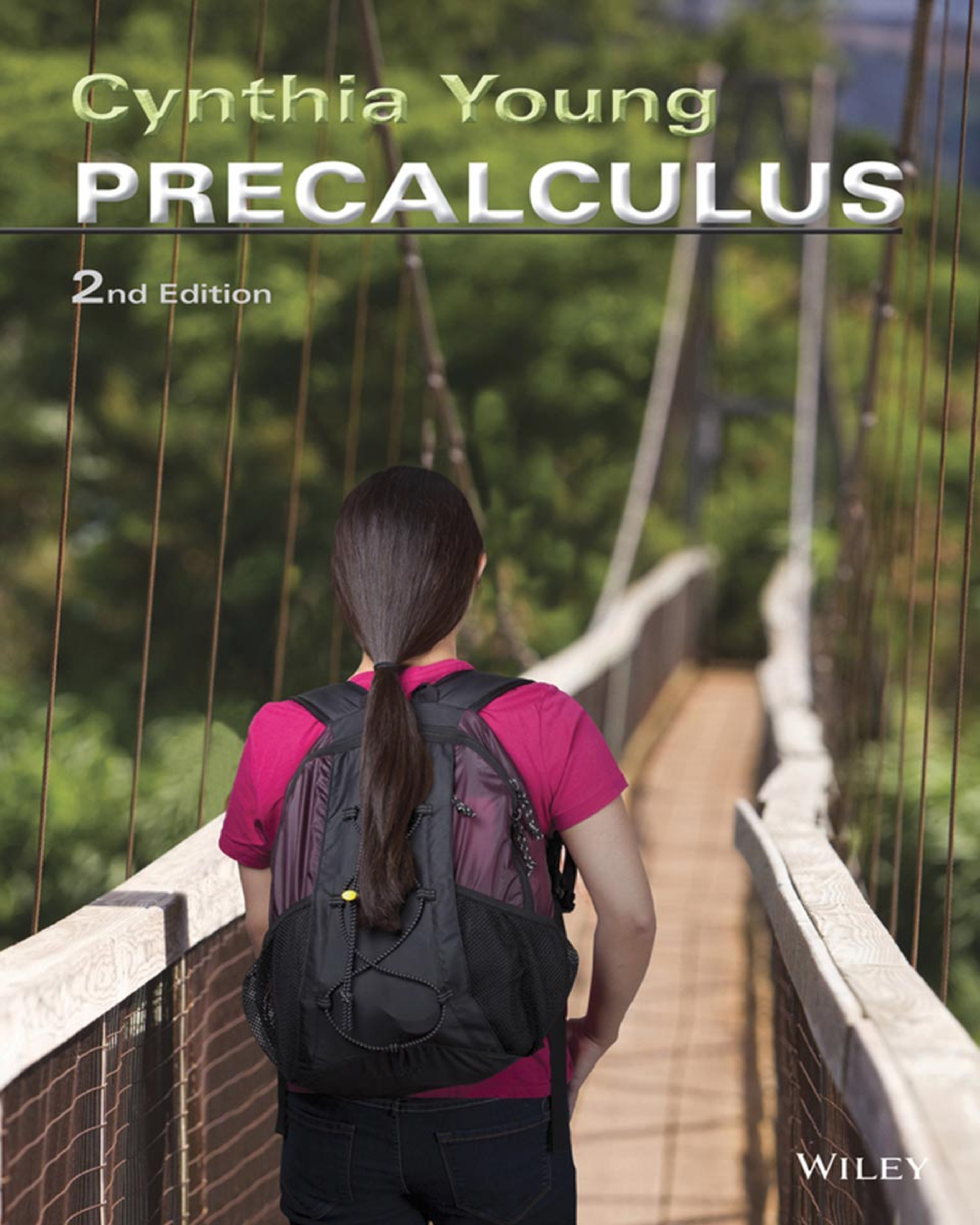
Cynthia Young

PRECALCULUS

2nd Edition



WILEY

Precalculus

Second Edition

Cynthia Y. Young | *Professor of Mathematics*
University of Central Florida

WILEY

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For Christopher and Caroline

About the Author

Cynthia Y. Young is a native of Tampa, Florida. She currently is a Professor of Mathematics at the University of Central Florida (UCF) and the author of *College Algebra*, *Trigonometry*, *Algebra and Trigonometry*, and *Precalculus*. She holds a B.A. degree in Secondary Mathematics Education from the University of North Carolina (Chapel Hill), an M.S. degree in Mathematical Sciences from UCF, and both an M.S. in Electrical Engineering and a Ph.D in Applied Mathematics from the University of Washington. She has taught high school in North Carolina and Florida, developmental mathematics at Shoreline Community College in Washington, and undergraduate and graduate students at UCF. Dr. Young's two main research interests are laser propagation through random media and improving student learning in STEM. She has authored or co-authored over 60 books and articles and been involved in over \$2.5M in external funding. Her atmospheric propagation research was recognized by the Office of Naval Research Young Investigator award, and in 2007 she was selected as a Fellow of the International Society for Optical Engineers. She is currently the co-director of UCF's EXCEL program whose goal is to improve the retention of STEM majors.

Although Dr. Young excels in research, she considers teaching her true calling. She has been the recipient of the UCF Excellence in Undergraduate Teaching Award, the UCF Scholarship of Teaching and Learning Award, and a two-time recipient of the UCF Teaching Incentive Program. Dr. Young is committed to improving student learning in mathematics and has shared her techniques and experiences with colleagues around the country through talks at colleges, universities, and conferences.

Dr. Young and her husband, Dr. Christopher Parkinson, enjoy spending time outdoors and competing in Field Trials with their Labrador Retrievers. *Laird's Cynful Wisdom* (call name "*Wiley*") is titled in Canada and currently pursuing her U.S. title. *Laird's Cynful Ellegance* (call name "*Ellie*") was a finalist in the Canadian National in 2009 and is retired (relaxing at home).

Dr. Young is pictured here with Ellie's 2011 litter of puppies!

Bonnie Farris

Preface

As a mathematics professor I would hear my students say, "I understand you in class, but when I get home I am lost." When I would probe further, students would continue with "I can't read the book." As a mathematician I always found mathematics textbooks quite easy to read—and then it dawned on me: don't look at this book through a mathematician's eyes; look at it through the eyes of students who might not view mathematics the same way that I do. What I found was that the books were not at all like my class. Students understood me in class, but when they got home they couldn't understand the book. It was then that the folks at Wiley lured me into writing. My goal was to write a book that is seamless with how we teach and is an ally (not an adversary) to student learning. I wanted to give students a book they could read without sacrificing the rigor needed for conceptual understanding. The following quote comes from a reviewer of this third edition when asked about the rigor of the book:

> *I would say that this text comes across as a little less rigorous than other texts, but I think that stems from how easy it is to read and how clear the author is. When one actually looks closely at the material, the level of rigor is high.*

Distinguishing Features

Four key features distinguish this book from others, and they came directly from my classroom.

Parallel Words and Math

Have you ever looked at your students' notes? I found that my students were only scribbling down the mathematics that I would write—never the words that I would say in class. I started passing out handouts that had two columns: one column for math and one column for words. Each Example would have one or the other; either the words were there and students had to fill in the math, or the math was there and students had to fill in the words. If you look at the Examples in this book, you will see that the words (your voice) are on the left and the mathematics is on the right. In most math books, when the author illustrates an Example, the mathematics is usually down the center of the page, and if the students don't know what mathematical operation was performed, they will look to the right for some brief statement of help. That's not how we teach; we don't write out an Example on the board and then say, "Class, guess what I just did!" Instead we lead our students, telling them what step is coming and then performing that mathematical step *together*—and reading naturally from left to right. Student reviewers have said that the Examples in this book are easy to read; that's because *your* voice is right there with them, working through problems *together*.

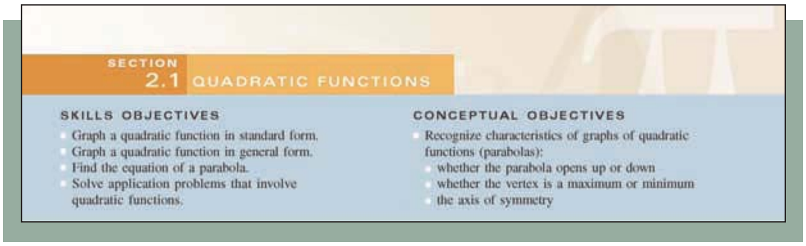
EXAMPLE 1 Graphing a Quadratic Function Given in Standard Form

Graph the quadratic function $f(x) = (x - 3)^2 - 1$.

Solution:

Step 1 The parabola opens up.	$a = 1$, so $a > 0$
Step 2 Determine the vertex.	$(h, k) = (3, -1)$
Step 3 Find the y-intercept.	$f(0) = (-3)^2 - 1 = 8$ $(0, 8)$ corresponds to the y-intercept

Skills and Concepts (Learning Objectives and Exercises)


SECTION 2.1 QUADRATIC FUNCTIONS

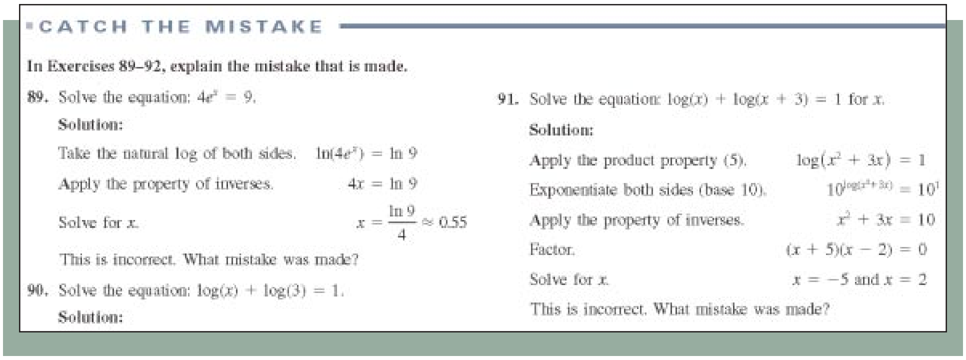
SKILLS OBJECTIVES
- Graph a quadratic function in standard form.
- Graph a quadratic function in general form.
- Find the equation of a parabola.
- Solve application problems that involve quadratic functions.

CONCEPTUAL OBJECTIVES
- Recognize characteristics of graphs of quadratic functions (parabolas):
  - whether the parabola opens up or down
  - whether the vertex is a maximum or minimum
  - the axis of symmetry

In my experience as a mathematics teacher/instructor/professor, I find skills to be on the micro level and concepts on the macro level of understanding mathematics. I believe that too often skills are emphasized at the expense of conceptual understanding. I have purposely separated *learning objectives* at the beginning of every section into two categories: *skills objectives*—what students should be able to do; and *conceptual objectives*—what students should understand. At the beginning of every class I discuss the learning objectives for the day—both skills and concepts. These are reinforced with both skills exercises and conceptual exercises.

Catch the Mistake

Have you ever made a mistake (or had a student bring you his or her homework with a mistake) and you go over it and over it and can't find the mistake? It's often easier to simply take out a new sheet of paper and solve it from scratch again than it is to actually find the mistake. Finding the mistake demonstrates a higher level of understanding. I include a few *Catch the Mistake* exercises in each section that demonstrate a common mistake that I have seen in my experience. I use these in class (either as a whole or often in groups), which leads to student discussion and offers an opportunity for formative assessment in real time.


CATCH THE MISTAKE

In Exercises 89–92, explain the mistake that is made.

89. Solve the equation: $4e^x = 9$.

Solution:

Take the natural log of both sides. $\ln(4e^x) = \ln 9$

Apply the property of inverses. $4x = \ln 9$

Solve for x. $x = \frac{\ln 9}{4} \approx 0.55$

This is incorrect. What mistake was made?

90. Solve the equation: $\log(x) + \log(3) = 1$.

Solution:

91. Solve the equation: $\log(x) + \log(x + 3) = 1$ for x.

Solution:

Apply the product property (5). $\log(x^2 + 3x) = 1$

Exponentiate both sides (base 10). $10^{\log(x^2+3x)} = 10^1$

Apply the property of inverses. $x^2 + 3x = 10$

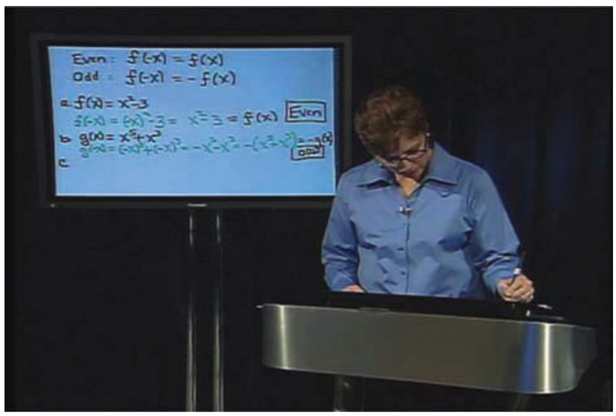
Factor. $(x + 5)(x - 2) = 0$

Solve for x. $x = -5$ and $x = 2$

This is incorrect. What mistake was made?

Lecture Videos by the Author

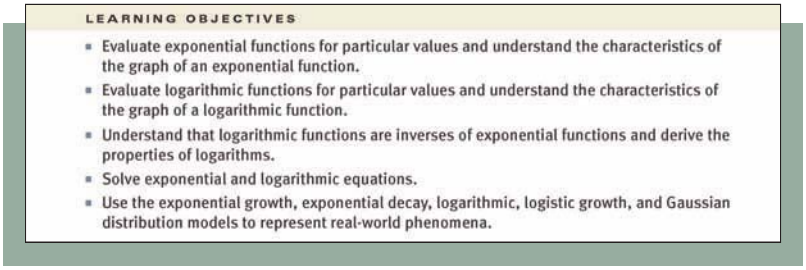
To ensure consistency in the students' learning experiences, I authored the videos myself. Throughout the book wherever a student sees the video icon, that indicates a video. These videos provide a mini lecture in that the chapter openers and chapter summaries are more like class discussion and selected Examples. Your Turns throughout the book also have an accompanying video of me working out that exact problem.



New to the Second Edition

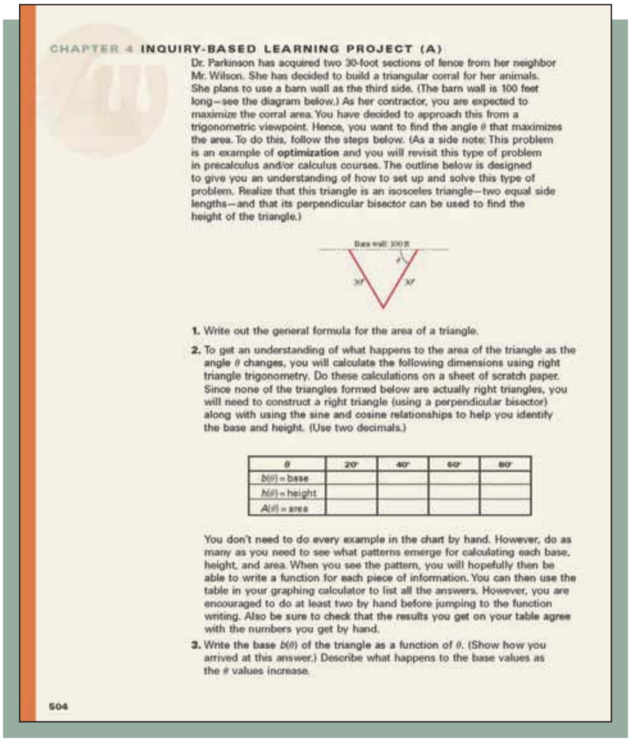
The first edition was *our* book, and this second edition is *our even better* book. I've incorporated some specific line-by-line suggestions from reviewers throughout the exposition, added some new Examples, and added over 200 new Exercises. The three main global upgrades to the second edition are a new Chapter Map with Learning Objectives, End-of-chapter Inquiry-Based Learning Projects, and additional Applications Exercises in areas such as Business, Economics, Life Sciences, Health Sciences, and Medicine. A section (0.8*) on Linear Regression was added, as well as some technology exercises on Quadratic, Exponential, and Logarithmic Regression.

Learning Objectives


LEARNING OBJECTIVES

- Evaluate exponential functions for particular values and understand the characteristics of the graph of an exponential function.
- Evaluate logarithmic functions for particular values and understand the characteristics of the graph of a logarithmic function.
- Understand that logarithmic functions are inverses of exponential functions and derive the properties of logarithms.
- Solve exponential and logarithmic equations.
- Use the exponential growth, exponential decay, logarithmic, logistic growth, and Gaussian distribution models to represent real-world phenomena.

Inquiry-Based Learning Projects


CHAPTER 4 INQUIRY-BASED LEARNING PROJECT (A)

Dr. Parkinson has acquired two 30-foot sections of fence from her neighbor Mr. Wilson. She has decided to build a triangular corral for her animals. She plans to use a barn wall as the third side. (The barn wall is 100 feet long—see the diagram below.) As her contractor, you are expected to maximize the corral area. You have decided to approach this from a trigonometric viewpoint. Hence, you want to find the angle θ that maximizes the area. To do this, follow the steps below. (As a side note: This problem is an example of **optimization** and you will revisit this type of problem in precalculus and/or calculus courses. The outline below is designed to give you an understanding of how to set up and solve this type of problem. Realize that this triangle is an isosceles triangle—two equal side lengths—and that its perpendicular bisector can be used to find the height of the triangle.)

1. Write out the general formula for the area of a triangle.
2. To get an understanding of what happens to the area of the triangle as the angle θ changes, you will calculate the following dimensions using right triangle trigonometry. Do these calculations on a sheet of scratch paper. Since none of the triangles formed below are actually right triangles, you will need to construct a right triangle (using a perpendicular bisector) along with using the sine and cosine relationships to help you identify the base and height. (Use two decimals.)

θ	20°	40°	60°	80°
$b(\theta)$ = base				
$h(\theta)$ = height				
$A(\theta)$ = area				

You don't need to do every example in the chart by hand. However, do as many as you need to see what patterns emerge for calculating each base, height, and area. When you see the pattern, you will hopefully then be able to write a function for each piece of information. You can then use the table in your graphing calculator to list all the answers. However, you are encouraged to do at least two by hand before jumping to the function writing. Also be sure to check that the results you get on your table agree with the numbers you get by hand.

3. Write the base $b(\theta)$ of the triangle as a function of θ. (Show how you arrived at this answer.) Describe what happens to the base values as the θ values increase.

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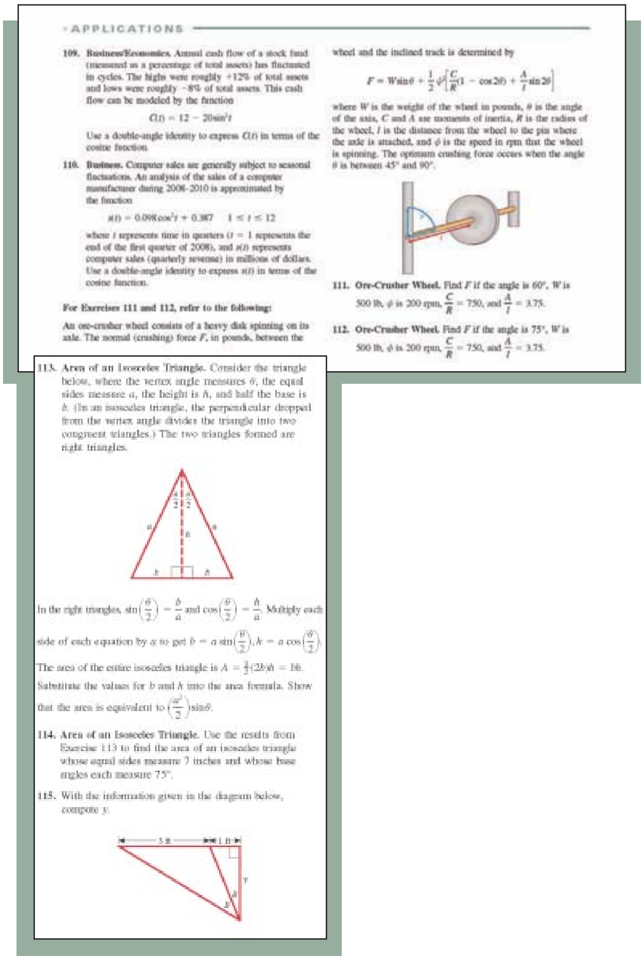
Applications to Business, Economics, Health Sciences, and Medicine


APPLICATIONS

109. **Business/Economics.** Annual cash flow of a stock fund (measured as a percentage of total assets) has fluctuated in cycles. The highs were roughly +12% of total assets and lows were roughly −8% of total assets. This cash flow can be modeled by the function $C(t) = 12 - 20\sin^2 t$. Use a double-angle identity to express $C(t)$ in terms of the cosine function.

110. **Business.** Computer sales are generally subject to seasonal fluctuations. An analysis of the sales of a computer manufacturer during 2008–2010 is approximated by the function $s(t) = 0.098\cos^2 t + 0.387 \quad 1 \le t \le 12$ where t represents time in quarters ($t = 1$ represents the end of the first quarter of 2008), and $s(t)$ represents computer sales (quarterly revenue) in millions of dollars. Use a double-angle identity to express $s(t)$ in terms of the cosine function.

For Exercises 111 and 112, refer to the following:

An ore-crusher wheel consists of a heavy disk spinning on its axle. The normal (crushing) force F, in pounds, between the wheel and the inclined track is determined by

$$F = W\sin\theta + \frac{1}{2}\phi^2\left[\frac{C}{R}(1-\cos 2\theta) + \frac{A}{l}\sin 2\theta\right]$$

where W is the weight of the wheel in pounds, θ is the angle of the axis, C and A are moments of inertia, R is the radius of the wheel, l is the distance from the wheel to the pin where the axle is attached, and ϕ is the speed in rpm that the wheel is spinning. The optimum crushing force occurs when the angle θ is between 45° and 90°.

111. **Ore-Crusher Wheel.** Find F if the angle is 60°, W is 500 lb, ϕ is 200 rpm, $\frac{C}{R} = 750$, and $\frac{A}{l} = 3.75$.

112. **Ore-Crusher Wheel.** Find F if the angle is 75°, W is 500 lb, ϕ is 200 rpm, $\frac{C}{R} = 750$, and $\frac{A}{l} = 3.75$.

113. **Area of an Isosceles Triangle.** Consider the triangle below, where the vertex angle measures θ, the equal sides measure a, the height is h, and half the base is b. (In an isosceles triangle, the perpendicular dropped from the vertex angle divides the triangle into two congruent triangles.) The two triangles formed are right triangles.

In the right triangles, $\sin\left(\frac{\theta}{2}\right) = \frac{b}{a}$ and $\cos\left(\frac{\theta}{2}\right) = \frac{h}{a}$. Multiply each side of each equation by a to get $b = a\sin\left(\frac{\theta}{2}\right)$, $h = a\cos\left(\frac{\theta}{2}\right)$. The area of the entire isosceles triangle is $A = \frac{1}{2}(2b)h = bh$. Substitute the values for b and h into the area formula. Show that the area is equivalent to $\left(\frac{a^2}{2}\right)\sin\theta$.

114. **Area of an Isosceles Triangle.** Use the results from Exercise 113 to find the area of an isosceles triangle whose equal sides measure 7 inches and whose base angles each measure 75°.

115. With the information given in the diagram below, compute y.

FEATURE	BENEFIT TO STUDENT
Chapter Opening Vignette	Piques the student's interest with a real-world application of material presented in the chapter. Later in the chapter, the same concept from the vignette is reinforced.
Chapter Overview, Flowchart, and Learning Objectives	Students see the big picture of how topics relate and overarching learning objectives are presented.
Skills and Conceptual Objectives	Skills objectives represent what students should be able to do. Conceptual objectives emphasize a higher level global perspective of concepts.
Clear, Concise, and Inviting Writing Style, Tone, and Layout	Students are able to *read* this book, which reduces math anxiety and promotes student success.
Parallel Words and Math	Increases students' ability to read and understand examples with a seamless representation of their instructor's class (instructor's voice and what they would write on the board).
Common Mistakes	Addresses a different learning style: teaching by counter-example. Demonstrates common mistakes so that students understand why a step is incorrect and reinforces the correct mathematics.
Color for Pedagogical Reasons	Particularly helpful for visual learners when they see a function written in red and then its corresponding graph in red or a function written in blue and then its corresponding graph in blue.
Study Tips	Reinforces specific notes that you would want to emphasize in class.
Author Videos	Gives students a mini class of several examples worked by the author.
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Catch the Mistake Exercises	Encourages students to assume the role of teacher—demonstrating a higher mastery level.
Conceptual Exercises	Teaches students to think more globally about a topic.
Inquiry-Based Learning Project	Lets students *discover* a mathematical identify, formula, etc. that is derived in the book.
Modeling OUR World	Engages students in a modeling project of a timely subject: global climate change.
Chapter Review	Key ideas and formulas are presented section by section in a chart. Improves study skills.
Chapter Review Exercises	Improves study skills.
Chapter Practice Test	Offers self-assessment and improves study skills.
Cumulative Test	Improves retention.

Supplements

Instructor Supplements

Instructor's Solutions Manual (ISBN vol. 1: 9781118640678; vol. 2: 9781118777909)

- Contains worked out solutions to all exercises in the text.

Instructor's Manual

Authored by Cynthia Young, the manual provides practical advice on teaching with the text, including:

- sample lesson plans and homework assignments
- suggestions for the effective utilization of additional resources and supplements
- sample syllabi
- Cynthia Young's Top 10 Teaching Tips & Tricks
- online component featuring the author presenting these Tips & Tricks

Annotated Instructor's Edition (ISBN: 9781118693087)

- Displays answers to all exercise questions, which can be found in the back of the book.
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PowerPoint Slides

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- Concept Questions
- Lecture Videos by Cynthia Young, including chapter introductions, chapter summaries, and selected video examples.

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WileyPLUS provides instant access to reports on trends in class performance, student use of course materials, and progress toward learning objectives, helping inform decisions and drive classroom discussions.

Acknowledgments

I want to express my sincerest gratitude to the entire Wiley team. I've said this before, and I will say it again: Wiley is the right partner for me. There is a reason that my dog is named Wiley—she's smart, competitive, a team player, and most of all, a joy to be around. There are several people within Wiley to whom I feel the need to express my appreciation: first and foremost to Laurie Rosatone who convinced Wiley Higher Ed to invest in a young assistant professor's vision for a series and who has been unwavering in her commitment to student learning. To my editor Joanna Dingle whose judgment I trust in both editorial and preschool decisions; thank you for surpassing my greatest expectations for an editor. To the rest of the ladies on the math editorial team (Jen Brady, Liz Baird, and Courtney Welsh), you are all first class! This revision was planned and executed exceptionally well thanks to you three. To the math marketing manager, Kimberly Kanakes, thank you for helping reps tell my story. To Ken Santor, thank you for your attention to detail. And finally, I'd like to thank all of the Wiley reps: thank you for your commitment to my series and your tremendous efforts to get professors to adopt this book for their students.

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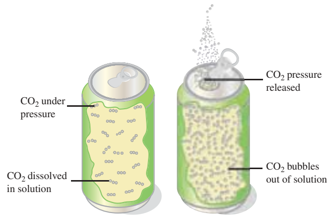

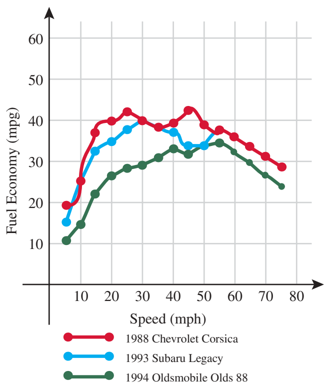

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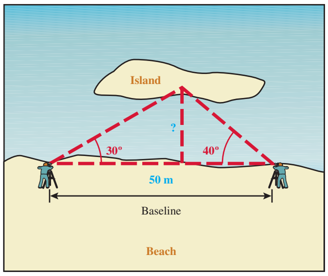

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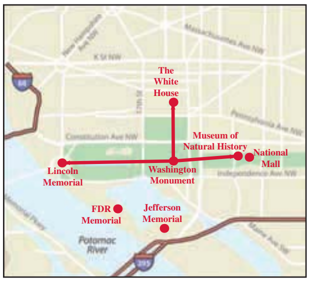

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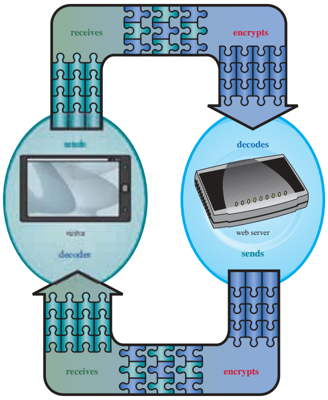

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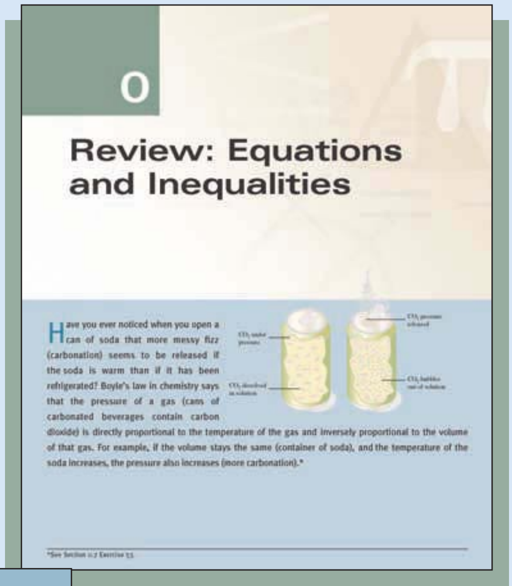
Chapter 11 Limits and the Appendix are available online at www.wiley.com/college/young. For print options including this material, please contact your local Wiley representative.

A Note from the Author to the Student

I wrote this text with careful attention to ways in which to make your learning experience more successful. If you take full advantage of the unique features and elements of this textbook, I believe your experience will be fulfilling and enjoyable. Let's walk through some of the special book features that will help you in your study of algebra and trigonometry.



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Review: Equations and Inequalities

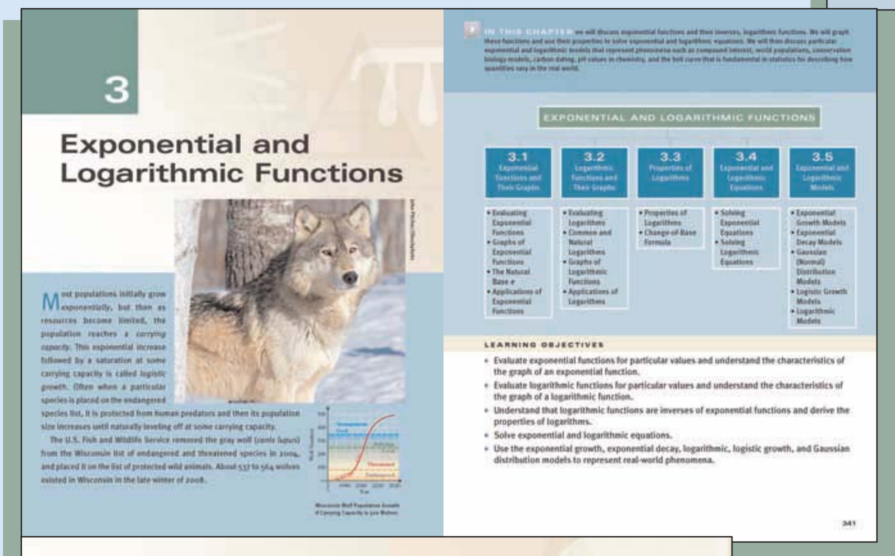
Have you ever noticed when you open a can of soda that more messy fizz (carbonation) seems to be released if the soda is warm than if it has been refrigerated? Boyle's law in chemistry says that the pressure of a gas (cans of carbonated beverages contain carbon dioxide) is directly proportional to the temperature of the gas and inversely proportional to the volume of that gas. For example, if the volume stays the same (container of soda), and the temperature of the soda increases, the pressure also increases (more carbonation).*

Prerequisites and Review (Chapter 0)

A comprehensive review of prerequisite knowledge (intermediate algebra topics) in Chapter 0 provides a brush up on knowledge and skills necessary for success in the course.

Clear, Concise, and Inviting Writing

Special attention has been made to present an engaging, clear, precise narrative in a layout that is easy to use and designed to reduce any math anxiety you may have.



3

Exponential and Logarithmic Functions

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

LEARNING OBJECTIVES

- Evaluate exponential functions for particular values and understand the characteristics of the graph of an exponential function.
- Evaluate logarithmic functions for particular values and understand the characteristics of the graph of a logarithmic function.
- Understand that logarithmic functions are inverses of exponential functions and derive the properties of logarithms.
- Solve exponential and logarithmic equations.
- Use the exponential growth, exponential decay, logarithmic, logistic growth, and Gaussian distribution models to represent real-world phenomena.

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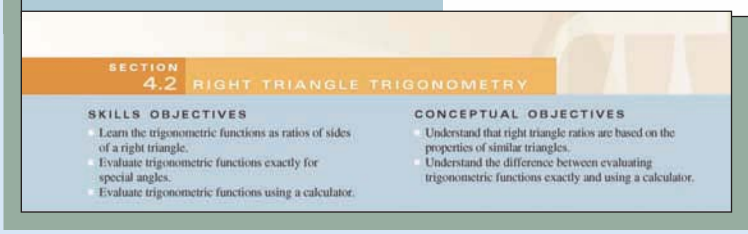
Chapter Introduction, Flow Chart, Section Headings, and Objectives

An opening vignette, flow chart, list of chapter sections, and chapter learning objectives give you an overview of the chapter.



SECTION 4.2 RIGHT TRIANGLE TRIGONOMETRY

SKILLS OBJECTIVES

- Learn the trigonometric functions as ratios of sides of a right triangle.
- Evaluate trigonometric functions exactly for special angles.
- Evaluate trigonometric functions using a calculator.

CONCEPTUAL OBJECTIVES

- Understand that right triangle ratios are based on the properties of similar triangles.
- Understand the difference between evaluating trigonometric functions exactly and using a calculator.

Skills and Conceptual Objectives

For every section, objectives are further divided by skills *and* concepts so you can see the difference between solving problems and truly understanding concepts.

Examples

Examples pose a specific problem using concepts already presented and then work through the solution. These serve to enhance your understanding of the subject matter.

EXAMPLE 9 Using the Change-of-Base Formula

Use the change-of-base formula to evaluate $\log_4 17$. Round to four decimal places.

Solution:

We will illustrate this in two ways (choosing common and natural logarithms) using a scientific calculator.

Common Logarithms

Use the change-of-base formula with base 10.	$\log_4 17 = \dfrac{\log 17}{\log 4}$
Approximate with a calculator.	≈ 2.043731421
	$\approx \boxed{2.0437}$

Natural Logarithms

Use the change-of-base formula with base e.	$\log_4 17 = \dfrac{\ln 17}{\ln 4}$
Approximate with a calculator.	≈ 2.043731421
	$\approx \boxed{2.0437}$

■ YOUR TURN Use the change-of-base formula to approximate $\log_7 34$. Round to four decimal places.

Your Turn

Immediately following many examples, you are given a similar problem to reinforce and check your understanding. This helps build confidence as you progress in the chapter. These are ideal for in-class activity or for preparing for homework later. Answers are provided in the margin for a quick check of your work.

COMMON MISTAKE

A common mistake is to write the sum of the logs as a log of the sum.

$$\log_b M + \log_b N \neq \log_b(M + N)$$

CORRECT	INCORRECT
Use the power property (7).	
$2\log_b 3 + 4\log_b u = \log_b 3^2 + \log_b u^4$	
Simplify.	
$\log_b 9 + \log_b u^4$	$\neq \log_b(9 + u^4)$ **ERROR**
Use the product property (5).	
$\boxed{= \log_b(9u^4)}$	

Common Mistake/ Correct vs. Incorrect

In addition to standard examples, some problems are worked out both correctly and incorrectly to highlight common errors students make. Counter examples like these are often an effective learning approach for many students.

Parallel Words and Math

This text reverses the common textbook presentation of examples by placing the explanation in words *on the left* and the mathematics in parallel *on the right*. This makes it easier for students to read through examples as the material flows more naturally from left to right and as commonly presented in class.

WORDS	MATH
For a point (x, y) that lies on the unit circle, $x^2 + y^2 = 1$.	$-1 \leq x \leq 1$ and $-1 \leq y \leq 1$
Since $(x, y) = (\cos\theta, \sin\theta)$, the following holds.	$-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$
State the **domain and range of the cosine and sine functions.**	Domain: $(-\infty, \infty)$ Range: $[-1, 1]$
Since $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\csc\theta = \dfrac{1}{\sin\theta}$, the values for θ that make $\sin\theta = 0$	

Study Tips and Caution Notes

These marginal reminders call out important hints or warnings to be aware of related to the topic or problem.

Study Tip

Both the initial side (initial ray) and the terminal side (terminal ray) of an angle are rays.

CAUTION

$$\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$$

$$\log_b M - \log_b N \neq \frac{\log_b M}{\log_b N}$$

Technology Tips

These marginal notes provide problem solving instructions and visual examples using graphing calculators.

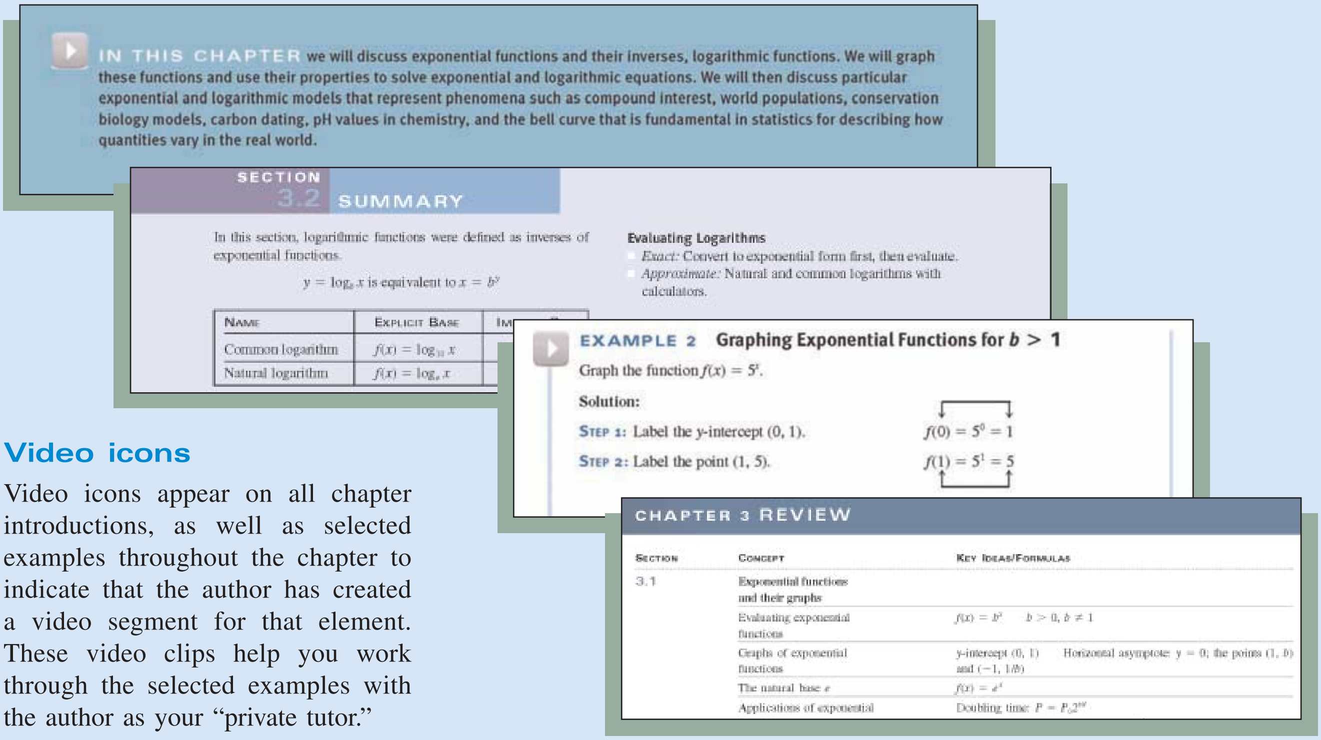
Technology Tip

Use the TI to evaluate the expression for s.

$$s = (6800 \text{ km})(45°)\left(\frac{\pi}{180°}\right)$$

Press 2nd ^ for π. Type 6800 x 45 x 2nd ^ ÷ 180 ENTER.

```
6800*45*π/180
         5340.707511
```

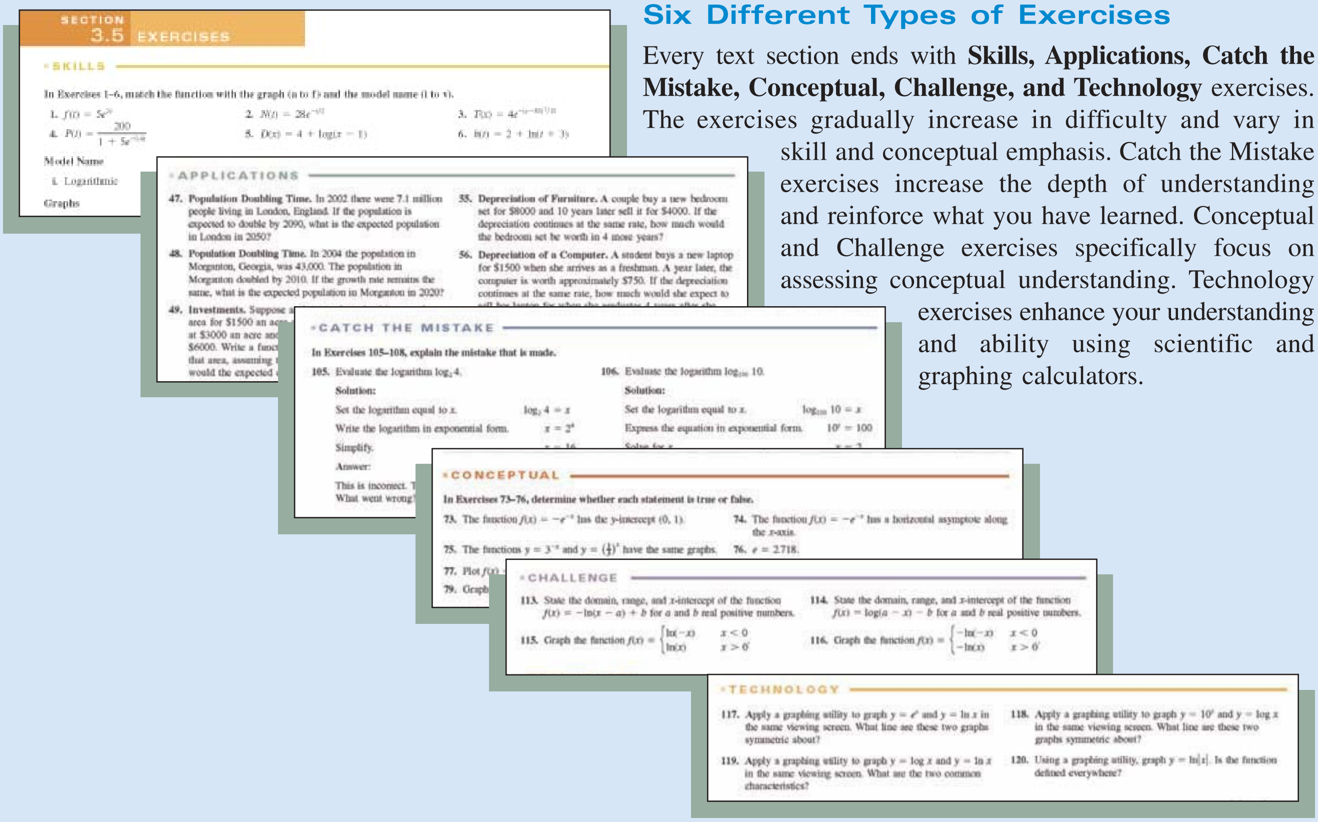


Video icons

Video icons appear on all chapter introductions, as well as selected examples throughout the chapter to indicate that the author has created a video segment for that element. These video clips help you work through the selected examples with the author as your "private tutor."

