-k)!k!} x^{n-1-k}$

Review Exercises

1. 1, 8, 27, 64 **3.** 5, 8, 11, 14

5. $a_5 = \frac{32}{243} = 0.13$ **7.** $a_{15} = -\frac{1}{3600}$

9. $a_n = (-1)^{n+1}3n$ **11.** $a_n = (-1)^n$

13. 56 **15.** $\frac{1}{n+1}$

17. 5, 3, 1, −1 **19.** 1, 2, 4, 32

21. 15 **23.** 69 **25.** $\sum_{n=1}^{7} \frac{(-1)^n}{2^{n-1}}$ **27.** $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

29. \$36,639.90; amount in the account after 5 yr

31. arithmetic, $d = -2$

33. arithmetic, $d = \frac{1}{2}$

35. arithmetic, $d = 1$

37. $a_n = -4 + (n - 1)(5) = 5n - 9$

39. $a_n = 1 + (n - 1)\left(-\frac{2}{3}\right) = -\frac{2}{3}n + \frac{5}{3}$

41. $a_1 = 5, d = 2, a_n = 5 + (n - 1)(2) = 2n + 3$

43. $a_1 = 10, d = 6, a_n = 10 + (n - 1)6 = 6n + 4$

45. 630 **47.** 420

49. Bob: \$885,000 Tania: \$990,000

51. geometric, $r = -2$ **53.** geometric, $r = \frac{1}{2}$

55. 3, 6, 12, 24, 48 **57.** 100, −400, 1600, −6400, 25,600

59. $a_n = a_1 r^{n-1} = 7 \cdot 2^{n-1}$ **61.** $a_n = 1(-2)^{n-1}$

63. $a_{25} = 33{,}554{,}432$ **65.** $a_{12} = -2.048 \times 10^{-6}$

67. 4920.50 **69.** 16,400 **71.** 3

73. \$60,875.61 **79.** 165 **81.** 1

83. $x^4 - 20x^3 + 150x^2 - 500x + 625$

85. $8x^3 - 60x^2 + 150x - 125$

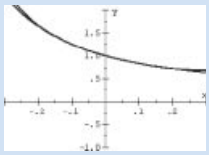
87. $x^{5/2} + 5x^2 + 10x^{3/2} + 10x + 5x^{1/2} + 1$

89. $r^5 - 5r^4s + 10r^3s^2 - 10r^2s^3 + 5rs^4 - s^5$

91. 112 **93.** 37,500 **95.** 22,957,480

97. $\frac{5369}{3600}$ **99.** $\frac{34{,}875}{14}$

101. $\frac{1}{1 + 2x}$



103. 99,900, yes

105. graphs become better approximations of the graph of $y = (1 + 2x)^4$ for $-0.1 < x < 0.1$; does not get closer for $0.1 < x < 1$

Practice Test

1. x^{n-1} **3.** $S_n = \frac{1 - x^n}{1 - x}$ **5.** $|x| < 1$

7. $1 - \left(\frac{1}{4}\right)^{10} \approx 1$ **9.** 24,950 **11.** 2520

13. 455 **15.** 20,254

17. $x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}$

21. $\frac{5}{3}$ **23.** $\sum_{n=1}^{\infty} \frac{5}{2(n+3)}$

25. $128 - 1344x + 6048x^2 - 15120x^3 + 22680x^4 - 20412x^5 + 10206x^6 - 2187x^7$

27. 184,756

Cumulative Test

1. $\frac{x^2 + hx + 2x + h}{(x + 1)(x + h + 1)}$

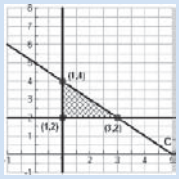
3. $f(x) = (x - 5)(x + 1)(x - i)(x + i)$

5. $x = -8$ **7.** $2x + h - 3$ **9.** (15, 6)

11. VA: $x = 3$ HA: $y = -5$ **13.** 2.585

15. no solution

17. $z(1, 4) = 24$



19. $\begin{bmatrix} 99 & 18 & -9 \\ 29 & 4 & 7 \end{bmatrix}$ **21.** $-16(x - 3) = (y - 5)^2$