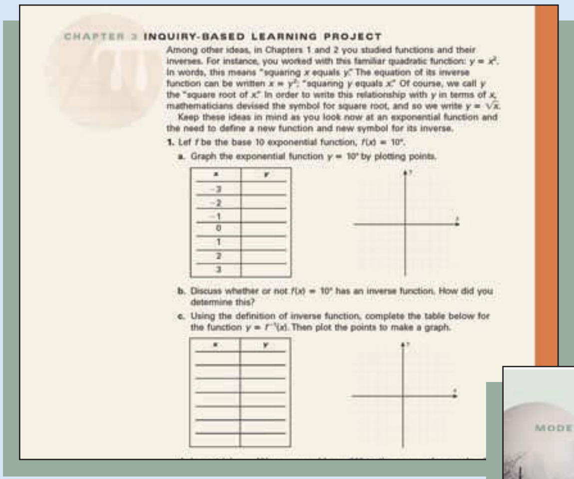
Six Different Types of Exercises

Every text section ends with **Skills, Applications, Catch the Mistake, Conceptual, Challenge, and Technology** exercises. The exercises gradually increase in difficulty and vary in skill and conceptual emphasis. Catch the Mistake exercises increase the depth of understanding and reinforce what you have learned. Conceptual and Challenge exercises specifically focus on assessing conceptual understanding. Technology exercises enhance your understanding and ability using scientific and graphing calculators.



Inquiry-Based Learning Projects

These end of chapter projects enable you to discover mathematical concepts on your own!


CHAPTER 3 INQUIRY-BASED LEARNING PROJECT

Among other ideas, in Chapters 1 and 2 you studied functions and their inverses. For instance, you worked with this familiar quadratic function: $y = x^2$. In words, this means "squaring x equals y." The equation of its inverse function can be written $x = y^2$: "squaring y equals x." Of course, we call y the "square root of x." In order to write this relationship with y in terms of x, mathematicians devised the symbol for square root, and so we write $y = \sqrt{x}$.

Keep these ideas in mind as you look now at an exponential function and the need to define a new function and new symbol for its inverse.

1. Let f be the base 10 exponential function, $f(x) = 10^x$.
   a. Graph the exponential function $y = 10^x$ by plotting points.

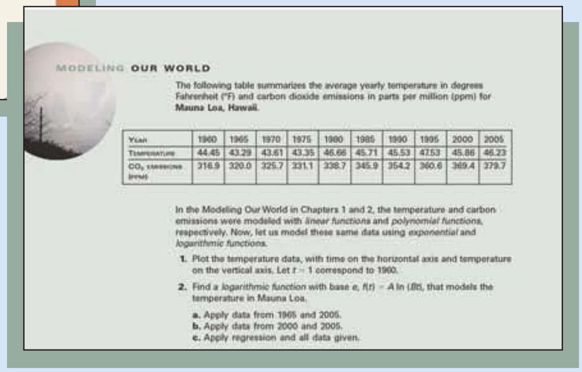
x	y
−3	
−2	
−1	
0	
1	
2	
3	

   b. Discuss whether or not $f(x) = 10^x$ has an inverse function. How did you determine this?
   c. Using the definition of inverse function, complete the table below for the function $y = f^{-1}(x)$. Then plot the points to make a graph.

x	y

Modeling Our World

These unique end-of-chapter exercises provide a fun and interesting way to take what you have learned and model a real world problem. By using climate change as the continuous theme, these exercises can help you to develop more advanced modeling skills with each chapter while seeing how modeling can help you better understand the world around you.

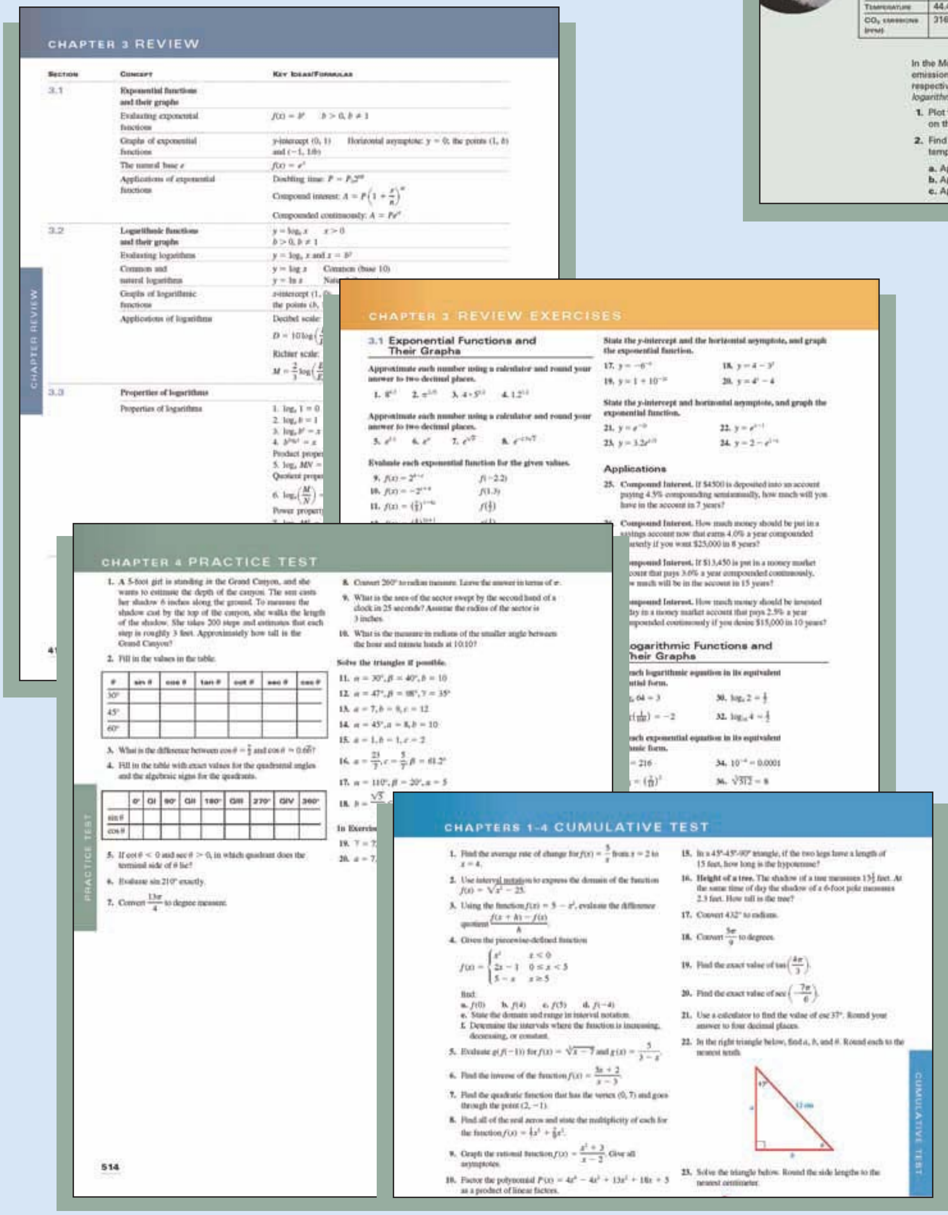

MODELING OUR WORLD

The following table summarizes the average yearly temperature in degrees Fahrenheit (°F) and carbon dioxide emissions in parts per million (ppm) for Mauna Loa, Hawaii.

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Temperature	44.45	43.29	43.61	43.35	46.66	45.71	45.53	47.53	45.86	46.23
CO_2 emissions (ppm)	316.9	320.0	325.7	331.1	338.7	345.9	354.2	360.6	369.4	379.7

In the Modeling Our World in Chapters 1 and 2, the temperature and carbon emissions were modeled with *linear functions* and *polynomial functions*, respectively. Now, let us model these same data using *exponential* and *logarithmic functions*.

1. Plot the temperature data, with time on the horizontal axis and temperature on the vertical axis. Let $t = 1$ correspond to 1960.
2. Find a *logarithmic function* with base e, $f(t) = A \ln(Bt)$, that models the temperature in Mauna Loa.
   a. Apply data from 1965 and 2005.
   b. Apply data from 2000 and 2005.
   c. Apply regression and all data given.


CHAPTER 3 REVIEW

CHAPTER 3 REVIEW EXERCISES

CHAPTER 4 PRACTICE TEST

CHAPTERS 1–4 CUMULATIVE TEST

Chapter Review, Review Exercises, Practice Test, Cumulative Test

At the end of every chapter, a summary review chart organizes the key learning concepts in an easy to use one or two-page layout. This feature includes key ideas and formulas, as well as indicating relevant pages and review exercises so that you can quickly summarize a chapter and study smarter. Review Exercises, arranged by section heading, are provided for extra study and practice. A Practice Test, without section headings, offers even more self-practice before moving on. A new Cumulative Test feature offers study questions based on all previous chapters' content, thus helping you build upon previously learned concepts.

Precalculus

Second Edition

CYNTHIA Y. YOUNG | *Professor of Mathematics*
UNIVERSITY OF CENTRAL FLORIDA

WILEY

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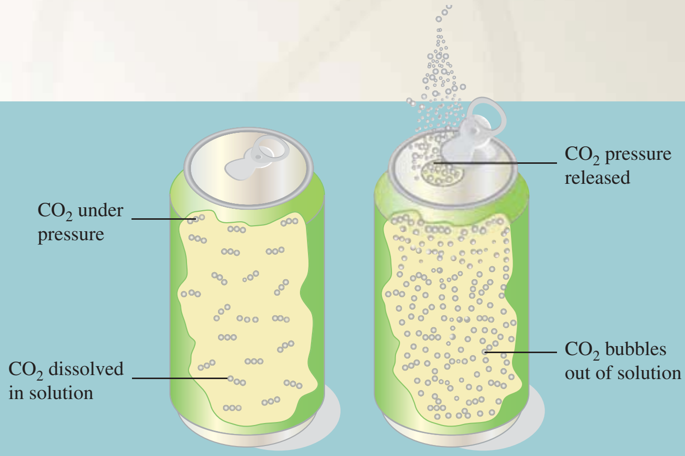
Review: Equations and Inequalities



Have you ever noticed when you open a can of soda that more messy fizz (carbonation) seems to be released if the soda is warm than if it has been refrigerated? Boyle's law in chemistry says that the pressure of a gas (cans of carbonated beverages contain carbon dioxide) is directly proportional to the temperature of the gas and inversely proportional to the volume of that gas. For example, if the volume stays the same (container of soda), and the temperature of the soda increases, the pressure also increases (more carbonation).*

*See Section 0.7 Exercise 53.

IN THIS CHAPTER we will review solving equations in one variable. We will start with linear and quadratic equations and then move on to other types of equations. We will review solving linear, polynomial, rational, and absolute value inequalities in one variable. We will discuss how to graph equations in two variables in the Cartesian plane and specifically discuss circles. Lastly, we will use equations to model variation.

REVIEW: EQUATIONS AND INEQUALITIES

0.1 Linear Equations
- Solving Linear Equations in One Variable
- Applications Involving Linear Equations

0.2 Quadratic Equations
- Factoring
- Square Root Method
- Completing the Square
- The Quadratic Formula

0.3 Other Types of Equations
- Rational Equations
- Radical Equations
- Equations Quadratic in Form: *u*-Substitution
- Factorable Equations
- Absolute Value Equations

0.4 Inequalities
- Graphing Inequalities and Interval Notation
- Linear Inequalities
- Polynomial Inequalities
- Rational Inequalities
- Absolute Value Inequalities

0.5 Graphing Equations
- Cartesian Plane
- The Distance and Midpoint Formulas
- Point-Plotting
- Using Intercepts and Symmetry as Graphing Aids
- Circles

0.6 Lines
- Graphing a Line
- Slope
- Equations of Lines
- Parallel and Perpendicular Lines

0.7 Modeling Variation
- Direct Variation
- Inverse Variation
- Joint Variation and Combined Variation

0.8* Linear Regression: Best Fit
- Scatterplots
- Identifying Patterns
- Linear Regression

LEARNING OBJECTIVES

- Solve linear equations in one variable.
- Solve quadratic equations in one variable.
- Solve other types of equations that can be transformed into linear or quadratic equations.
- Solve inequalities in one variable.
- Graph equations in two variables in the Cartesian plane.
- Find the equation of a line.
- Use equations to model variation.
- Find the line of best fit for a given data set.*

*Optional Technology Required Section.

SECTION 0.1 LINEAR EQUATIONS

SKILLS OBJECTIVES

- Solve linear equations in one variable.
- Solve application problems involving linear equations.

CONCEPTUAL OBJECTIVE

- Understand the mathematical modeling process.

Solving Linear Equations in One Variable

An **algebraic expression** (see Appendix) consists of one or more terms that are combined through basic operations such as addition, subtraction, multiplication, or division; for example,

$$3x + 2 \qquad 5 - 2y \qquad x + y$$

An **equation** is a statement that says two expressions are equal. For example, the following are all equations in one variable, x:

$$x + 7 = 11 \qquad x^2 = 9 \qquad 7 - 3x = 2 - 3x \qquad 4x + 7 = x + 2 + 3x + 5$$

To **solve** an equation in one variable means to find all the values of that variable that make the equation true. These values are called **solutions**, or **roots**, of the equation. The first of these statements shown above, $x + 7 = 11$, is true when $x = 4$ and false for any other values of x. We say that $x = 4$ is the solution to the equation. Sometimes an equation can have more than one solution, as in $x^2 = 9$. In this case, there are actually two values of x that make this equation true, $x = -3$ and $x = 3$. We say the **solution set** of this equation is $\{-3, 3\}$. In the third equation, $7 - 3x = 2 - 3x$, no values of x make the statement true. Therefore, we say this equation has **no solution**. And the fourth equation, $4x + 7 = x + 2 + 3x + 5$, is true for any values of x. An equation that is true for any value of the variable x is called an **identity**. In this case, we say the solution set is the **set of all real numbers**.

Two equations that have the same solution set are called **equivalent equations**. For example,

$$3x + 7 = 13 \qquad 3x = 6 \qquad x = 2$$

are all equivalent equations because each of them has the solution set $\{2\}$. Note that $x^2 = 4$ is not equivalent to these three equations because it has the solution set $\{-2, 2\}$.

When solving equations, it helps to find a simpler equivalent equation in which the variable is isolated (alone). The following table summarizes the procedures for generating equivalent equations.

Generating Equivalent Equations

Original Equation	Description	Equivalent Equation
$3(x - 6) = 6x - x$	■ Eliminate the parentheses. ■ Combine like terms on one or both sides of the equation.	$3x - 18 = 5x$
$7x + 8 = 29$	Add (or subtract) the same quantity to (from) *both* sides of the equation. $7x + 8 - \mathbf{8} = 29 - \mathbf{8}$	$7x = 21$
$5x = 15$	Multiply (or divide) both sides of the equation by the same nonzero quantity: $\frac{5x}{5} = \frac{15}{5}$.	$x = 3$
$-7 = x$	Interchange the two sides of the equation.	$x = -7$

You probably already know how to solve simple linear equations. Solving a linear equation in one variable is done by finding an equivalent equation. In generating an equivalent equation, remember that whatever operation is performed on one side of an equation must also be performed on the other side of the equation.

EXAMPLE 1 Solving a Linear Equation

Solve the equation $3x + 4 = 16$.

Solution:

Subtract 4 from both sides of the equation.

$$\begin{array}{rcr} 3x + 4 & = & 16 \\ -4 & & -4 \\ \hline 3x & = & 12 \end{array}$$

Divide both sides by 3.

$$\frac{3x}{3} = \frac{12}{3}$$

The solution is $x = 4$.

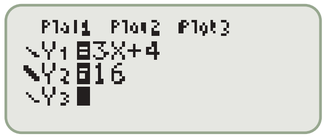
$$\boxed{x = 4}$$

The solution set is $\{4\}$.

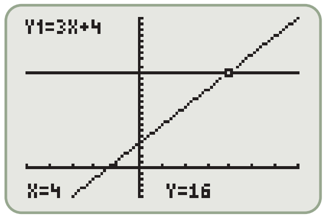
■ **YOUR TURN** Solve the equation $2x + 3 = 9$.

Technology Tip

Use a graphing utility to display graphs of $y_1 = 3x + 4$ and $y_2 = 16$.



The x-coordinate of the point of intersection is the solution to the equation $3x + 4 = 16$.



■ **Answer:** The solution is $x = 3$. The solution set is $\{3\}$.

Example 1 illustrates solving linear equations in one variable. What is a linear equation in one variable?

DEFINITION Linear Equation

A **linear equation in one variable**, x, can be written in the form

$$ax + b = 0$$

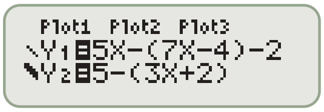
where a and b are real numbers and $a \neq 0$.

What makes this equation linear is that x is raised to the first power. We can also classify a linear equation as a **first-degree** equation.

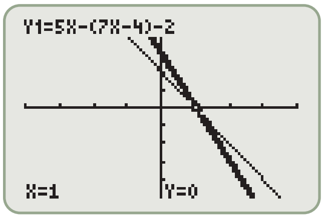
Equation	Degree	General Name
$x - 7 = 0$	First	Linear
$x^2 - 6x - 9 = 0$	Second	Quadratic
$x^3 + 3x^2 - 8 = 0$	Third	Cubic

Technology Tip

Use a graphing utility to display graphs of $y_1 = 5x - (7x - 4) - 2$ and $y_2 = 5 - (3x + 2)$.



The x-coordinate of the point of intersection is the solution to the equation $5x - (7x - 4) - 2 = 5 - (3x + 2)$.



EXAMPLE 2 Solving a Linear Equation

Solve the equation $5x - (7x - 4) - 2 = 5 - (3x + 2)$.

Solution:

Eliminate the parentheses.

$$5x - (7x - 4) - 2 = 5 - (3x + 2)$$

Don't forget to distribute the negative sign through *both* terms inside the parentheses.

$$5x - 7x + 4 - 2 = 5 - 3x - 2$$

Combine like terms on each side.

$$-2x + 2 = 3 - 3x$$

Add $3x$ to both sides.

$$\begin{aligned} +3x \qquad & \quad + 3x \\ \hline x + 2 &= 3 \end{aligned}$$

Subtract 2 from both sides.

$$\begin{aligned} -2 \quad & \; -2 \\ \hline x &= 1 \end{aligned}$$

Check to verify that $x = 1$ is a solution to the original equation.

$$\begin{aligned} 5 \cdot 1 - (7 \cdot 1 - 4) - 2 &= 5 - (3 \cdot 1 + 2) \\ 5 - (7 - 4) - 2 &= 5 - (3 + 2) \\ 5 - (3) - 2 &= 5 - (5) \\ 0 &= 0 \end{aligned}$$

Since the solution $x = 1$ makes the equation true, the solution set is $\{1\}$.

■ **YOUR TURN** Solve the equation $4(x - 1) - 2 = x - 3(x - 2)$.

■ **Answer:** The solution is $x = 2$. The solution set is $\{2\}$.

Study Tip

Prime Factors

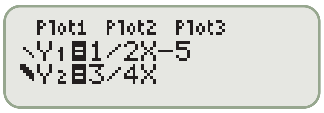
$$\begin{aligned} 2 &= 2 \\ 6 &= 2 \cdot 3 \\ 5 &= \cdot 5 \\ \hline \text{LCD} &= 2 \cdot 3 \cdot 5 = 30 \end{aligned}$$

To solve a linear equation involving fractions, find the least common denominator (LCD) of all terms and multiply both sides of the equation by the LCD. We will first review how to find the LCD.

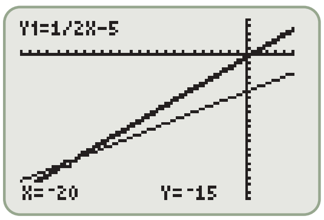
To add the fractions $\frac{1}{2} + \frac{1}{6} + \frac{2}{5}$, we must first find a common denominator. Some people are taught to find the lowest number that 2, 6, and 5 all divide evenly into. Others prefer a more systematic approach in terms of prime factors.

Technology Tip

Use a graphing utility to display graphs of $y_1 = \frac{1}{2}p - 5$ and $y_2 = \frac{3}{4}p$.



The x-coordinate of the point of intersection is the solution.



EXAMPLE 3 Solving a Linear Equation Involving Fractions

Solve the equation $\frac{1}{2}p - 5 = \frac{3}{4}p$.

Solution:

Write the equation.

$$\frac{1}{2}p - 5 = \frac{3}{4}p$$

Multiply each term in the equation by the LCD, 4.

$$(4)\frac{1}{2}p - (4)5 = (4)\frac{3}{4}p$$

The result is a linear equation with no fractions.

$$2p - 20 = 3p$$

Subtract $2p$ from both sides.

$$\begin{aligned} -2p \qquad & \; -2p \\ \hline -20 &= p \end{aligned}$$

$$\boxed{p = -20}$$

Since $p = -20$ satisfies the original equation, the solution set is $\{-20\}$.

■ **YOUR TURN** Solve the equation $\frac{1}{4}m = \frac{1}{12}m - 3$.

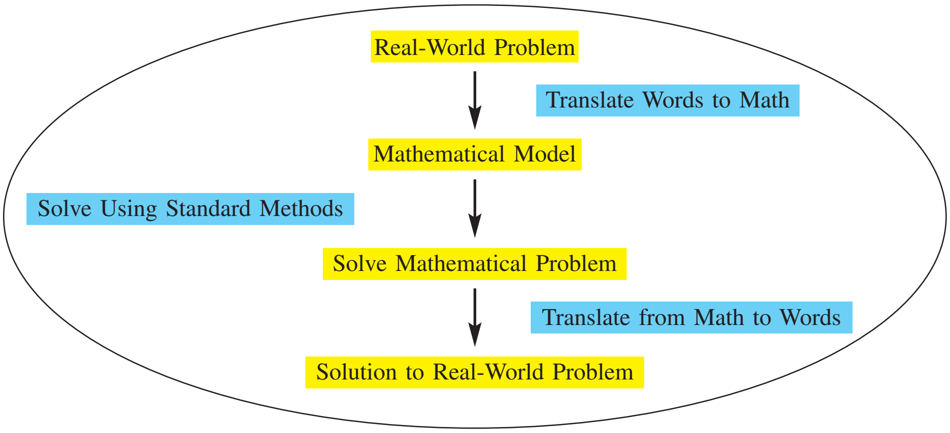
■ **Answer:** The solution is $m = -18$. The solution set is $\{-18\}$.

Solving a Linear Equation in One Variable

Step	Description	Example
1	Simplify the algebraic expressions on both sides of the equation.	$-3(x - 2) + 5 = 7(x - 4) - 1$ $-3x + 6 + 5 = 7x - 28 - 1$ $-3x + 11 = 7x - 29$
2	Gather all variable terms on one side of the equation and all constant terms on the other side.	$-3x + 11 = 7x - 29$ $+3x \qquad +3x$ $11 = 10x - 29$ $+29 \qquad +29$ $40 = 10x$
3	Isolate the variable.	$10x = 40$ $\boxed{x = 4}$

Applications Involving Linear Equations

We now use linear equations to solve problems that occur in our day-to-day lives. You typically will read the problem in words, develop a mathematical model (equation) for the problem, solve the equation, and write the answer in words.



You will have to come up with a unique formula to solve each kind of word problem, but there is a universal *procedure* for approaching all word problems.

PROCEDURE FOR SOLVING WORD PROBLEMS

Step 1: **Identify the question.** Read the problem *one* time and note what you are asked to find.
Step 2: **Make notes.** Read until you can note something (an amount, a picture, anything). Continue reading and making notes until you have read the problem a second* time.
Step 3: **Assign a variable to whatever is being asked for.** If there are two choices, then let it be the smaller of the two.
Step 4: **Set up an equation.** Assign a variable to represent what you are asked to find.
Step 5: **Solve the equation.**
Step 6: **Check the solution.** Substitute the solution for the variable in the equation, and also run the solution past the "common sense department" using estimation.

*Step 2 often requires multiple readings of the problem.

EXAMPLE 4 How Long Was the Trip?

During a camping trip in North Bay, Ontario, a couple went one-third of the way by boat, 10 miles by foot, and one-sixth of the way by horse. How long was the trip?

Solution:

STEP 1 Identify the question.
How many miles was the trip?

STEP 2 Make notes.

Read	Write
... one-third of the way by boat	BOAT: $\frac{1}{3}$ of the trip
... 10 miles by foot	FOOT: 10 miles
... one-sixth of the way by horse	HORSE: $\frac{1}{6}$ of the trip

STEP 3 Assign a variable.
Distance of total trip in miles $= x$

STEP 4 Set up an equation.
The total distance of the trip is the sum of all the distances by boat, foot, and horse.

Distance by boat + Distance by foot + Distance by horse = Total distance of trip

Distance by boat $= \frac{1}{3}x$

Distance by foot $= 10$ miles

Distance by horse $= \frac{1}{6}x$

$$\overbrace{\frac{1}{3}x}^{\text{boat}} + \overbrace{10}^{\text{foot}} + \overbrace{\frac{1}{6}x}^{\text{horse}} = \overbrace{x}^{\text{total}}$$

STEP 5 Solve the equation.

$$\frac{1}{3}x + 10 + \frac{1}{6}x = x$$

Multiply by the LCD, 6. $\quad 2x + 60 + x = 6x$

Collect x terms on the right. $\quad 60 = 3x$

Divide by 3. $\quad 20 = x$

$\boxed{\text{The trip was 20 miles.}}$ $\quad x = 20$

STEP 6 Check the solution.
Estimate: The boating distance, $\frac{1}{3}$ of 20 miles, is approximately 7 miles; the riding distance on horse, $\frac{1}{6}$ of 20 miles, is approximately 3 miles. Adding these two distances to the 10 miles by foot gives a trip distance of 20 miles.

■ **YOUR TURN** A family arrives at the Walt Disney World parking lot. To get from their car in the parking lot to the gate at the Magic Kingdom, they walk $\frac{1}{4}$ mile, take a tram for $\frac{1}{3}$ of their total distance, and take a monorail for $\frac{1}{2}$ of their total distance. How far is it from their car to the gate of the Magic Kingdom?

■ **Answer:** The distance from their car to the gate is 1.5 miles.

Geometry Problems

Some problems require geometric formulas in order to be solved.

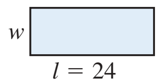
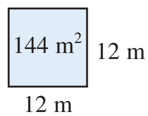
EXAMPLE 5 Geometry

A rectangle 24 meters long has the same area as a square with 12-meter sides. What are the dimensions of the rectangle?

Solution:

STEP 1 Identify the question.
What are the dimensions (length and width) of the rectangle?

STEP 2 Make notes.

Read	Write/Draw
A rectangle 24 meters long	 area of rectangle $= l \cdot w = 24w$
A square with 12-meter sides	 area of square $= 12 \cdot 12 = 144$

STEP 3 Assign a variable. Let $w =$ width of the rectangle.

STEP 4 Set up an equation.

The area of the rectangle is equal to the area of the square. rectangle area = square area

Substitute in known quantities. $24w = 144$

STEP 5 Solve the equation.

Divide by 24. $w = \frac{144}{24} = 6$

$\boxed{\text{The rectangle is 24 meters long and 6 meters wide.}}$

STEP 6 Check the solution.
A 24 meter by 6 meter rectangle has an area of 144 square meters.

■ **YOUR TURN** A rectangle 3 inches wide has the same area as a square with 9-inch sides. What are the dimensions of the rectangle?

■ **Answer:** The rectangle is 27 in. long and 3 in. wide.

Interest Problems

In our personal or business financial planning, a particular concern we have is interest. **Interest** is money paid for the use of money; it is the cost of borrowing money. The total amount borrowed is called the **principal**. The principal can be the price of our new car; we pay the bank interest for loaning us money to buy the car. The principal can also be the amount we keep in a CD or money market account; the bank uses this money and pays us interest. Typically, interest rate, expressed as a percentage, is the amount charged for the use of the principal for a given time, usually in years.

Simple interest is interest that is paid only on the principal during a period of time. Later we will discuss *compound interest*, which is interest paid on both the principal and the interest accrued over a period of time.

DEFINITION **Simple Interest**

If a principal of P dollars is borrowed for a period of t years at an annual interest rate r (expressed in decimal form), the interest I charged is

$$I = Prt$$

This is the formula for **simple interest**.

EXAMPLE 6 Multiple Investments

Theresa earns a full athletic scholarship for college. Her parents give her the $20,000 they had saved to pay for her college tuition. She decides to invest that money with an overall goal of earning 11% interest. She wants to put some of the money in a low-risk investment that has been earning 8% a year and the rest of the money in a medium-risk investment that typically earns 12% a year. How much money should she put in each investment to reach her goal?

Solution:

STEP 1 Identify the question.
How much money is invested in each (the 8% and the 12%) account?

STEP 2 Make notes.

Read	**Write/Draw**
Theresa has $20,000 to invest.	$20,000
If part is invested at 8% and the rest at 12%, how much should be invested at each rate to yield 11% on the total amount invested?	$\underbrace{\text{Some at 8\%} \quad \text{Some at 12\%}}_{\$20{,}000 \text{ at } 11\%}$

STEP 3 Assign a variable.
If we let x represent the amount Theresa puts into the 8% investment, how much of the $20,000 is left for her to put in the 12% investment?

Amount in the 8% investment: x

Amount in the 12% investment: $20{,}000 - x$

STEP 4 Set up an equation.

Simple interest formula: $I = Prt$

INVESTMENT	PRINCIPAL	RATE	TIME (YR)	INTEREST
8% Account	x	0.08	1	$0.08x$
12% Account	$20{,}000 - x$	0.12	1	$0.12(20{,}000 - x)$
Total	20,000	0.11	1	$0.11(20{,}000)$

Adding the interest earned in the 8% investment to the interest earned in the 12% investment should earn an average of 11% on the total investment.

$$0.08x + 0.12(20{,}000 - x) = 0.11(20{,}000)$$

STEP 5 Solve the equation.

Eliminate the parentheses. $0.08x + 2400 - 0.12x = 2200$

Collect x terms on the left, constants on the right. $-0.04x = -200$

Divide by -0.04. $x = \boxed{5000}$

Calculate the amount at 12%. $20{,}000 - 5000 = \boxed{15{,}000}$

Theresa should invest \$5000 at 8% and \$15,000 at 12% to reach her goal.

STEP 6 Check the solution.

If money is invested at 8% and 12% with a goal of averaging 11%, our intuition tells us that more should be invested at 12% than 8%, which is what we found. The exact check is as follows:

$$0.08(5000) + 0.12(15{,}000) = 0.11(20{,}000)$$
$$400 + 1800 = 2200$$
$$2200 = 2200$$

YOUR TURN You win \$24,000 and you decide to invest the money in two different investments: one paying 18% and the other paying 12%. A year later you have \$27,480 total. How much did you originally invest in each account?

Answer: \$10,000 is invested at 18% and \$14,000 is invested at 12%.

Mixture Problems

Mixtures are something we come across every day. Different candies that sell for different prices may make up a movie snack. New blends of coffees are developed by coffee connoisseurs. Chemists mix different concentrations of acids in their labs. Whenever two or more distinct ingredients are combined, the result is a **mixture**.

Our choice at a gas station is typically 87, 89, and 93 octane. The octane number is the number that represents the percentage of iso-octane in fuel. 89 octane is significantly overpriced. Therefore, if your car requires 89 octane, it would be more cost-effective to mix 87 and 93 octane.

EXAMPLE 7 Mixture Problem

The manual for your new car suggests using gasoline that is 89 octane. In order to save money, you decide to use some 87 octane and some 93 octane in combination with the 89 octane currently in your tank in order to have an approximate 89 octane mixture. Assuming you have 1 gallon of 89 octane remaining in your tank (your tank capacity is 16 gallons), how many gallons of 87 and 93 octane should be used to fill up your tank to achieve a mixture of 89 octane?

Solution:

STEP 1 Identify the question.

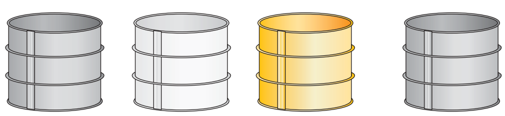
How many gallons of 87 octane and how many gallons of 93 octane should be used?

STEP 2 Make notes.

Read

Assuming you have 1 gallon of 89 octane remaining in your tank (your tank capacity is 16 gallons), how many gallons of 87 and 93 octane should you add?

Write/Draw



89 octane [1 gallon] + 87 octane [? gallons] + 93 octane [? gallons] = 89 octane [16 gallons]

STEP 3 Assign variables.

$$x = \text{gallons of 87 octane gasoline added at the pump}$$

$$15 - x = \text{gallons of 93 octane gasoline added at the pump}$$

$$1 = \text{gallons of 89 octane gasoline already in the tank}$$

STEP 4 Set up an equation.

$$0.89(1) + 0.87x + 0.93(15 - x) = 0.89(16)$$

STEP 5 Solve the equation. $0.89(1) + 0.87x + 0.93(15 - x) = 0.89(16)$

Eliminate the parentheses. $0.89 + 0.87x + 13.95 - 0.93x = 14.24$

Collect x terms on the left side. $-0.06x + 14.84 = 14.24$

Subtract 14.84 from both sides of the equation. $-0.06x = -0.6$

Divide both sides by -0.06. $x = 10$

Calculate the amount of 93 octane. $15 - 10 = 5$

$$\boxed{\text{Add 10 gallons of 87 octane and 5 gallons of 93 octane.}}$$

STEP 6 Check the solution.
Estimate: Our intuition tells us that if the desired mixture is 89 octane, then we should add approximately 1 part 93 octane and 2 parts 87 octane. The solution we found, 10 gallons of 87 octane and 5 gallons of 93 octane, agrees with this.

YOUR TURN For a certain experiment, a student requires 100 ml of a solution that is 11% HCl (hydrochloric acid). The storeroom has only solutions that are 5% HCl and 15% HCl. How many milliliters of each available solution should be mixed to get 100 ml of 11% HCl?

Answer: 40 ml of 5% HCl and 60 ml of 15% HCl

Distance–Rate–Time Problems

The next example deals with distance, rate, and time. On a road trip, you see a sign that says your destination is 90 miles away, and your speedometer reads 60 miles per hour. Dividing 90 miles by 60 miles per hour tells you that if you continue at this speed, your arrival will be in 1.5 hours. Here is how you know.

If the rate, or speed, is assumed to be constant, then the equation that relates distance (d), rate (r), and time (t) is given by $d = r \cdot t$. In the above driving example,

$$d = 90 \text{ miles} \qquad r = 60\,\frac{\text{miles}}{\text{hour}}$$

Substituting these into $d = r \cdot t$, we arrive at $$90 \text{ miles} = \left[60\,\frac{\text{miles}}{\text{hour}}\right] \cdot t$$

Solving for t, we get $$t = \frac{90 \text{ miles}}{60\,\frac{\text{miles}}{\text{hour}}} = 1.5 \text{ hours}$$

EXAMPLE 8 Distance–Rate–Time

It takes 8 hours to fly from Orlando to London and 9.5 hours to return. If an airplane averages 550 miles per hour (mph) in still air, what is the average rate of the wind blowing in the direction from Orlando to London? Assume the wind speed is constant for both legs of the trip. Round your answer to the nearest mph.

Solution:

STEP 1 Identify the question.
At what rate in miles per hour is the wind blowing?

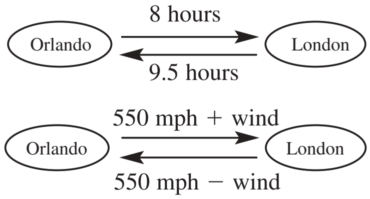
STEP 2 Make notes.

Read

It takes 8 hours to fly from Orlando to London and 9.5 hours to return.

If the airplane averages 550 mph in still air ...

Write/Draw



STEP 3 Assign a variable. w = wind speed

STEP 4 Set up an equation.
The formula relating distance, rate, and time is $d = r \cdot t$. The distance d of each flight is the same. On the Orlando to London flight, the time is 8 hours due to an increased speed from a tailwind. On the London to Orlando flight, the time is 9.5 hours and the speed is decreased due to the headwind.

Orlando to London: $d = (550 + w)8$

London to Orlando: $d = (550 - w)9.5$

These distances are the same, so set them equal to each other:

$$(550 + w)8 = (550 - w)9.5$$

STEP 5 Solve the equation.

Eliminate the parentheses. $4400 + 8w = 5225 - 9.5w$

Collect w terms on the left, constants on the right. $17.5w = 825$

Divide by 17.5. $w = 47.1429 \approx 47$

The wind is blowing approximately $\boxed{47 \text{ mph}}$ in the direction from Orlando to London.

STEP 6 Check the solution.
Estimate: Going from Orlando to London, the tailwind is approximately 50 mph, which when added to the plane's 550 mph speed yields a ground speed of 600 mph. The Orlando to London route took 8 hours. The distance of that flight is (600 mph) (8 hr), which is 4800 miles. The return trip experienced a headwind of approximately 50 mph, so subtracting the 50 from 550 gives an average speed of 500 mph. That route took 9.5 hours, so the distance of the London to Orlando flight was (500 mph)(9.5 hr), which is 4750 miles. Note that the estimates of 4800 and 4750 miles are close.

YOUR TURN A Cessna 150 averages 150 mph in still air. With a tailwind it is able to make a trip in $2\frac{1}{3}$ hours. Because of the headwind, it is only able to make the return trip in $3\frac{1}{2}$ hours. What is the average wind speed?

Answer: The wind is blowing 30 mph.

EXAMPLE 9 Work

Connie can clean her house in 2 hours. If Alvaro helps her, together they can clean the house in 1 hour and 15 minutes. How long would it take Alvaro to clean the house by himself?

Solution:

STEP 1 Identify the question.

How long does it take Alvaro to clean the house?

STEP 2 Make notes.

- Connie can clean her house in 2 hours, so Connie can clean $\frac{1}{2}$ of the house per hour.
- Together Connie and Alvaro can clean the house in 1 hour and 15 minutes, or $\frac{5}{4}$ of an hour. Therefore together, they can clean $\frac{1}{5/4} = \frac{4}{5}$ of the house per hour.
- Let x = number of hours it takes Alvaro to clean the house by himself. So Alvaro can clean $\frac{1}{x}$ of the house per hour.

	AMOUNT OF TIME TO DO ONE JOB	AMOUNT OF JOB DONE PER UNIT OF TIME
Connie	2	$\frac{1}{2}$
Alvaro	x	$\frac{1}{x}$
Together	$\frac{5}{4}$	$\frac{4}{5}$

STEP 3 Set up an equation.

$$\underbrace{\frac{1}{2}}_{\text{Amount of house Connie can clean per hour}} + \underbrace{\frac{1}{x}}_{\text{Amount of house Alvaro can clean per hour}} = \underbrace{\frac{4}{5}}_{\text{Amount of house they can clean per hour if they work together}}$$

STEP 4 Solve the equation.

Multiply by the LCD, $10x$. $\quad 5x + 10 = 8x$

Solve for x. $\quad x = \frac{10}{3} = 3\frac{1}{3}$

It takes Alvaro $\boxed{\text{3 hours and 20 minutes}}$ to clean the house by himself.

STEP 5 Check the solution.

Estimate: Since Connie can clean the house in 2 hours and together with Alvaro it takes 1.25 hours, we know it takes Alvaro longer than 2 hours to clean the house himself.

SECTION 0.1 SUMMARY

To solve a linear equation

1. Simplify the algebraic expressions on both sides of the equation.
2. Gather all variable terms on one side of the equation and all constant terms on the other side.
3. Isolate the variable.

In the real world, many kinds of application problems can be solved through modeling with linear equations. Some problems require the development of a mathematical model, while others rely on common formulas. The following procedure will guide you:

1. Identify the question.
2. Make notes.
3. Assign a variable.
4. Set up an equation.
5. Solve the equation.
6. Check the solution against your intuition.

SECTION 0.1 EXERCISES

SKILLS

In Exercises 1–26, solve for the indicated variable.

1. $9m - 7 = 11$
2. $2x + 4 = 5$
3. $5t + 11 = 18$
4. $7x + 4 = 21 + 24x$
5. $3x - 5 = 25 + 6x$
6. $5x + 10 = 25 + 2x$
7. $20n - 30 = 20 - 5n$
8. $14c + 15 = 43 + 7c$
9. $4(x - 3) = 2(x + 6)$
10. $5(2y - 1) = 2(4y - 3)$
11. $-3(4t - 5) = 5(6 - 2t)$
12. $2(3n + 4) = -(n + 2)$
13. $2(x - 1) + 3 = x - 3(x + 1)$
14. $4(y + 6) - 8 = 2y - 4(y + 2)$
15. $5p + 6(p + 7) = 3(p + 2)$
16. $3(z + 5) - 5 = 4z + 7(z - 2)$
17. $7x - (2x + 3) = x - 2$
18. $3x - (4x + 2) = x - 5$
19. $2 - (4x + 1) = 3 - (2x - 1)$
20. $5 - (2x - 3) = 7 - (3x + 5)$
21. $2a - 9(a + 6) = 6(a + 3) - 4a$
22. $25 - [2 + 5y - 3(y + 2)] = -3(2y - 5) - [5(y - 1) - 3y + 3]$
23. $32 - [4 + 6x - 5(x + 4)] = 4(3x + 4) - [6(3x - 4) + 7 - 4x]$
24. $12 - [3 + 4m - 6(3m - 2)] = -7(2m - 8) - 3[(m - 2) + 3m - 5]$
25. $20 - 4[c - 3 - 6(2c + 3)] = 5(3c - 2) - [2(7c - 8) - 4c + 7]$
26. $46 - [7 - 8y + 9(6y - 2)] = -7(4y - 7) - 2[6(2y - 3) - 4 + 6y]$

Exercises 27–38 involve fractions. Clear the fractions by first multiplying by the least common denominator, and then solve the resulting linear equation.

27. $\frac{1}{5}m = \frac{1}{60}m + 1$
28. $\frac{1}{12}z = \frac{1}{24}z + 3$
29. $\frac{x}{7} = \frac{2x}{63} + 4$
30. $\frac{a}{11} = \frac{a}{22} + 9$
31. $\frac{1}{3}p = 3 - \frac{1}{24}p$
32. $\frac{3x}{5} - x = \frac{x}{10} - \frac{5}{2}$
33. $\frac{5y}{3} - 2y = \frac{2y}{84} + \frac{5}{7}$
34. $2m - \frac{5m}{8} = \frac{3m}{72} + \frac{4}{3}$
35. $p + \frac{p}{4} = \frac{5}{2}$
36. $\frac{c}{4} - 2c = \frac{5}{4} - \frac{c}{2}$
37. $\frac{x - 3}{3} - \frac{x - 4}{2} = 1 - \frac{x - 6}{6}$
38. $1 - \frac{x - 5}{3} = \frac{x + 2}{5} - \frac{6x - 1}{15}$

APPLICATIONS

39. **Puzzle.** Angela is on her way from home in Jersey City to New York City for dinner. She walks 1 mile to the train station, takes the train $\frac{3}{4}$ of the way, and takes a taxi $\frac{1}{6}$ of the way to the restaurant. How far does Angela live from the restaurant?

40. **Puzzle.** An employee at Kennedy Space Center (KSC) lives in Daytona Beach and works in the vehicle assembly building (VAB). She carpools to work with a colleague. On the days that her colleague drives the car pool, she drives 7 miles to the park-and-ride, continues with her colleague to the KSC headquarters building, and then takes the KSC shuttle from the headquarters building to the VAB. The drive from the park-and-ride to the headquarters building is $\frac{5}{6}$ of her total trip and the shuttle ride is $\frac{1}{20}$ of her total trip. How many miles does she travel from her house to the VAB on days when her colleague drives?

41. **Budget.** A company has a total of $20,000 allocated for monthly costs. Fixed costs are $15,000 per month and variable costs are $18.50 per unit. How many units can be manufactured in a month?

42. **Budget.** A woman decides to start a small business making monogrammed cocktail napkins. She can set aside $1870 for monthly costs. Fixed costs are $1329.50 per month and variable costs are $3.70 per set of napkins. How many sets of napkins can she afford to make per month?

43. **Geometry.** Consider two circles, a smaller one and a larger one. If the larger one has a radius that is 3 feet larger than that of the smaller circle and the ratio of the circumferences is 2:1, what are the radii of the two circles?

44. **Geometry.** The length of a rectangle is 2 more than 3 times the width, and the perimeter is 28 inches. What are the dimensions of the rectangle?

45. **Biology: Alligators.** It is common to see alligators in ponds, lakes, and rivers in Florida. The ratio of head size (back of the head to the end of the snout) to the full body length of an alligator is typically constant. If a $3\frac{1}{2}$-foot alligator has a head length of 6 inches, how long would you expect an alligator to be whose head length is 9 inches?

46. **Biology: Snakes.** In the African rainforest there is a snake called a Gaboon viper. The fang size of this snake is proportional to the length of the snake. A 3-foot snake typically has 2-inch fangs. If a herpetologist finds Gaboon viper fangs that are 2.6 inches long, how big a snake would she expect to find?

47. **Investing.** Ashley has $120,000 to invest and decides to put some in a CD that earns 4% interest per year and the rest in a low-risk stock that earns 7%. How much did she invest in each to earn $7800 interest in the first year?

48. **Investing.** You inherit $13,000 and you decide to invest the money in two different investments: one paying 10% and the other paying 14%. A year later your investments are worth $14,580. How much did you originally invest in each account?

49. **Investing.** Wendy was awarded a volleyball scholarship to the University of Michigan, so on graduation her parents gave her the $14,000 they had saved for her college tuition. She opted to invest some money in a privately held company that pays 10% per year and evenly split the remaining money between a money market account yielding 2% and a high-risk stock that yielded 40%. At the end of the first year she had $16,610 total. How much did she invest in each of the three?

50. **Interest.** A high school student was able to save $5000 by working a part-time job every summer. He invested half the money in a money market account and half the money in a stock that paid three times as much interest as the money market account. After a year he earned $150 in interest. What were the interest rates of the money market account and the stock?

51. **Chemistry.** For a certain experiment, a student requires 100 ml of a solution that is 8% HCl (hydrochloric acid). The storeroom has only solutions that are 5% HCl and 15% HCl. How many milliliters of each available solution should be mixed to get 100 ml of 8% HCl?

52. **Chemistry.** How many gallons of pure alcohol must be mixed with 5 gallons of a solution that is 20% alcohol to make a solution that is 50% alcohol?

53. **Communications.** The speed of light is approximately 3.0×10^8 meters per second (670,616,629 miles per hour). The distance from Earth to Mars varies because their orbits around the Sun are independent. On average, Mars is 100 million miles from Earth. If we use laser communication systems, what will be the delay between Houston and NASA astronauts on Mars?

54. **Speed of Sound.** The speed of sound is approximately 760 miles per hour in air. If a gun is fired $\frac{1}{2}$ mile away, how long will it take the sound to reach you?

55. **Business.** During the month of February 2011, the average price of gasoline rose 4.7% in the United States. If the average price of gasoline at the end of February 2011 was $3.21 per gallon, what was the price of gasoline at the beginning of February?

56. **Business.** During the Christmas shopping season of 2010, the average price of a flat screen television fell by 40%. A shopper purchased a 42-inch flat screen television for $299 in late November 2010. How much would the shopper have paid, to the nearest dollar, for the same television if it was purchased in September 2010?

57. Medicine. A patient requires an IV of 0.9% saline solution, also known as normal saline solution. How much distilled water, to the nearest milliliter, must be added to 100 milliliters of a 3% saline solution to produce normal saline?

58. Medicine. A patient requires an IV of D5W, a 5% solution of Dextrose (sugar) in water. To the nearest milliliter, how much D20W, a 20% solution of Dextrose in water, must be added to 100 milliliters of distilled water to produce a D5W solution?

59. Boating. A motorboat can maintain a constant speed of 16 miles per hour relative to the water. The boat makes a trip upstream to a marina in 20 minutes. The return trip takes 15 minutes. What is the speed of the current?

60. Aviation. A Cessna 175 can average 130 miles per hour. If a trip takes 2 hours one way and the return takes 1 hour and 15 minutes, find the wind speed, assuming it is constant.

61. Distance–Rate–Time. A jogger and a walker cover the same distance. The jogger finishes in 40 minutes. The walker takes an hour. How fast is each exerciser moving if the jogger runs 2 miles per hour faster than the walker?

62. Distance–Rate–Time. A high school student in Seattle, Washington, attended the University of Central Florida. On the way to UCF he took a southern route. After graduation he returned to Seattle via a northern trip. On both trips he had the same average speed. If the southern trek took 45 hours and the northern trek took 50 hours, and the northern trek was 300 miles longer, how long was each trip?

63. Distance–Rate–Time. College roommates leave for their first class in the same building. One walks at 2 miles per hour and the other rides his bike at a slow 6 miles per hour pace. How long will it take each to get to class if the walker takes 12 minutes longer to get to class and they travel on the same path?

64. Distance–Rate–Time. A long-distance delivery service sends out a truck with a package at 7 A.M. At 7:30 the manager realizes there was another package going to the same location. He sends out a car to catch the truck. If the truck travels at an average speed of 50 miles per hour and the car travels at 70 miles per hour, how long will it take the car to catch the truck?

65. Work. Christopher can paint the interior of his house in 15 hours. If he hires Cynthia to help him, together they can do the same job in 9 hours. If he lets Cynthia work alone, how long will it take her to paint the interior of his house?

66. Work. Jay and Morgan work in the summer for a landscaper. It takes Jay 3 hours to complete the company's largest yard alone. If Morgan helps him, it takes only 1 hour. How much time would it take Morgan alone?

67. Work. Tracey and Robin deliver Coke products to local convenience stores. Tracey can complete the deliveries in 4 hours alone. Robin can do it in 6 hours alone. If they decide to work together on a Saturday, how long will it take?

68. Work. Joshua can deliver his newspapers in 30 minutes. It takes Amber 20 minutes to do the same route. How long would it take them to deliver the newspapers if they worked together?

69. Sports. In Super Bowl XXXVII, the Tampa Bay Buccaneers scored a total of 48 points. All of their points came from field goals and touchdowns. Field goals are worth 3 points and each touchdown was worth 7 points (Martin Gramatica was successful in every extra point attempt). They scored a total of 8 times. How many field goals and touchdowns were scored?

70. Sports. A tight end can run the 100-yard dash in 12 seconds. A defensive back can do it in 10 seconds. The tight end catches a pass at his own 20 yard line with the defensive back at the 15 yard line. If no other players are nearby, at what yard line will the defensive back catch up to the tight end?

71. Recreation. How do two children of different weights balance on a seesaw? The heavier child sits closer to the center and the lighter child sits further away. When the product of the weight of the child and the distance from the center is equal on both sides, the seesaw should be horizontal to the ground. Suppose Max weighs 42 pounds and Maria weighs 60 pounds. If Max sits 5 feet from the center, how far should Maria sit from the center in order to balance the seesaw horizontal to the ground?

72. Recreation. Refer to Exercise 71. Suppose Martin, who weighs 33 pounds, sits on the side of the seesaw with Max. If their average distance to the center is 4 feet, how far should Maria sit from the center in order to balance the seesaw horizontal to the ground?

73. Recreation. If a seesaw has an adjustable bench, then the board can be positioned over the fulcrum. Maria and Max in Exercise 71 decide to sit on the very edge of the board on each side. Where should the fulcrum be placed along the board in order to balance the seesaw horizontally to the ground? Give the answer in terms of the distance from each child's end.

74. Recreation. Add Martin (Exercise 72) to Max's side of the seesaw and recalculate Exercise 73.

CATCH THE MISTAKE

In Exercises 75–76, explain the mistake that is made.

75. Solve the equation $4x + 3 = 6x - 7$.

Solution:

Subtract $4x$ and add 7 to the equation. $\quad 3 = 6x$

Divide by 3. $\quad x = 2$

This is incorrect. What mistake was made?

76. Solve the equation $3(x + 1) + 2 = x - 3(x - 1)$.

Solution:

$$3x + 3 + 2 = x - 3x - 3$$
$$3x + 5 = -2x - 3$$
$$5x = -8$$
$$x = -\frac{8}{5}$$

This is incorrect. What mistake was made?

CONCEPTUAL

77. Solve for x, given that a, b, and c are real numbers and $a \neq 0$:

$$ax + b = c$$

78. Find the number a for which $y = 2$ is a solution of the equation $y - a = y + 5 - 3ay$.

In Exercises 79–86, solve each formula for the specified variable.

79. $P = 2l + 2w$ for w

80. $P = 2l + 2w$ for l

81. $A = \frac{1}{2}bh$ for h

82. $C = 2\pi r$ for r

83. $A = lw$ for w

84. $d = rt$ for t

85. $V = lwh$ for h

86. $V = \pi r^2 h$ for h

CHALLENGE

87. Tricia and Janine are roommates and leave Houston on Interstate 10 at the same time to visit their families for a long weekend. Tricia travels west and Janine travels east. If Tricia's average speed is 12 miles per hour faster than Janine's, find the speed of each if they are 320 miles apart in 2 hours and 30 minutes.

88. Rick and Mike are roommates and leave Gainesville on Interstate 75 at the same time to visit their girlfriends for a long weekend. Rick travels north and Mike travels south. If Mike's average speed is 8 miles per hour faster than Rick's, find the speed of each if they are 210 miles apart in 1 hour and 30 minutes.

TECHNOLOGY

In Exercises 89–92, graph the function represented by each side of the equation in the same viewing rectangle and solve for *x*.

89. $3(x + 2) - 5x = 3x - 4$

90. $-5(x - 1) - 7 = 10 - 9x$

91. $2x + 6 = 4x - 2x + 8 - 2$

92. $10 - 20x = 10x - 30x + 20 - 10$

93. Suppose you bought a house for \$132,500 and sold it 3 years later for \$168,190. Plot these points using a graphing utility. Assuming a linear relationship, how much could you have sold the house for had you waited 2 additional years?

94. Suppose you bought a house for \$132,500 and sold it 3 years later for \$168,190. Plot these points using a graphing utility. Assuming a linear relationship, how much could you have sold the house for had you sold it 1 year after buying it?

95. A golf club membership has two options. Option A is a \$300 monthly fee plus \$15 cart fee every time you play. Option B has a \$150 monthly fee and a \$42 fee every time you play. Write a mathematical model for monthly costs for each plan and graph both in the same viewing rectangle using a graphing utility. Explain when Option A is the better deal and when Option B is the better deal.

96. A phone provider offers two calling plans. Plan A has a \$30 monthly charge and a \$0.10 per minute charge on every call. Plan B has a \$50 monthly charge and a \$0.03 per minute charge on every call. Explain when Plan A is the better deal and when Plan B is the better deal.

SECTION 0.2 QUADRATIC EQUATIONS

SKILLS OBJECTIVES

- Solve quadratic equations by factoring.
- Use the square root method to solve quadratic equations.
- Solve quadratic equations by completing the square.
- Use the quadratic formula to solve quadratic equations.

CONCEPTUAL OBJECTIVES

- Choose appropriate methods for solving quadratic equations.
- Interpret different types of solution sets (real, imaginary, complex conjugates, repeated roots).
- Derive the quadratic formula.

Factoring

In a linear equation, the variable is raised only to the first power in any term where it occurs. In a *quadratic equation*, the variable is raised to the second power in at least one term. Examples of *quadratic equations*, also called second-degree equations, are

$$x^2 + 3 = 7 \qquad 5x^2 + 4x - 7 = 0 \qquad x^2 - 3 = 0$$

DEFINITION **Quadratic Equation**

A **quadratic equation** in x is an equation that can be written in the **standard form**

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

Study Tip

In a quadratic equation the variable is raised to the power of 2, which is the highest power present in the equation.

There are several methods for solving quadratic equations: *factoring*, the *square root method*, *completing the square*, and the *quadratic formula.*

FACTORING METHOD

The **factoring method** applies the **zero product property**:

WORDS	MATH
If a product is zero, then at least one of its factors has to be zero.	If $B \cdot C = 0$, then $B = 0$ or $C = 0$ or both.

Consider $(x - 3)(x + 2) = 0$. The zero product property says that $x - 3 = 0$ or $x + 2 = 0$, which leads to $x = -2$ or $x = 3$. The solution set is $\{-2, 3\}$.

When a quadratic equation is written in the standard form $ax^2 + bx + c = 0$, it may be possible to factor the left side of the equation as a product of two first-degree polynomials. We use the zero product property and set each linear factor equal to zero. We solve the resulting two linear equations to obtain the solutions of the quadratic equation.

EXAMPLE 1 Solving a Quadratic Equation by Factoring

Solve the equation $x^2 - 6x - 16 = 0$.

Solution:

The quadratic equation is already in standard form. $\quad x^2 - 6x - 16 = 0$

Factor the left side into a product of two linear factors. $\quad (x - 8)(x + 2) = 0$

If a product equals zero, one of its factors has to be equal to zero. $\quad x - 8 = 0 \quad \text{or} \quad x + 2 = 0$

Solve both linear equations. $\quad \boxed{x = 8 \quad \text{or} \quad x = -2}$

The solution set is $\boxed{\{-2, 8\}}$.

■ **YOUR TURN** Solve the quadratic equation $x^2 + x - 20 = 0$ by factoring.

■ **Answer:** The solution is $x = -5, 4$. The solution set is $\{-5, 4\}$.

▼ **CAUTION**

Do not divide by a variable (because the value of that variable may be zero). Bring all terms to one side first and then factor.

EXAMPLE 2 Solving a Quadratic Equation by Factoring

Solve the equation $2x^2 = 3x$.

COMMON MISTAKE

The common mistake here is dividing both sides by x, which is not allowed because x might be 0.

CORRECT

Write the equation in standard form by subtracting $3x$.

$$2x^2 - 3x = 0$$

Factor the left side.

$$x(2x - 3) = 0$$

Use the zero product property and set each factor equal to zero.

$$x = 0 \text{ or } 2x - 3 = 0$$

Solve each linear equation.

$$\boxed{x = 0 \text{ or } x = \frac{3}{2}}$$

The solution set is $\boxed{\left\{0, \frac{3}{2}\right\}}$.

INCORRECT

Write the original equation.

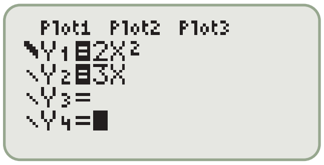
$$2x^2 = 3x$$

The **error** occurs here when both sides are divided by x.

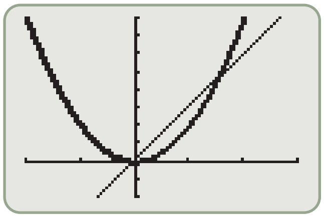
$$2x = 3$$

Technology Tip

Use a graphing utility to display graphs of $y_1 = 2x^2$ and $y_2 = 3x$.



The x-coordinates of the points of intersection are the solutions to this equation.



In Example 2, the root $x = 0$ is lost when the original quadratic equation is divided by x. Remember to put the equation in standard form first and then factor.

Square Root Method

The square root of 16, $\sqrt{16}$, is 4, *not* ± 4. In the Appendix, the **principal square root** is discussed. The solutions to $x^2 = 16$, however, are $x = -4$ and $x = 4$. Let us now investigate quadratic equations that do not have a first-degree term. They have the form

$$ax^2 + c = 0 \quad a \neq 0$$

The method we use to solve such equations uses the square root property.

Square root property

Words

If an expression squared is equal to a constant, then that expression is equal to the positive or negative square root of the constant.

Math

If $x^2 = P$, then $x = \pm\sqrt{P}$.

Note: The variable squared must be isolated first (coefficient equal to 1).

Example 3 Using the Square Root Property

Solve the equation $3x^2 - 27 = 0$.

Solution:

Add 27 to both sides. $3x^2 = 27$

Divide both sides by 3. $x^2 = 9$

Apply the square root property. $x = \pm\sqrt{9} = \pm 3$

The solution set is $\boxed{\{-3, 3\}}$.

If we alter Example 3 by changing subtraction to addition, we see in Example 4 that we get imaginary roots, as opposed to real roots which is reviewed in the Appendix.

Example 4 Using the Square Root Property

Solve the equation $3x^2 + 27 = 0$.

Solution:

Subtract 27 from both sides. $3x^2 = -27$

Divide by 3. $x^2 = -9$

Apply the square root property. $x = \pm\sqrt{-9}$

Simplify. $x = \pm i\sqrt{9} = \pm 3i$

The solution set is $\boxed{\{-3i, 3i\}}$.

■ **Your turn** Solve the equations:

a. $y^2 - 147 = 0$ **b.** $v^2 + 64 = 0$

■ **Answer:**

a. The solution is $y = \pm 7\sqrt{3}$. The solution set is $\{-7\sqrt{3}, 7\sqrt{3}\}$.

b. The solution is $v = \pm 8i$. The solution set is $\{-8i, 8i\}$.

EXAMPLE 5 **Using the Square Root Property**

Solve the equation $(x - 2)^2 = 16$.

Solution:

Approach 1: If an expression squared is 16, then the expression equals $\pm\sqrt{16}$. $\quad (x - 2) = \pm\sqrt{16}$

Separate into two equations. $\quad x - 2 = \sqrt{16}$ or $x - 2 = -\sqrt{16}$

$x - 2 = 4 \qquad x - 2 = -4$

$x = 6 \qquad x = -2$

The solution set is $\boxed{\{-2, 6\}}$.

Approach 2: It is acceptable notation to keep the equations together.

$$(x - 2) = \pm\sqrt{16}$$
$$x - 2 = \pm 4$$
$$x = 2 \pm 4$$
$$\boxed{x = -2, 6}$$

Completing the Square

Factoring and the square root method are two efficient, quick procedures for solving many quadratic equations. However, some equations, such as $x^2 - 10x - 3 = 0$, cannot be solved directly by these methods. A more general procedure to solve this kind of equation is called **completing the square**. The idea behind completing the square is to transform any standard quadratic equation $ax^2 + bx + c = 0$ into the form $(x + A)^2 = B$, where A and B are constants and the left side, $(x + A)^2$, has the form of a **perfect square**. This last equation can then be solved by the square root method. How do we transform the first equation into the second equation?

Note that the above-mentioned example, $x^2 - 10x - 3 = 0$, cannot be factored into expressions in which all numbers are integers (or even rational numbers). We can, however, transform this quadratic equation into a form that contains a perfect square.

Words	Math
Write the original equation.	$x^2 - 10x - 3 = 0$
Add 3 to both sides.	$x^2 - 10x = 3$
Add 25 to both sides.*	$x^2 - 10x + 25 = 3 + 25$
The left side can be written as a perfect square.	$(x - 5)^2 = 28$
Apply the square root method.	$x - 5 = \pm\sqrt{28}$
Add 5 to both sides.	$x = 5 \pm 2\sqrt{7}$

*Why did we add 25 to both sides? Recall that $(x - c)^2 = x^2 - 2xc + c^2$. In this case $c = 5$ in order for $-2xc = -10x$. Therefore, the desired perfect square $(x - 5)^2$ results in $x^2 - 10x + 25$. Applying this product, we see that $+25$ is needed.

If the coefficient of x^2 is 1, a systematic approach is to take the coefficient of the first degree term of $x^2 - 10x - 3 = 0$, which is -10. Divide -10 by 2 to get -5; then square -5 to get 25.

Solving a Quadratic Equation by Completing the Square

Words	**Math**
Express the quadratic equation in the following form.	$x^2 + bx = c$
Divide b by 2 and square the result, then add the square to both sides.	$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$
Write the left side of the equation as a perfect square.	$\left(x + \frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$
Solve using the square root method.	

EXAMPLE 6 Completing the Square

Solve the quadratic equation $x^2 + 8x - 3 = 0$ by completing the square.

Solution:

Add 3 to both sides. $x^2 + 8x = 3$

Add $\left(\frac{1}{2} \cdot 8\right)^2 = 4^2$ to both sides. $x^2 + 8x + 4^2 = 3 + 4^2$

Write the left side as a perfect square and simplify the right side. $(x + 4)^2 = 19$

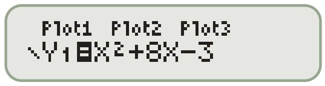
Apply the square root method to solve. $x + 4 = \pm\sqrt{19}$

Subtract 4 from both sides. $\boxed{x = -4 \pm \sqrt{19}}$

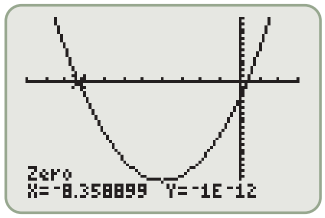
The solution set is $\boxed{\left\{-4 - \sqrt{19}, -4 + \sqrt{19}\right\}}$.

Technology Tip

Graph $y_1 = x^2 + 8x - 3$.



The x-intercepts are the solutions to this equation.



In Example 6, the leading coefficient (the coefficient of the x^2 term) is 1. When the leading coefficient is not 1, start by first dividing the equation by that leading coefficient.

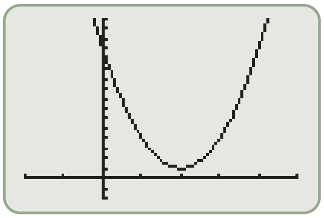
Study Tip

When the leading coefficient is not 1, start by first dividing the equation by that leading coefficient.

Technology Tip

Graph $y_1 = 3x^2 - 12x + 13$.

The graph does not cross the x-axis, so there is no real solution to this equation.



EXAMPLE 7 Completing the Square When the Leading Coefficient Is Not Equal to 1

Solve the equation $3x^2 - 12x + 13 = 0$ by completing the square.

Solution:

Divide by the leading coefficient, 3. $\quad x^2 - 4x + \frac{13}{3} = 0$

Collect the variables to one side of the equation and constants to the other side. $\quad x^2 - 4x = -\frac{13}{3}$

Add $\left(-\frac{4}{2}\right)^2 = 4$ to both sides. $\quad x^2 - 4x + 4 = -\frac{13}{3} + 4$

Write the left side of the equation as a perfect square and simplify the right side. $\quad (x-2)^2 = -\frac{1}{3}$

Solve using the square root method. $\quad x - 2 = \pm\sqrt{-\frac{1}{3}}$

Simplify. $\quad x = 2 \pm i\sqrt{\frac{1}{3}}$

Rationalize the denominator (Appendix). $\quad x = 2 \pm \frac{i}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

Simplify. $\quad x = 2 \pm \frac{i\sqrt{3}}{3}$

$$\boxed{x = 2 - \frac{i\sqrt{3}}{3},\ x = 2 + \frac{i\sqrt{3}}{3}}$$

The solution set is $\boxed{\left\{2 - \frac{i\sqrt{3}}{3}, 2 + \frac{i\sqrt{3}}{3}\right\}}$.

YOUR TURN Solve the equation $2x^2 - 4x + 3 = 0$ by completing the square.

Answer: The solution is $x = 1 \pm \frac{i\sqrt{2}}{2}$. The solution set is $\left\{1 - \frac{i\sqrt{2}}{2}, 1 + \frac{i\sqrt{2}}{2}\right\}$.

The Quadratic Formula

Let us now consider the most general quadratic equation:

$$ax^2 + bx + c = 0 \quad a \neq 0$$

We can solve this equation by completing the square.

Words	**Math**
Divide the equation by the leading coefficient a.	$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
Subtract $\frac{c}{a}$ from both sides.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
Square half of $\frac{b}{a}$ and add the result $\left(\frac{b}{2a}\right)^2$ to both sides.	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$
Write the left side of the equation as a perfect square and the right side as a single fraction.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Solve using the square root method.	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$
Subtract $\frac{b}{2a}$ from both sides and simplify the radical.	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
Write as a single fraction.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

We have derived the **quadratic formula**.

The Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then the solution is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The quadratic equation must be in standard form ($ax^2 + bx + c = 0$) in order to identify the parameters:

a—coefficient of x^2 $\qquad$ b—coefficient of x $\qquad$ c—constant

Study Tip

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Read as "negative b plus or minus the square root of the quantity b squared minus $4ac$ all over $2a$."

We read this formula as *negative b plus or minus the square root of the quantity b squared minus 4ac all over 2a.* It is important to note that negative b could be positive (if b is negative). For this reason, an alternate form is "opposite b. . . ." The quadratic formula should be memorized and used when simpler methods (factoring and the square root method) cannot be used. The quadratic formula works for *any* quadratic equation.

Study Tip

The quadratic formula works for *any* quadratic equation.

Study Tip

Using parentheses as placeholders helps avoid ± errors.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$$

EXAMPLE 8 Using the Quadratic Formula and Finding Two Distinct Real Roots

Use the quadratic formula to solve the quadratic equation $x^2 - 4x - 1 = 0$.

Solution:

For this problem, $a = 1$, $b = -4$, and $c = -1$.

Write the quadratic formula. $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use parentheses to avoid losing a minus sign. $$x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$$

Substitute values for a, b, and c into the parentheses. $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$

Simplify. $$x = \frac{4 \pm \sqrt{16 + 4}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = \frac{4}{2} \pm \frac{2\sqrt{5}}{2} = \boxed{2 \pm \sqrt{5}}$$

The solution set $\boxed{\{2 - \sqrt{5}, 2 + \sqrt{5}\}}$ contains two distinct real numbers.

■ **YOUR TURN** Use the quadratic formula to solve the quadratic equation $x^2 + 6x - 2 = 0$.

■ **Answer:** The solution is $x = -3 \pm \sqrt{11}$. The solution set is $\{-3 - \sqrt{11}, -3 + \sqrt{11}\}$.

EXAMPLE 9 Using the Quadratic Formula and Finding Two Complex Roots

Use the quadratic formula to solve the quadratic equation $x^2 + 8 = 4x$.

Solution:

Write this equation in standard form $x^2 - 4x + 8 = 0$ in order to identify $a = 1$, $b = -4$, and $c = 8$.

Write the quadratic formula. $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use parentheses to avoid overlooking a minus sign. $$x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$$

Substitute the values for a, b, and c into the parentheses. $$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)}$$

Simplify. $$x = \frac{4 \pm \sqrt{16 - 32}}{2} = \frac{4 \pm \sqrt{-16}}{2} = \frac{4 \pm 4i}{2} = \frac{4}{2} \pm \frac{4i}{2} = \boxed{2 \pm 2i}$$

The solution set $\boxed{\{2 - 2i, 2 + 2i\}}$ contains two complex numbers. Note that they are complex conjugates of each other.

■ **YOUR TURN** Use the quadratic formula to solve the quadratic equation $x^2 + 2 = 2x$.

■ **Answer:** The solution set is $\{1 - i, 1 + i\}$.

EXAMPLE 10 **Using the Quadratic Formula and Finding One Repeated Real Root**

Use the quadratic formula to solve the quadratic equation $4x^2 - 4x + 1 = 0$.

Solution:

Identify a, b, and c. $a = 4, b = -4, c = 1$

Write the quadratic formula. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Use parentheses to avoid losing a minus sign. $x = \dfrac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$

Substitute values $a = 4$, $b = -4$, $c = 1$. $x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(1)}}{2(4)}$

Simplify. $x = \dfrac{4 \pm \sqrt{16 - 16}}{8} = \dfrac{4 \pm 0}{8} = \dfrac{1}{2}$

The solution set is a repeated real root $\boxed{\left\{\frac{1}{2}\right\}}$.

Note: This quadratic also could have been solved by factoring: $(2x - 1)^2 = 0$.

YOUR TURN Use the quadratic formula to solve the quadratic equation $9x^2 - 6x + 1 = 0$.

Answer: $\left\{\frac{1}{3}\right\}$

Types of Solutions

The expression inside the radical, $b^2 - 4ac$, is called the **discriminant**. The discriminant gives important information about the corresponding solutions or roots of $ax^2 + bx + c = 0$, where a, b, and c are real numbers.

$b^2 - 4ac$	Solutions (roots)
Positive	Two distinct real roots
0	One real root (a double or repeated root)
Negative	Two complex roots (complex conjugates)

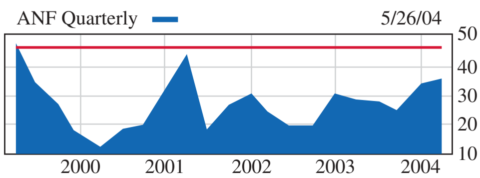
In Example 8, the discriminant is positive and the solution has two distinct real roots. In Example 9, the discriminant is negative and the solution has two complex (conjugate) roots. In Example 10, the discriminant is zero and the solution has one repeated real root.

Applications Involving Quadratic Equations

In Section 0.1, we developed a procedure for solving word problems involving linear equations. The procedure is the same for applications involving quadratic equations. The only difference is that the mathematical equations will be quadratic, as opposed to linear.

EXAMPLE 11 Stock Value

From 1999 to 2001 the price of Abercrombie & Fitch's (ANF) stock was approximately given by $P = 0.2t^2 - 5.6t + 50.2$, where P is the price of stock in dollars, t is in months, and $t = 1$ corresponds to January 1999. When was the value of the stock worth \$30?



Technology Tip

The graphing utility screen for $\dfrac{-(-5.6) \pm \sqrt{(-5.6)^2 - 4(0.2)(20.2)}}{2(0.2)}$

```
(-(-5.6)-√((-5.6
)²-4*.2*20.2))/(
2*.2)
       4.253205655
```

```
(-(-5.6)+√((-5.6
)²-4*.2*20.2))/(
2*.2)
       23.74679434
█
```

Study Tip

Dimensions such as length and width are distances, which are defined as positive quantities. Although the mathematics may yield both positive and negative values, the negative values are excluded.

Solution:

STEP 1 Identify the question.
When is the price of the stock equal to \$30?

STEP 2 Make notes.
Stock price: $P = 0.2t^2 - 5.6t + 50.2$

$P = 30$

STEP 3 Set up an equation. $0.2t^2 - 5.6t + 50.2 = 30$

STEP 4 Solve the equation.
Subtract 30 from both sides. $0.2t^2 - 5.6t + 20.2 = 0$

Solve for t using the quadratic formula.

$$t = \frac{-(-5.6) \pm \sqrt{(-5.6)^2 - 4(0.2)(20.2)}}{2(0.2)}$$

Simplify.

$$t \approx \frac{5.6 \pm 3.9}{0.4} \approx 4.25, 23.75$$

Rounding these two numbers, we find that $t \approx 4$ and $t \approx 24$. Since $t = 1$ corresponds to January 1999, these two solutions correspond to $\boxed{\text{April 1999 and December 2000}}$.

STEP 5 Check the solution.
Look at the figure. The horizontal axis represents the year (2000 corresponds to January 2000), and the vertical axis represents the stock price. Estimating when the stock price is approximately \$30, we find April 1999 and December 2000.

SECTION 0.2 SUMMARY

The four methods for solving quadratic equations,

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

are *factoring*, the *square root method*, *completing the square*, and the *quadratic formula*. Factoring and the square root method are the quickest and easiest but cannot always be used. The quadratic formula and completing the square work for all quadratic equations.

Quadratic Formula: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A quadratic equation can have three types of solutions: two distinct real roots, one real root (repeated), or two complex roots (conjugates of each other).

SECTION 0.2 EXERCISES

SKILLS

In Exercises 1–22, solve by factoring.

1. $x^2 - 5x + 6 = 0$ **2.** $v^2 + 7v + 6 = 0$ **3.** $p^2 - 8p + 15 = 0$ **4.** $u^2 - 2u - 24 = 0$

5. $x^2 = 12 - x$ **6.** $11x = 2x^2 + 12$ **7.** $16x^2 + 8x = -1$ **8.** $3x^2 + 10x - 8 = 0$

9. $9y^2 + 1 = 6y$ **10.** $4x = 4x^2 + 1$ **11.** $8y^2 = 16y$ **12.** $3A^2 = -12A$

13. $9p^2 = 12p - 4$ **14.** $4u^2 = 20u - 25$ **15.** $x^2 - 9 = 0$ **16.** $16v^2 - 25 = 0$

17. $x(x + 4) = 12$ **18.** $3t^2 - 48 = 0$ **19.** $2p^2 - 50 = 0$ **20.** $5y^2 - 45 = 0$

21. $3x^2 = 12$ **22.** $7v^2 = 28$

In Exercises 23–34, solve using the square root method.

23. $p^2 - 8 = 0$ **24.** $y^2 - 72 = 0$ **25.** $x^2 + 9 = 0$ **26.** $v^2 + 16 = 0$

27. $(x - 3)^2 = 36$ **28.** $(x - 1)^2 = 25$ **29.** $(2x + 3)^2 = -4$ **30.** $(4x - 1)^2 = -16$

31. $(5x - 2)^2 = 27$ **32.** $(3x + 8)^2 = 12$ **33.** $(1 - x)^2 = 9$ **34.** $(1 - x)^2 = -9$

In Exercises 35–46, solve by completing the square.

35. $x^2 + 2x = 3$ **36.** $y^2 + 8y - 2 = 0$ **37.** $t^2 - 6t = -5$ **38.** $x^2 + 10x = -21$

39. $y^2 - 4y + 3 = 0$ **40.** $x^2 - 7x + 12 = 0$ **41.** $2p^2 + 8p = -3$ **42.** $2x^2 - 4x + 3 = 0$

43. $2x^2 - 7x + 3 = 0$ **44.** $3x^2 - 5x - 10 = 0$ **45.** $\dfrac{x^2}{2} - 2x = \dfrac{1}{4}$ **46.** $\dfrac{t^2}{3} + \dfrac{2t}{3} + \dfrac{5}{6} = 0$

In Exercises 47–58, solve using the quadratic formula.

47. $t^2 + 3t - 1 = 0$ **48.** $t^2 + 2t = 1$ **49.** $s^2 + s + 1 = 0$ **50.** $2s^2 + 5s = -2$

51. $3x^2 - 3x - 4 = 0$ **52.** $4x^2 - 2x = 7$ **53.** $x^2 - 2x + 17 = 0$ **54.** $4m^2 + 7m + 8 = 0$

55. $5x^2 + 7x = 3$ **56.** $3x^2 + 5x = -11$ **57.** $\frac{1}{4}x^2 + \frac{2}{3}x - \frac{1}{2} = 0$ **58.** $\frac{1}{4}x^2 - \frac{2}{3}x - \frac{1}{3} = 0$

In Exercises 59–74, solve using any method.

59. $v^2 - 8v = 20$ **60.** $v^2 - 8v = -20$ **61.** $t^2 + 5t - 6 = 0$ **62.** $t^2 + 5t + 6 = 0$

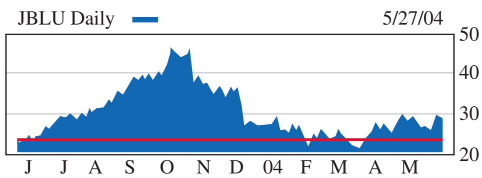
63. $(x + 3)^2 = 16$ **64.** $(x + 3)^2 = -16$ **65.** $(p - 2)^2 = 4p$ **66.** $(u + 5)^2 = 16u$

67. $8w^2 + 2w + 21 = 0$ **68.** $8w^2 + 2w - 21 = 0$ **69.** $3p^2 - 9p + 1 = 0$ **70.** $3p^2 - 9p - 1 = 0$

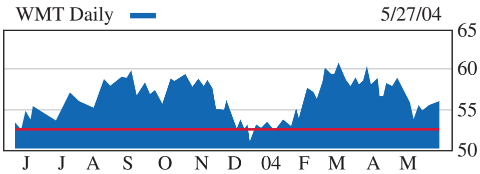
71. $\frac{2}{3}t^2 - \frac{4}{3}t = \frac{1}{5}$ **72.** $\frac{1}{2}x^2 + \frac{2}{3}x = \frac{2}{5}$ **73.** $x^2 - 0.1x = 0.12$ **74.** $y^2 - 0.5y = -0.06$

APPLICATIONS

75. Stock Value. From June 2003 until April 2004 JetBlue airlines stock (JBLU) was approximately worth $P = -4t^2 + 80t - 360$, where P denotes the price of the stock in dollars and t corresponds to months, with $t = 1$ corresponding to January 2003. During what months was the stock equal to \$24?