23. 3 **25.** 35

Applications Index

Subject Index

D

E

H

I

S

T

Definitions, Rules, Formulas, and Graphs

Exponents and Radicals

$a^0 = 1,\ a \neq 0$

$a^{-x} = \dfrac{1}{a^x}$

$a^x a^y = a^{x+y}$

$\dfrac{a^x}{a^y} = a^{x-y}$

$(a^x)^y = a^{xy}$

$(ab)^x = a^x b^x$

$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

$\sqrt{a} = a^{1/2}$

$\sqrt[n]{a} = a^{1/n}$

$\sqrt[n]{a^m} = a^{m/n} = (\sqrt[n]{a})^m$

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

$\sqrt[n]{\left(\dfrac{a}{b}\right)} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Inequalities

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

If $a < b$ and $c > 0$, then $ca < cb$.

If $a < b$ and $c < 0$, then $ca > cb$.

Absolute Value

1. $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x > 0 \end{cases}$
2. If $|x| = c$, then $x = c$ or $x = -c$. $(c > 0)$
3. If $|x| < c$, then $-c < x < c$. $(c > 0)$
4. If $|x| > c$, then $x < -c$ or $x > c$. $(c > 0)$

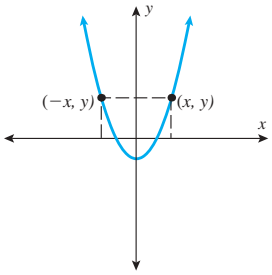
Special Factorizations

1. *Difference of two squares:*
   $A^2 - B^2 = (A + B)(A - B)$
2. *Perfect square trinomials:*
   $A^2 + 2AB + B^2 = (A + B)^2$
   $A^2 - 2AB + B^2 = (A - B)^2$
3. *Sum of two cubes:*
   $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$
4. *Difference of two cubes:*
   $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

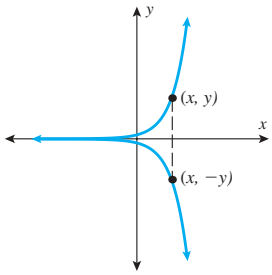
Properties of Logarithms

1. $\log_b(MN) = \log_b M + \log_b N$
2. $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$
3. $\log_b M^p = p\log_b M$
4. $\log_b M = \dfrac{\log_a M}{\log_a b} = \dfrac{\ln M}{\ln b} = \dfrac{\log M}{\log b}$
5. $\log_b b^x = x;\ \ln e^x = x$
6. $b^{\log_b x} = x;\ e^{\ln x} = x \quad x > 0$

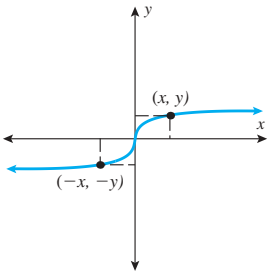
Symmetry



y-Axis Symmetry



x-Axis Symmetry



Origin Symmetry

Formulas/Equations

Distance Formula The distance from (x_1, y_1) to (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Midpoint Formula The midpoint of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Standard Equation of a Circle The standard equation of a circle of radius r with center at (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

Slope Formula The slope m of the line containing the points (x_1, y_1) and (x_2, y_2) is

$$\text{slope } (m) = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_1 \neq x_2)$$

m is undefined if $x_1 = x_2$

Slope-Intercept Equation of a Line The equation of a line with slope m and y-intercept $(0, b)$ is $y = mx + b$

Point-Slope Equation of a Line The equation of a line with slope m containing the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Quadratic Formula The solutions of the equation $ax^2 + bx + c = 0$, $a \neq 0$, are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac > 0$, there are two distinct real solutions.

If $b^2 - 4ac = 0$, there is a repeated real solution.

If $b^2 - 4ac < 0$, there are two complex solutions (complex conjugates).