76. Stock Value. From November 2003 until March 2004 Wal-Mart Stock (WMT) was approximately worth $P = 2t^2 - 12t + 70$, where P denotes the price of the stock in dollars and t corresponds to months, with $t = 1$ corresponding to November 2003. During what months was the stock equal to \$60?



In Exercises 77 and 78 refer to the following:

Research indicates that monthly profit for Widgets R Us is modeled by the function

$$P = -100 + (0.2q - 3)q$$

where P is profit measured in millions of dollars and q is the quantity of widgets produced measured in thousands.

77. Business. Find the break-even point for a month to the nearest unit.

78. Business. Find the production level that produces a monthly profit of \$40 million.

In Exercises 79 and 80 refer to the following:

In response to economic conditions, a local business explores the effect of a price increase on weekly profit. The function

$$P = -5(x + 3)(x - 24)$$

models the effect that a price increase of x dollars on a bottle of wine will have on the profit P measured in dollars.

79. Business/Economics. What is the smallest price increase that will produce a weekly profit of \$460?

80. Business/Economics. What is the smallest price increase that will produce a weekly profit of \$630?

In Exercises 81 and 82 refer to the following:

An epidemiological study of the spread of the flu in a small city finds that the total number P of people who contracted the flu t days into an outbreak is modeled by the function

$$P = -t^2 + 13t + 130 \quad 1 \le t \le 6$$

81. Health/Medicine. After approximately how many days will 160 people have contracted the flu?

82. Health/Medicine. After approximately how many days will 172 people have contracted the flu?

83. Environment: Reduce Your Margins, Save a Tree. Let's define the *usable area* of an 8.5-inch by 11-inch piece of paper as the rectangular space between the margins of that piece of paper. Assume the default margins in a word processor in a college's computer lab are set up to be 1.25 inches wide (top and bottom) and 1 inch wide (left and right). Answer the following questions using this information.

a. Determine the amount of usable space, in square inches, on one side of an 8.5-inch by 11-inch piece of paper with the default margins of 1.25 inch and 1 inch.

b. The Green Falcons, a campus environmental club, has convinced their college's computer lab to reduce the default margins in their word-processing software by x inches. Create and simplify the quadratic expression that represents the new usable area, in square inches, of one side of an 8.5-inch by 11-inch piece of paper if the default margins at the computer lab are each reduced by x inches.

c. Subtract the usable space in part (a) from the expression in part (b). Explain what this difference represents.

d. If 10 pages are printed using the new margins and as a result the computer lab saved one whole sheet of paper, then by how much did the computer lab reduce the margins? Round to the nearest tenth of an inch.

84. Environment: Reduce Your Margins, Save a Tree. Repeat Exercise 83 assuming the computer lab's default margins are 1 inch all the way around (left, right, top, and bottom). If 15 pages are printed using the new margins and as a result the computer lab saved one whole sheet of paper, then by how much did the computer lab reduce the margins? Round to the nearest tenth of an inch.

85. Television. A standard 32-inch television has a 32 inch diagonal and a 25 inch width. What is the height of the 32-inch television?

86. Television. A 42-inch LCD television has a 42 inch diagonal and a 20 inch height. What is the width of the 42-inch LCD television?

87. Numbers. Find two consecutive numbers such that their product is 306.

88. Numbers. Find two consecutive odd integers such that their product is 143.

89. Geometry. The area of a rectangle is 135 square feet. The width is 6 feet less than the length. Find the dimensions of the rectangle.

90. Geometry. A rectangle has an area of 31.5 square meters. If the length is 2 meters more than twice the width, find the dimensions of the rectangle.

91. Geometry. A triangle has a height that is 2 more than 3 times the base and an area of 60 square units. Find the base and height.

92. Geometry. A square's side is increased by 3 yards, which corresponds to an increase in the area by 69 square yards. How many yards is the side of the initial square?

93. Falling Objects. If a person drops a water balloon off the rooftop of a 100-foot building, the height of the water balloon is given by the equation $h = -16t^2 + 100$, where t is in seconds. When will the water balloon hit the ground?

94. Falling Objects. If the person in Exercise 85 throws the water balloon downward with a speed of 5 feet per second, the height of the water balloon is given by the equation $h = -16t^2 - 5t + 100$, where t is in seconds. When will the water balloon hit the ground?

95. Gardening. A square garden has an area of 900 square feet. If a sprinkler (with a circular pattern) is placed in the center of the garden, what is the minimum radius of spray the sprinkler would need in order to water all of the garden?

96. Sports. A baseball diamond is a square. The distance from base to base is 90 feet. What is the distance from home plate to second base?

97. Volume. A flat square piece of cardboard is used to construct an open box. Cutting a 1 foot by 1 foot square off of each corner and folding up the edges will yield an open box (assuming these edges are taped together). If the desired volume of the box is 9 cubic feet, what are the dimensions of the original square piece of cardboard?

98. Volume. A rectangular piece of cardboard whose length is twice its width is used to construct an open box. Cutting a 1 foot by 1 foot square off of each corner and folding up the edges will yield an open box. If the desired volume is 12 cubic feet, what are the dimensions of the original rectangular piece of cardboard?

99. Gardening. A landscaper has planted a rectangular garden that measures 8 feet by 5 feet. He has ordered 1 cubic yard (27 cubic feet) of stones for a border along the outside of the garden. If the border needs to be 4 inches deep and he wants to use all of the stones, how wide should the border be?

100. Gardening. A gardener has planted a semicircular rose garden with a radius of 6 feet, and 2 cubic yards of mulch (1 cu yd = 27 cu ft) is being delivered. Assuming she uses all of the mulch, how deep will the layer of mulch be?

101. Work. Lindsay and Kimmie, working together, can balance the financials for the Kappa Kappa Gamma sorority in 6 days. Lindsay by herself can complete the job in 5 days less than Kimmie. How long will it take Lindsay to complete the job by herself?

102. Work. When Jack cleans the house, it takes him 4 hours. When Ryan cleans the house, it takes him 6 hours. How long would it take both of them if they worked together?

CATCH THE MISTAKE

In Exercises 103–106, explain the mistake that is made.

103.
$$\begin{aligned} t^2 - 5t - 6 &= 0 \\ (t-3)(t-2) &= 0 \\ t &= 2, 3 \end{aligned}$$

104.
$$\begin{aligned} (2y-3)^2 &= 25 \\ 2y - 3 &= 5 \\ 2y &= 8 \\ y &= 4 \end{aligned}$$

105.
$$\begin{aligned} 16a^2 + 9 &= 0 \\ 16a^2 &= -9 \\ a^2 &= -\frac{9}{16} \\ a &= \pm\sqrt{\frac{9}{16}} \\ a &= \pm\frac{3}{4} \end{aligned}$$

106.
$$\begin{aligned} 2x^2 - 4x &= 3 \\ 2(x^2 - 2x) &= 3 \\ 2(x^2 - 2x + 1) &= 3 + 1 \\ 2(x-1)^2 &= 4 \\ (x-1)^2 &= 2 \\ x - 1 &= \pm\sqrt{2} \\ x &= 1 \pm \sqrt{2} \end{aligned}$$

CONCEPTUAL

In Exercises 107–110, determine whether the following statements are true or false.

107. The equation $(3x + 1)^2 = 16$ has the same solution set as the equation $3x + 1 = 4$.

108. The quadratic equation $ax^2 + bx + c = 0$ can be solved by the square root method only if $b = 0$.

109. All quadratic equations can be solved exactly.

110. The quadratic formula can be used to solve any quadratic equation.

111. Write a quadratic equation in standard form that has $x = a$ as a repeated real root. Alternate solutions are possible.

112. Write a quadratic equation in standard form that has $x = bi$ as a root. Alternate solutions are possible.

113. Write a quadratic equation in standard form that has the solution set $\{2, 5\}$. Alternate solutions are possible.

114. Write a quadratic equation in standard form that has the solution set $\{-3, 0\}$. Alternate solutions are possible.

In Exercises 115–118, solve for the indicated variable in terms of other variables.

115. Solve $s = \frac{1}{2}gt^2$ for t.

116. Solve $A = P(1 + r)^2$ for r.

117. Solve $a^2 + b^2 = c^2$ for c.

118. Solve $P = EI - RI^2$ for I.

119. Solve the equation by factoring: $x^4 - 4x^2 = 0$.

120. Solve the equation by factoring: $3x - 6x^2 = 0$.

121. Solve the equation using factoring by grouping: $x^3 + x^2 - 4x - 4 = 0$.

122. Solve the equation using factoring by grouping: $x^3 + 2x^2 - x - 2 = 0$.

CHALLENGE

123. Show that the sum of the roots of a quadratic equation is equal to $-\frac{b}{a}$.

124. Show that the product of the roots of a quadratic equation is equal to $\frac{c}{a}$.

125. Write a quadratic equation in standard form whose solution set is $\{3 - \sqrt{5}, 3 + \sqrt{5}\}$. Alternate solutions are possible.

126. Write a quadratic equation in standard form whose solution set is $\{2 - i, 2 + i\}$. Alternate solutions are possible.

127. Aviation. An airplane takes 1 hour longer to go a distance of 600 miles flying against a headwind than on the return trip with a tailwind. If the speed of the wind is a constant 50 miles per hour for both legs of the trip, find the speed of the plane in still air.

128. Boating. A speedboat takes 1 hour longer to go 24 miles up a river than to return. If the boat cruises at 10 miles per hour in still water, what is the rate of the current?

129. Find a quadratic equation whose two distinct real roots are the negatives of the two distinct real roots of the equation $ax^2 + bx + c = 0$.

130. Find a quadratic equation whose two distinct real roots are the reciprocals of the two distinct real roots of the equation $ax^2 + bx + c = 0$.

131. A small jet and a 757 leave Atlanta at 1 P.M. The small jet is traveling due west. The 757 is traveling due south. The speed of the 757 is 100 miles per hour faster than that of the small jet. At 3 P.M. the planes are 1000 miles apart. Find the average speed of each plane. (Assume there is no wind.)

132. Two boats leave Key West at noon. The smaller boat is traveling due west. The larger boat is traveling due south. The speed of the larger boat is 10 miles per hour faster than that of the smaller boat. At 3 P.M. the boats are 150 miles apart. Find the average speed of each boat. (Assume there is no current.)

TECHNOLOGY

133. Solve the equation $x^2 - x = 2$ by first writing in standard form and then factoring. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 - x$ and $y_2 = 2$). At what x-values do these two graphs intersect? Do those points agree with the solution set you found?

134. Solve the equation $x^2 - 2x = -2$ by first writing in standard form and then using the quadratic formula. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 - 2x$ and $y_2 = -2$). Do these graphs intersect? Does this agree with the solution set you found?

135. a. Solve the equation $x^2 - 2x = b$, $b = 8$ by first writing in standard form. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 - 2x$ and $y_2 = b$). At what x-values do these two graphs intersect? Do those points agree with the solution set you found?

b. Repeat (a) for $b = -3, -1, 0$, and 5.

136. a. Solve the equation $x^2 + 2x = b$, $b = 8$ by first writing in standard form. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 + 2x$ and $y_2 = b$). At what x-values do these two graphs intersect? Do those points agree with the solution set you found?

b. Repeat (a) for $b = -3, -1, 0$, and 5.

SECTION 0.3 OTHER TYPES OF EQUATIONS

SKILLS OBJECTIVES

- Solve rational equations.
- Solve radical equations.
- Solve equations that are quadratic in form.
- Solve equations that are factorable.
- Solve absolute value equations.

CONCEPTUAL OBJECTIVES

- Transform a difficult equation into a simpler linear or quadratic equation.
- Recognize the need to check solutions when the transformation process may produce extraneous solutions.
- Realize that not all polynomial equations are factorable.

Rational Equations

A **rational equation** is an equation that contains one or more rational expressions (Appendix). Some rational equations can be transformed into linear or quadratic equations that you can then solve, but as you will see momentarily, you must be certain that the solution to the resulting linear or quadratic equation also satisfies the original rational equation.

EXAMPLE 1 **Solving a Rational Equation That Can Be Reduced to a Linear Equation**

Solve the equation $\frac{2}{3x} + \frac{1}{2} = \frac{4}{x} + \frac{4}{3}$.

Solution:

State the excluded values (those which make any denominator equal 0). $\quad \frac{2}{3x} + \frac{1}{2} = \frac{4}{x} + \frac{4}{3} \quad x \neq 0$

Multiply *each term* by the LCD, $6x$. $\quad 6x\left(\frac{2}{3x}\right) + 6x\left(\frac{1}{2}\right) = 6x\left(\frac{4}{x}\right) + 6x\left(\frac{4}{3}\right)$

Simplify both sides. $\quad 4 + 3x = 24 + 8x$

Subtract 4. $\quad \underline{-4 \qquad\quad -4}$

$3x = 20 + 8x$

Subtract $8x$. $\quad \underline{-8x \qquad\quad -8x}$

$-5x = 20$

Divide by -5. $\quad \boxed{x = -4}$

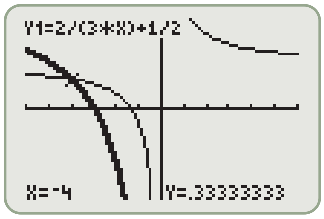
Since $x = -4$ satisfies the original equation, the solution set is $\{-4\}$.

YOUR TURN Solve the equation $\frac{3}{y} + 2 = \frac{7}{2y}$.

Technology Tip

Use a graphing utility to display graphs of $y_1 = \frac{2}{3x} + \frac{1}{2}$ and $y_2 = \frac{4}{x} + \frac{4}{3}$.

The x-coordinate of the point of intersection is the solution to the equation $\frac{2}{3x} + \frac{1}{2} = \frac{4}{x} + \frac{4}{3}$.



Study Tip

Since dividing by 0 is not defined, we exclude values of the variable that correspond to a denominator equaling 0.

Answer: The solution is $y = \frac{1}{4}$. The solution set is $\{\frac{1}{4}\}$.

EXAMPLE 2 Solving Rational Equations: Eliminating Extraneous Solutions

Solve the equation $\dfrac{3x}{x-1} + 2 = \dfrac{3}{x-1}$.

Solution:

State the excluded values (those which make any denominator equal 0).
$$\frac{3x}{x-1} + 2 = \frac{3}{x-1} \qquad x \neq 1$$

Eliminate the fractions by multiplying each term by the LCD, $x - 1$.
$$\frac{3x}{x-1}\cdot(x-1) + 2\cdot(x-1) = \frac{3}{x-1}\cdot(x-1)$$

Simplify.
$$\frac{3x}{\cancel{x-1}}\cdot\cancel{(x-1)} + 2\cdot(x-1) = \frac{3}{\cancel{x-1}}\cdot\cancel{(x-1)}$$
$$3x + 2(x-1) = 3$$

Distribute the 2.
$$3x + 2x - 2 = 3$$

Combine x terms on the left.
$$5x - 2 = 3$$

Add 2 to both sides.
$$5x = 5$$

Divide both sides by 5.
$$x = 1$$

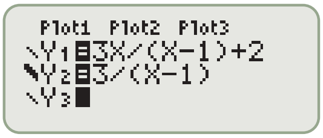
It may seem that $x = 1$ is the solution. However, the original equation had the restriction $x \neq 1$. Therefore, $x = 1$ is an extraneous solution and must be eliminated as a possible solution.

Thus, the equation $\dfrac{3x}{x-1} + 2 = \dfrac{3}{x-1}$ has $\boxed{\text{no solution}}$.

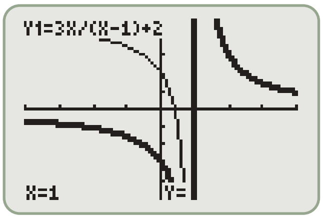
■ **YOUR TURN** Solve the equation $\dfrac{2x}{x-2} - 3 = \dfrac{4}{x-2}$.

Technology Tip

Use a graphing utility to display graphs of $y_1 = \dfrac{3x}{x-1} + 2$ and $y_2 = \dfrac{3}{x-1}$.



The x-coordinate of the point of intersection is the solution to the equation $\dfrac{3x}{x-1} + 2 = \dfrac{3}{x-1}$.



No intersection implies no solution.

■ **Answer:** no solution

Study Tip

When a variable is in the denominator of a fraction, the LCD will contain the variable. This sometimes results in an extraneous solution.

In order to find a *least* common denominator of more complicated expressions, it is useful to first factor the denominators to identify common multiples.

Rational equation:
$$\frac{1}{3x-3} + \frac{1}{2x-2} = \frac{1}{x^2-x}$$

Factor the denominators:
$$\frac{1}{3(x-1)} + \frac{1}{2(x-1)} = \frac{1}{x(x-1)}$$

LCD: $6x(x-1)$

EXAMPLE 3 Solving Rational Equations

Solve the equation $\frac{1}{3x+18} - \frac{1}{2x+12} = \frac{1}{x^2+6x}$.

Solution:

Factor the denominators. $\frac{1}{3(x+6)} - \frac{1}{2(x+6)} = \frac{1}{x(x+6)}$

State the excluded values. $x \neq 0, -6$

Multiply the equation by the LCD, $6x(x+6)$.

$$6x(x+6) \cdot \frac{1}{3(x+6)} - 6x(x+6) \cdot \frac{1}{2(x+6)} = 6x(x+6) \cdot \frac{1}{x(x+6)}$$

Divide out the common factors.

$$6x\cancel{(x+6)} \cdot \frac{1}{3\cancel{(x+6)}} - 6x\cancel{(x+6)} \cdot \frac{1}{2\cancel{(x+6)}} = 6\cancel{x(x+6)} \cdot \frac{1}{\cancel{x(x+6)}}$$

Simplify. $2x - 3x = 6$

Solve the linear equation. $x = -6$

Since one of the excluded values is $x \neq -6$, we say that $x = -6$ is an extraneous solution. Therefore, this rational equation has $\boxed{\text{no solution}}$.

■ **YOUR TURN** Solve the equation $\frac{2}{x} + \frac{1}{x+1} = -\frac{1}{x(x+1)}$.

■ **Answer:** no solution

EXAMPLE 4 Solving a Rational Equation That Can Be Reduced to a Quadratic Equation

Solve the equation $1 + \frac{3}{x^2-2x} = \frac{2}{x-2}$.

Solution:

Factor the denominators. $1 + \frac{3}{x(x-2)} = \frac{2}{x-2}$

State the excluded values (those which make any denominator equal to 0). $x \neq 0, 2$

Multiply each term by the LCD, $x(x-2)$. $1 \cdot x(x-2) + \frac{3}{x(x-2)} \cdot x(x-2) = \frac{2}{(x-2)} \cdot x(x-2)$

Divide out the common factors. $x(x-2) + \frac{3}{\cancel{x(x-2)}}\cancel{x(x-2)} = \frac{2}{\cancel{(x-2)}}x\cancel{(x-2)}$

Simplify. $x(x-2) + 3 = 2x$

Eliminate the parentheses. $x^2 - 2x + 3 = 2x$

Write the quadratic equation in standard form. $x^2 - 4x + 3 = 0$

Factor. $(x-3)(x-1) = 0$

Apply the zero product property. $\boxed{x = 3 \quad \text{or} \quad x = 1}$

Since $x = 3$ or $x = 1$ both satisfy the original equation, the solution set is $\{1, 3\}$.

Radical Equations

Radical equations are equations in which the variable is inside a radical (that is, under a square root, cube root, or higher root). Examples of radical equations follow:

$$\sqrt{x-3} = 2 \qquad \sqrt{2x+3} = x \qquad \sqrt{x+2} + \sqrt{7x+2} = 6$$

Often you can transform a radical equation into a simple linear or quadratic equation. Sometimes the transformation process yields **extraneous solutions**, or apparent solutions that may solve the transformed problem but are not solutions of the original radical equation. Therefore, it is very important to check your answers in the original equation.

EXAMPLE 5 Solving an Equation Involving a Radical

Solve the equation $\sqrt{x-3} = 2$.

Solution:

Square both sides of the equation. $\left(\sqrt{x-3}\right)^2 = 2^2$

Simplify. $x - 3 = 4$

Solve the resulting linear equation. $\boxed{x = 7}$

The solution set is $\{7\}$.

Check: $\sqrt{7-3} = \sqrt{4} = 2$

■ **YOUR TURN** Solve the equation $\sqrt{3p+4} = 5$.

Study Tip

Extraneous solutions are common when we deal with radical equations, so remember to check your answers.

■ **Answer:** $p = 7$ or $\{7\}$

EXAMPLE 6 Solving an Equation Involving a Radical

Solve the equation $\sqrt{2x+3} = x$.

Solution:

Square both sides of the equation. $\left(\sqrt{2x+3}\right)^2 = x^2$

Simplify. $2x + 3 = x^2$

Write the quadratic equation in standard form. $x^2 - 2x - 3 = 0$

Factor. $(x-3)(x+1) = 0$

Use the zero product property. $x = 3 \quad \text{or} \quad x = -1$

Check these values to see whether they *both* make the original equation statement true.

$x = 3$: $\sqrt{2(3)+3} = 3 \Rightarrow \sqrt{6+3} = 3 \Rightarrow \sqrt{9} = 3 \Rightarrow 3 = 3$ ✓

$x = -1$: $\sqrt{2(-1)+3} = -1 \Rightarrow \sqrt{-2+3} = -1 \Rightarrow \sqrt{1} = -1 \Rightarrow 1 \neq -1$ ✗

The solution is $\boxed{x = 3}$. The solution set is $\{3\}$.

■ **YOUR TURN** Solve the equation $\sqrt{12+t} = t$.

■ **YOUR TURN** Solve the equation $\sqrt{2x+6} = x + 3$.

Technology Tip

Use a graphing utility to display graphs of $y_1 = \sqrt{2x+3}$ and $y_2 = x$.

The x-coordinate of the point of intersection is the solution to the equation $\sqrt{2x+3} = x$.

■ **Answer:** $t = 4$ or $\{4\}$

■ **Answer:** $x = -1$ and $x = -3$ or $\{-3, -1\}$

Examples 5 and 6 contained only one radical each. We transformed the radical equation into a linear (Example 5) or quadratic (Example 6) equation with one step. The next example contains two radicals. Our technique will be to isolate one radical on one side of the equation with the other radical on the other side of the equation.

CORRECT

Square the expression.

$$\left(3 + \sqrt{x+2}\right)^2$$

Write the square as a product of two factors.

$$\left(3 + \sqrt{x+2}\right)\left(3 + \sqrt{x+2}\right)$$

Use the FOIL method.

$$9 + 6\sqrt{x+2} + (x+2)$$

INCORRECT

Square the expression.

$$\left(3 + \sqrt{x+2}\right)^2$$

The **error** occurs here when only individual terms are squared.

$$\neq 9 + (x+2)$$

EXAMPLE 7 Solving an Equation with More Than One Radical

Solve the equation $\sqrt{x+2} + \sqrt{7x+2} = 6$.

Solution:

Subtract $\sqrt{x+2}$ from both sides.	$\sqrt{7x+2} = 6 - \sqrt{x+2}$
Square both sides.	$\left(\sqrt{7x+2}\right)^2 = \left(6 - \sqrt{x+2}\right)^2$
Simplify.	$7x + 2 = \left(6 - \sqrt{x+2}\right)\left(6 - \sqrt{x+2}\right)$
Multiply the expressions on the right side of the equation.	$7x + 2 = 36 - 12\sqrt{x+2} + (x+2)$
Isolate the term with the radical on the left side.	$12\sqrt{x+2} = 36 + x + 2 - 7x - 2$
Combine like terms on the right side.	$12\sqrt{x+2} = 36 - 6x$
Divide by 6.	$2\sqrt{x+2} = 6 - x$
Square both sides.	$4(x+2) = (6-x)^2$
Simplify.	$4x + 8 = 36 - 12x + x^2$
Rewrite the quadratic equation in standard form.	$x^2 - 16x + 28 = 0$
Factor.	$(x-14)(x-2) = 0$
Solve.	$x = 14$ and $x = 2$

The apparent solutions are 2 and 14. Note that $x = 14$ does not satisfy the original equation; therefore, it is extraneous. The solution is $\boxed{x = 2}$. The solution set is $\{2\}$.

■ **YOUR TURN** Solve the equation $\sqrt{x-4} = 5 - \sqrt{x+1}$.

Technology Tip

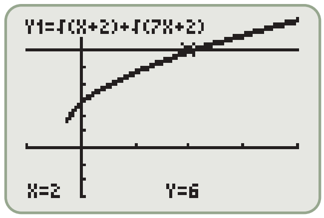
Use a graphing utility to display graphs of

$y_1 = \sqrt{x+2} + \sqrt{7x+2}$

and $y_2 = 6$.

The x-coordinate of the point of intersection is the solution to the equation

$\sqrt{x+2} + \sqrt{7x+2} = 6$.



■ **Answer:** $x = 8$ or $\{8\}$

PROCEDURE FOR SOLVING RADICAL EQUATIONS

Step 1: Isolate the term with a radical on one side.
Step 2: Raise both (*entire*) sides of the equation to the power that will eliminate this radical, and simplify the equation.
Step 3: If a radical remains, repeat Steps 1 and 2.
Step 4: Solve the resulting linear or quadratic equation.
Step 5: Check the solutions and eliminate any extraneous solutions.

Note: If there is more than one radical in the equation, it does not matter which radical is isolated first.

Equations Quadratic in Form: *u*-Substitution

Equations that are higher order or that have fractional powers often can be transformed into a quadratic equation by introducing a u-substitution. When this is the case, we say that equations are **quadratic in form**. In the table below, the two original equations are quadratic in form because they can be transformed into a quadratic equation given the correct substitution.

ORIGINAL EQUATION	SUBSTITUTION	NEW EQUATION
$x^4 - 3x^2 - 4 = 0$	$u = x^2$	$u^2 - 3u - 4 = 0$
$t^{2/3} + 2t^{1/3} + 1 = 0$	$u = t^{1/3}$	$u^2 + 2u + 1 = 0$
$\frac{2}{y} - \frac{1}{\sqrt{y}} + 1 = 0$	$u = y^{-1/2}$	$2u^2 - u + 1 = 0$

For example, the equation $x^4 - 3x^2 - 4 = 0$ is a fourth-degree equation in x. How did we know that $u = x^2$ would transform the original equation into a quadratic equation? If we rewrite the original equation as $(x^2)^2 - 3(x^2) - 4 = 0$, the expression in parentheses is the u-substitution.

Let us introduce the substitution $u = x^2$. Note that squaring both sides implies $u^2 = x^4$. We then replace x^2 in the original equation with u, and x^4 in the original equation with u^2, which leads to a quadratic equation in u: $u^2 - 3u - 4 = 0$.

WORDS	MATH
Solve for x.	$x^4 - 3x^2 - 4 = 0$
Introduce u-substitution.	$u = x^2$ (Note that $u^2 = x^4$.)
Write the quadratic equation in u.	$u^2 - 3u - 4 = 0$
Factor.	$(u - 4)(u + 1) = 0$
Solve for u.	$u = 4$ or $u = -1$
Transform back to x, $u = x^2$.	$x^2 = 4$ or $x^2 = -1$
Solve for x.	$\boxed{x = \pm 2 \quad \text{or} \quad x = \pm i}$

The solution set is $\{\pm 2, \pm i\}$.

It is important to correctly determine the appropriate substitution in order to arrive at an equation quadratic in form. For example, $t^{2/3} + 2t^{1/3} + 1 = 0$ is an original equation given in the above table. If we rewrite this equation as $\left(t^{1/3}\right)^2 + 2\left(t^{1/3}\right) + 1 = 0$, then it becomes apparent that the correct substitution is $u = t^{1/3}$, which transforms the equation in t into a quadratic equation in u: $u^2 + 2u + 1 = 0$.

Procedure for Solving Equations Quadratic in Form

Step 1: Identify the substitution.
Step 2: Transform the equation into a quadratic equation.
Step 3: Solve the quadratic equation.
Step 4: Apply the substitution to rewrite the solution in terms of the original variable.
Step 5: Solve the resulting equation.
Step 6: Check the solutions in the original equation.

EXAMPLE 8 Solving an Equation Quadratic in Form with Negative Exponents

Find the solutions to the equation $x^{-2} - x^{-1} - 12 = 0$.

Solution:

Rewrite the original equation.	$(x^{-1})^2 - (x^{-1}) - 12 = 0$
Determine the u-substitution.	$u = x^{-1}$ (Note that $u^2 = x^{-2}$.)
The original equation in x corresponds to a quadratic equation in u.	$u^2 - u - 12 = 0$
Factor.	$(u - 4)(u + 3) = 0$
Solve for u.	$u = 4$ or $u = -3$

The most common mistake is forgetting to transform back to x.

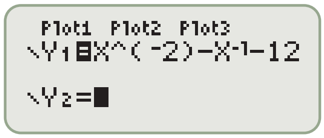
Transform back to x. Let $u = x^{-1}$.	$x^{-1} = 4$ or $x^{-1} = -3$
Write x^{-1} as $\frac{1}{x}$.	$\frac{1}{x} = 4$ or $\frac{1}{x} = -3$
Solve for x.	$\boxed{x = \frac{1}{4}}$ or $\boxed{x = -\frac{1}{3}}$

The solution set is $\left\{-\frac{1}{3}, \frac{1}{4}\right\}$.

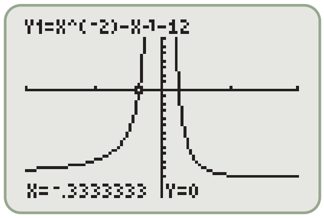
YOUR TURN Find the solutions to the equation $x^{-2} - x^{-1} - 6 = 0$.

Technology Tip

Use a graphing utility to graph $y_1 = x^{-2} - x^{-1} - 12$.



The x-intercepts are the solutions to this equation.



Study Tip

Remember to transform back to the original variable.

Answer: The solution is $x = -\frac{1}{2}$ or $x = \frac{1}{3}$. The solution set is $\left\{-\frac{1}{2}, \frac{1}{3}\right\}$.

EXAMPLE 9 Solving an Equation Quadratic in Form with Fractional Exponents

Find the solutions to the equation $x^{2/3} - 3x^{1/3} - 10 = 0$.

Solution:

Rewrite the original equation.	$\left(x^{1/3}\right)^2 - 3x^{1/3} - 10 = 0$
Identify the substitution as $u = x^{1/3}$.	$u^2 - 3u - 10 = 0$
Factor.	$(u - 5)(u + 2) = 0$
Solve for u.	$u = 5$ or $u = -2$
Let $u = x^{1/3}$ again.	$x^{1/3} = 5$ $\quad$ $x^{1/3} = -2$
Cube both sides of the equations.	$\left(x^{1/3}\right)^3 = (5)^3$ $\quad$ $\left(x^{1/3}\right)^3 = (-2)^3$
Simplify.	$\boxed{x = 125}$ $\quad$ $\boxed{x = -8}$

The solution set is $\boxed{\{-8, 125\}}$, which a check will confirm.

■ **Answer:** $t = 9$ or $\{9\}$

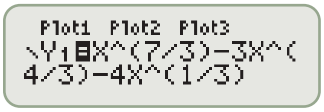
■ **YOUR TURN** Find the solution to the equation $2t - 5t^{1/2} - 3 = 0$.

Factorable Equations

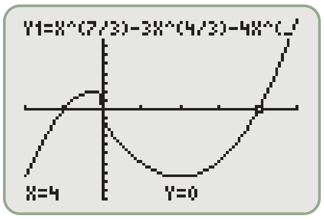
Some equations (both polynomial and with rational exponents) that are factorable can be solved using the zero product property.

Technology Tip

Use a graphing utility to graph $y_1 = x^{7/3} - 3x^{4/3} - 4x^{1/3}$.



The x-intercepts are the solutions to this equation.



EXAMPLE 10 Solving an Equation with Rational Exponents by Factoring

Solve the equation $x^{7/3} - 3x^{4/3} - 4x^{1/3} = 0$.

Solution:

Factor the left side of the equation.	$x^{1/3}\left(x^2 - 3x - 4\right) = 0$
Factor the quadratic expression.	$x^{1/3}(x - 4)(x + 1) = 0$
Apply the zero product property.	$x^{1/3} = 0$ or $x - 4 = 0$ or $x + 1 = 0$
Solve for x.	$\boxed{x = 0}$ or $\boxed{x = 4}$ or $\boxed{x = -1}$

The solution set is $\{-1, 0, 4\}$.

EXAMPLE 11 **Solving a Polynomial Equation Using Factoring by Grouping**

Solve the equation $x^3 + 2x^2 - x - 2 = 0$.

Solution:

Factor by grouping (Appendix). $(x^3 - x) + (2x^2 - 2) = 0$

Identify the common factors. $x(x^2 - 1) + 2(x^2 - 1) = 0$

Factor. $(x + 2)(x^2 - 1) = 0$

Factor the quadratic expression. $(x + 2)(x - 1)(x + 1) = 0$

Apply the zero product property. $x + 2 = 0$ or $x - 1 = 0$ or $x + 1 = 0$

Solve for x. $\boxed{x = -2}$ or $\boxed{x = 1}$ or $\boxed{x = -1}$

The solution set is $\{-2, -1, 1\}$.

■ **YOUR TURN** Solve the equation $x^3 + x^2 - 4x - 4 = 0$.

■ **Answer:** $x = -1$ or $x = \pm 2$ or $\{-2, -1, 2\}$

Absolute Value Equations

The **absolute value** of a real number can be interpreted algebraically and graphically.

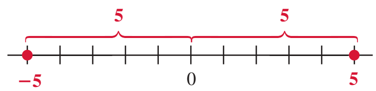
DEFINITION **Absolute Value**

The **absolute value** of a real number a, denoted by the symbol $|a|$, is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Study Tip

Algebraically: $|x| = 5$ implies $x = -5$ or $x = 5$.
Graphically: -5 and 5 are five units from 0.



When absolute value is involved in algebraic equations, we interpret the definition of absolute value as follows.

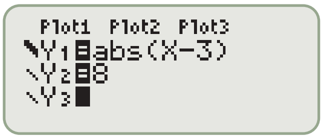
DEFINITION **Absolute Value Equation**

If $|x| = a$, then $x = -a$ or $x = a$, where $a \geq 0$.

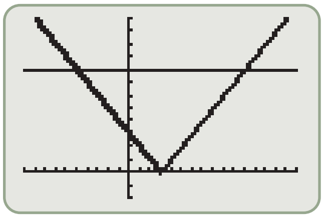
In words, "If the absolute value of a number is a, then that number equals $-a$ or a." For example, the equation $|x| = 7$ is true if $x = -7$ or $x = 7$. We say the equation $|x| = 7$ has the solution set $\{-7, 7\}$. *Note:* $|x| = -3$ does not have a solution because there is no value of x such that its absolute value is -3.

Technology Tip

Use a graphing utility to display graphs of $y_1 = |x - 3|$ and $y_2 = 8$.



The x-coordinates of the points of intersection are the solutions to $|x - 3| = 8$.



EXAMPLE 12 Solving an Absolute Value Equation

Solve the equation $|x - 3| = 8$ algebraically and graphically.

Solution:

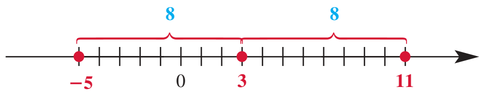
Using the absolute value equation definition, we see that if the absolute value of an expression is 8, then that expression is either -8 or 8. Rewrite as two equations:

$$x - 3 = -8 \qquad \text{or} \qquad x - 3 = 8$$

$$x = -5 \qquad\qquad\qquad x = 11$$

The solution set is $\boxed{\{-5, 11\}}$.

Graph: The absolute value equation $|x - 3| = 8$ is interpreted as "What numbers are eight units away from 3 on the number line?" We find that eight units to the right of 3 is 11 and eight units to the left of 3 is -5.



■ **Answer:** $x = 2$ or $x = -12$. The solution set is $\{-12, 2\}$.

■ **YOUR TURN** Solve the equation $|x + 5| = 7$.

EXAMPLE 13 Solving an Absolute Value Equation

Solve the equation $2 - 3|x - 1| = -4|x - 1| + 7$.

Solution:

Isolate the absolute value expressions to one side.

Add $4|x - 1|$ to both sides. $\qquad 2 + |x - 1| = 7$

Subtract 2 from both sides. $\qquad |x - 1| = 5$

If the absolute value of an expression is equal to 5, then the expression is equal to either -5 or 5.

$$x - 1 = -5 \qquad \text{or} \qquad x - 1 = 5$$

$$x = -4 \qquad\qquad\qquad x = 6$$

The solution set is $\boxed{\{-4, 6\}}$.

■ **Answer:** $x = -4$ or $x = 12$. The solution set is $\{-4, 12\}$.

■ **YOUR TURN** Solve the equation $3 - 2|x - 4| = -3|x - 4| + 11$.

EXAMPLE 14 **Solving a Quadratic Absolute Value Equation**

Solve the equation $|5 - x^2| = 1$.

Solution:

If the absolute value of an expression is 1, that expression is either -1 or 1, which leads to two equations.

$$5 - x^2 = -1 \quad \text{or} \quad 5 - x^2 = 1$$
$$-x^2 = -6 \qquad\qquad -x^2 = -4$$
$$x^2 = 6 \qquad\qquad x^2 = 4$$
$$x = \pm\sqrt{6} \qquad\qquad x = \pm\sqrt{4} = \pm 2$$

The solution set is $\boxed{\{\pm 2, \pm\sqrt{6}\}}$.

■ **YOUR TURN** Solve the equation $|7 - x^2| = 2$.

■ **Answer:** $x = \pm\sqrt{5}$ or $x = \pm 3$. The solution set is $\{\pm\sqrt{5}, \pm 3\}$.

SECTION 0.3 SUMMARY

Rational equations, radical equations, equations quadratic in form, factorable equations, and absolute value equations can often be solved by transforming them into simpler linear or quadratic equations.

- **Rational Equations:** Multiply the entire equation by the LCD. Solve the resulting equation (if it is linear or quadratic). Check for extraneous solutions.
- **Radical Equations:** Isolate the term containing a radical and raise it to the appropriate power that will eliminate the radical. If there is more than one radical, it does not matter which radical is isolated first. Raising radical equations to powers may cause extraneous solutions, so check each solution.
- **Equations Quadratic in Form:** Identify the u-substitution that transforms the equation into a quadratic equation. Solve the quadratic equation and then remember to transform back to the original variable.
- **Factorable Equations:** Look for a factor common to all terms or factor by grouping.
- **Absolute Value Equations:** Transform the absolute value equation into two equations that do not involve absolute value.

SECTION 0.3 EXERCISES

SKILLS

In Exercises 1–20, specify any values that must be excluded from the solution set and then solve the rational equation.

1. $\dfrac{x}{x-2} + 5 = \dfrac{2}{x-2}$

2. $\dfrac{n}{n-5} + 2 = \dfrac{n}{n-5}$

3. $\dfrac{2p}{p-1} = 3 + \dfrac{2}{p-1}$

4. $\dfrac{4t}{t+2} = 3 - \dfrac{8}{t+2}$

5. $\dfrac{3x}{x+2} - 4 = \dfrac{2}{x+2}$

6. $\dfrac{5y}{2y-1} - 3 = \dfrac{12}{2y-1}$

7. $\dfrac{1}{n} + \dfrac{1}{n+1} = \dfrac{-1}{n(n+1)}$

8. $\dfrac{1}{x} + \dfrac{1}{x-1} = \dfrac{1}{x(x-1)}$

9. $\dfrac{3}{a} - \dfrac{2}{a+3} = \dfrac{9}{a(a+3)}$

10. $\dfrac{1}{c-2} + \dfrac{1}{c} = \dfrac{2}{c(c-2)}$

11. $\dfrac{n-5}{6n-6} = \dfrac{1}{9} - \dfrac{n-3}{4n-4}$

12. $\dfrac{5}{m} + \dfrac{3}{m-2} = \dfrac{6}{m(m-2)}$

13. $\dfrac{2}{5x+1} = \dfrac{1}{2x-1}$

14. $\dfrac{3}{4n-1} = \dfrac{2}{2n-5}$

15. $\dfrac{t-1}{1-t} = \dfrac{3}{2}$

16. $\dfrac{2-x}{x-2} = \dfrac{3}{4}$

17. $x + \dfrac{12}{x} = 7$

18. $x - \dfrac{10}{x} = -3$

19. $\dfrac{4(x-2)}{x-3} + \dfrac{3}{x} = \dfrac{-3}{x(x-3)}$

20. $\dfrac{5}{y+4} = 4 + \dfrac{3}{y-2}$

In Exercises 21–60, solve the radical equation for the given variable.

21. $\sqrt{u+1} = -4$

22. $-\sqrt{3-2u} = 9$

23. $\sqrt[3]{5x+2} = 3$

24. $\sqrt[3]{1-x} = -2$

25. $(4y+1)^{1/3} = -1$

26. $(5x-1)^{1/3} = 4$

27. $(x+3)^{5/3} = 32$

28. $(x+2)^{4/3} = 16$

29. $(x+1)^{2/3} = 4$

30. $(x-7)^{4/3} = 81$

31. $\sqrt{12+x} = x$

32. $x = \sqrt{56-x}$

33. $y = 5\sqrt{y}$

34. $\sqrt{y} = \dfrac{y}{4}$

35. $s = 3\sqrt{s-2}$

36. $-2s = \sqrt{3-s}$

37. $\sqrt{2x+6} = x+3$

38. $\sqrt{8-2x} = 2x-2$

39. $\sqrt{1-3x} = x+1$

40. $\sqrt{2-x} = x-2$

41. $3x - 6\sqrt{x-1} = 3$

42. $5x - 10\sqrt{x+2} = -10$

43. $3x - 6\sqrt{x+2} = 3$

44. $2x - 4\sqrt{x+1} = 4$

45. $3\sqrt{x+4} - 2x = 9$

46. $2\sqrt{x+1} - 3x = -5$

47. $\sqrt{x^2-4} = x-1$

48. $\sqrt{25-x^2} = x+1$

49. $\sqrt{x^2-2x-5} = x+1$

50. $\sqrt{2x^2-8x+1} = x-3$

51. $\sqrt{3x+1} - \sqrt{6x-5} = 1$

52. $\sqrt{2-x} + \sqrt{6-5x} = 6$

53. $\sqrt{x+12} + \sqrt{8-x} = 6$

54. $\sqrt{5-x} + \sqrt{3x+1} = 4$

55. $\sqrt{2x-1} - \sqrt{x-1} = 1$

56. $\sqrt{8-x} = 2 + \sqrt{2x+3}$

57. $\sqrt{3x-5} = 7 - \sqrt{x+2}$

58. $\sqrt{x+5} = 1 + \sqrt{x-2}$

59. $\sqrt{2+\sqrt{x}} = \sqrt{x}$

60. $\sqrt{2-\sqrt{x}} = \sqrt{x}$

In Exercises 61–80, solve the equations by introducing a substitution that transforms these equations to quadratic form.

61. $x^{2/3} + 2x^{1/3} = 0$

62. $x^{1/2} - 2x^{1/4} = 0$

63. $x^4 - 3x^2 + 2 = 0$

64. $x^4 - 8x^2 + 16 = 0$

65. $2x^4 + 7x^2 + 6 = 0$

66. $x^8 - 17x^4 + 16 = 0$

67. $4(t-1)^2 - 9(t-1) = -2$

68. $2(1-y)^2 + 5(1-y) - 12 = 0$

69. $x^{-8} - 17x^{-4} + 16 = 0$

70. $2u^{-2} + 5u^{-1} - 12 = 0$

71. $3y^{-2} + y^{-1} - 4 = 0$

72. $5a^{-2} + 11a^{-1} + 2 = 0$

73. $z^{2/5} - 2z^{1/5} + 1 = 0$

74. $2x^{1/2} + x^{1/4} - 1 = 0$

75. $6t^{-2/3} - t^{-1/3} - 1 = 0$

76. $t^{-2/3} - t^{-1/3} - 6 = 0$

77. $3 = \dfrac{1}{(x+1)^2} + \dfrac{2}{(x+1)}$

78. $\dfrac{1}{(x+1)^2} + \dfrac{4}{(x+1)} + 4 = 0$

79. $u^{4/3} - 5u^{2/3} = -4$

80. $u^{4/3} + 5u^{2/3} = -4$

In Exercises 81–96, solve by factoring.

81. $x^3 - x^2 - 12x = 0$

82. $2y^3 - 11y^2 + 12y = 0$

83. $4p^3 - 9p = 0$

84. $25x^3 = 4x$

85. $u^5 - 16u = 0$

86. $t^5 - 81t = 0$

87. $x^3 - 5x^2 - 9x + 45 = 0$

88. $2p^3 - 3p^2 - 8p + 12 = 0$

89. $y(y-5)^3 - 14(y-5)^2 = 0$

90. $v(v+3)^3 - 40(v+3)^2 = 0$

91. $x^{9/4} - 2x^{5/4} - 3x^{1/4} = 0$

92. $u^{7/3} + u^{4/3} - 20u^{1/3} = 0$

93. $t^{5/3} - 25t^{-1/3} = 0$

94. $4x^{9/5} - 9x^{-1/5} = 0$

95. $y^{3/2} - 5y^{1/2} + 6y^{-1/2} = 0$

96. $4p^{5/3} - 5p^{2/3} - 6p^{-1/3} = 0$

In Exercises 97–118, solve the absolute value equation.

97. $|p - 7| = 3$

98. $|p + 7| = 3$

99. $|4 - y| = 1$

100. $|2 - y| = 11$

101. $|3t - 9| = 3$

102. $|4t + 2| = 2$

103. $|7 - 2x| = 9$

104. $|6 - 3y| = 12$

105. $|1 - 3y| = 1$

106. $|5 - x| = 2$

107. $\left|\frac{2}{3}x - \frac{4}{7}\right| = \frac{5}{3}$

108. $\left|\frac{1}{2}x + \frac{3}{4}\right| = \frac{1}{16}$

109. $|x - 5| + 4 = 12$

110. $|x + 3| - 9 = 2$

111. $2|p + 3| - 15 = 5$

112. $8 - 3|p - 4| = 2$

113. $5|y - 2| - 10 = 4|y - 2| - 3$

114. $3 - |y + 9| = 11 - 3|y + 9|$

115. $|4 - x^2| = 1$

116. $|7 - x^2| = 3$

117. $|x^2 + 1| = 5$

118. $|x^2 - 1| = 5$

APPLICATIONS

In Exercises 119 and 120 refer to the following:

An analysis of sales indicates that demand for a product during a calendar year is modeled by

$$d = 3\sqrt{t + 1} - 0.75t$$

where d is demand in millions of units and t is the month of the year where $t = 0$ represents January.

119. Economics. During which month(s) is demand 3 million units?

120. Economics. During which month(s) is demand 4 million units?

In Exercises 121 and 122 refer to the following:

Body Surface Area (BSA) is used in physiology and medicine for many clinical purposes. BSA can be modeled by the function

$$BSA = \sqrt{\frac{wh}{3600}}$$

where w is weight in kilograms and h is height in centimeters.

121. Health. The BSA of a 72-kilogram female is 1.8. Find the height of the female to the nearest centimeter.

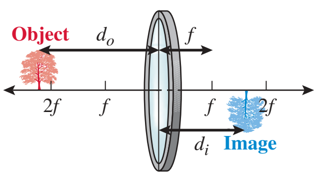
122. Health. The BSA of a 177-centimeter tall-male is 2.1. Find the weight of the male to the nearest kilogram.

For Exercises 123–126, refer to this lens law.

The position of the image is found using the thin lens equation

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

where d_o is the distance from the object to the lens, d_i is the distance from the lens to the image, and f is the focal length of the lens.



123. Optics. If the focal length of a lens is 3 centimeters and the image distance is 5 centimeters from the lens, what is the distance from the object to the lens?

124. Optics. If the focal length of the lens is 8 centimeters and the image distance is 2 centimeters from the lens, what is the distance from the object to the lens? *Note:* A negative d_i implies that the image is behind the lens.

125. Optics. The focal length of a lens is 2 centimeters. If the image distance from the lens is half the distance from the object to the lens, find the object distance.

126. Optics. The focal length of a lens is 8 centimeters. If the image distance from the lens is half the distance from the object to the lens, find the object distance.

127. Speed of Sound. A man buys a house with an old well but does not know how deep the well is. To get an estimate, he decides to drop a rock at the opening of the well and count how long it takes until he hears the splash. The total elapsed time T given by $T = t_1 + t_2$, is the sum of the time it takes for the rock to reach the water, t_1, and the time it takes for the sound of the splash to travel to the top of the well, t_2. The time (seconds) that it takes for the rock to reach the water is given by $t_1 = \frac{\sqrt{d}}{4}$, where d is the depth of the well in feet. Since the speed of sound is 1100 feet per second, the time (seconds) it takes for the sound to reach the top of the well is $t_2 = \frac{d}{1100}$. If the splash is heard after 3 seconds, how deep is the well?

128. Speed of Sound. If the owner of the house in Exercise 127 forgot to account for the speed of sound $\left(T = t_1 = \frac{\sqrt{d}}{4}\right)$, what would he have calculated the depth of the well to be?

129. Physics: Pendulum. The period (T) of a pendulum is related to the length (L) of the pendulum and acceleration due to gravity (g) by the formula $T = 2\pi\sqrt{\frac{L}{g}}$. If gravity is 9.8 m/s^2 and the period is 1 second, find the approximate length of the pendulum. Round to the nearest centimeter. *Note:* 100 cm = 1 m.

130. Physics: Pendulum. The period (T) of a pendulum is related to the length (L) of the pendulum and acceleration due to gravity (g) by the formula $T = 2\pi\sqrt{\frac{L}{g}}$. If gravity is 32 ft/s^2 and the period is 1 second, find the approximate length of the pendulum. Round to the nearest inch. *Note:* 12 in. = 1 ft.

For Exercises 131 and 132, refer to the following:

Einstein's special theory of relativity states that time is relative: Time speeds up or slows down, depending on how fast one object is moving with respect to another. For example, a space probe traveling at a velocity v near the speed of light c will have "clocked" a time t hours, but for a stationary observer on Earth that corresponds to a time t_0. The formula governing this relativity is given by

$$t = t_0\sqrt{1 - \frac{v^2}{c^2}}$$

131. Physics: Special Theory of Relativity. If the time elapsed on a space probe mission is 18 years but the time elapsed on Earth during that mission is 30 years, how fast is the space probe traveling? Give your answer relative to the speed of light.

132. Physics: Special Theory of Relativity. If the time elapsed on a space probe mission is 5 years but the time elapsed on Earth during that mission is 30 years, how fast is the space probe traveling? Give your answer relative to the speed of light.

CATCH THE MISTAKE

In Exercises 133–136, explain the mistake that is made.

133. Solve the equation $\sqrt{3t + 1} = -4$.

Solution:

$$3t + 1 = 16$$
$$3t = 15$$
$$t = 5$$

This is incorrect. What mistake was made?

134. Solve the equation $x = \sqrt{x + 2}$.

Solution:

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = -1, x = 2$$

This is incorrect. What mistake was made?

135. Solve the equation $\frac{4}{p} - 3 = \frac{2}{5p}$.

Solution:

Cross multiply.

$$(p - 3)2 = 4(5p)$$
$$2p - 6 = 20p$$
$$-6 = 18p$$
$$p = -\frac{6}{18}$$
$$p = -\frac{1}{3}$$

This is incorrect. What mistake was made?

136. Solve the equation $\frac{1}{x} + \frac{1}{x - 1} = \frac{1}{x(x - 1)}$.

Solution:

Multiply by the LCD, $x(x - 1)$.

$$\frac{x(x-1)}{x} + \frac{x(x-1)}{x-1} = \frac{x(x-1)}{x(x-1)}$$

Simplify.

$$(x - 1) + x = 1$$
$$x - 1 + x = 1$$
$$2x = 2$$
$$x = 1$$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 137–138, determine whether each statement is true or false.

137. The solution to the equation $x = \frac{1}{1/x}$ is the set of all real numbers.

138. The solution to the equation $\frac{1}{(x-1)(x+2)} = \frac{1}{x^2 + x - 2}$ is the set of all real numbers.

139. Solve for x, given that a, b, and c are real numbers and $c \neq 0$.

$$\frac{a}{x} - \frac{b}{x} = c$$

140. Solve the equation for y: $\frac{1}{y - a} + \frac{1}{y + a} = \frac{2}{y - 1}$. Does y have any restrictions?

CHALLENGE

141. Solve the equation $\sqrt{x+6} + \sqrt{11+x} = 5\sqrt{3+x}$.

142. Solve the equation $3x^{7/12} - x^{5/6} - 2x^{1/3} = 0$.

143. Solve the equation for x in terms of y: $y = \dfrac{a}{1 + \dfrac{b}{x} + c}$.

144. Solve for t: $\dfrac{t + \dfrac{1}{t}}{\dfrac{1}{t} - 1} = 1$.

TECHNOLOGY

145. Solve the equation $\sqrt{x-3} = 4 - \sqrt{x+2}$. Plot both sides of the equation in the same viewing screen, $y_1 = \sqrt{x-3}$ and $y_2 = 4 - \sqrt{x+2}$, and zoom in on the x-coordinate of the point of intersection. Does the graph agree with your solution?

146. Solve the equation $2\sqrt{x+1} = 1 + \sqrt{3-x}$. Plot both sides of the equation in the same viewing screen, $y_1 = 2\sqrt{x+1}$ and $y_2 = 1 + \sqrt{3-x}$, and zoom in on the x-coordinate of the points of intersection. Does the graph agree with your solution

147. Solve the equation $x^{1/2} = -4x^{1/4} + 21$. Plot both sides of the equation in the same viewing screen, $y_1 = x^{1/2}$ and $y_2 = -4x^{1/4} + 21$. Does the point(s) of intersection agree with your solution?

148. Solve the equation $x^{-1} = 3x^{-2} - 10$. Plot both sides of the equation in the same viewing screen, $y_1 = x^{-1}$ and $y_2 = 3x^{-2} - 10$. Does the point(s) of intersection agree with your solution?

SECTION 0.4 INEQUALITIES

SKILLS OBJECTIVES

- Use interval notation.
- Solve linear inequalities.
- Solve polynomial inequalities.
- Solve rational inequalities.
- Solve absolute value inequalities.

CONCEPTUAL OBJECTIVES

- Apply intersection and union concepts.
- Understand that inequalities may have one solution, no solution, or an interval solution.
- Understand zeros and test intervals.
- Realize that a rational inequality has an implied domain restriction on the variable.

Graphing Inequalities and Interval Notation

We will express solutions to inequalities four ways: an inequality, a solution set, an interval, and a graph. The following are ways of expressing all real numbers greater than or equal to a and less than b:

Inequality Notation	Solution Set	Interval Notation	Graph/Number Line
$a \le x < b$	$\{x \mid a \le x < b\}$	$[a, b)$	 or 

In this example, a is referred to as the **left endpoint** and b is referred to as the **right endpoint**. If an inequality is a strict inequality ($<$ or $>$), then the graph and interval notation use *parentheses*. If it includes an endpoint ($\ge$ or $\le$), then the graph and interval notation use *brackets*. Number lines are drawn with either closed/open circles or brackets/parentheses. In this text the brackets/parentheses notation will be used. Intervals are classified as follows:

Open (,) Closed [,] Half open (,] or [,)

Let x be a real number. x is...	Inequality	Set Notation	Interval	Graph
greater than a and less than b	$a < x < b$	$\{x \mid a < x < b\}$	(a, b)	
greater than or equal to a and less than b	$a \le x < b$	$\{x \mid a \le x < b\}$	$[a, b)$	
greater than a and less than or equal to b	$a < x \le b$	$\{x \mid a < x \le b\}$	$(a, b]$	
greater than or equal to a and less than or equal to b	$a \le x \le b$	$\{x \mid a \le x \le b\}$	$[a, b]$	
less than a	$x < a$	$\{x \mid x < a\}$	$(-\infty, a)$	
less than or equal to a	$x \le a$	$\{x \mid x \le a\}$	$(-\infty, a]$	
greater than b	$x > b$	$\{x \mid x > b\}$	(b, ∞)	
greater than or equal to b	$x \ge b$	$\{x \mid x \ge b\}$	$[b, \infty)$	
all real numbers	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, \infty)$	

1. *Infinity* (∞) is not a number. It is a symbol that means continuing indefinitely to the right on the number line. Similarly, *negative infinity* ($-\infty$) means continuing indefinitely to the left on the number line. Since both are unbounded, we use a parenthesis, never a bracket.
2. In interval notation, the lower number is always written to the left.

Write the inequality in interval notation: $-1 \le x < 3$.

CORRECT $[-1, 3)$ **INCORRECT** $(3, -1]$

EXAMPLE 1 Expressing Inequalities Using Interval Notation and a Graph

Express the following as an inequality, an interval, and a graph:

a. x is greater than -3.
b. x is less than or equal to 5.
c. x is greater than or equal to -1 and less than 4.
d. x is greater than or equal to 0 and less than or equal to 4.

Solution:

Inequality	Interval	Graph
a. $x > -3$	$(-3, \infty)$	
b. $x \le 5$	$(-\infty, 5]$	
c. $-1 \le x < 4$	$[-1, 4)$	
d. $0 \le x \le 4$	$[0, 4]$	

Since the solutions to inequalities are sets of real numbers, it is useful to discuss two operations on sets called **intersection** and **union**.

DEFINITION **Union and Intersection**

The **union** of sets A and B, denoted $A \cup B$, is the set formed by combining all the elements in A with all the elements in B.

$$A \cup B = \{x \mid x \text{ is in } A \textbf{ or } B \textbf{ or } \text{both}\}$$

The **intersection** of sets A and B, denoted $A \cap B$, is the set formed by the elements that are in both A and B.

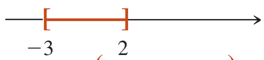
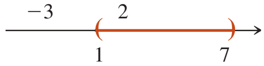
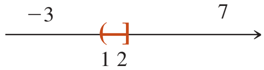
$$A \cap B = \{x \mid x \text{ is in } A \textbf{ and } B\}$$

The notation "$x \mid x$ is in" is read "all x such that x is in." The vertical line represents "such that."

EXAMPLE 2 **Determining Unions and Intersections: Intervals and Graphs**

If $A = [-3, 2]$ and $B = (1, 7)$, determine $A \cup B$ and $A \cap B$. Write these sets in interval notation, and graph.

Solution:

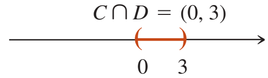
Set	Interval notation	Graph
A	$[-3, 2]$	
B	$(1, 7)$	
$A \cup B$	$[-3, 7)$	
$A \cap B$	$(1, 2]$	

■ **YOUR TURN** If $C = [-3, 3)$ and $D = (0, 5]$, find $C \cup D$ and $C \cap D$. Express the intersection and union in interval notation, and graph.

■ **Answer:**

$C \cup D = [-3, 5]$

$C \cap D = (0, 3)$



Linear Inequalities

If we were to solve the linear equation $3x - 2 = 7$, we would add 2 to both sides, divide by 3, and find that $x = 3$ is the solution, the *only* value that makes the equation true. If we were to solve the *linear inequality* $3x - 2 \leq 7$, we would follow the same procedure: Add 2 to both sides, divide by 3, and find that $x \leq 3$, which is an *interval* or *range* of numbers that make the inequality true.

In solving linear inequalities, we follow the same procedures that we used in solving linear equations with one general exception: *If you multiply or divide an inequality by a negative number, then you must change the direction of the inequality sign.*

Study Tip

If you multiply or divide an inequality by a negative number, remember to change the direction of the inequality sign.

Inequality Properties

Procedures That Do Not Change the Inequality Sign

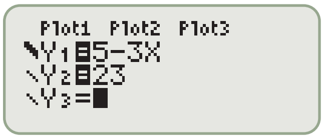
1. Simplifying by eliminating parentheses and collecting like terms.	$3(x - 6) < 6x - x$ $3x - 18 < 5x$
2. Adding or subtracting the same quantity on both sides.	$7x + 8 \geq 29$ $7x \geq 21$
3. Multiplying or dividing by the same *positive* real number.	$5x \leq 15$ $x \leq 3$

Procedures That Change (Reverse) the Inequality Sign

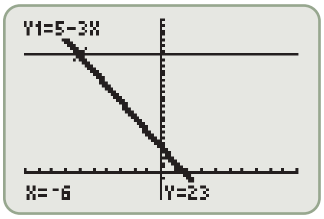
1. Interchanging the two sides of the inequality.	$x \leq 4$ is equivalent to $4 \geq x$
2. Multiplying or dividing by the same *negative* real number.	$-5x \leq 15$ is equivalent to $x \geq -3$

Technology Tip

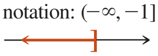
Use a graphing utility to display graphs of $y_1 = 5 - 3x$ and $y_2 = 23$.



The solutions are the x-values such that the graph of $y_1 = 5 - 3x$ is below that of $y_2 = 23$.



■ **Answer:** Solution set: $\{x \mid x \leq -1\}$
Interval notation: $(-\infty, -1]$
Graph:



EXAMPLE 3 Solving a Linear Inequality

Solve and graph the inequality $5 - 3x < 23$.

Solution:

Write the original inequality.	$5 - 3x < 23$
Subtract 5 from both sides.	$-3x < 18$
Divide both sides by -3 and reverse the inequality sign.	$\frac{-3x}{-3} > \frac{18}{-3}$
Simplify.	$x > -6$

Solution set: $\boxed{\{x \mid x > -6\}}$ Interval notation: $\boxed{(-6, \infty)}$ Graph:

■ **YOUR TURN** Solve the inequality $5 \leq 3 - 2x$. Express the solution in set and interval notation, and graph.

EXAMPLE 4 Solving a Double Linear Inequality

Solve the inequality $-2 < 3x + 4 \leq 16$.

Solution:

This double inequality can be written as two inequalities.	$-2 < 3x + 4 \leq 16$
Both inequalities must be satisfied.	$-2 < 3x + 4$ and $3x + 4 \leq 16$
Subtract 4 from both sides of each inequality.	$-6 < 3x$ and $3x \leq 12$
Divide each inequality by 3.	$-2 < x$ and $x \leq 4$

Combining these two inequalities gives us $-2 < x \leq 4$ in inequality notation; in interval notation we have $(-2, \infty) \cap (-\infty, 4)$ or $(-2, 4]$.

Notice that the steps we took in solving these inequalities individually were identical. This leads us to a **shortcut method** in which we solve them together:

Write the combined inequality. $-2 < 3x + 4 \le 16$

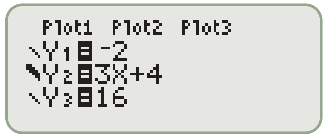
Subtract 4 from each part. $-6 < 3x \le 12$

Divide each part by 3. $-2 < x \le 4$

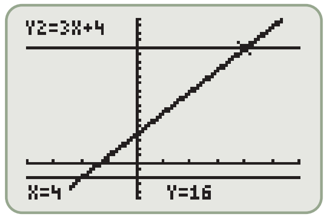
Interval notation: $\boxed{(-2, 4]}$

Technology Tip

Use a graphing utility to display graphs of $y_1 = -2$, $y_2 = 3x + 4$, and $y_3 = 16$.



The solutions are the x-values such that the graph of $y_2 = 3x + 4$ is between the graphs of $y_1 = -2$ and $y_3 = 16$ and overlaps that of $y_3 = 16$.



EXAMPLE 5 Comparative Shopping

Two car rental companies have advertised weekly specials on full-size cars. Hertz is advertising an \$80 rental fee plus an additional \$0.10 per mile. Thrifty is advertising \$60 and \$0.20 per mile. How many miles must you drive for the rental car from Hertz to be the better deal?

Solution:

Let x = number of miles driven during the week.

Write the cost for the Hertz rental. $80 + 0.1x$

Write the cost for the Thrifty rental. $60 + 0.2x$

Write the inequality if Hertz is less than Thrifty. $80 + 0.1x < 60 + 0.2x$

Subtract $0.1x$ from both sides. $80 < 60 + 0.1x$

Subtract 60 from both sides. $20 < 0.1x$

Divide both sides by 0.1. $200 < x$

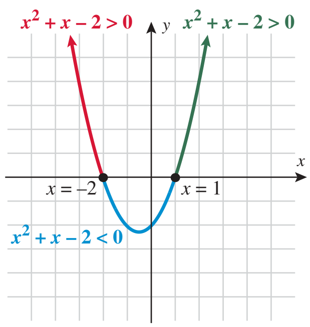
You must drive more than 200 miles for Hertz to be the better deal.

Polynomial Inequalities

A polynomial must pass through zero before its value changes from positive to negative or from negative to positive. **Zeros** of a polynomial are the values of x that make the polynomial equal to zero. These zeros divide the real number line into **test intervals** where the value of the polynomial is either positive or negative. For $x^2 + x - 2 < 0$, if we set the polynomial equal to zero and solve:

$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \quad \text{or} \quad x = 1$$

we find that $x = -2$ and $x = 1$ are the zeros. These zeros divide the real number line into three test intervals: $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$.



Since the polynomial is equal to zero at $x = -2$ and $x = 1$, we select one real number that lies in each of the three intervals and test to see whether the value of the polynomial at each point is either positive or negative. In this example, we select the real numbers $x = -3$, $x = 0$, and $x = 2$. At this point, there are two ways we can determine whether the value of the polynomial is positive or negative on the interval. One approach is to substitute each of the test points into the polynomial $x^2 + x - 2$.

$x = -3$: $(-3)^2 + (-3) - 2 = 9 - 3 - 2 = 4$ Positive

$x = 0$: $(0)^2 + (0) - 2 = 0 - 0 - 2 = -2$ Negative

$x = 2$: $(2)^2 + (2) - 2 = 4 + 2 - 2 = 4$ Positive

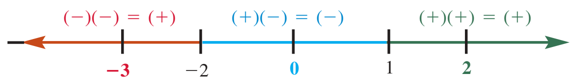
The second approach is to simply determine the sign of the result as opposed to actually calculating the exact number. This alternate approach is often used when the expressions or test points get more complicated to evaluate. The polynomial is written as the product $(x + 2)(x - 1)$; therefore, we simply look for the sign in each set of parentheses.

$$(x + 2)(x - 1)$$

$$\mathbf{x = -3:}\quad (-3 + 2)(-3 - 1) = (-1)(-4) \rightarrow (-)(-) = (+)$$

$$\mathbf{x = \quad :}\quad (0 + 2)(0 - 1) = (2)(-1) \rightarrow (+)(-) = (-)$$

$$\mathbf{x = 2:}\quad (2 + 2)(2 - 1) = (4)(1) \rightarrow (+)(+) = (+)$$



In this second approach we find the same result: $(-\infty, -2)$ and $(1, \infty)$ correspond to a positive value of the polynomial, and $(-2, 1)$ corresponds to a negative value of the polynomial.

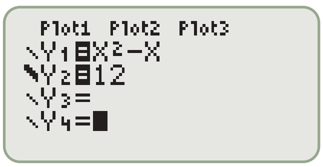
In this example, the statement $x^2 + x - 2 < 0$ is true when the value of the polynomial (in factored form), $(x + 2)(x - 1)$, is negative. In the interval $(-2, 1)$, the value of the polynomial is negative. Thus, the solution to the inequality $x^2 + x - 2 < 0$ is $(-2, 1)$. To check the solution, select any number in the interval and substitute it into the original inequality to make sure it makes the statement true. The value $x = -1$ lies in the interval $(-2, 1)$. Upon substituting into the original inequality, we find that $x = -1$ satisfies the inequality $(-1)^2 + (-1) - 2 = -2 < 0$.

Study Tip

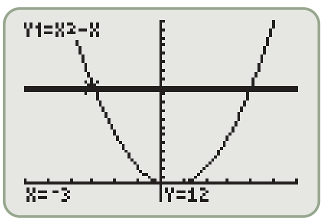
If the original polynomial is <0, then the interval(s) that yield(s) *negative* products should be selected. If the original polynomial is >0, then the interval(s) that yield(s) *positive* products should be selected.

Technology Tip

Use a graphing utility to display graphs of $y_1 = x^2 - x$ and $y_2 = 12$.



The solutions are the x-values such that the graph of y_1 lies above the graph of y_2.



PROCEDURE FOR SOLVING POLYNOMIAL INEQUALITIES

Step 1: Write the inequality in *standard form*.
Step 2: Identify zeros of the polynomial.
Step 3: Draw the number line with zeros labeled.
Step 4: Determine the sign of the polynomial in each interval.
Step 5: Identify which interval(s) make the inequality true.
Step 6: Write the solution in interval notation.

Note: Be careful in Step 5. If the original polynomial is <0, then the interval(s) that correspond(s) to the value of the polynomial being negative should be selected. If the original polynomial is >0, then the interval(s) that correspond(s) to the value of the polynomial being positive should be selected.

EXAMPLE 6 Solving a Quadratic Inequality

Solve the inequality $x^2 - x > 12$.

Solution:

STEP 1: Write the inequality in standard form. $\quad x^2 - x - 12 > 0$

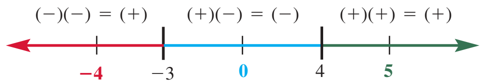
Factor the left side. $\quad (x + 3)(x - 4) > 0$

STEP 2: Identify the zeros. $\quad (x + 3)(x - 4) = 0$

$$x = -3 \quad \text{or} \quad x = 4$$

STEP 3: Draw the number line with the zeros labeled.

STEP 4: Determine the sign of $(x + 3)(x - 4)$ in each interval.



STEP 5: Intervals in which the value of the polynomial is *positive* make this inequality true. $(-\infty, -3)$ or $(4, \infty)$

STEP 6: Write the solution in interval notation. $\boxed{(-\infty, -3) \cup (4, \infty)}$

■ **YOUR TURN** Solve the inequality $x^2 - 5x \le 6$ and express the solution in interval notation.

■ **Answer:** $[-1, 6]$

The inequality in Example 6, $x^2 - x > 12$, is a strict inequality, so we use parentheses when we express the solution in interval notation $(-\infty, -3) \cup (4, \infty)$. It is important to note that if we change the inequality sign from $>$ to $\ge$, then the zeros $x = -3$ and $x = 4$ also make the inequality true. Therefore, the solution to $x^2 - x \ge 12$ is $(-\infty, -3] \cup [4, \infty)$.

EXAMPLE 7 Solving a Quadratic Inequality

Solve the inequality $x^2 > -5x$.

COMMON MISTAKE

A common mistake is to divide by x. Never divide by a variable, because the value of the variable might be zero. Always start by writing the inequality in standard form and then factor to determine the zeros.

★ CORRECT

STEP 1: Write the inequality in standard form.

$$x^2 + 5x > 0$$

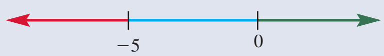
Factor.

$$x(x + 5) > 0$$

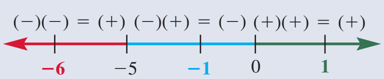
STEP 2: Identify the zeros.

$$x = 0, x = -5$$

STEP 3: Draw the number line with the zeros labeled.



STEP 4: Determine the sign of $x(x + 5)$ in each interval.



STEP 5: Intervals in which the value of the polynomial is *positive* satisfy the inequality.

$(-\infty, -5)$ and $(0, \infty)$

STEP 6: Express the solution in interval notation.

$$\boxed{(-\infty, -5) \cup (0, \infty)}$$

☒ INCORRECT

Write the original inequality.

$$x^2 > -5x$$

ERROR:
Divide both sides by x.

$$x > -5$$

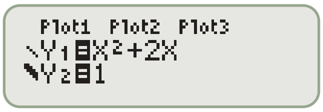
Dividing by x is the mistake. If x is negative, the inequality sign must be reversed. What if x is zero?

▼ **CAUTION**

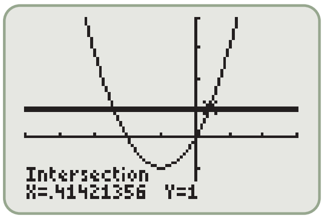
Do not divide inequalities by a variable.

Technology Tip

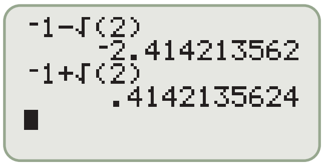
Using a graphing utility, graph $y_1 = x^2 + 2x$ and $y_2 = 1$.



The solutions are the x-values such that the graph of y_1 lies below the graph of y_2.



Note that



■ **Answer:** $(-\infty, 1 - \sqrt{2}\,] \cup [1 + \sqrt{2}, \infty)$

EXAMPLE 8 Solving a Quadratic Inequality

Solve the inequality $x^2 + 2x < 1$.

Solution:

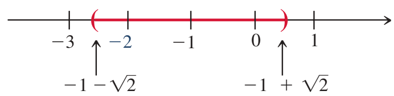
Write the inequality in standard form. $\quad x^2 + 2x - 1 < 0$

Identify the zeros. $\quad x^2 + 2x - 1 = 0$

Apply the quadratic formula.
$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$

Simplify.
$$x = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

Draw the number line with the intervals labeled.



Note: $-1 - \sqrt{2} \approx -2.41$
$-1 + \sqrt{2} \approx 0.41$

Test each interval.

$(-\infty, -1 - \sqrt{2})$	$x = -3$:	$(-3)^2 + 2(-3) - 1 = 2 > 0$
$(-1 - \sqrt{2}, -1 + \sqrt{2})$	$x = 0$:	$(0)^2 + 2(0) - 1 = -1 < 0$
$(-1 + \sqrt{2}, \infty)$	$x = 1$:	$(1)^2 + 2(1) - 1 = 2 > 0$

Intervals in which the value of the polynomial is *negative* make this inequality true. $\quad \boxed{(-1 - \sqrt{2}, -1 + \sqrt{2})}$

■ **YOUR TURN** Solve the inequality $x^2 - 2x \geq 1$.

■ **Answer:** $(-\infty, -2\,) \cup (0, 3)$

EXAMPLE 9 Solving a Polynomial Inequality

Solve the inequality $x^3 - 3x^2 \geq 10x$.

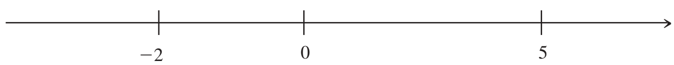
Solution:

Write the inequality in standard form. $\quad x^3 - 3x^2 - 10x \geq 0$

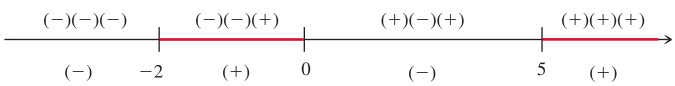
Factor. $\quad x(x - 5)(x + 2) \geq 0$

Identify the zeros. $\quad x = 0, x = 5, x = -2$

Draw the number line with the zeros (intervals) labeled.



Test each interval.



Intervals in which the value of the polynomial is *positive* make this inequality true. $\quad \boxed{[-2, 0] \cup [5, \infty)}$

■ **YOUR TURN** Solve the inequality $x^3 - x^2 - 6x < 0$.

Rational Inequalities

A rational expression can change signs if either the numerator or denominator changes signs. In order to go from positive to negative or vice versa, you must pass through zero. To *solve* rational inequalities such as $\frac{x-3}{x^2-4} \geq 0$, we use a similar procedure to the one used for solving polynomial inequalities, with one exception. You must eliminate from the solution set values for x that make the denominator equal to zero. In this example, we must eliminate $x = -2$ and $x = 2$ because these values make the denominator equal to zero. Rational inequalities have implied domains. In this example, $x \neq \pm 2$ is a domain restriction and these values ($x = -2$ and $x = 2$) must be eliminated from a possible solution.

We will proceed with a similar procedure involving zeros and test intervals that was outlined for polynomial inequalities. However, in rational inequalities once expressions are combined into a single fraction, any values that make *either* the numerator *or* the denominator equal to zero divide the number line into intervals.

Study Tip

Values that make the denominator equal to zero are always excluded.

EXAMPLE 10 **Solving a Rational Inequality**

Solve the inequality $\frac{x-3}{x^2-4} \geq 0$.

Solution:

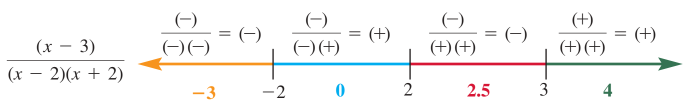
Factor the denominator. $\qquad \frac{(x-3)}{(x-2)(x+2)} \geq 0$

State the domain restrictions on the variable. $\qquad x \neq 2, x \neq -2$

Identify the zeros of numerator and denominator. $\qquad x = -2, x = 2, x = 3$

Draw the number line and divide into intervals.

Test the intervals.



Intervals in which the value of the rational expression is *positive* satisfy this inequality. $\qquad$ **$(-2, 2)$** and **$(3, \infty)$**

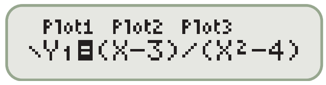
Since this inequality is greater than or equal to, we include $x = 3$ in our solution because it satisfies the inequality. However, $x = -2$ and $x = 2$ are not included in the solution because they make the denominator equal to zero.

The solution is $\boxed{(-2, 2) \cup [3, \infty)}$.

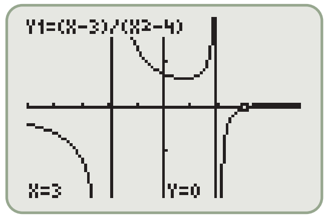
■ **YOUR TURN** Solve the inequality $\frac{x+2}{x-1} \leq 0$.

Technology Tip

Use a graphing utility to display the graph of $y_1 = \frac{x-3}{x^2-4}$.



The solutions are the x-values such that the graph of y_1 lies on top and above the x-axis, excluding $x = \pm 2$.



■ **Answer:** $[-2, 1)$

EXAMPLE 11 Solving a Rational Inequality

Solve the inequality $\frac{x}{x+2} \le 3$.

▼ CAUTION

Rational inequalities should not be solved using cross multiplication.

COMMON MISTAKE

Do not cross multiply. The LCD or expression by which you are multiplying might be negative for some values of x, and that would require the direction of the inequality sign to be reversed.

★ CORRECT

Subtract 3 from both sides.

$$\frac{x}{x+2} - 3 \le 0$$

Write as a single rational expression.

$$\frac{x - 3(x+2)}{x+2} \le 0$$

Eliminate the parentheses.

$$\frac{x - 3x - 6}{x+2} \le 0$$

Simplify the numerator.

$$\frac{-2x - 6}{x+2} \le 0$$

Factor the numerator.

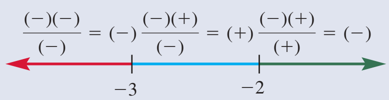
$$\frac{-2(x+3)}{x+2} \le 0$$

Identify the zeros of the numerator and the denominator.

$$x = -3 \text{ and } x = -2$$

Draw the number line and test the intervals.

$$\frac{-2(x+3)}{x+2} \le 0$$



Intervals in which the value of the rational expression is *negative* satisfy the inequality. $(-\infty, -3]$ and $(-2, \infty)$. Note that $x = -2$ is not included in the solution because it makes the denominator zero, and $x = -3$ is included because it satisfies the inequality.

The solution is

$$\boxed{(-\infty, -3] \cup (-2, \infty)}$$

✗ INCORRECT

ERROR:

Do not cross multiply.

$$x \le 3(x+2)$$

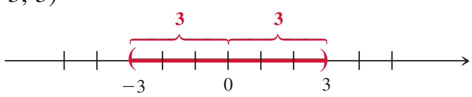
Absolute Value Inequalities

To solve the inequality $|x| < 3$, look for all real numbers that make this statement true. If we interpret this inequality as distance, we ask *what numbers are less than three units from the origin?* We can represent the solution in the following ways:

Inequality notation: $-3 < x < 3$

Interval notation: $(-3, 3)$

Graph:

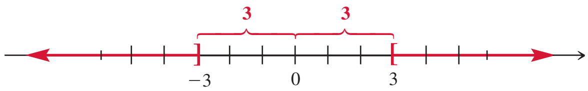


Similarly, to solve the inequality $|x| \geq 3$, look for all real numbers that make the statement true. If we interpret this inequality as a distance, we ask *what numbers are at least three units from the origin?* We can represent the solution in the following three ways:

Inequality notation: $x \leq -3$ or $x \geq 3$

Interval notation: $(-\infty, -3] \cup [3, \infty)$

Graph:



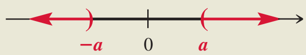
This discussion leads us to the following equivalence relations.

PROPERTIES OF ABSOLUTE VALUE INEQUALITIES

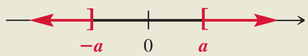
1. $|x| < a$ is equivalent to $-a < x < a$

2. $|x| \leq a$ is equivalent to $-a \leq x \leq a$

3. $|x| > a$ is equivalent to $x < -a$ or $x > a$



4. $|x| \geq a$ is equivalent to $x \leq -a$ or $x \geq a$



It is important to realize that in the above four properties, the variable x can be any algebraic expression.

EXAMPLE 12 Solving an Inequality Involving an Absolute Value

Solve the inequality $|3x - 2| \leq 7$.

Solution:

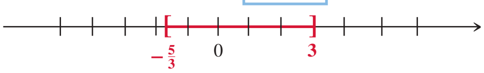
We apply property (2) and squeeze the absolute value expression between -7 and 7. $\qquad -7 \leq 3x - 2 \leq 7$

Add 2 to all three parts. $\qquad -5 \leq 3x \leq 9$

Divide all three parts by 3. $\qquad -\dfrac{5}{3} \leq x \leq 3$

The solution in interval notation is $\boxed{\left[-\frac{5}{3}, 3\right]}$.