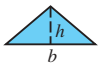
Geometry Formulas

Circle

r = Radius, A = Area, C = Circumference

$A = \pi r^2 \quad C = 2\pi r$

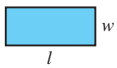
Triangle



b = Base, h = Height (Altitude), A = area

$A = \frac{1}{2}bh$

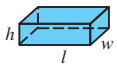
Rectangle



l = Length, w = Width, A = area, P = perimeter

$A = lw \quad P = 2l + 2w$

Rectangular Box



l = Length, w = Width, h = Height, V = Volume, S = Surface area

$V = lwh \quad S = 2lw + 2lh + 2wh$

Sphere

r = Radius, V = Volume, S = Surface area

$V = \frac{4}{3}\pi r^3 \quad S = 4\pi r^2$

Right Circular Cylinder

r = Radius, h = Height, V = Volume, S = Surface area

$V = \pi r^2 h \quad S = 2\pi r^2 + 2\pi rh$

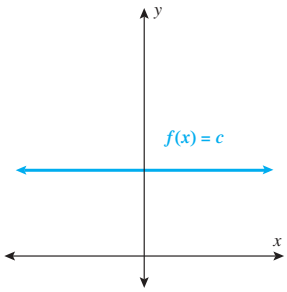
Conversion Table

1 centimeter ≈ 0.394 inch	1 joule ≈ 0.738 foot-pound	1 mile ≈ 1.609 kilometers
1 meter ≈ 39.370 inches	1 gram ≈ 0.035 ounce	1 gallon ≈ 3.785 liters
≈ 3.281 feet	1 kilogram ≈ 2.205 pounds	1 pound ≈ 4.448 newtons
1 kilometer ≈ 0.621 mile	1 inch ≈ 2.540 centimeters	1 foot-lb ≈ 1.356 Joules
1 liter ≈ 0.264 gallon	1 foot ≈ 30.480 centimeters	1 ounce ≈ 28.350 grams
1 newton ≈ 0.225 pound	≈ 0.305 meter	1 pound ≈ 0.454 kilogram

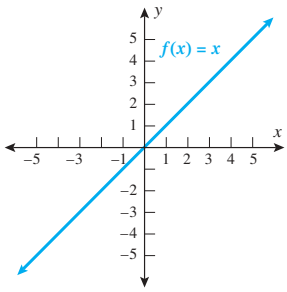
Functions

Constant Function	$f(x) = b$
Linear Function	$f(x) = mx + b$, where m is the slope and b is the y-intercept
Quadratic Function	$f(x) = ax^2 + bx + c$, $a \neq 0$ or $f(x) = a(x - h)^2 + k$ parabola vertex (h, k)
Polynomial Function	$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
Rational Function	$R(x) = \dfrac{n(x)}{d(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + a_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$
Exponential Function	$f(x) = b^x, b > 0, b \neq 1$
Logarithmic Function	$f(x) = \log_b x, b > 0, b \neq 1$

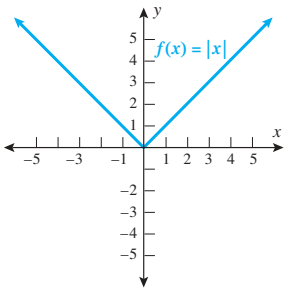
Graphs of Common Functions



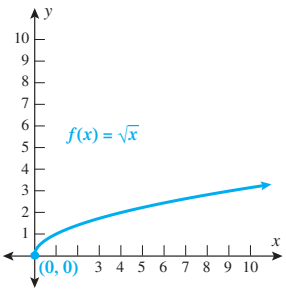
Constant Function



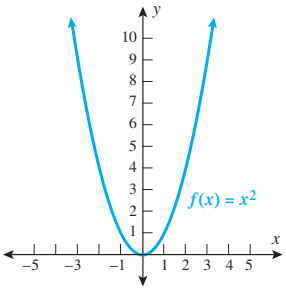
Identity Function



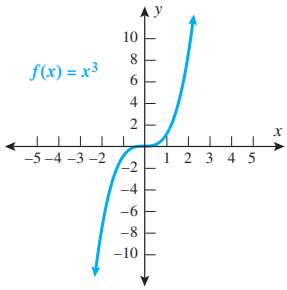
Absolute Value Function



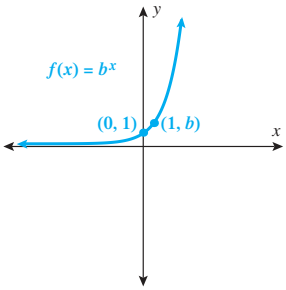
Square Root Function



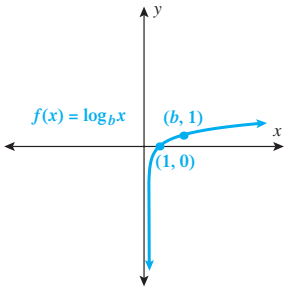
Square Function



Cube Function



Exponential Function



Logarithmic Function

Transformations

In each case, c represents a positive real number.