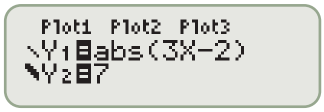
Graph:



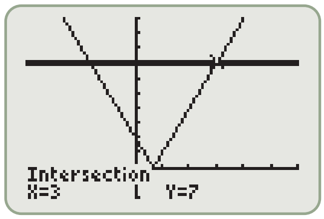
■ **YOUR TURN** Solve the inequality $|2x + 1| < 11$.

Technology Tip

Use a graphing utility to display graphs of $y_1 = |3x - 2|$ and $y_2 = 7$.



The values of x where the graph of y_1 lies on top and below the graph of y_2 are the solutions to this inequality.



■ **Answer:**
Inequality notation: $-6 < x < 5$
Interval notation: $(-6, 5)$

It is often helpful to note that for absolute value inequalities,

- *less than* inequalities can be written as a single statement (see Example 12).
- *greater than* inequalities must be written as two statements (see Example 13).

Study Tip

Less than inequalities can be written as a single statement.

Greater than inequalities must be written as two statements.

EXAMPLE 13 Solving an Inequality Involving an Absolute Value

Solve the inequality $|1 - 2x| > 5$.

Solution:

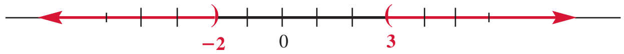
Apply property (3). $1 - 2x < -5$ or $1 - 2x > 5$

Subtract 1 from all expressions. $-2x < -6$ $\quad$ $-2x > 4$

Divide by -2 and reverse the inequality sign. $x > 3$ $\quad$ $x < -2$

Express the solution in interval notation. $\boxed{(-\infty, -2) \cup (3, \infty)}$

Graph:



■ **YOUR TURN** Solve the inequality $|5 - 2x| \geq 1$.

■ **Answer:**
Inequality notation: $x \leq 2$ or $x \geq 3$
Interval notation: $(-\infty, 2] \cup [3, \infty)$

Notice that if we change the problem in Example 13 to $|1 - 2x| > -5$, the answer is all real numbers because **the absolute value of any expression is greater than or equal to zero**. Similarly, $|1 - 2x| < -5$ would have no solution because **the absolute value of an expression can never be negative**.

EXAMPLE 14 Solving an Inequality Involving an Absolute Value

Solve the inequality $2 - |3x| < 1$.

Solution:

Subtract 2 from both sides. $-|3x| < -1$

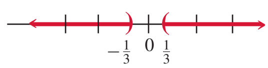
Multiply by (-1) and reverse the inequality sign. $|3x| > 1$

Apply property (3). $3x < -1$ or $3x > 1$

Divide both inequalities by 3. $x < -\frac{1}{3}$ or $x > \frac{1}{3}$

Express in interval notation. $\boxed{\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)}$

Graph.



SECTION 0.4 SUMMARY

In this section, we used interval notation to represent the solution to inequalities.

- **Linear Inequalities:** Solve linear inequalities similarly to how we solve linear equations with one exception—when you multiply or divide by a negative number, you must reverse the inequality sign.
- **Polynomial Inequalities:** First write the inequality in standard form (zero on one side). Determine the zeros, draw the number line, test the intervals, select the intervals according to the sign of the inequality, and write the solution in interval notation.
- **Rational Inequalities:** Write as a single fraction and then proceed with a similar approach as used in ploynomial inequalities—only the test intervals are determined by finding the zeros of either the numerator or the denominator. Exclude any values from the solution that result in the denominator being equal to zero.
- **Absolute Value Inequalities:** Write an absolute value inequality in terms of two inequalities that do not involve absolute value:
  - $|x| < A$ is equivalent to $-A < x < A$.
  - $|x| > A$ is equivalent to $x < -A$ or $x > A$.

SECTION 0.4 EXERCISES

SKILLS

In Exercises 1–10, rewrite in interval notation and graph.

1. $-2 \le x < 3$
2. $-4 \le x \le -1$
3. $-3 < x \le 5$
4. $0 < x < 6$
5. $x \le 6$ and $x \ge 4$
6. $x > -3$ and $x \le 2$
7. $x \le -6$ and $x \ge -8$
8. $x < 8$ and $x < 2$
9. $x > 4$ and $x \le -2$
10. $x \ge -5$ and $x < -6$

In Exercises 11–20, graph the indicated set and write as a single interval, if possible.

11. $(-\infty, 4) \cap [1, \infty)$
12. $(-3, \infty) \cap [-5, \infty)$
13. $[-5, 2) \cap [-1, 3]$
14. $[-4, 5) \cap [-2, 7)$
15. $(-\infty, 4) \cup (4, \infty)$
16. $(-\infty, -3] \cup [-3, \infty)$
17. $(-\infty, -3] \cup [3, \infty)$
18. $(-2, 2) \cap [-3, 1]$
19. $(-\infty, \infty) \cap (-3, 2]$
20. $(-\infty, \infty) \cup (-4, 7)$

In Exercises 21–36, solve each linear inequality and express the solution set in interval notation.

21. $3(t + 1) > 2t$
22. $2(y + 5) \le 3(y - 4)$
23. $7 - 2(1 - x) > 5 + 3(x - 2)$
24. $4 - 3(2 + x) < 5$
25. $\frac{2}{3}y - \frac{1}{2}(5 - y) < \frac{5y}{3} - (2 + y)$
26. $\frac{s}{2} - \frac{(s - 3)}{3} > \frac{s}{4} - \frac{1}{12}$
27. $-1.8x + 2.5 > 3.4$
28. $2.7x - 1.3 < 6.8$
29. $-3 < 1 - x \le 9$
30. $3 \le -2 - 5x \le 13$
31. $0 < 2 - \frac{1}{3}y < 4$
32. $3 < \frac{1}{2}A - 3 < 7$
33. $\frac{1}{2} \le \frac{1 + y}{3} \le \frac{3}{4}$
34. $-1 < \frac{2 - z}{4} \le \frac{1}{5}$
35. $-0.7 \le 0.4x + 1.1 \le 1.3$
36. $7.1 > 4.7 - 1.2x > 1.1$

In Exercises 37–54, solve each polynomial inequality and express the solution set in interval notation.

37. $2t^2 - 3 \le t$
38. $3t^2 \ge -5t + 2$
39. $5v - 1 > 6v^2$
40. $12t^2 < 37t + 10$
41. $2s^2 - 5s \ge 3$
42. $8s + 12 \le -s^2$
43. $y^2 + 2y \ge 4$
44. $y^2 + 3y \le 1$
45. $x^2 - 4x < 6$
46. $x^2 - 2x > 5$
47. $u^2 \ge 3u$
48. $u^2 \le -4u$
49. $x^2 > 9$
50. $t^2 \le 49$
51. $x^3 + x^2 - 2x \le 0$
52. $x^3 + 2x^2 - 3x > 0$
53. $x^3 + x > 2x^2$
54. $x^3 + 4x \le 4x^2$

In Exercises 55–70, solve each rational inequality and express the solution set in interval notation.

55. $\frac{s+1}{4-s^2} \geq 0$

56. $\frac{s+5}{4-s^2} \leq 0$

57. $\frac{3t^2}{t+2} \geq 5t$

58. $\frac{-2t-t^2}{4-t} \geq t$

59. $\frac{3p-2p^2}{4-p^2} < \frac{3+p}{2-p}$

60. $-\frac{7p}{p^2-100} \leq \frac{p+2}{p+10}$

61. $\frac{x^2+10}{x^2+16} > 0$

62. $-\frac{x^2+2}{x^2+4} < 0$

63. $\frac{v^2-9}{v-3} \geq 0$

64. $\frac{v^2-1}{v+1} \leq 0$

65. $\frac{2}{t-3} + \frac{1}{t+3} \geq 0$

66. $\frac{1}{t-2} + \frac{1}{t+2} \leq 0$

67. $\frac{3}{x+4} - \frac{1}{x-2} \leq 0$

68. $\frac{2}{x-5} - \frac{1}{x-1} \geq 0$

69. $\frac{1}{p-2} - \frac{1}{p+2} \geq \frac{3}{p^2-4}$

70. $\frac{2}{2p-3} - \frac{1}{p+1} \leq \frac{1}{2p^2-p-3}$

In Exercises 71–86, solve the absolute value inequality and express the solution set in interval notation.

71. $|x-4| > 2$

72. $|x-1| < 3$

73. $|4-x| \leq 1$

74. $|1-y| < 3$

75. $|2x| > -3$

76. $|2x| < -3$

77. $|7-2y| \geq 3$

78. $|6-5y| \leq 1$

79. $|4-3x| \geq 0$

80. $|4-3x| \geq 1$

81. $2|4x| - 9 \geq 3$

82. $5|x-1| + 2 \leq 7$

83. $9 - |2x| < 3$

84. $4 - |x+1| > 1$

85. $|x^2-1| \leq 8$

86. $|x^2+4| \geq 29$

APPLICATIONS

87. Lasers. A circular laser beam with a radius r_T is transmitted from one tower to another tower. If the received beam radius r_R fluctuates 10% from the transmitted beam radius due to atmospheric turbulence, write an inequality representing the received beam radius.

88. Electronics: Communications. Communication systems are often evaluated based on their signal-to-noise ratio (SNR), which is the ratio of the average power of received signal, S, to average power of noise, N, in the system. If the SNR is required to be at least 2 at all times, write an inequality representing the received signal power if the noise can fluctuate 10%.

The following table is the 2007 Federal Tax Rate Schedule for people filing as *single:*

TAX BRACKET #	IF TAXABLE INCOME IS OVER–	BUT NOT OVER–	THE TAX IS:
I	$0	$7,825	10% of the amount over $0
II	$7,825	$31,850	$782.50 plus 15% of the amount over $7,825
III	$31,850	$77,100	$4,386.25 plus 25% of the amount over $31,850
IV	$77,100	$160,850	$15,698.75 plus 28% of the amount over $77,100
V	$160,850	$349,700	$39,148.75 plus 33% of the amount over $160,850
VI	$349,700	No limit	$101,469.25 plus 35% of the amount over $349,700

89. Federal Income Tax. What was the range of federal income taxes a person in tax bracket III would pay the IRS?

90. Federal Income Tax. What was the range of federal income taxes a person in tax bracket IV would pay the IRS?

In Exercises 91 and 92 refer to the following:

The annual revenue for a small company is modeled by

$$R = 5000 + 1.75x$$

where x is hundreds of units sold and R is revenue in thousands of dollars.

91. Business. Find the number of units (to the nearest 100) that must be sold to generate at least $10 million in revenue.

92. Business. Find the number of units (to the nearest 100) that must be sold to generate at least $7.5 million in revenue.

In Exercises 93 and 94 refer to the following:

The Target or Training Heart Rate (THR) is a range of heart rate (measured in beats per minute) that enables a person's heart and lungs to benefit the most from an aerobic workout. THR can be modeled by the formula

$$THR = (HR_{max} - HR_{rest}) \times I + HR_{rest}$$

where HR_{max} is the maximum heart rate that is deemed safe for the individual, HR_{rest} is the resting heart rate, and I is the intensity of the workout that is reported as a percentage.

93. Health. A female with a resting heart rate of 65 beats per minute has a maximum safe heart rate of 170 beats per minute. If her target heart rate is between 100 and 140 beats per minute, what percent intensities of workout can she consider?

94. Health. A male with a resting heart rate of 75 beats per minute has a maximum safe heart rate of 175 beats per minute. If his target heart rate is between 110 and 150 beats per minute, what percent intensities of workout can he consider?

95. Profit. A Web-based embroidery company makes monogrammed napkins. The profit associated with producing x orders of napkins is governed by the equation

$$P(x) = -x^2 + 130x - 3000$$

Determine the range of orders the company should accept in order to make a profit.

96. Profit. Repeat Exercise 95 using $P(x) = x^2 - 130x + 3600$.

97. Car Value. The term "upside down" on car payments refers to owing more than a car is worth. Assume you buy a new car and finance 100% over 5 years. The difference between the value of the car and what is owed on the car is governed by the expression $\frac{t}{t-3}$, where t is age (in years) of the car. Determine the time period when the car is worth more than you owe $\left(\frac{t}{t-3} > 0\right)$. When do you owe more than it's worth $\left(\frac{t}{t-3} < 0\right)$?

98. Car Value. Repeat Exercise 97 using the expression $-\frac{2-t}{4-t}$.

99. Bullet Speed. A .22 caliber gun fires a bullet at a speed of 1200 feet per second. If a .22 caliber gun is fired straight upward into the sky, the height of the bullet in feet is given by the equation $h = -16t^2 + 1200t$, where t is the time in seconds with $t = 0$ corresponding to the instant the gun is fired. How long is the bullet in the air?

100. Bullet Speed. A .38 caliber gun fires a bullet at a speed of 600 feet per second. If a .38 caliber gun is fired straight upward into the sky, the height of the bullet in feet is given by the equation $h = -16t^2 + 600t$. How many seconds is the bullet in the air?

In Exercises 101 and 102 refer to the following:

In response to economic conditions, a local business explores the effect of a price increase on weekly profit. The function

$$P = -5(x + 3)(x - 24)$$

models the effect that a price increase of x dollars on a bottle of wine will have on the profit P measured in dollars.

101. Economics. What price increase will lead to a weekly profit of less than \$460?

102. Economics. What price increases will lead to a weekly profit of more than \$550?

103. Sports. Two women tee off of a par-3 hole on a golf course. They are playing "closest to the pin." If the first woman tees off and lands exactly 4 feet from the hole, write an inequality that describes where the second woman lands to the hole in order to win the hole. What equation would suggest a tie? Let d = the distance from where the second woman lands to the tee.

104. Electronics. A band-pass filter in electronics allows certain frequencies within a range (or band) to pass through to the receiver and eliminates all other frequencies. Write an absolute value inequality that allows any frequency f within 15 hertz of the carrier frequency f_c to pass.

In Exercises 105 and 106 refer to the following:

A company is reviewing revenue for the prior sales year. The model for projected revenue and the model for actual revenue are

$$R_{\text{projected}} = 200 + 5x$$
$$R_{\text{actual}} = 210 + 4.8x$$

where x represents the number of units sold and R represents the revenue in thousands of dollars. Since the two revenue models are not identical, an error in projected revenue occurred. This error is represented by

$$E = |R_{\text{projected}} - R_{\text{actual}}|$$

105. Business. For what number of units sold was the error in projected revenue less than \$5000?

106. Business. For what number of units sold was the error in projected revenue less than \$3000?

CATCH THE MISTAKE

In Exercises 107–110, explain the mistake that is made.

107. Solve the inequality $2 - 3p \le -4$ and express the solution in interval notation.

Solution:

$$\begin{aligned} 2 - 3p &\le -4 \\ -3p &\le -6 \\ p &\le 2 \\ &(-\infty, 2] \end{aligned}$$

This is incorrect. What mistake was made?

108. Solve the inequality $u^2 < 25$.

Solution:

Take the square root of both sides. $u < -5$

Write the solution in interval notation. $(-\infty, -5)$

This is incorrect. What mistake was made?

109. Solve the inequality $3x < x^2$.

Solution:

Divide by x.	$3 < x$
Write the solution in interval notation.	$(3, \infty)$

This is incorrect. What mistake was made?

110. Solve the inequality $\frac{x+4}{x} < -\frac{1}{3}$.

Solution:

Cross multiply.	$3(x + 4) < -1(x)$
Eliminate the parentheses.	$3x + 12 < -x$
Combine like terms.	$4x < -12$
Divide both sides by 4.	$x < -3$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 111–114, determine whether each statement is true or false. Assume that *a* is a positive real number.

111. If $x < a$, then $a > x$.

112. If $-x \geq a$, then $x \geq -a$.

113. If $x < a^2$, then the solution is $(-\infty, a)$.

114. If $x \geq a^2$, then the solution is $[a, \infty)$.

CHALLENGE

In Exercises 115 and 116, solve for *x* given that *a* and *b* are both positive real numbers.

115. $\frac{x^2 + a^2}{x^2 + b^2} \geq 0$

116. $\frac{x^2 - b^2}{x + b} < 0$

117. For what values of x does the absolute value equation $|x + 1| = 4 + |x - 2|$ hold?

118. Solve the inequality $|3x^2 - 7x + 2| > 8$.

TECHNOLOGY

119. **a.** Solve the inequality $x - 3 < 2x - 1 < x + 4$.
b. Graph all three expressions of the inequality in the same viewing screen. Find the range of x-values when the graph of the middle expression lies above the graph of the left side and below the graph of the right side.
c. Do (a) and (b) agree?

120. **a.** Solve the inequality $x - 2 < 3x + 4 \leq 2x + 6$.
b. Graph all three expressions of the inequality in the same viewing screen. Find the range of x-values when the graph of the middle expression lies above the graph of the left side and on top of and below the graph of the right side.
c. Do (a) and (b) agree?

121. Solve the inequality $\left|\frac{x}{x+1}\right| < 1$ by graphing both sides of the inequality, and identify which x-values make this statement true.

122. Solve the inequality $\left|\frac{x}{x+1}\right| < 2$ by graphing both sides of the inequality, and identify which x-values make this statement true.

SECTION 0.5

GRAPHING EQUATIONS

SKILLS OBJECTIVES

- Calculate the distance between two points and the midpoint of a line segment joining two points.
- Graph equations in two variables by point-plotting.
- Use intercepts and symmetry as graphing aids.
- Graph circles.

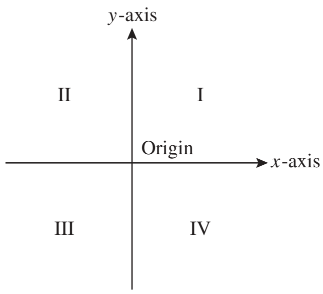
CONCEPTUAL OBJECTIVES

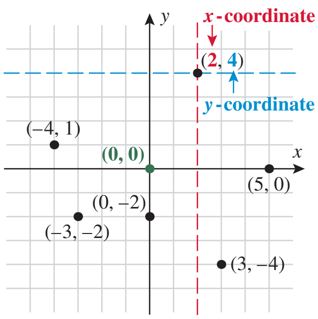
- Expand the concept of a one-dimensional number line to a two-dimensional plane.
- Relate symmetry graphically and algebraically.

Cartesian Plane

HIV infection rates, stock prices, and temperature conversions are all examples of relationships between two quantities that can be expressed in a two-dimensional graph. Because it is two-dimensional, such a graph lies in a **plane**.

Two perpendicular real number lines, known as the **axes** in the plane, intersect at a point we call the **origin**. Typically, the horizontal axis is called the ***x*-axis** and the vertical axis is denoted as the ***y*-axis**. The axes divide the plane into four **quadrants**, numbered by Roman numerals and ordered counterclockwise.

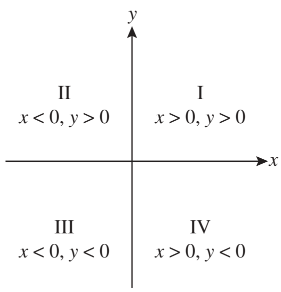




Points in the plane are represented by **ordered pairs**, denoted (x, y). The first number of the ordered pair indicates the position in the horizontal direction and is often called the ***x*-coordinate** or **abscissa**. The second number indicates the position in the vertical direction and is often called the ***y*-coordinate** or **ordinate**. The **origin** is denoted **(0, 0)**.

Examples of other coordinates are given on the graph to the left.

The point (2, 4) lies in quadrant I. To **plot** this point, start at the origin **(0, 0)** and move to the right two units and up four units.



All points in quadrant I have positive coordinates, and all points in quadrant III have negative coordinates. Quadrant II has negative x-coordinates and positive y-coordinates; quadrant IV has positive x-coordinates and negative y-coordinates.

This representation is called the **rectangular coordinate system** or **Cartesian coordinate system**, named after the French mathematician René Descartes.

The Distance and Midpoint Formulas

DEFINITION **Distance Formula**

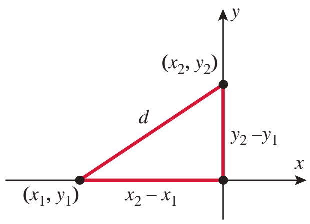
The **distance *d*** between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance between two points is the square root of the sum of the square of the difference between the x-coordinates and the square of the difference between the y-coordinates.

Study Tip

It does not matter which point is taken to be the first point or the second point.



EXAMPLE 1 **Using the Distance Formula to Find the Distance Between Two Points**

Find the distance between $(-3, 7)$ and $(5, -2)$.

Solution:

Write the distance formula. $d = \sqrt{[x_2 - x_1]^2 + [y_2 - y_1]^2}$

Substitute $(x_1, y_1) = (-3, 7)$ and $(x_2, y_2) = (5, -2)$. $d = \sqrt{[5 - (-3)]^2 + [-2 - 7]^2}$

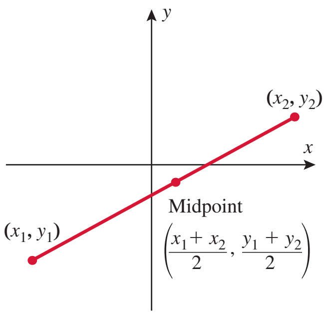
Simplify. $d = \sqrt{[5 + 3]^2 + [-2 - 7]^2}$

$d = \sqrt{8^2 + (-9)^2} = \sqrt{64 + 81} = \sqrt{145}$

Solve for d. $\boxed{d = \sqrt{145}}$

■ **Answer:** $d = \sqrt{58}$

■ **YOUR TURN** Find the distance between $(4, -5)$ and $(-3, -2)$.



DEFINITION **Midpoint Formula**

The **midpoint**, (x_m, y_m), of the line segment with endpoints (x_1, y_1) and (x_2, y_2) is given by

$$(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The midpoint can be found by averaging the x-coordinates and averaging the y-coordinates.

EXAMPLE 2 **Finding the Midpoint of a Line Segment**

Find the midpoint of the line segment joining the points $(2, 6)$ and $(-4, -2)$.

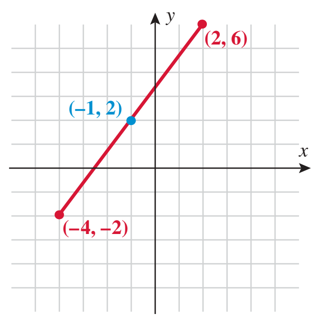
Solution:

Write the midpoint formula. $(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Substitute $(x_1, y_1) = (2, 6)$ and $(x_2, y_2) = (-4, -2)$. $(x_m, y_m) = \left(\frac{2 + (-4)}{2}, \frac{6 + (-2)}{2}\right)$

Simplify. $\boxed{(x_m, y_m) = (-1, 2)}$

One way to verify your answer is to plot the given points and the midpoint to make sure your answer looks reasonable.



■ **Answer:** midpoint = $(4, 2)$

■ **YOUR TURN** Find the midpoint of the line segment joining the points $(3, -4)$ and $(5, 8)$.

Point-Plotting

The **graph of an equation** in two variables, x and y, consists of all the points in the xy-plane whose coordinates (x, y) satisfy the equation. A procedure for plotting the graphs of equations is outlined below and is illustrated with the example $y = x^2$.

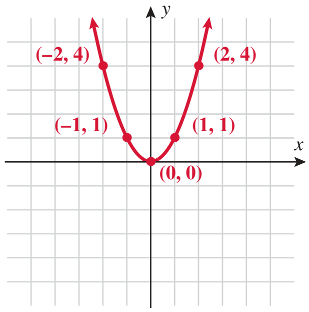
Words

Step 1: In a table, list several pairs of coordinates that make the equation true.

Math

x	$y = x^2$	(x, y)
0	0	(0, 0)
−1	1	(−1, 1)
1	1	(1, 1)
−2	4	(−2, 4)
2	4	(2, 4)

Step 2: Plot these points on a graph and connect the points with a smooth curve. Use arrows to indicate that the graph continues.



In graphing an equation, first select arbitrary values for x and then use the equation to find the corresponding value of y, or vice versa.

EXAMPLE 3 Graphing an Equation by Plotting Points

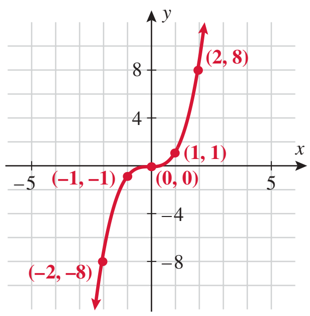
Graph the equation $y = x^3$.

Solution:

Step 1 In a table, list several pairs of coordinates that satisfy the equation.

x	$y = x^3$	(x, y)
0	0	(0, 0)
−1	−1	(−1, −1)
1	1	(1, 1)
−2	−8	(−2, −8)
2	8	(2, 8)

Step 2 Plot these points on a graph and connect the points with a smooth curve, indicating with arrows that the curve continues in both the positive and negative directions.



Using Intercepts and Symmetry as Graphing Aids

Intercepts

Study Tip

Identifying the intercepts helps define the graph unmistakably.

When point-plotting graphs of equations, which points should be selected? Points where a graph crosses (or touches) either the x-axis or y-axis are called **intercepts** and identifying these points helps define the graph unmistakably.

An ***x*-intercept** of a graph is a point where the graph intersects the x-axis. Specifically, an x-intercept is the x-coordinate of such a point. For example, if a graph intersects the x-axis at the point (3, 0), then we say that 3 is the x-intercept. Since the value for y along the x-axis is zero, all points corresponding to x-intercepts have the form $(a, 0)$.

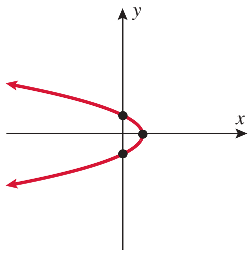
Study Tip

- Either the x-coordinate, say a, or the point $(a, 0)$ can be used to denote an x-intercept.
- Either the y-coordinate, say b, or the point $(0, b)$ can be used to denote a y-intercept.

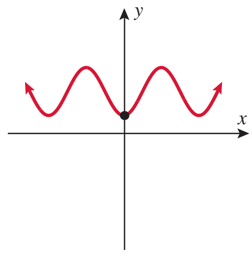
A ***y*-intercept** of a graph is a point where the graph intersects the y-axis. Specifically, a y-intercept is the y-coordinate of such a point. For example, if a graph intersects the y-axis at the point (0, 2), then we say that 2 is the y-intercept. Since the value for x along the y-axis is zero, all points corresponding to y-intercepts have the form $(0, b)$.

It is important to note that graphs of equations do not have to have intercepts, and if they do have intercepts, they can have one or more of each type.

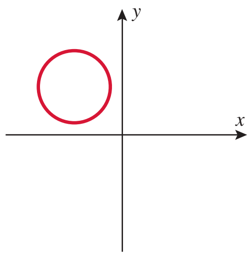
One x-intercept
Two y-intercepts



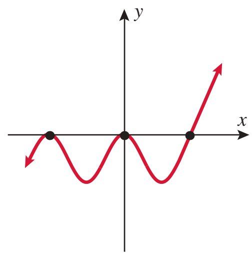
No x-intercepts
One y-intercept



No x-intercepts
No y-intercepts



Three x-intercepts
One y-intercept



Note: The origin (0, 0) corresponds to both an x-intercept and a y-intercept.

EXAMPLE 4 Finding Intercepts from an Equation

Given the equation $y = x^2 + 1$, find the indicated intercepts of its graph, if any.

a. x-intercept(s) **b.** y-intercept(s)

Solution (a):

Let $y = 0$. $\quad 0 = x^2 + 1$

Solve for x. $\quad x^2 = -1 \quad$ no real solution

$\boxed{\text{There are no } x\text{-intercepts.}}$

Solution (b):

Let $x = 0$. $\quad y = 0^2 + 1$

Solve for y. $\quad y = 1$

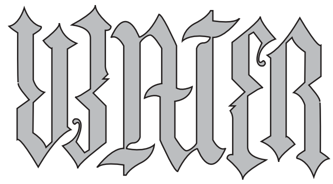
The y-intercept is located at the point $\boxed{(0, 1)}$.

YOUR TURN For the equation $y = x^2 - 4$

a. find the x-intercept(s), if any. **b.** find the y-intercept(s), if any.

Answer:
a. x-intercepts: -2 and 2
b. y-intercept: -4

Symmetry



The word **symmetry** conveys balance. Suppose you have two pictures to hang on a wall. If you space them equally apart on the wall, then you prefer a symmetric décor. This is an example of symmetry about a line. The word (water) written in the margin is identical if you rotate the word 180 degrees (or turn the page upside down). This is an example of symmetry about a point. Symmetric graphs have the characteristic that their mirror image can be obtained about a reference, typically a line or a point.

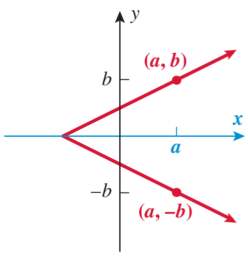
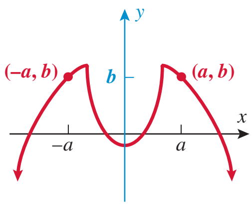
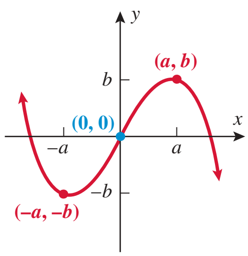
Symmetry aids in graphing by giving information "for free." For example, if a graph is symmetric about the y-axis, then once the graph to the right of the y-axis is found, the left side of the graph is the mirror image of that. If a graph is symmetric about the origin, then once the graph is known in quadrant I, the graph in quadrant III is found by rotating the known graph 180 degrees.

It would be beneficial to know whether a graph of an equation is symmetric about a line or point before the graph of the equation is sketched. Although a graph can be symmetric about any line or point, we will discuss only symmetry about the x-axis, y-axis, and origin. These types of symmetry and the algebraic procedures for testing for symmetry are outlined below.

Study Tip

Symmetry gives us information about the graph "for free."

Types and Tests for Symmetry

Type of Symmetry	Graph	If the Point (a, b) Is on the Graph, Then the Point . . .	Algebraic Test for Symmetry
Symmetric with respect to the **x-axis**		$(a, -b)$ is on the graph. .	Replacing y with $-y$ leaves the equation unchanged.
Symmetric with respect to the **y-axis**		$(-a, b)$ is on the graph.	Replacing x with $-x$ leaves the equation unchanged.
Symmetric with respect to the **origin**		$(-a, -b)$ is on the graph.	Replacing x with $-x$ and y with $-y$ leaves the equation unchanged.

EXAMPLE 5 Testing for Symmetry

Determine what type of symmetry (if any) the graphs of the equations exhibit.

a. $y = x^2 + 1$ **b.** $y = x^3 + 1$

Solution (a):

Replace x with $-x$. $y = (-x)^2 + 1$

Simplify. $y = x^2 + 1$

The resulting equation is equivalent to the original equation, so the graph of the equation $y = x^2 + 1$ is **symmetric with respect to the *y*-axis**.

Replace y with $-y$. $(-y) = x^2 + 1$

Simplify. $y = -x^2 - 1$

The resulting equation $y = -x^2 - 1$ is not equivalent to the original equation $y = x^2 + 1$, so the graph of the equation $y = x^2 + 1$ is **not symmetric with respect to the *x*-axis**.

Replace x with $-x$ and y with $-y$. $(-y) = (-x)^2 + 1$

Simplify. $-y = x^2 + 1$

$y = -x^2 - 1$

The resulting equation $y = -x^2 - 1$ is not equivalent to the original equation $y = x^2 + 1$, so the graph of the equation $y = x^2 + 1$ is **not symmetric with respect to the origin**.

The graph of the equation $y = x^2 + 1$ is **symmetric with respect to the *y*-axis**.

Solution (b):

Replace x with $-x$. $y = (-x)^3 + 1$

Simplify. $y = -x^3 + 1$

The resulting equation $y = -x^3 + 1$ is not equivalent to the original equation $y = x^3 + 1$. Therefore, the graph of the equation $y = x^3 + 1$ is **not symmetric with respect to the *y*-axis**.

Replace y with $-y$. $(-y) = x^3 + 1$

Simplify. $y = -x^3 - 1$

The resulting equation $y = -x^3 - 1$ is not equivalent to the original equation $y = x^3 + 1$. Therefore, the graph of the equation $y = x^3 + 1$ is **not symmetric with respect to the *x*-axis**.

Replace x with $-x$ and y with $-y$. $(-y) = (-x)^3 + 1$

Simplify. $-y = -x^3 + 1$

$y = x^3 - 1$

The resulting equation $y = x^3 - 1$ is not equivalent to the original equation $y = x^3 + 1$. Therefore, the graph of the equation $y = x^3 + 1$ is **not symmetric with respect to the origin**.

The graph of the equation $y = x^3 + 1$ exhibits **no symmetry**.

YOUR TURN Determine the symmetry (if any) for $x = y^2 - 1$.

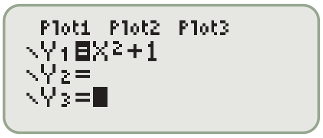
Study Tip

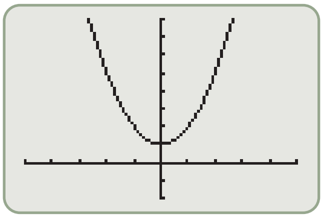
When testing for symmetry about the x-axis, y-axis, and origin, there are *five* possibilities:

- No symmetry.
- Symmetry with respect to the x-axis
- Symmetry with respect to the y-axis
- Symmetry with respect to the origin
- Symmetry with respect to the x-axis, y-axis, and origin

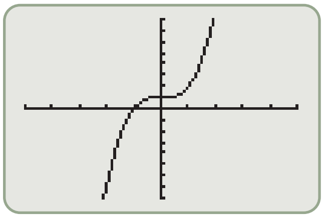
Technology Tip

Graph of $y_1 = x^2 + 1$ is shown.





Graph of $y_1 = x^3 + 1$ is shown.



Answer: The graph of the equation is symmetric with respect to the x-axis.

EXAMPLE 6 Using Intercepts and Symmetry as Graphing Aids

For the equation $x^2 + y^2 = 25$, use intercepts and symmetry to help you graph the equation using the point-plotting technique.

Solution:

STEP 1 Find the intercepts.

For the x-intercepts, let $y = 0$. $\quad x^2 + 0^2 = 25$

Solve for x. $\quad x = \pm 5$

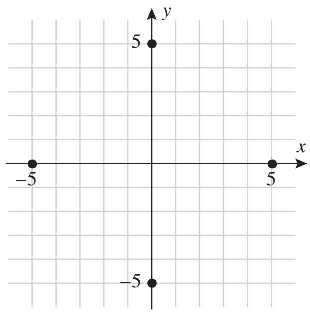
The two x-intercepts correspond to the points $(-5, 0)$ and $(5, 0)$.

For the y-intercepts, let $x = 0$. $\quad 0^2 + y^2 = 25$

Solve for y. $\quad y = \pm 5$

The two y-intercepts correspond to the points $(0, -5)$ and $(0, 5)$.

STEP 2 Identify the points on the graph corresponding to the intercepts.



STEP 3 Test for symmetry with respect to the y-axis, x-axis, and origin.

Test for symmetry with respect to the ***y*-axis**.

Replace x with $-x$. $\quad (-x)^2 + y^2 = 25$

Simplify. $\quad x^2 + y^2 = 25$

The resulting equation is equivalent to the original, so the graph of $x^2 + y^2 = 25$ is **symmetric with respect to the *y*-axis**.

Test for symmetry with respect to the ***x*-axis**.

Replace y with $-y$. $\quad x^2 + (-y)^2 = 25$

Simplify. $\quad x^2 + y^2 = 25$

The resulting equation is equivalent to the original, so the graph of $x^2 + y^2 = 25$ is **symmetric with respect to the *x*-axis**.

Test for symmetry with respect to the **origin**.

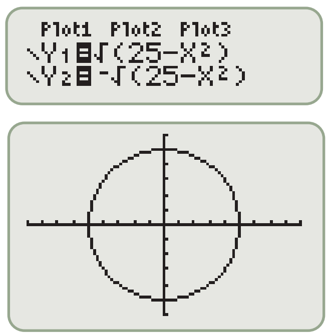
Replace x with $-x$ and y with $-y$. $\quad (-x)^2 + (-y)^2 = 25$

Simplify. $\quad x^2 + y^2 = 25$

The resulting equation is equivalent to the original, so the graph of $x^2 + y^2 = 25$ is **symmetric with respect to the origin**.

Technology Tip

To enter the graph of $x^2 + y^2 = 25$, solve for y first. The graphs of $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ are shown.



We need to determine solutions to the equation on only the positive x- and y-axes and in quadrant I because of the following symmetries:

- **Symmetry with respect to the y-axis gives the solutions in quadrant II.**
- **Symmetry with respect to the origin gives the solutions in quadrant III.**
- **Symmetry with respect to the x-axis yields solutions in quadrant IV.**

Solutions to $x^2 + y^2 = 25$.

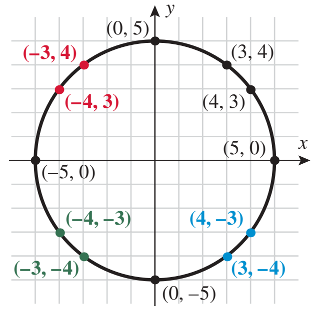
Quadrant I: (3, 4), (4, 3)

Additional points due to symmetry:

Quadrant II: $(-3, 4)$, $(-4, 3)$

Quadrant III: $(-3, -4)$, $(-4, -3)$

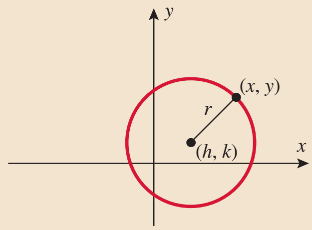
Quadrant IV: $(3, -4)$, $(4, -3)$



Circles

DEFINITION Circle

A **circle** is the set of all points in a plane that are a fixed distance from a point, the **center**. The center, C, is typically denoted by (h, k), and the fixed distance, or **radius**, is denoted by r.



EQUATION OF A CIRCLE

The standard form of the equation of a **circle** with **radius** r and **center** (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

For the special case of a circle with center at the origin (0, 0), the equation simplifies to $x^2 + y^2 = r^2$.

UNIT CIRCLE

A circle with radius 1 and center (0, 0) is called the **unit circle**:

$$x^2 + y^2 = 1$$

The unit circle plays an important role in the study of trigonometry. Note that if $x^2 + y^2 = 0$, the radius is 0, so the "circle" is just a point.

EXAMPLE 7 Finding the Center and Radius of a Circle

Identify the center and radius of the given circle and graph.

$$(x-2)^2+(y+1)^2=4$$

Solution:

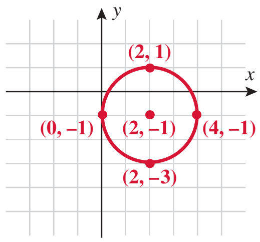
Rewrite this equation in standard form. $[x-2]^2+[y-(-1)]^2=2^2$

Identify h, k, and r by comparing this equation with the standard form of a circle: $(x-h)^2+(y-k)^2=r^2$. $h=2$, $k=-1$, and $r=2$

$\boxed{\text{Center } (2, -1) \text{ and } r = 2}$

To draw the circle, label the center $(2, -1)$. Label four additional points 2 units (the radius) away from the center: $(4, -1)$, $(0, -1)$, $(2, 1)$, and $(2, -3)$.

Note that the easiest four points to get are those obtained by going out from the center both horizontally and vertically. Connect those four points with a smooth curve.



■ **YOUR TURN** Identify the center and radius of the given circle and graph.

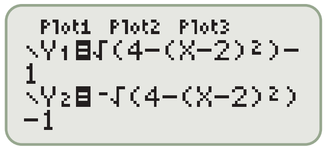
$$(x+1)^2+(y+2)^2=9$$

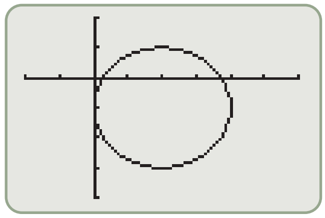
Technology Tip

To enter the graph of $(x-2)^2+(y+1)^2=4$, solve for y first. The graphs of

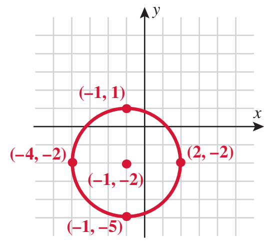
$y_1=\sqrt{4-(x-2)^2}-1$ and

$y_2=-\sqrt{4-(x-2)^2}-1$ are shown.





■ **Answer:** Center: $(-1, -2)$
Radius: 3



Let's change the look of the equation given in Example 7.

In Example 7, the equation of the circle was given as $(x-2)^2+(y+1)^2=4$

Eliminate the parentheses. $x^2-4x+4+y^2+2y+1=4$

Group like terms and subtract 4 from both sides. $x^2+y^2-4x+2y+1=0$

We have written the *general form* of the equation of the circle in Example 7.

> The **general form** of the **equation of a circle** is
>
> $$x^2+y^2+ax+by+c=0$$

Suppose you are given a point that lies on a circle and the center of the circle. Can you find the equation of the circle?

EXAMPLE 8 Finding the Equation of a Circle Given Its Center and One Point

The point $(10, -4)$ lies on a circle centered at $(7, -8)$. Find the equation of the circle in general form.