Function		Draw the graph of f and:
Vertical translations	$y = f(x) + c$	Shift f upward c units.
	$y = f(x) - c$	Shift f downward c units.
Horizontal translations	$y = f(x - c)$	Shift f to the right c units.
	$y = f(x + c)$	Shift f to the left c units.
Reflections	$y = -f(x)$	Reflect f about the x-axis.
	$y = f(-x)$	Reflect f about the y-axis.

Heron's Formula for Area

If the semiperimeter, s, of a triangle is

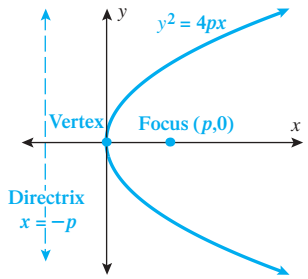
$$s = \frac{a + b + c}{2}$$

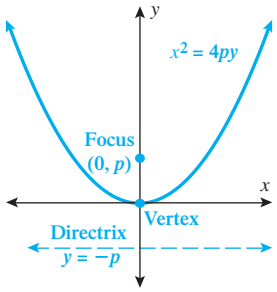
then the area of that triangle is

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

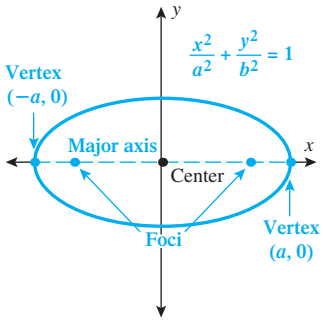
Conic Sections

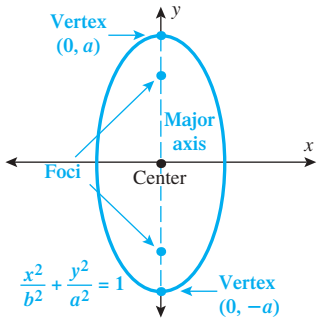
Parabola



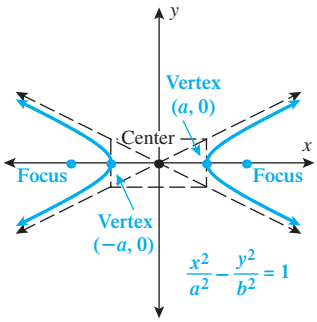


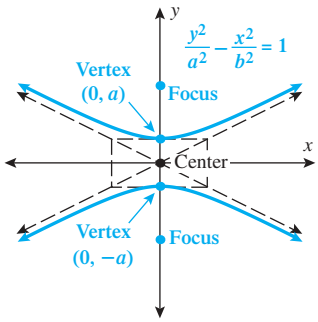
Ellipse





Hyperbola





Sequences

1. Infinite Sequence:

$$\{a_n\} = a_1, a_2, a_3, \ldots, a_n, \ldots$$

2. Summation Notation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

3. nth Term of an Arithmetic Sequence:

$$a_n = a_1 + (n - 1)d$$

4. Sum of First n Terms of an Arithmetic Sequence:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

5. nth Term of a Geometric Sequence:

$$a_n = a_1 r^{n-1}$$

6. Sum of First n Terms of a Geometric Sequence:

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1)$$

7. Sum of an Infinite Geometric Series with $|r| < 1$:

$$S = \frac{a_1}{1 - r}$$

The Binomial Theorem

1. $n! = n(n - 1)(n - 2)\cdots 3 \cdot 2 \cdot 1$;
$1! = 1$; $0! = 1$

2. $\binom{n}{r} = \dfrac{n!}{r!(n - r)!}$

3. Binomial theorem:

$$(a + b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

Permutations, Combinations, and Probability

1. ${}_nP_r$, the number of permutations of n elements taken r at a time, is given by

$${}_nP_r = \frac{n!}{(n - r)!}.$$

2. ${}_nC_r$, the number of combinations of n elements taken r at a time, is given by

$${}_nC_r = \frac{n!}{(n - r)!r!}.$$

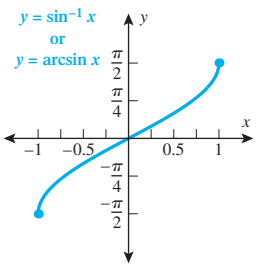
3. *Probability of an Event:* $P(E) = \dfrac{n(E)}{n(S)}$, where