Solution:

This circle is centered at $(7, -8)$, so its standard equation is $(x - 7)^2 + (y + 8)^2 = r^2$.

Since the point $(10, -4)$ lies on the circle, it must satisfy the equation of the circle.

Substitute $(x, y) = (10, -4)$. $\qquad (10 - 7)^2 + (-4 + 8)^2 = r^2$

Simplify. $\qquad 3^2 + 4^2 = r^2$

The distance from $(10, -4)$ to $(7, -8)$ is 5 units. $\qquad r = 5$

Substitute $r = 5$ into the standard equation. $\qquad (x - 7)^2 + (y + 8)^2 = 5^2$

Eliminate the parentheses and simplify. $\qquad x^2 - 14x + 49 + y^2 + 16y + 64 = 25$

Write in **general form**. $\qquad \boxed{x^2 + y^2 - 14x + 16y + 88 = 0}$

■ **YOUR TURN** The point $(1, 11)$ lies on a circle centered at $(-5, 3)$. Find the equation of the circle in general form.

■ **Answer:** $x^2 + y^2 + 10x - 6y - 66 = 0$

If the equation of a circle is given in general form, it must be rewritten in standard form in order to identify its center and radius. To transform equations of circles from general to standard form, complete the square on both the x- and y-variables.

Technology Tip

To graph $x^2 - 8x + y^2 + 20y + 107 = 0$ without transforming it into a standard form, solve for y using the quadratic formula:

$$y = \frac{-20 \pm \sqrt{20^2 - 4(1)(x^2 - 8x + 107)}}{2}$$

Next, set the window to $[-5, 30]$ by $[-30, 5]$ and use $\boxed{\text{ZSquare}}$ under $\boxed{\text{ZOOM}}$ to adjust the window variable to make the circle look circular.

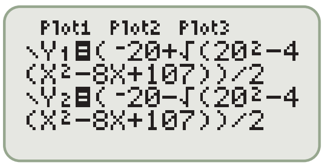
The graphs of

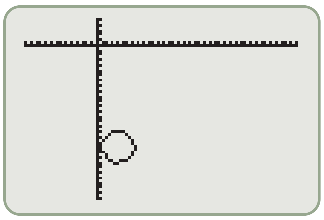
$$y_1 = \frac{-20 + \sqrt{20^2 - 4(x^2 - 8x + 107)}}{2}$$

and

$$y_2 = \frac{-20 - \sqrt{20^2 - 4(x^2 - 8x + 107)}}{2}$$

are shown.





EXAMPLE 9 Finding the Center and Radius of a Circle by Completing the Square

Find the center and radius of the circle with the equation:

$$x^2 - 8x + y^2 + 20y + 107 = 0$$

Solution:

Our goal is to transform this equation into standard form. $\qquad (x - h)^2 + (y - k)^2 = r^2$

Group x and y terms, respectively, on the left side of the equation; move constants to the right side. $\qquad (x^2 - 8x) + (y^2 + 20y) = -107$

Complete the square on both the x and y expressions. $\qquad (x^2 - 8x + \square) + (y^2 + 20y + \square) = -107$

Add $\left(-\frac{8}{2}\right)^2 = 16$ and $\left(\frac{20}{2}\right)^2 = 100$ to both sides.

$$(x^2 - 8x + 16) + (y^2 + 20y + 100) = -107 + 16 + 100$$

Factor the perfect squares on the left side and simplify the right side. $\qquad (x - 4)^2 + (y + 10)^2 = 9$

Write in standard form. $\qquad (x - 4)^2 + [y - (-10)]^2 = 3^2$

$\boxed{\text{The center is } (4, -10) \text{ and the radius is } 3.}$

■ **YOUR TURN** Find the center and radius of the circle with the equation:

$$x^2 + y^2 + 4x - 6y - 12 = 0$$

■ **Answer:** Center: $(-2, 3)$
Radius: 5

SECTION 0.5 SUMMARY

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of segment joining two points

$$(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Intercepts

- x-intercept: Let $y = 0$ and solve for x.
- y-intercept: Let $x = 0$ and solve for y.

Symmetry

- About the x-axis: Replace y with $-y$ and the resulting equation is the same.
- About the y-axis: Replace x with $-x$ and the resulting equation is the same.
- About the origin: Replace x with $-x$ and y with $-y$ and the resulting equation is the same.

Circles

$(x - h)^2 + (y - k)^2 = r^2$ center (h, k) and radius r

SECTION 0.5 EXERCISES

SKILLS

In Exercises 1–12, calculate the distance between the given points, and find the midpoint of the segment joining them.

1. $(1, 3)$ and $(5, 3)$
2. $(-2, 4)$ and $(-2, -4)$
3. $(-1, 4)$ and $(3, 0)$
4. $(-3, -1)$ and $(1, 3)$
5. $(-10, 8)$ and $(-7, -1)$
6. $(-2, 12)$ and $(7, 15)$
7. $(-3, -1)$ and $(-7, 2)$
8. $(-4, 5)$ and $(-9, -7)$
9. $(-6, -4)$ and $(-2, -8)$
10. $(0, -7)$ and $(-4, -5)$
11. $\left(-\frac{1}{2}, \frac{1}{3}\right)$ and $\left(\frac{7}{2}, \frac{10}{3}\right)$
12. $\left(\frac{1}{5}, \frac{7}{3}\right)$ and $\left(\frac{9}{5}, -\frac{2}{3}\right)$

In Exercises 13–18, graph the equation by plotting points.

13. $y = -3x + 2$
14. $y = 4 - x$
15. $y = x^2 - x - 2$
16. $y = x^2 - 2x + 1$
17. $x = y^2 - 1$
18. $x = |y + 1| + 2$

In Exercises 19–24, find the x-intercept(s) and y-intercepts(s) (if any) of the graphs of the given equations.

19. $2x - y = 6$
20. $y = 4x^2 - 1$
21. $y = \sqrt{x - 4}$
22. $y = \dfrac{x^2 - x - 12}{x}$
23. $4x^2 + y^2 = 16$
24. $x^2 - y^2 = 9$

In Exercises 25–30, test algebraically to determine whether the equation's graph is symmetric with respect to the x-axis, y-axis, or origin.

25. $x = y^2 + 4$
26. $y = x^5 + 1$
27. $x = |y|$
28. $x^2 + 2y^2 = 30$
29. $y = x^{2/3}$
30. $xy = 1$

In Exercises 31–36, plot the graph of the given equation.

31. $y = x^2 - 1$
32. $x = y^2 + 1$
33. $y = \dfrac{1}{x}$
34. $|x| = |y|$
35. $x^2 - y^2 = 16$
36. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

In Exercises 37–44, write the equation of the circle in standard form.

37. Center $(5, 7)$
$r = 9$

38. Center $(2, 8)$
$r = 6$

39. Center $(-11, 12)$
$r = 13$

40. Center $(6, -7)$
$r = 8$

41. Center $(5, -3)$
$r = 2\sqrt{3}$

42. Center $(-4, -1)$
$r = 3\sqrt{5}$

43. Center $\left(\frac{2}{3}, -\frac{3}{5}\right)$
$r = \frac{1}{4}$

44. Center $\left(-\frac{1}{3}, -\frac{2}{7}\right)$
$r = \frac{2}{5}$

In Exercises 45–50, state the center and radius of the circle with the given equations.

45. $(x - 2)^2 + (y + 5)^2 = 49$

46. $(x + 3)^2 + (y - 7)^2 = 81$

47. $(x - 4)^2 + (y - 9)^2 = 20$

48. $(x + 1)^2 + (y + 2)^2 = 8$

49. $\left(x - \frac{2}{5}\right)^2 + \left(y - \frac{1}{7}\right)^2 = \frac{4}{9}$

50. $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{3}\right)^2 = \frac{9}{25}$

In Exercises 51–60, find the center and radius of each circle.

51. $x^2 + y^2 - 10x - 14y - 7 = 0$

52. $x^2 + y^2 - 4x - 16y + 32 = 0$

53. $x^2 + y^2 - 2x - 6y + 1 = 0$

54. $x^2 + y^2 - 8x - 6y + 21 = 0$

55. $x^2 + y^2 - 10x + 6y + 22 = 0$

56. $x^2 + y^2 + 8x + 2y - 28 = 0$

57. $x^2 + y^2 - 6x - 4y + 1 = 0$

58. $x^2 + y^2 - 2x - 10y + 2 = 0$

59. $x^2 + y^2 - x + y + \dfrac{1}{4} = 0$

60. $x^2 + y^2 - \dfrac{x}{2} - \dfrac{3y}{2} + \dfrac{3}{8} = 0$

APPLICATIONS

In Exercises 61–62, refer to the following:

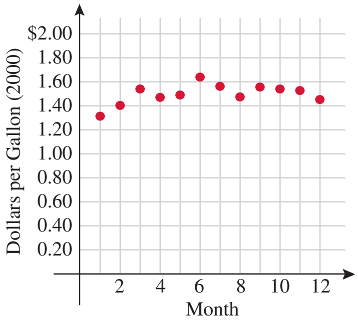
It is often useful to display data in visual form by plotting the data as a set of points. This provides a graphical display between the two variables. The following table contains data on the average monthly price of gasoline.

U.S. All Grades Conventional Retail Gasoline Prices, 1994–2010 (Dollars per Gallon)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1994											1.175	1.112
1995	1.107	1.099	1.099	1.143	1.213	1.226	1.189	1.161	1.148	1.122	1.098	1.105
1996	1.123	1.121	1.169	1.259	1.302	1.282	1.254	1.238	1.238	1.243	1.273	1.273
1997	1.270	1.263	1.237	1.228	1.229	1.227	1.206	1.250	1.254	1.222	1.198	1.159
1998	1.115	1.082	1.055	1.064	1.088	1.086	1.078	1.049	1.033	1.045	1.020	0.964
1999	0.957	0.940	1.000	1.137	1.143	1.134	1.177	1.237	1.279	1.271	1.280	1.302
2000	1.319	1.409	1.538	1.476	1.496	1.645	1.568	1.480	1.562	1.546	1.533	1.458
2001	1.467	1.471	1.423	1.557	1.689	1.586	1.381	1.422	1.539	1.312	1.177	1.111
2002	1.134	1.129	1.259	1.402	1.394	1.380	1.402	1.398	1.403	1.466	1.424	1.389
2003	1.164	1.622	1.675	1.557	1.477	1.489	1.519	1.625	1.654	1.551	1.512	1.488
2004	1.595	1.654	1.728	1.794	1.981	1.950	1.902	1.880	1.880	1.993	1.973	1.843
2005	1.852	1.927	2.102	2.251	2.155	2.162	2.287	2.489	2.907	2.736	2.265	2.216
2006	2.343	2.293	2.454	2.762	2.873	2.849	2.964	2.952	2.548	2.258	2.254	2.328
2007	2.237	2.276	2.546	2.831	3.157	3.067	2.989	2.821	2.858	2.838	3.110	3.032
2008	3.068	3.064	3.263	3.468	3.783	4.038	4.051	3.789	3.760	3.065	2.153	1.721
2009	1.821	1.942	1.987	2.071	2.289	2.645	2.530	2.613	2.530	2.549	2.665	2.620
2010	2.730	2.657	2.793	2.867	2.847	2.733	2.728	2.733	2.727	2.816	2.866	3.004

Source: http://www.eia.doe.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=EMM_EPM0U_PTE_NUS_DPG&f=M

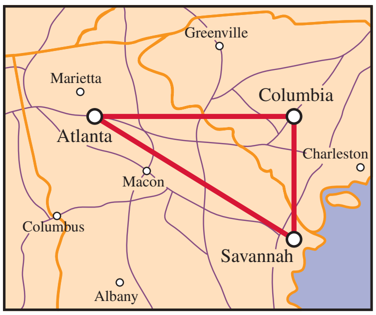
The following graph displays the data for the year 2000.



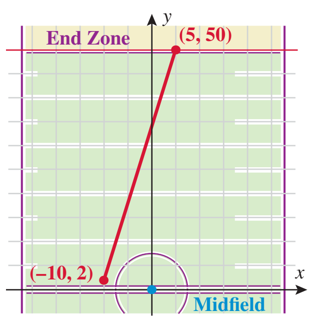
61. **Economics.** Create a graph displaying the price of gasoline for the year 2008.

62. **Economics.** Create a graph displaying the price of gasoline for the year 2009.

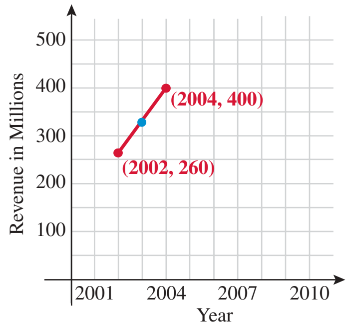
63. **Travel.** A retired couple who live in Columbia, South Carolina, decide to take their motor home and visit their two children who live in Atlanta and in Savannah, Georgia. Savannah is 160 miles south of Columbia and Atlanta is 215 miles west of Columbia. How far apart do the children live from each other?



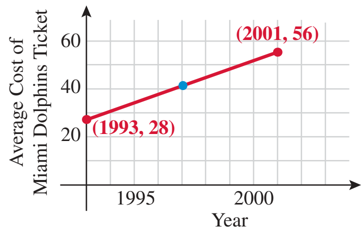
64. **Sports.** In the 1984 Orange Bowl, Doug Flutie, the 5 foot 9 inch quarterback for Boston College, shocked the world as he threw a "hail Mary" pass that was caught in the end zone with no time left on the clock, defeating the Miami Hurricanes 47–45. Although the record books have it listed as a 48-yard pass, what was the actual distance the ball was thrown? The following illustration depicts the path of the ball.



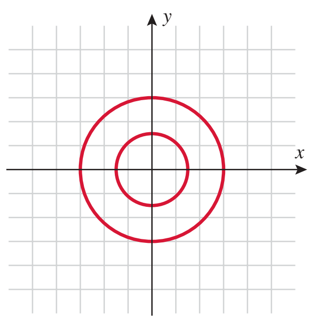
65. **NASCAR Revenue.** Action Performance, Inc., the leading seller of NASCAR merchandise, recorded $260 million in revenue in 2002 and $400 million in revenue in 2004. Calculate the midpoint to estimate the revenue Action Performance, Inc. recorded in 2003. Assume the horizontal axis represents the year and the vertical axis represents the revenue in millions.



66. **Ticket Price.** In 1993 the average Miami Dolphins ticket price was $28 and in 2001 the average price was $56. Find the midpoint of the segment joining these two points to estimate the ticket price in 1997.



67. **Design.** A university designs its campus with a master plan of two concentric circles. All of the academic buildings are within the inner circle (so that students can get between classes in less than 10 minutes), and the outer circle contains all the dormitories, the Greek park, cafeterias, the gymnasium, and intramural fields. Assuming the center of campus is the origin, write an equation for the inner circle if the diameter is 3000 feet.



68. **Design.** Repeat Exercise 67 for the outer circle with a diameter of 6000 feet.

69. Cell Phones. A cellular phone tower has a reception radius of 200 miles. Assuming the tower is located at the origin, write the equation of the circle that represents the reception area.

70. Environment. In a state park, a fire has spread in the form of a circle. If the radius is 2 miles, write an equation for the circle.

71. Economics. The demand for an electronic device is modeled by

$$p = 2.95 - \sqrt{0.01x - 0.01}$$

where x is thousands of units demanded per day and p is the price (in dollars) per unit.

a. Find the domain of the demand equation. Interpret your result.

b. Plot the demand equation.

72. Economics. The demand for a new electronic game is modeled by

$$p = 39.95 - \sqrt{0.01x - 0.4}$$

where x is thousands of units demanded per day and p is the price (in dollars) per unit.

a. Find the domain of the demand equation. Interpret your result.

b. Plot the demand equation.

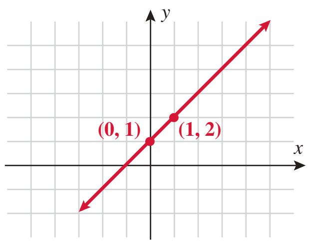
CATCH THE MISTAKE

In Exercises 73–76, explain the mistake that is made.

73. Graph the equation $y = x^2 + 1$.

Solution:

x	$y = x^2 + 1$	(x, y)
0	1	(0, 1)
1	2	(1, 2)



This is incorrect. What mistake was made?

74. Use symmetry to help you graph $x^2 = y - 1$.

Solution:

Replace x with $-x$. $\quad (-x)^2 = y - 1$

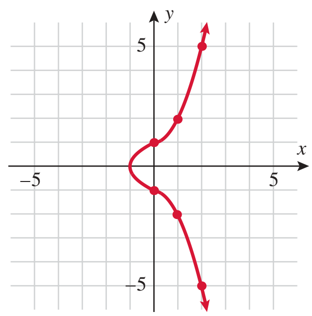
Simplify. $\quad x^2 = y - 1$

$x^2 = y - 1$ is symmetric with respect to the x-axis.

Determine points that lie on the graph in quadrant I.

y	$x^2 = y - 1$	(x, y)
1	0	(0, 1)
2	1	(1, 2)
5	2	(2, 5)

Symmetry with respect to the x-axis implies that $(0, -1)$, $(1, -2)$, and $(2, -5)$ are also points that lie on the graph.



This is incorrect. What mistake was made?

75. Identify the center and radius of the circle with equation $(x - 4)^2 + (y + 3)^2 = 25$.

Solution: The center is (4, 3) and the radius is 5.

This is incorrect. What mistake was made?

76. Identify the center and radius of the circle with equation $(x - 2)^2 + (y + 3)^2 = 2$.

Solution: The center is $(2, -3)$ and the radius is 2.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 77–80, determine whether each statement is true or false.

77. If the point (a, b) lies on a graph that is symmetric about the x-axis, then the point $(-a, b)$ also must lie on the graph.

78. If the point (a, b) lies on a graph that is symmetric about the y-axis, then the point $(-a, b)$ also must lie on the graph.

79. If the point $(a, -b)$ lies on a graph that is symmetric about the x-axis, y-axis, and origin, then the points (a, b), $(-a, -b)$, and $(-a, b)$ must also lie on the graph.

80. Two points are all that is needed to plot the graph of an equation.

81. Describe the graph (if it exists) of

$$x^2 + y^2 + 10x - 6y + 34 = 0$$

82. Describe the graph (if it exists) of

$$x^2 + y^2 - 4x + 6y + 49 = 0$$

CHALLENGE

83. Determine whether the graph of $y = \dfrac{ax^2 + b}{cx^3}$ has any symmetry, where a, b, and c are real numbers.

84. Find the intercepts of $y = (x - a)^2 - b^2$, where a and b are real numbers.

85. Find the equation of a circle that has a diameter with endpoints $(5, 2)$ and $(1, -6)$.

86. Find the equation of a circle that has a diameter with endpoints $(3, 0)$ and $(-1, -4)$.

87. For the equation $x^2 + y^2 + ax + by + c = 0$, specify conditions on a, b, and c so that the graph is a single point.

88. For the equation $x^2 + y^2 + ax + by + c = 0$, specify conditions on a, b, and c so that there is no corresponding graph.

TECHNOLOGY

In Exercises 89–90, graph the equation using a graphing utility and state whether there is any symmetry.

89. $y = 16.7x^4 - 3.3x^2 + 7.1$

90. $y = 0.4x^5 + 8.2x^3 - 1.3x$

In Exercises 91 and 92, (a) with the equation of the circle in standard form, state the center and radius, and graph; (b) use the quadratic formula to solve for *y*; and (c) use a graphing utility to graph each equation found in (b). Does the graph in (a) agree with the graphs in (c)?

91. $x^2 + y^2 - 11x + 3y - 7.19 = 0$

92. $x^2 + y^2 + 1.2x - 3.2y + 2.11 = 0$

SECTION 0.6 LINES

SKILLS OBJECTIVE

- Graph a line.
- Calculate the slope of a line.
- Find the equation of a line using slope–intercept form.
- Find the equation of a line using point–slope form.
- Find the equation of a line that is parallel or perpendicular to a given line.

CONCEPTUAL OBJECTIVES

- Classify lines as increasing, decreasing, horizontal, or vertical.
- Understand slope as a rate of change.

Graphing a Line

First-degree equations such as

$$y = -2x + 4 \qquad 3x + y = 6 \qquad y = 2 \qquad x = -3$$

have graphs that are straight lines. The first two equations given represent inclined or "slant" lines, whereas $y = 2$ represents a horizontal line and $x = -3$ represents a vertical line. One way of writing an equation of a straight line is called *general form.*

Equation of a straight line: General* form

If A, B, and C are constants and x and y are variables, then the equation

$$Ax + By = C$$

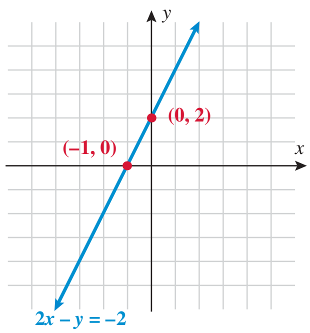
is in **general form** and its graph is a straight line.

Note: A or B (but not both) can be zero.

The equation $2x - y = -2$ is a first-degree equation, so its graph is a straight line. To graph this line, find the two intercepts, plot those points, and use a straight edge to draw the **line**.

$$2x - y = -2$$

Intercept	x	y	(x, y)
x-intercept	-1	0	$(-1, 0)$
y-intercept	0	2	$(0, 2)$



Slope

If the graph of $2x - y = -2$ represented an incline that you were about to walk on, would you classify that incline as steep? In the language of mathematics, we use the word **slope** as a measure of steepness. Slope is the ratio of the change in y over the change in x. An easy way to remember this is *rise over run.*

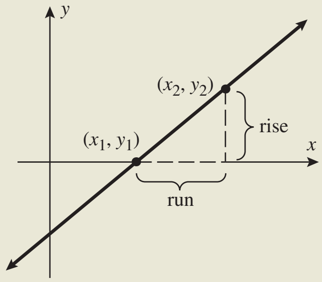
*Some books refer to this as standard form.

SLOPE OF A LINE

A nonvertical line passing through two points (x_1, y_1) and (x_2, y_2) has slope m given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_1 \neq x_2 \text{ or}$$

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical change}}{\text{horizontal change}}$$



Note: Always start with the same point for both the x-coordinates and the y-coordinates.

Study Tip

To get the correct sign ($\pm$) for the slope, remember to start with the same point for both x and y.

Let's find the slope of our graph $2x - y = -2$. We'll let $(x_1, y_1) = (-2, -2)$ and $(x_2, y_2) = (1, 4)$ in the slope formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{[4 - (-2)]}{[1 - (-2)]} = \frac{6}{3} = 2$$

Notice that if we had chosen the two intercepts $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (-1, 0)$ instead, we still would have found the slope to be $m = 2$.

COMMON MISTAKE

The most common mistake in calculating slope is writing the coordinates in the wrong order, which results in the slope being opposite in sign.

Find the slope of the line containing the two points (1, 2) and (3, 4).

CORRECT

Label the points.

$$(x_1, y_1) = (1, 2)$$
$$(x_2, y_2) = (3, 4)$$

Write the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute the coordinates.

$$m = \frac{4 - 2}{3 - 1}$$

Simplify. $m = \frac{2}{2} = \boxed{1}$

INCORRECT

The **ERROR** is interchanging the coordinates of the first and second points.

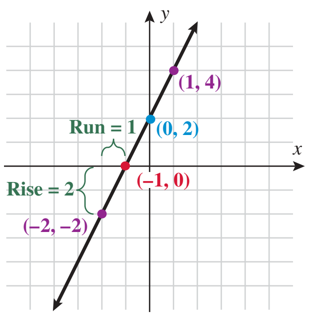
$$m = \frac{4 - 2}{1 - 3}$$

The calculated slope is **INCORRECT** by a negative sign.

$$m = \frac{2}{-2} = -1$$

CAUTION

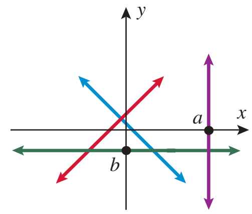
Interchanging the coordinates will result in a sign error in a nonzero slope.



When interpreting slope, always read the graph from *left to right*. Since we have determined the slope to be 2, or $\frac{2}{1}$, we can interpret this as rising two units and running (to the right) one unit. If we start at the point **(−2, −2)** and move two units up and one unit to the right, we end up at the x-intercept, **(−1, 0)**. Again, moving two units up and one unit to the right put us at the y-intercept, **(0, 2)**. Another rise of 2 and run of 1 take us to the point **(1, 4)**. See the figure on the left.

Lines fall into one of four categories: increasing, decreasing, horizontal, or vertical.

Line	Slope
Increasing	**Positive ($m > 0$)**
Decreasing	**Negative ($m < 0$)**
Horizontal	**Zero ($m = 0$), hence $y = b$**
Vertical	**Undefined, hence $x = a$**



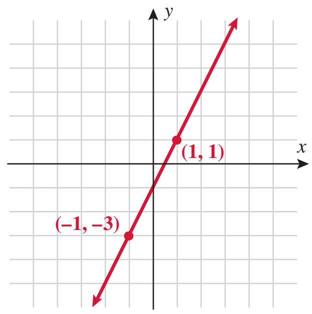
The slope of a horizontal line is 0 because the y-coordinates of any two points are the same. The change in y in the slope formula's numerator is 0, hence $m = 0$. The slope of a vertical line is undefined because the x-coordinates of any two points are the same. The change in x in the slope formula's denominator is zero; hence m is undefined.

EXAMPLE 1 Graph, Classify the Line, and Determine the Slope

Sketch a line through each pair of points, classify the line as increasing, decreasing, vertical, or horizontal, and determine its slope.

a. $(-1, -3)$ and $(1, 1)$ **b.** $(-3, 3)$ and $(3, 1)$

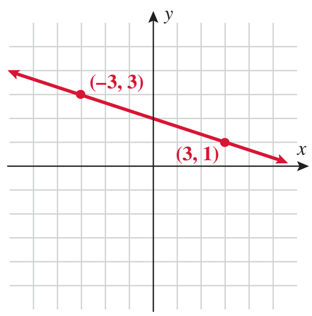
c. $(-1, -2)$ and $(3, -2)$ **d.** $(1, -4)$ and $(1, 3)$



Solution (a): $(-1, -3)$ and $(1, 1)$

This line is **increasing**, so its slope is positive.

$$m = \frac{1 - (-3)}{1 - (-1)} = \frac{4}{2} = \frac{2}{1} = 2.$$



Solution (b): $(-3, 3)$ and $(3, 1)$

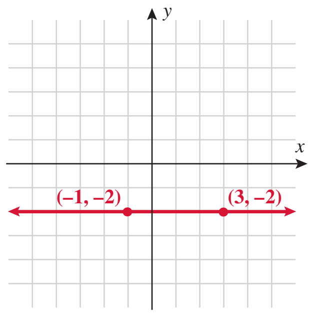
This line is **decreasing**, so its slope is negative.

$$m = \frac{3 - 1}{-3 - 3} = -\frac{2}{6} = -\frac{1}{3}.$$

Solution (c): $(-1, -2)$ and $(3, -2)$

This is a **horizontal** line, so its slope is zero.

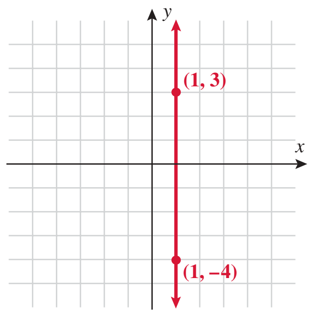
$$m = \frac{-2 - (-2)}{3 - (-1)} = \frac{0}{4} = 0.$$



Solution (d): $(1, -4)$ and $(1, 3)$

This is a **vertical** line, so its slope is undefined.

$$m = \frac{3 - (-4)}{1 - 1} = \frac{7}{0}, \text{ which is undefined.}$$



■ **YOUR TURN** For each pair of points, classify the line that passes through them as increasing, decreasing, vertical, or horizontal, and determine its slope. Do not graph.

a. $(2, 0)$ and $(1, 5)$ **b.** $(-2, -3)$ and $(2, 5)$

c. $(-3, -1)$ and $(-3, 4)$ **d.** $(-1, 2)$ and $(3, 2)$

■ **Answer:**
a. $m = -5$, increasing
b. $m = 2$, decreasing
c. slope is undefined, vertical
d. $m = 0$, horizontal

Equations of Lines

Slope–Intercept Form

As mentioned earlier, the general form for an equation of a line is $Ax + By = C$. A more standard way to write an equation of a line is in slope–intercept form, because it identifies the slope and the y-intercept.

EQUATION OF A STRAIGHT LINE: SLOPE–INTERCEPT FORM

The **slope–intercept form** for the equation of a nonvertical line is

$$y = mx + b$$

Its graph has slope m and y-intercept b.

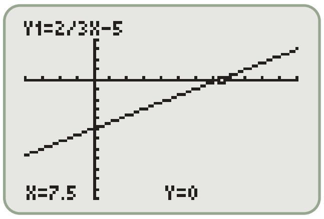
For example, $2x - y = -3$ is in general form. To write this equation in **slope–intercept form**, we isolate the y variable:

$$y = 2x + 3$$

The **slope** of this line is **2** and the ***y*-intercept** is **3**.

Technology Tip

To graph the equation $2x - 3y = 15$, solve for y first. The graph of $y_1 = \frac{2}{3}x - 5$ is shown.



EXAMPLE 2 Using Slope–Intercept Form to Graph an Equation of a Line

Write $2x - 3y = 15$ in slope–intercept form and graph it.

Solution:

STEP 1 *Write in slope–intercept form.*

Subtract $2x$ from both sides. $\quad -3y = -2x + 15$

Divide both sides by -3. $\quad \boxed{y = \frac{2}{3}x - 5}$

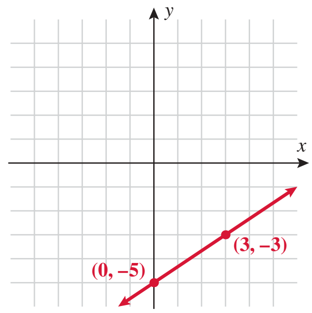
STEP 2 *Graph.*

Identify the slope and y-intercept. $\quad$ Slope: $m = \frac{2}{3}$ $\quad$ y-intercept: $b = -5$

Plot the point corresponding to the y-intercept $(0, -5)$.

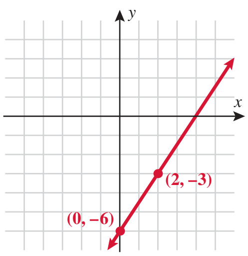
From the point $(0, -5)$, rise two units and run (to the right) three units, which corresponds to the point $(3, -3)$.

Draw the line passing through the two points.



■ **YOUR TURN** Write $3x - 2y = 12$ in slope–intercept form and graph it.

■ **Answer:** $y = \frac{3}{2}x - 6$



Instead of starting with equations of lines and characterizing them, let us now start with particular features of a line and derive its governing equation. Suppose that you are given the y-intercept and the slope of a line. Using the slope–intercept form of an equation of a line, $y = mx + b$, you could find its equation.

EXAMPLE 3 Using Slope–Intercept Form to Find the Equation of a Line

Find the equation of a line that has slope $\frac{2}{3}$ and y-intercept $(0, 1)$.

Solution:

Write the slope–intercept form of an equation of a line. $\quad y = mx + b$

Label the slope. $\quad m = \frac{2}{3}$

Label the y-intercept. $\quad b = 1$

The equation of the line in slope–intercept form is $\boxed{y = \frac{2}{3}x + 1}$.

■ **YOUR TURN** Find the equation of the line that has slope $-\frac{3}{2}$ and y-intercept $(0, 2)$.

■ **Answer:** $y = -\frac{3}{2}x + 2$

Point–Slope Form

Now, suppose that the two pieces of information you are given about an equation are its slope and one point that lies on its graph. You still have enough information to write an equation of the line. Recall the formula for slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{where } x_2 \neq x_1$$

We are given the slope m, and we know a particular point that lies on the line (x_1, y_1). We refer to all other points that lie on the line as (x, y). Substituting these values into the slope formula gives us

$$m = \frac{y - y_1}{x - x_1}$$

Cross multiplying yields

$$y - y_1 = m(x - x_1)$$

This is called the *point–slope form* of an equation of a line.

Equation of a Straight Line: Point–Slope Form

The **point–slope form** for the equation of a line is

$$y - y_1 = m(x - x_1)$$

Its graph passes through the point (x_1, y_1), and its slope is m.

Note: This formula does not hold for vertical lines since their slope is undefined.

EXAMPLE 4 Using Point–Slope Form to Find the Equation of a Line

Find the equation of the line that has slope $-\frac{1}{2}$ and passes through the point $(-1, 2)$.

Solution:

Write the point–slope form of an equation of a line. $\quad y - y_1 = m(x - x_1)$

Substitute the values $m = -\frac{1}{2}$ and $(x_1, y_1) = (-1, 2)$. $\quad y - 2 = -\frac{1}{2}[x - (-1)]$

Distribute. $\quad y - 2 = -\frac{1}{2}x - \frac{1}{2}$

Isolate y. $\quad y = -\frac{1}{2}x + \frac{3}{2}$

We can also express the equation in general form $\boxed{x + 2y = 3}$.

■ **YOUR TURN** Find the equation of the line that has slope $\frac{1}{4}$ and passes through the point $\left(1, -\frac{1}{2}\right)$.

■ **Answer:** $y = \frac{1}{4}x - \frac{3}{4}$ or $-x + 4y = -3$

Finding the Equation of a Line Given Two Points

Suppose the slope of a line is not given at all. Instead, two points that lie on the line are given. If we know two points that lie on the line, then we can calculate the slope. Then, using the slope and *either* of the two points, we can derive the equation of the line.

EXAMPLE 5 Finding the Equation of a Line Given Two Points

Find the equation of the line that passes through the points $(-2, -1)$ and $(3, 2)$.

Solution:

Write the equation of a line. $y = mx + b$

Calculate the slope. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Substitute $(x_1, y_1) = (-2, -1)$ and $(x_2, y_2) = (3, 2)$. $m = \dfrac{2 - (-1)}{3 - (-2)} = \dfrac{3}{5}$

Proceed using either *slope–intercept* or *point–slope* form (see Study Tip).

Substitute $\frac{3}{5}$ for the slope. $y = \dfrac{3}{5}x + b$

Let $(x, y) = (3, 2)$. (Either point satisfies the equation.) $2 = \dfrac{3}{5}(3) + b$

Solve for b. $b = \dfrac{1}{5}$

Write the equation in slope–intercept form. $y = \dfrac{3}{5}x + \dfrac{1}{5}$

Write the equation in general form. $\boxed{-3x + 5y = 1}$

YOUR TURN Find the equation of the line that passes through the points $(-1, 3)$ and $(2, -4)$.

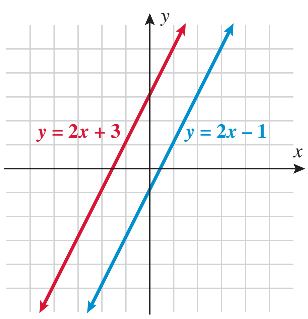
Study Tip

When two points that lie on a line are given, first calculate the slope of the line, then use either point and the slope–intercept form (shown in Example 5) or the point–slope form:

$$\begin{aligned} m &= \tfrac{3}{5}, (3, 2) \\ y - y_1 &= m(x - x_1) \\ y - 2 &= \tfrac{3}{5}(x - 3) \\ 5y - 10 &= 3(x - 3) \\ 5y - 10 &= 3x - 9 \\ \end{aligned}$$
$$\boxed{-3x + 5y = 1}$$

Answer: $y = -\frac{7}{3}x + \frac{2}{3}$ or $7x + 3y = 2$

Parallel and Perpendicular Lines



Two distinct nonintersecting lines in a plane are *parallel*. How can we tell whether the two lines in the graph on the left are parallel? Parallel lines must have the same steepness. In other words, parallel lines must have the same slope. The two lines shown on the left are parallel because they have the same slope, 2.

DEFINITION **Parallel Lines**

Two distinct lines in a plane are **parallel** if and only if their slopes are equal.

In other words, if two lines in a plane are parallel, then their slopes are equal, and if the slopes of two lines in a plane are equal, then the lines are parallel.

Words	Math
Lines L_1 and L_2 are parallel.	$L_1 \parallel L_2$
Two parallel lines have the same slope.	$m_1 = m_2$

EXAMPLE 6 Finding an Equation of a Parallel Line

Find the equation of the line that passes through the point (1, 1) and is parallel to the line $y = 3x + 1$.

Solution:

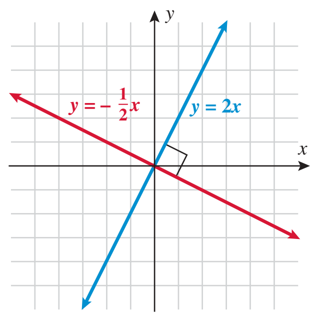
Write the slope–intercept equation of a line.	$y = mx + b$
Parallel lines have equal slope.	$m = 3$
Substitute the slope into the equation of the line.	$y = 3x + b$
Since the line passes through (1, 1), this point must satisfy the equation.	$1 = 3(1) + b$
Solve for b.	$b = -2$

The equation of the line is $\boxed{y = 3x - 2}$.

YOUR TURN Find the equation of the line parallel to $y = 2x - 1$ that passes through the point $(-1, 3)$.

■ **Answer:** $y = 2x + 5$

Two *perpendicular* lines form a right angle at their point of intersection. Notice the slopes of the two perpendicular lines in the figure to the right. They are $-\frac{1}{2}$ and 2, negative reciprocals of each other. It turns out that almost all perpendicular lines share this property. Horizontal ($m = 0$) and vertical (m undefined) lines do not share this property.



DEFINITION **Perpendicular Lines**

Except for the special case of a vertical and a horizontal line, two lines in a plane are **perpendicular** if and only if their slopes are negative reciprocals of each other.

In other words, if two lines in a plane are perpendicular, their slopes are negative reciprocals, provided their slopes are defined. Similarly, if the slopes of two lines in a plane are negative reciprocals, then the lines are perpendicular.

Words	Math
Lines L_1 and L_2 are perpendicular.	$L_1 \perp L_2$
Two perpendicular lines have negative reciprocal slopes.	$m_1 = -\dfrac{1}{m_2} \quad m_1 \neq 0, m_2 \neq 0$

Study Tip

If a line has slope equal to 3, then a line perpendicular to it has slope $-\frac{1}{3}$.

EXAMPLE 7 **Finding an Equation of a Line That Is Perpendicular to Another Line**

Find the equation of the line that passes through the point (3, 0) and is perpendicular to the line $y = 3x + 1$.

Solution:

Identify the slope of the given line $y = 3x + 1$. $\quad m_1 = 3$

The slope of a line perpendicular to the given line is the negative reciprocal of the slope of the given line. $\quad m_2 = -\dfrac{1}{m_1} = -\dfrac{1}{3}$

Write the equation of the line we are looking for in slope–intercept form. $\quad y = m_2x + b$

Substitute $m_2 = -\frac{1}{3}$ into $y = m_2x + b$. $\quad y = -\dfrac{1}{3}x + b$

Since the desired line passes through (3, 0), this point must satisfy the equation.

$$0 = -\frac{1}{3}(3) + b$$

$$0 = -1 + b$$

Solve for b. $\quad b = 1$

The equation of the line is $\boxed{y = -\frac{1}{3}x + 1}$.

■ **Answer:** $y = 2x - 7$

■ **YOUR TURN** Find the equation of the line that passes through the point $(1, -5)$ and is perpendicular to the line $y = -\frac{1}{2}x + 4$.

SECTION 0.6 SUMMARY

Lines are often expressed in two forms:

- General Form: $Ax + By = C$
- Slope–Intercept Form: $y = mx + b$

All lines (except horizontal and vertical) have exactly one x-intercept and exactly one y-intercept. The slope of a line is a measure of steepness.

- Slope of a line passing through (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}} \qquad x_1 \neq x_2$$

- Horizontal lines: $m = 0$
- Vertical lines: m is undefined

An equation of a line can be found if either two points or the slope and a point are given. The point–slope form $y - y_1 = m(x - x_1)$ is useful when the slope and a point are given. Parallel lines have the same slope. Perpendicular lines have negative reciprocal (opposite) slopes, provided their slopes are defined.

SECTION 0.6 EXERCISES

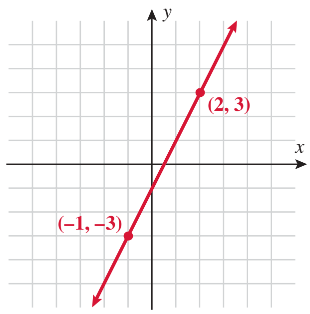
SKILLS

In Exercises 1–10, find the slope of the line that passes through the given points.

1. $(1, 3)$ and $(2, 6)$
2. $(2, 1)$ and $(4, 9)$
3. $(-2, 5)$ and $(2, -3)$
4. $(-1, -4)$ and $(4, 6)$
5. $(-7, 9)$ and $(3, -10)$
6. $(11, -3)$ and $(-2, 6)$
7. $(0.2, -1.7)$ and $(3.1, 5.2)$
8. $(-2.4, 1.7)$ and $(-5.6, -2.3)$
9. $\left(\frac{2}{3}, -\frac{1}{4}\right)$ and $\left(\frac{5}{6}, -\frac{3}{4}\right)$
10. $\left(\frac{1}{2}, \frac{3}{5}\right)$ and $\left(-\frac{3}{4}, \frac{7}{5}\right)$

For each graph in Exercises 11–16, identify (by inspection) the *x*- and *y*-intercepts and slope if they exist, and classify the line as increasing, decreasing, horizontal, or vertical.