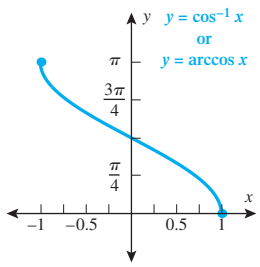
$n(E)$ = the number of outcomes in event E and
$n(S)$ = the number of outcomes in the sample space.

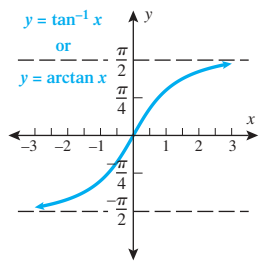
Inverse Trigonometric Functions

$y = \sin^{-1} x$	$x = \sin y$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	$-1 \le x \le 1$
$y = \cos^{-1} x$	$x = \cos y$	$0 \le y \le \pi$	$-1 \le x \le 1$
$y = \tan^{-1} x$	$x = \tan y$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	x is any real number
$y = \cot^{-1} x$	$x = \cot y$	$0 < y < \pi$	x is any real number
$y = \sec^{-1} x$	$x = \sec y$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$	$x \le -1$ or $x \ge 1$
$y = \csc^{-1} x$	$x = \csc y$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$	$x \le -1$ or $x \ge 1$

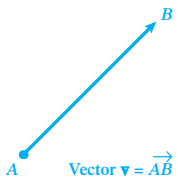
Graphs of the Inverse Trigonometric Functions



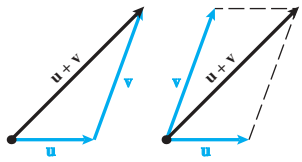




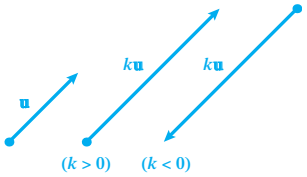
Vectors



Vector Addition



Scalar Multiplication



For vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$, and real number k,

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j}$$

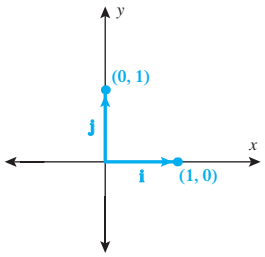
$$|\mathbf{u}| = \sqrt{a^2 + b^2}$$

$$\mathbf{u} + \mathbf{v} = \langle a + c, b + d \rangle$$

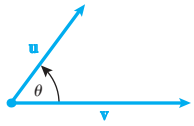
$$k\mathbf{u} = \langle ka, kb \rangle$$

$$\mathbf{u} \cdot \mathbf{v} = ac + bd$$

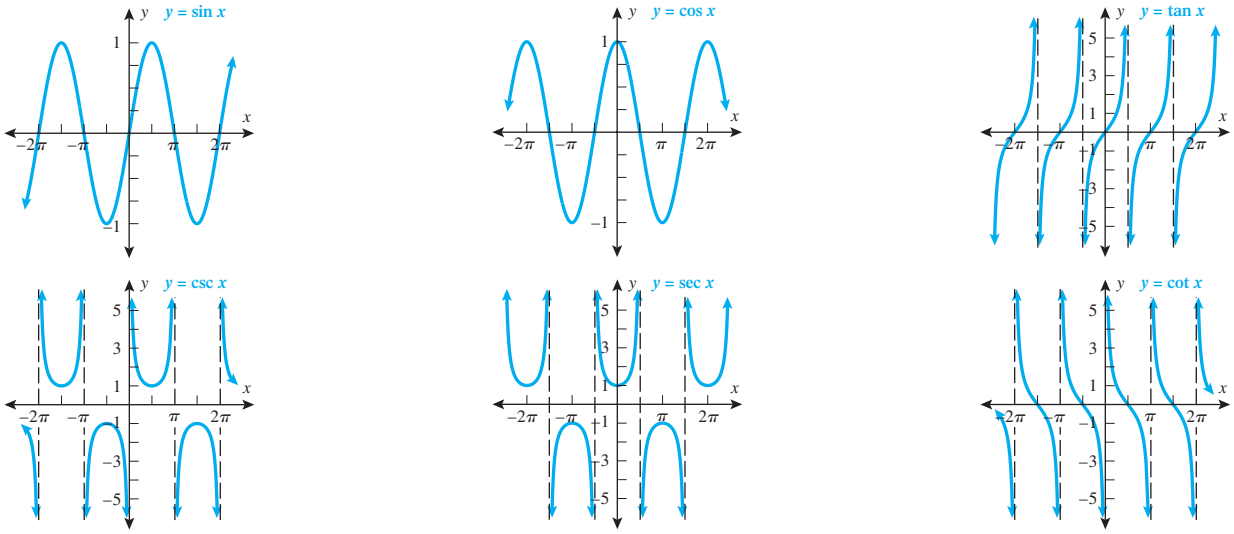
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$



$$\text{Comp}_{\mathbf{v}}\, \mathbf{u} = |\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}$$



Graphs of the Trigonometric Functions



Amplitude, Period, and Phase Shift

$y = A\sin(Bx + C)$ $\qquad$ $y = A\cos(Bx + C)$

$$\text{Amplitude} = |A| \qquad \text{Period} = \frac{2\pi}{B}$$

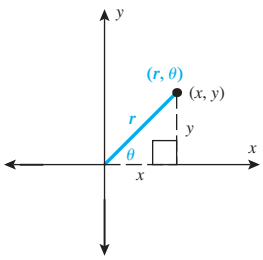
$$\text{Phase shift} = \frac{C}{B}\begin{cases}\text{left} & \text{if } C/B > 0\\ \text{right} & \text{if } C/B < 0\end{cases}$$

$y = A\tan(Bx + C)$ $\qquad$ $y = A\cot(Bx + C)$

$$\text{Period} = \frac{\pi}{B}$$

$$\text{Phase shift} = \frac{C}{B}\begin{cases}\text{left} & \text{if } C/B > 0\\ \text{right} & \text{if } C/B < 0\end{cases}$$

Polar Coordinates



$x = r\cos\theta$

$y = r\sin\theta$

$r^2 = x^2 + y^2$

$\tan\theta = \dfrac{y}{x}$

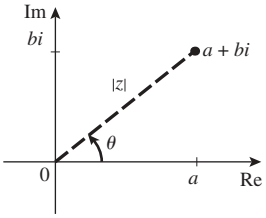
Complex Numbers

For the complex number $z = a + bi$

the **conjugate** is $\bar{z} = a - bi$

the **modulus** is $|z| = \sqrt{a^2 + b^2}$

the **argument** is θ, where $\tan\theta = b/a$



Polar (Trigonometric) form of a complex number

For $z = a + bi$, the **polar form** is

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ is the modulus of z and θ is the argument of z

DeMoivre's Theorem

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

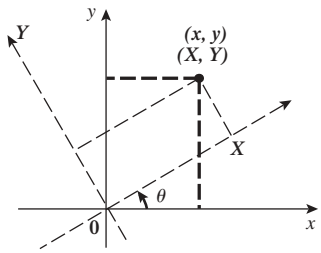
Nth Root Theorem

$$z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$$

$$= r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

where $k = 0, 1, 2, \ldots n - 1$

Rotation of Axes



Rotation of axes formulas

$$x = X\cos\theta - Y\sin\theta$$

$$y = X\sin\theta + Y\cos\theta$$

Angle-of-rotation formula for conic sections

$$\cot(2\theta) = \frac{A - C}{B} \quad \text{or}$$

$$\tan(2\theta) = \frac{B}{A - C}$$

Sums of Powers of Integers

$$\sum_{k=1}^{n} 1 = n \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

The Derivative of a Function

The **average rate of change** of f between a and b is

$$\frac{f(b) - f(a)}{b - a}$$

The **derivative** of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

The Area Under the Graph a Curve

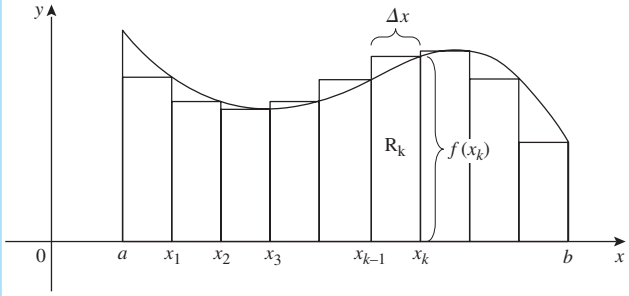
The **area under the graph of f** on the interval $[a, b]$ is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \overbrace{\underbrace{f(x_k)}_{\text{height}}\underbrace{\Delta x}_{\text{width}}}^{\text{Area of rectangle, } R_k}$$

where

$$\Delta x = \frac{b - a}{n}$$

$$x_k = a + k\Delta x$$

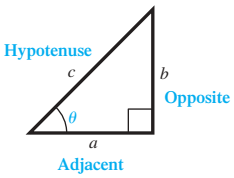


Right Triangle Trigonometry

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

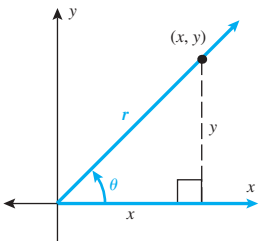


Trigonometric Functions in the Cartesian Plane

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



Exact Values of Trigonometric Functions

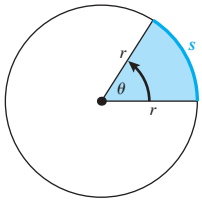
x degrees	x radians	$\sin x$	$\cos x$	$\tan x$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	—

Angle Measurement

π radians $= 180°$

$s = r\theta \quad A = \frac{1}{2}r^2\theta \quad (\theta \text{ in radians})$

To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.



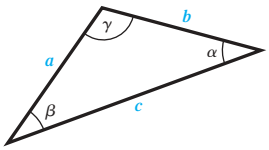
To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

Oblique Triangles

Law of Sines

In any triangle,

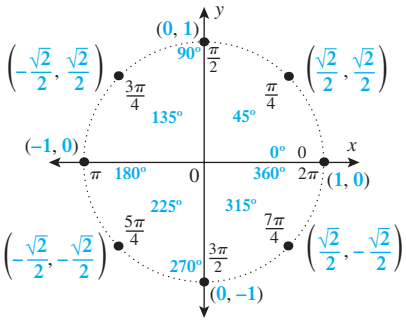
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

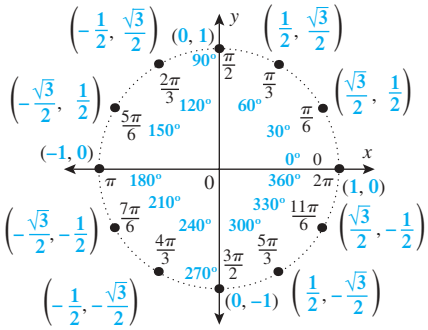


Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Circular Functions ($\cos\theta$, $\sin\theta$)





Trigonometric Identities

Sum Identities

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Difference Identities

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double-Angle Identities

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 1 - 2\sin^2 x \\ 2\cos^2 x - 1 \end{cases}$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2\cot x}{\cot^2 x - 1} = \frac{2}{\cot x - \tan x}$$

Half-Angle Identities

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$$

Sign (+/−) is determined by quadrant in which $x/2$ lies

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x} = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

Identities for Reducing Powers

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$
$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Cofunction Identities

(Replace $\pi/2$ with 90° if x is in degree measure.)

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$
$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$
$$\sec\left(\frac{\pi}{2} - x\right) = \csc x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

Product–Sum Identities

$$\sin x \cos y = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)]$$
$$\cos x \sin y = \tfrac{1}{2}[\sin(x + y) - \sin(x - y)]$$
$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)]$$

Sum–Product Identities

$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x + y}{2}\right)\sin\left(\frac{x - y}{2}\right)$$

Basic Trigonometric Identities

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Identities for Negatives

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$