11.



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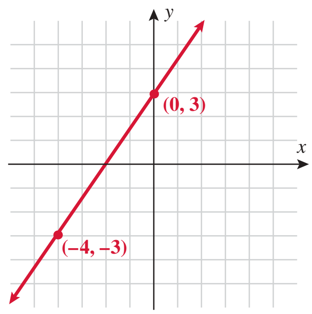

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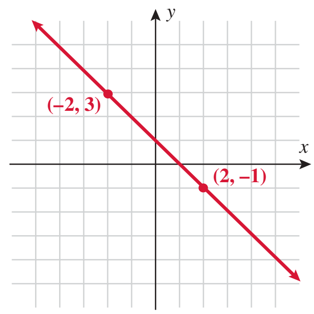

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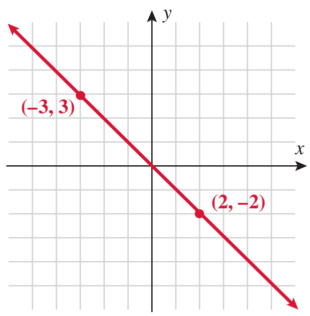

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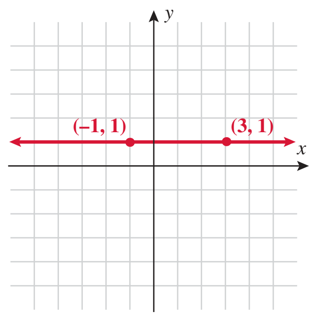

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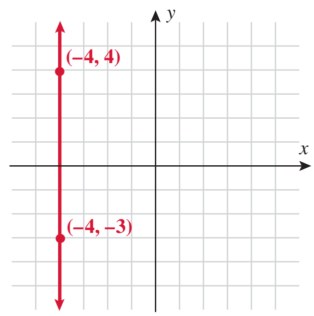


In Exercises 17–30, find the *x*- and *y*-intercepts if they exist and graph the corresponding line.

17. $y = 2x - 3$
18. $y = -3x + 2$
19. $y = -\frac{1}{2}x + 2$
20. $y = \frac{1}{3}x - 1$
21. $2x - 3y = 4$
22. $-x + y = -1$
23. $\frac{1}{2}x + \frac{1}{2}y = -1$
24. $\frac{1}{3}x - \frac{1}{4}y = \frac{1}{12}$
25. $x = -1$
26. $y = -3$
27. $y = 1.5$
28. $x = -7.5$
29. $x = -\frac{7}{2}$
30. $y = \frac{5}{3}$

In Exercises 31–42, write the equation in slope–intercept form. Identify the slope and the *y*-intercept.

31. $2x - 5y = 10$
32. $3x - 4y = 12$
33. $x + 3y = 6$
34. $x + 2y = 8$
35. $4x - y = 3$
36. $x - y = 5$
37. $12 = 6x + 3y$
38. $4 = 2x - 8y$
39. $0.2x - 0.3y = 0.6$
40. $0.4x + 0.1y = 0.3$
41. $\frac{1}{2}x + \frac{2}{3}y = 4$
42. $\frac{1}{4}x + \frac{2}{5}y = 2$

In Exercises 43–50, write the equation of the line, given the slope and intercept.

43. Slope: $m = 2$
y-intercept: $(0, 3)$

44. Slope: $m = -2$
y-intercept: $(0, 1)$

45. Slope: $m = -\frac{1}{3}$
y-intercept: $(0, 0)$

46. Slope: $m = \frac{1}{2}$
y-intercept: $(0, -3)$

47. Slope: $m = 0$
y-intercept: $(0, 2)$

48. Slope: $m = 0$
y-intercept: $(0, -1.5)$

49. Slope: undefined
x-intercept: $\left(\frac{3}{2}, 0\right)$

50. Slope: undefined
x-intercept: $(-3.5, 0)$

In Exercises 51–60, write an equation of the line in slope–intercept form, if possible, given the slope and a point that lies on the line.

51. Slope: $m = 5$ $(-1, -3)$

52. Slope: $m = 2$ $(1, -1)$

53. Slope: $m = -3$ $(-2, 2)$

54. Slope: $m = -1$ $(3, -4)$

55. Slope: $m = \frac{3}{4}$ $(1, -1)$

56. Slope: $m = -\frac{1}{7}$ $(-5, 3)$

57. Slope: $m = 0$ $(-2, 4)$

58. Slope: $m = 0$ $(3, -3)$

59. Slope: undefined $(-1, 4)$

60. Slope: undefined $(4, -1)$

In Exercises 61–80, write the equation of the line that passes through the given points. Express the equation in slope–intercept form or in the form $x = a$ or $y = b$.

61. $(-2, -1)$ and $(3, 2)$

62. $(-4, -3)$ and $(5, 1)$

63. $(-3, -1)$ and $(-2, -6)$

64. $(-5, -8)$ and $(7, -2)$

65. $(20, -37)$ and $(-10, -42)$

66. $(-8, 12)$ and $(-20, -12)$

67. $(-1, 4)$ and $(2, -5)$

68. $(-2, 3)$ and $(2, -3)$

69. $\left(\frac{1}{2}, \frac{3}{4}\right)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$

70. $\left(-\frac{2}{3}, -\frac{1}{2}\right)$ and $\left(\frac{7}{3}, \frac{1}{2}\right)$

71. $(3, 5)$ and $(3, -7)$

72. $(-5, -2)$ and $(-5, 4)$

73. $(3, 7)$ and $(9, 7)$

74. $(-2, -1)$ and $(3, -1)$

75. $(0, 6)$ and $(-5, 0)$

76. $(0, -3)$ and $(0, 2)$

77. $(-6, 8)$ and $(-6, -2)$

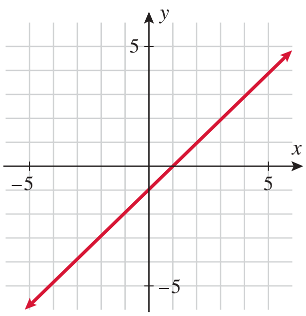
78. $(-9, 0)$ and $(-9, 2)$

79. $\left(\frac{2}{5}, -\frac{3}{4}\right)$ and $\left(\frac{2}{5}, \frac{1}{2}\right)$

80. $\left(\frac{1}{3}, \frac{2}{5}\right)$ and $\left(\frac{1}{3}, \frac{1}{2}\right)$

In Exercises 81–86, write the equation corresponding to each line. Express the equation in slope–intercept form.

81.



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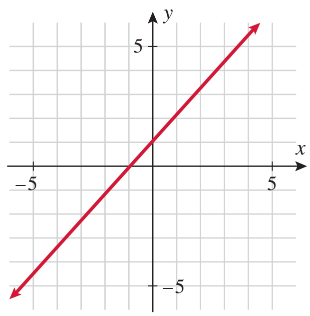

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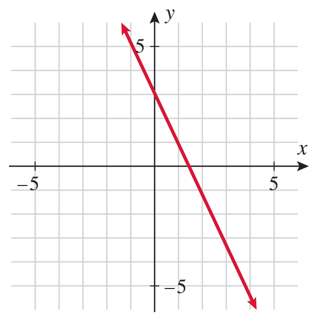
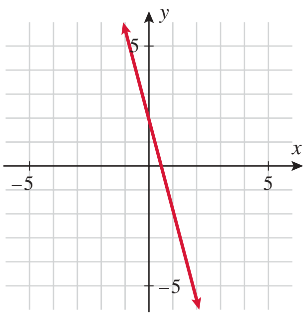
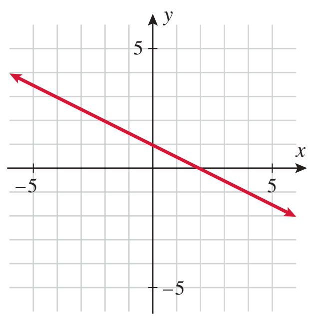


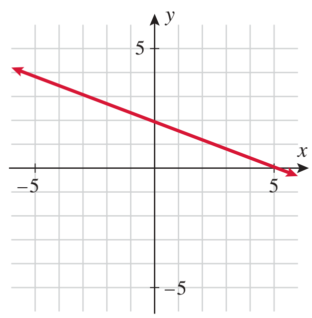
84.



85.



86.



In Exercises 87–96, find the equation of the line that passes through the given point and also satisfies the additional piece of information. Express your answer in slope–intercept form, if possible.

87. $(-3, 1)$; parallel to the line $y = 2x - 1$

88. $(1, 3)$; parallel to the line $y = -x + 2$

89. $(0, 0)$; perpendicular to the line $2x + 3y = 12$

90. $(0, 6)$; perpendicular to the line $x - y = 7$

91. $(3, 5)$; parallel to the x-axis

92. $(3, 5)$; parallel to the y-axis

93. $(-1, 2)$; perpendicular to the y-axis

94. $(-1, 2)$; perpendicular to the x-axis

95. $(-2, -7)$; parallel to the line $\frac{1}{2}x - \frac{1}{3}y = 5$

96. $(1, 4)$; perpendicular to the line $-\frac{2}{3}x + \frac{3}{2}y = -2$

97. $\left(-\frac{2}{3}, \frac{2}{3}\right)$; perpendicular to the line $8x + 10y = -45$

98. $\left(\frac{6}{5}, 3\right)$; perpendicular to the line $6x + 14y = 7$

99. $\left(\frac{7}{2}, 4\right)$; parallel to the line $-15x + 35y = 7$

100. $\left(-\frac{1}{4}, -\frac{13}{9}\right)$; parallel to the line $10x + 45y = -9$

APPLICATIONS

101. Budget: Home Improvement. The cost of having your bathroom remodeled is the combination of material costs and labor costs. The materials (tile, grout, toilet, fixtures, etc.) cost is $1200 and the labor cost is $25 per hour. Write an equation that models the total cost C of having your bathroom remodeled as a function of hours h. How much will the job cost if the worker estimates 32 hours?

102. Budget: Rental Car. The cost of a one-day car rental is the sum of the rental fee, $50, plus $0.39 per mile. Write an equation that models the total cost associated with the car rental.

103. Budget: Monthly Driving Costs. The monthly costs associated with driving a new Honda Accord are the monthly loan payment plus $25 every time you fill up with gasoline. If you fill up 5 times in a month, your total monthly cost is $500. How much is your loan payment?

104. Budget: Monthly Driving Costs. The monthly costs associated with driving a Ford Explorer are the monthly loan payment plus the cost of filling up your tank with gasoline. If you fill up 3 times in a month, your total monthly cost is $520. If you fill up 5 times in a month, your total monthly cost is $600. How much is your monthly loan, and how much does it cost every time you fill up with gasoline?

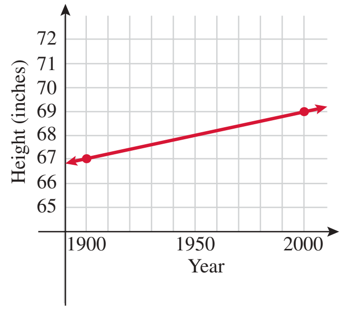
105. Business. The operating costs for a local business are a fixed amount of $1300 plus $3.50 per unit sold, while revenue is $7.25 per unit sold. How many units does the business have to sell in order to break even?

106. Business. The operating costs for a local business are a fixed amount of $12,000 plus $13.50 per unit sold, while revenue is $27.25 per unit sold. How many units does the business have to sell in order to break even?

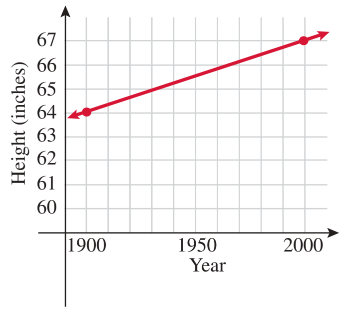
107. Weather: Temperature. The National Oceanic and Atmospheric Administration (NOAA) has an online conversion chart that relates degrees Fahrenheit, °F, to degrees Celsius, °C. 77°F is equivalent to 25°C, and 68°F is equivalent to 20°C. Assuming the relationship is linear, write the equation relating degrees Celsius to degrees Fahrenheit. What temperature is the same in both degrees Celsius and degrees Fahrenheit?

108. Weather: Temperature. According to NOAA, a "standard day" is 15°C at sea level, and every 500-feet elevation above sea level corresponds to a 1°C temperature drop. Assuming the relationship between temperature and elevation is linear, write an equation that models this relationship. What is the expected temperature at 2500 feet on a "standard day"?

109. Life Sciences: Height. The average height of a man has increased over the last century. What is the rate of change in inches per year of the average height of men?



110. Life Sciences: Height. The average height of a woman has increased over the last century. What is the rate of change in inches per year of the average height of women?



111. Life Sciences: Weight. The average weight of a baby born in 1900 was 6 pounds 4 ounces. In 2000 the average weight of a newborn was 6 pounds 10 ounces. What is the rate of change of birth weight in ounces per year? What do we expect babies to weigh at birth in 2040?

112. Sports. The fastest a man could run a mile in 1906 was 4 minutes and 30 seconds. In 1957 Don Bowden became the first American to break the 4-minute mile. Calculate the rate of change in mile speed per year.

113. Monthly Phone Costs. Mike's home phone plan charges a flat monthly fee plus a charge of $0.05 per minute for long-distance calls. The total monthly charge is represented by $y = 0.05x + 35$, $x \geq 0$, where y is the total monthly charge and x is the number of long-distance minutes used. Interpret the meaning of the y-intercept.

114. Cost: Automobile. The value of a Daewoo car is given by $y = 11{,}100 - 1850x$, $x \geq 0$, where y is the value of the car and x is the age of the car in years. Find the x-intercept and y-intercept and interpret the meaning of each.

115. Weather: Rainfall. The average rainfall in Norfolk, Virginia, for July was 5.2 inches in 2003. The average July rainfall for Norfolk was 3.8 inches in 2007. What is the rate of change of rainfall in inches per year? If this trend continues, what is the expected average rainfall in 2010?

116. Weather: Temperature. The average temperature for Boston in January 2005 was 43°F. In 2007 the average January temperature was 44.5°F. What is the rate of change of the temperature per year? If this trend continues, what is the expected average temperature in January 2010?

117. Environment. In 2000 Americans used approximately 380 billion plastic bags. In 2005 approximately 392 billion were used. What is the rate of change of plastic bags used per year? How many plastic bags are expected to be used in 2010?

118. Finance: Debt. According to the Federal Reserve, Americans individually owed \$744 in revolving credit in 2004. In 2006 they owed approximately \$788. What is the rate of change of the amount of revolving credit owed per year? How much were Americans expected to owe in 2008?

119. Business. A website that supplies Asian specialty foods to restaurants advertises a 64-ounce bottle of Hoisin Sauce for \$16.00. Shipping cost for one bottle is \$15.93. The shipping cost for two bottles is \$19.18. The cost for five bottles, including shipping, is \$111.83. Answer the following questions based on this scenario. Round to the nearest cent, when necessary.

a. Write the three ordered pairs where x represents the number of bottles purchased and y represents the total cost (including shipping) for one, two, or five bottles purchased.

b. Calculate the slope between the origin and the ordered pair that represents the purchase of one bottle of Hoisin Sauce. Explain what this amount means in terms of the sauce purchase.

c. Calculate the slope between the origin and the ordered pair that represents the purchase of two bottles of Hoisin (including shipping). Explain what this amount means in terms of the sauce purchase.

d. Calculate the slope between the origin and the ordered pair that represents the purchase of five bottles of Hoisin (including shipping). Explain what this amount means in terms of the sauce purchase.

120. Business. A website that supplies Asian specialty foods to restaurants advertises an 8-ounce bottle of Plum Sauce for \$4.00, but shipping for one bottle is \$14.27. The shipping cost for two bottles is \$14.77. The cost for five bottles, including shipping, is \$35.93. Answer the following questions based on this scenario. Round to the nearest cent, when necessary.

a. Write the three ordered pairs where x represents the number of bottles purchased and y represents the total cost, including shipping for one, two, or five bottles purchased.

b. Calculate the slope between the origin and the ordered pair that represents the purchase of one bottle of Plum Sauce. Explain what this amount means in terms of the sauce purchase.

c. Calculate the slope between the origin and the ordered pair that represents the purchase of two bottles of Plum sauce (including shipping). Explain what this amount means in terms of the sauce purchase.

d. Calculate the slope between the origin and the ordered pair that represents the purchase of five bottles of Plum sauce (including shipping). Explain what this amount means in terms of the sauce purchase.

CATCH THE MISTAKE

In Exercises 121–124, explain the mistake that is made.

121. Find the x- and y-intercepts of the line with equation $2x - 3y = 6$.

Solution:

x-intercept: set $x = 0$ and solve for y. $\quad -3y = 6$

$y = -2$

The x-intercept is $(0, -2)$.

y-intercept: set $y = 0$ and solve for x. $\quad 2x = 6$

$x = 3$

The y-intercept is $(3, 0)$.

This is incorrect. What mistake was made?

122. Find the slope of the line that passes through the points $(-2, 3)$ and $(4, 1)$.

Solution:

Write the slope formula. $\quad m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Substitute $(-2, 3)$ and $(4, 1)$. $\quad m = \dfrac{1 - 3}{-2 - 4} = \dfrac{-2}{-6} = \dfrac{1}{3}$

This is incorrect. What mistake was made?

123. Find the slope of the line that passes through the points $(-3, 4)$ and $(-3, 7)$.

Solution:

Write the slope formula. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Substitute $(-3, 4)$ and $(-3, 7)$. $m = \dfrac{-3 - (-3)}{4 - 7} = 0$

This is incorrect. What mistake was made?

124. Given the slope, classify the line as increasing, decreasing, horizontal, or vertical.

a. $m = 0$ **b.** m undefined

c. $m = 2$ **d.** $m = -1$

Solution:

a. vertical line **b.** horizontal line

c. increasing **d.** decreasing

These are incorrect. What mistakes were made?

CONCEPTUAL

In Exercises 125–130, determine whether each statement is true or false.

125. A nonhorizontal line can have at most one x-intercept.

126. A line must have at least one y-intercept.

127. If the slopes of two lines are $-\frac{1}{5}$ and 5, then the lines are parallel.

128. If the slopes of two lines are -1 and 1, then the lines are perpendicular.

129. If a line has slope equal to zero, describe a line that is perpendicular to it.

130. If a line has no slope (undefined slope), describe a line that is parallel to it.

CHALLENGE

131. Find an equation of a line that passes through the point $(-B, A + 1)$ and is parallel to the line $Ax + By = C$. Assume that B is not equal to zero.

132. Find an equation of a line that passes through the point $(B, A - 1)$ and is parallel to the line $Ax + By = C$. Assume that B is not equal to zero.

133. Find an equation of a line that passes through the point $(-A, B - 1)$ and is perpendicular to the line $Ax + By = C$. Assume that A and B are both nonzero.

134. Find an equation of a line that passes through the point $(A, B + 1)$ and is perpendicular to the line $Ax + By = C$.

135. Show that two lines with equal slopes and different y-intercepts have no point in common. *Hint:* Let $y_1 = mx + b_1$ and $y_2 = mx + b_2$ with $b_1 \neq b_2$. What equation must be true for there to be a point of intersection? Show that this leads to a contradiction.

136. Let $y_1 = m_1x + b_1$ and $y_2 = m_2x + b_2$ be two nonparallel lines $(m_1 \neq m_2)$. What is the x-coordinate of the point where they intersect?

TECHNOLOGY

In Exercises 137–142, determine whether the lines are parallel, perpendicular, or neither, and then graph both lines in the same viewing screen using a graphing utility to confirm your answer.

137. $y_1 = 17x + 22$
$y_2 = -\frac{1}{17}x - 13$

138. $y_1 = 0.35x + 2.7$
$y_2 = 0.35x - 1.2$

139. $y_1 = 0.25x + 3.3$
$y_2 = -4x + 2$

140. $y_1 = \frac{1}{2}x + 5$
$y_2 = 2x - 3$

141. $y_1 = 0.16x + 2.7$
$y_2 = 6.25x - 1.4$

142. $y_1 = -3.75x + 8.2$
$y_2 = \frac{4}{15}x + \frac{5}{6}$

SECTION 0.7 MODELING VARIATION

SKILLS OBJECTIVES

- Develop mathematical models using direct variation.
- Develop mathematical models using inverse variation.
- Develop mathematical models using combined variation.
- Develop mathematical models using joint variation.

CONCEPTUAL OBJECTIVES

- Understand the difference between direct variation and inverse variation.
- Understand the difference between combined variation and joint variation.

In this section, we discuss mathematical models for different applications. Two quantities in the real world often *vary* with respect to one another. Sometimes, they vary *directly*. For example, the more money we make, the more total dollars of federal income tax we expect to pay. Sometimes, quantities vary *inversely*. For example, when interest rates on mortgages decrease, we expect the number of homes purchased to increase, because a buyer can afford "more house" with the same mortgage payment when rates are lower. In this section, we discuss quantities varying *directly*, *inversely*, or *jointly*.

Direct Variation

When one quantity is a constant multiple of another quantity, we say that the quantities are *directly proportional* to one another.

DIRECT VARIATION

Let x and y represent two quantities. The following are equivalent statements:

- $y = kx$, where k is a nonzero constant.
- y **varies directly** with x.
- y is **directly proportional** to x.

The constant k is called the **constant of variation** or the **constant of proportionality**.

In 2005 the national average cost of residential electricity was 9.53 ¢/kWh (cents per kilowatt-hour). For example, if a residence used 3400 kWh, then the bill would be \$324, and if a residence used 2500 kWh, then the bill would be \$238.25.

EXAMPLE 1 Finding the Constant of Variation

In the United States, the cost of electricity is directly proportional to the number of kilowatt · hours (kWh) used. If a household in Tennessee on average used 3098 kWh per month and had an average monthly electric bill of \$179.99, find a mathematical model that gives the cost of electricity in Tennessee in terms of the number of kilowatt · hours used.

Solution:

Write the direct variation model. $y = kx$

Label the variables and constant.

$x =$ number of kWh
$y =$ cost (dollars)
$k =$ cost per kWh

Substitute the given data $x = 3098$ kWh and $y = \$179.99$ into $y = kx$. $179.99 = 3098k$

Solve for k. $k = \dfrac{179.99}{3098} \approx 0.05810$

$y = 0.0581x$

In Tennessee the cost of electricity is $\boxed{5.81\ \text{¢/kWh}}$.

■ **YOUR TURN** Find a mathematical model that describes the cost of electricity in California if the cost is directly proportional to the number of kWh used and a residence that consumes 4000 kWh is billed \$480.

■ **Answer:** $y = 0.12x$; the cost of electricity in California is 12 ¢/kWh.

Not all variation we see in nature is direct variation. Isometric growth, where the various parts of an organism grow in direct proportion to each other, is rare in living organisms. If organisms grew isometrically, young children would look just like adults, only smaller. In contrast, most organisms grow nonisometrically; the various parts of organisms do not increase in size in a one-to-one ratio. The relative proportions of a human body change dramatically as the human grows. Children have proportionately larger heads and shorter legs than adults. *Allometric growth* is the pattern of growth whereby different parts of the body grow at different rates with respect to each other. Some human body characteristics vary directly, and others can be mathematically modeled by *direct variation with powers*.

DIRECT VARIATION WITH POWERS

Let x and y represent two quantities. The following are equivalent statements:

- $y = kx^n$, where k is a nonzero constant.
- y **varies directly with the** n**th power** of x.
- y **is directly proportional to the** n**th power** of x.

One example of direct variation with powers is height and weight of humans. Statistics show that weight (in pounds) is directly proportional to the cube of height (feet):

$$W = kH^3$$

EXAMPLE 2 Direct Variation with Powers

The following is a personal ad:

Single professional male (6 ft/194 lb) seeks single professional female for long-term relationship. Must be athletic, smart, like the movies and dogs, and have height and weight similarly proportioned to mine.

Find a mathematical equation that describes the height and weight of the male who wrote the ad. How much would a 5 feet 6 inches woman weigh who has the same proportionality as the male?

Solution:

Write the direct variation (cube) model for height versus weight.	$W = kH^3$
Substitute the given data $W = 194$ and $H = 6$ into $W = kH^3$.	$194 = k(6)^3$
Solve for k.	$k = \frac{194}{216} = 0.898148 \approx 0.90$
	$W = 0.9H^3$
Let $H = 5.5$ ft.	$W = 0.9(5.5)^3 \approx 149.73$

A woman 5 feet 6 inches tall with the same height and weight proportionality as the male would weigh approximately $\boxed{\text{150 pounds}}$.

■ **Answer:** ≈ 200 pounds

■ **YOUR TURN** A brother and sister both have weight (pounds) that varies as the cube of height (feet) and they share the same proportionality constant. The sister is 6 feet tall and weighs 170 pounds. Her brother is 6 feet 4 inches. How much does he weigh?

Inverse Variation

Two fundamental topics covered in economics are supply and demand. Supply is the quantity that producers are willing to sell at a given price. For example, an artist may be willing to paint and sell 5 portraits if each sells for $50, but that same artist may be willing to sell 100 portraits if each sells for $10,000. Demand is the quantity of a good that consumers are not only willing to purchase but also have the capacity to buy at a given price. For example, consumers may purchase 1 billion Big Macs from McDonald's every year, but perhaps only 1 million filets mignons are sold at Outback. There may be 1 billion people who want to buy the filet mignon but don't have the financial means to do so. Economists study the equilibrium between supply and demand.

Demand can be modeled with an *inverse variation* of price: When the price increases, demand decreases, and vice versa.

INVERSE VARIATION

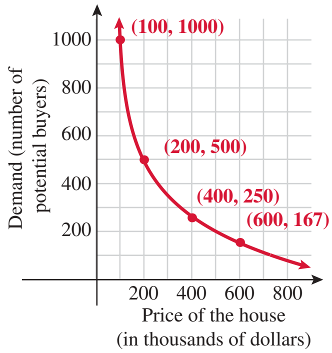
Let x and y represent two quantities. The following are equivalent statements:

- $y = \frac{k}{x}$, where k is a nonzero constant.
- y **varies inversely** with x.
- y is **inversely proportional** to x.

The constant k is called the **constant of variation** or the **constant of proportionality**.

EXAMPLE 3 Inverse Variation

The number of potential buyers of a house decreases as the price of the house increases (see graph on the right). If the number of potential buyers of a house in a particular city is inversely proportional to the price of the house, find a mathematical equation that describes the demand for houses as it relates to price. How many potential buyers will there be for a $2 million house?



Solution:

Write the inverse variation model. $y = \frac{k}{x}$

Label the variables and constant. x = price of house in thousands of dollars
y = number of buyers

Select *any* point that lies on the curve. (200, 500)

Substitute the given data $x = 200$ and $y = 500$ into $y = \frac{k}{x}$. $500 = \frac{k}{200}$

Solve for k. $k = 200 \cdot 500 = 100{,}000$

$$y = \frac{100{,}000}{x}$$

Let $x = 2000$. $y = \frac{100{,}000}{2000} = 50$

There are only 50 potential buyers for a $2 million house in this city.

YOUR TURN In New York City, the number of potential buyers in the housing market is inversely proportional to the price of a house. If there are 12,500 potential buyers for a $2 million condominium, how many potential buyers are there for a $5 million condominium?

Answer: 5000

Two quantities can vary inversely with the nth power of x.

If x and y are related by the equation $y = \frac{k}{x^n}$, then we say that y varies **inversely with the nth power of x**, or y is inversely **proportional to the nth power of x**.

Joint Variation and Combined Variation

We now discuss combinations of variations. When one quantity is directly proportional to the product of two or more other quantities, the variation is called **joint variation**. When direct variation and inverse variation occur at the same time, the variation is called **combined variation**.

An example of a **joint variation** is simple interest (Section 0.1), which is defined as

$$I = Prt$$

where

- I is the interest in dollars.
- P is the principal (initial) in dollars.
- r is the interest rate (expressed in decimal form).
- t is time in years.

The interest earned is directly proportional to the product of three quantities (principal, interest rate, and time). Note that if the interest rate increases, then the interest earned also increases. Similarly, if either the initial investment (principal) or the time the money is invested increases, then the interest earned also increases.

An example of **combined variation** is the combined gas law in chemistry:

$$P = k\frac{T}{V}$$

where

- P is pressure.
- T is temperature (kelvins).
- V is volume.
- k is a gas constant.

This relation states that the pressure of a gas is directly proportional to the temperature and inversely proportional to the volume containing the gas. For example, as the temperature increases, the pressure increases, but when the volume decreases, pressure increases.

As an example, the gas in the headspace of a soda bottle has a fixed volume. Therefore, as temperature increases, the pressure increases. Compare the different pressures of opening a twist-off cap on a bottle of soda that is cold versus one that is hot. The hot one feels as though it "releases more pressure."

EXAMPLE 4 Combined Variation

The gas in the headspace of a soda bottle has a volume of 9.0 milliliters, pressure of 2 atm (atmospheres), and a temperature of 298 K (standard room temperature of 77°F). If the soda bottle is stored in a refrigerator, the temperature drops to approximately 279 K (42°F). What is the pressure of the gas in the headspace once the bottle is chilled?

Solution:

Write the combined gas law. $P = k\frac{T}{V}$

Let $P = 2$ atm, $T = 298$ K, and $V = 9.0$ ml. $2 = k\frac{298}{9}$

Solve for k. $k = \frac{18}{298}$

Let $k = \frac{18}{298}$, $T = 279$, and $V = 9.0$ in $P = k\frac{T}{V}$. $P = \frac{18}{298} \cdot \frac{279}{9} \approx 1.87$

Since we used the same physical units for both the chilled and room-temperature soda bottles, the pressure is in atmospheres. $\boxed{P = 1.87 \text{ atm}}$

SECTION 0.7 SUMMARY

Direct, inverse, joint, and combined variation can be used to model the relationship between two quantities. For two quantities x and y, we say that

- y is directly proportional to x if $y = kx$.
- y is inversely proportional to x if $y = \frac{k}{x}$.

Joint variation occurs when one quantity is directly proportional to two or more quantities. Combined variation occurs when one quantity is directly proportional to one or more quantities and inversely proportional to one or more other quantities.

SECTION 0.7 EXERCISES

SKILLS

In Exercises 1–16, write an equation that describes each variation. Use k as the constant of variation.

1. y varies directly with x.
2. s varies directly with t.
3. V varies directly with x^3.
4. A varies directly with x^2.
5. z varies directly with m.
6. h varies directly with $\sqrt{t}$.
7. f varies inversely with λ.
8. P varies inversely with r^2.
9. F varies directly with w and inversely with L.
10. V varies directly with T and inversely with P.
11. v varies directly with both g and t.
12. S varies directly with both t and d.
13. R varies inversely with both P and T.
14. y varies inversely with both x and z.
15. y is directly proportional to the square root of x.
16. y is inversely proportional to the cube of t.

In Exercises 17–36, write an equation that describes each variation.

17. d is directly proportional to t; $d = r$ when $t = 1$.
18. F is directly proportional to m; $F = a$ when $m = 1$.
19. V is directly proportional to both l and w; $V = 6h$ when $w = 3$ and $l = 2$.
20. A is directly proportional to both b and h; $A = 10$ when $b = 5$ and $h = 4$.
21. A varies directly with the square of r; $A = 9\pi$ when $r = 3$.
22. V varies directly with the cube of r; $V = 36\pi$ when $r = 3$.
23. V varies directly with both h and r^2; $V = 1$ when $r = 2$ and $h = \frac{4}{\pi}$.
24. W is directly proportional to both R and the square of I; $W = 4$ when $R = 100$ and $I = 0.25$.
25. V varies inversely with P; $V = 1000$ when $P = 400$.
26. I varies inversely with the square of d; $I = 42$ when $d = 16$.
27. F varies inversely with both λ and L; $F = 20\pi$ when $\lambda = 1$ μm (micrometers or microns) and $L = 100$ km.
28. y varies inversely with both x and z; $y = 32$ when $x = 4$ and $z = 0.05$.
29. t varies inversely with s; $t = 2.4$ when $s = 8$.
30. W varies inversely with the square of d; $W = 180$ when $d = 0.2$.
31. R varies inversely with the square of I; $R = 0.4$ when $I = 3.5$.
32. y varies inversely with both x and the square root of z; $y = 12$ when $x = 0.2$ and $z = 4$.
33. R varies directly with L and inversely with A; $R = 0.5$ when $L = 20$ and $A = 0.4$.
34. F varies directly with m and inversely with d; $F = 32$ when $m = 20$ and $d = 8$.
35. F varies directly with both m_1 and m_2 and inversely with the square of d; $F = 20$ when $m_1 = 8$, $m_2 = 16$, and $d = 0.4$.
36. w varies directly with the square root of g and inversely with the square of t; $w = 20$ when $g = 16$ and $t = 0.5$.

APPLICATIONS

37. **Wages.** Jason and Valerie both work at Panera Bread and have the following paycheck information for a certain week. Find an equation that shows their wages W varying directly with the number of hours worked H.

Employee	Hours Worked	Wages
Jason	23	\$172.50
Valerie	32	\$240.00

38. Sales Tax. The sales tax in Orange and Seminole Counties in Florida differs by only 0.5%. A new resident knows this but doesn't know which of the counties has the higher tax. The resident lives near the border of the counties and is in the market for a new plasma television and wants to purchase it in the county with the lower tax. If the tax on a pair of \$40 sneakers is \$2.60 in Orange County and the tax on a \$12 T-shirt is \$0.84 in Seminole County, write two equations: one for each county that describes the tax T, which is directly proportional to the purchase price P.

For Exercises 39 and 40, refer to the following:

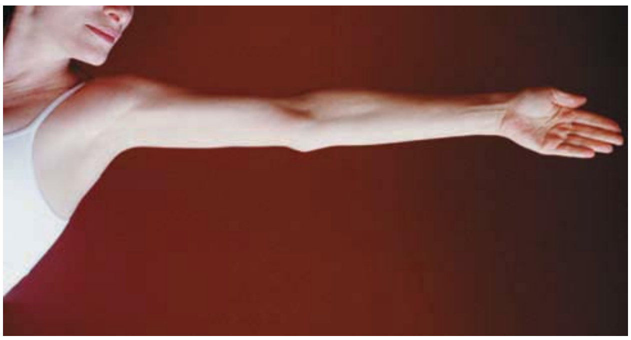
The ratio of the speed of an object to the speed of sound determines the Mach number. Aircraft traveling at a subsonic speed (less than the speed of sound) have a Mach number less than 1. In other words, the speed of an aircraft is directly proportional to its Mach number. Aircraft traveling at a supersonic speed (greater than the speed of sound) have a Mach number greater than 1. The speed of sound at sea level is approximately 760 miles per hour.

39. Military. The U.S. Navy Blue Angels fly F-18 Hornets that are capable of Mach 1.7. How fast can F-18 Hornets fly at sea level?

40. Military. The U.S. Air Force's newest fighter aircraft is the F-22A Raptor, which is capable of Mach 1.5. How fast can a F-22A Raptor fly at sea level?

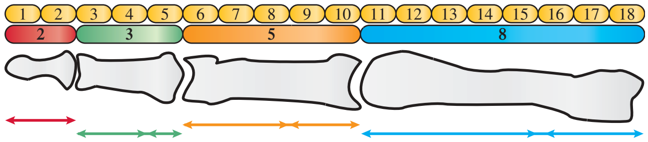
Exercises 41 and 42 are examples of the golden ratio, or phi, a proportionality constant that appears in nature. The numerical approximate value of phi is 1.618 (from www.goldenratio.net).

41. Human Anatomy. The length of your forearm F (wrist to elbow) is directly proportional to the length of your hand H (length from wrist to tip of middle finger). Write the equation that describes this relationship if the length of your forearm is 11 inches and the length of your hand is 6.8 inches.



Kim Steele/Getty Images, Inc.

42. Human Anatomy. Each section of your index finger, from the tip to the base of the wrist, is larger than the preceding one by about the golden (Fibonacci) ratio. Find an equation that represents the ratio of each section of your finger related to the previous one if one section is eight units long and the next section is five units long.

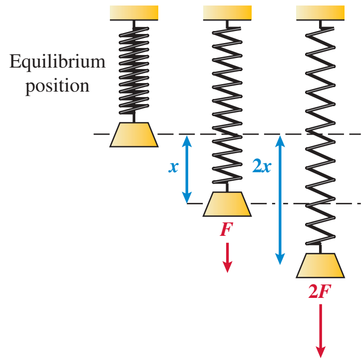


For Exercises 43 and 44, refer to the following:

Hooke's law in physics states that if a spring at rest (equilibrium position) has a weight attached to it, then the distance the spring stretches is directly proportional to the force (weight), according to the formula:

$$F = kx$$

where F is the force in Newtons (N), x is the distance stretched in meters (m), and k is the spring constant (N/m).



43. Physics. A force of 30 N will stretch the spring 10 centimeters. How far will a force of 72 N stretch the spring?

44. Physics. A force of 30 N will stretch the spring 10 centimeters. How much force is required to stretch the spring 18 centimeters?

45. Business. A cell phone company develops a pay-as-you-go cell phone plan in which the monthly cost varies directly as the number of minutes used. If the company charges \$17.70 in a month when 236 minutes are used, what should the company charge for a month in which 500 minutes are used?

46. Economics. Demand for a product varies inversely with the price per unit of the product. Demand for the product is 10,000 units when the price is \$5.75 per unit. Find the demand for the product (to the nearest hundred units) when the price is \$6.50.

47. Sales. Levi's makes jeans in a variety of price ranges for juniors. The Flare 519 jeans sell for about \$20, whereas the 646 Vintage Flare jeans sell for \$300. The demand for Levi's jeans is inversely proportional to the price. If 300,000 pairs of the 519 jeans were bought, approximately how many of the Vintage Flare jeans were bought?

48. Sales. Levi's makes jeans in a variety of price ranges for men. The Silver Tab Baggy jeans sell for about \$30, whereas the Offender jeans sell for about \$160. The demand for Levi's jeans is inversely proportional to the price. If 400,000 pairs of the Silver Tab Baggy jeans were bought, approximately how many of the Offender jeans were bought?

For Exercises 49 and 50, refer to the following:

In physics, the inverse square law states that any physical force or energy flow is inversely proportional to the square of the distance from the source of that physical quantity. In particular, the intensity of light radiating from a point source is inversely proportional to the square of the distance from the source. Below is a table of average distances from the Sun:

Planet	Distance to the Sun
Mercury	58,000 km
Earth	150,000 km
Mars	228,000 km

49. Solar Radiation. The solar radiation on Earth is approximately 1400 watts per square meter (W/m^2). How much solar radiation is there on Mars? Round to the nearest hundred watts per square meter.

50. Solar Radiation. The solar radiation on Earth is approximately 1400 watts per square meter. How much solar radiation is there on Mercury? Round to the nearest hundred watts per square meter.

51. Investments. Marilyn receives a $25,000 bonus from her company and decides to put the money toward a new car that she will need in 2 years. Simple interest is directly proportional to the principal and the time invested. She compares two different banks' rates on money market accounts. If she goes with Bank of America, she will earn $750 in interest, but if she goes with the Navy Federal Credit Union, she will earn $1500. What is the interest rate on money market accounts at both banks?

52. Investments. Connie and Alvaro sell their house and buy a fixer-upper house. They made $130,000 on the sale of their previous home. They know it will take 6 months before the general contractor can start their renovation, and they want to take advantage of a 6-month CD that pays simple interest. What is the rate of the 6-month CD if they will make $3250 in interest?

53. Chemistry. A gas contained in a 4-milliliter container at a temperature of 300 K has a pressure of 1 atm. If the temperature decreases to 275 K, what is the resulting pressure?

54. Chemistry. A gas contained in a 4-milliliter container at a temperature of 300 K has a pressure of 1 atm. If the container changes to a volume of 3 milliliters, what is the resulting pressure?

CATCH THE MISTAKE

In Exercises 55 and 56, explain the mistake that is made.

55. y varies directly with t and inversely with x. When $x = 4$ and $t = 2$, then $y = 1$. Find an equation that describes this variation.

Solution:

Write the variation equation. $y = ktx$

Let $x = 4$, $t = 2$, and $y = 1$. $1 = k(2)(4)$

Solve for k. $k = \frac{1}{8}$

Substitute $k = \frac{1}{8}$ into $y = ktx$. $y = \frac{1}{8}tx$

This is incorrect. What mistake was made?

56. y varies directly with t and the square of x. When $x = 4$ and $t = 1$, then $y = 8$. Find an equation that describes this variation.

Solution:

Write the variation equation. $y = kt\sqrt{x}$

Let $x = 4$, $t = 1$, and $y = 8$. $8 = k(1)\sqrt{4}$

Solve for k. $k = 4$

Substitute $k = 4$ into $y = kt\sqrt{x}$. $y = 4t\sqrt{x}$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 57 and 58, determine whether each statement is true or false.

57. The area of a triangle is directly proportional to both the base and the height of the triangle (joint variation).

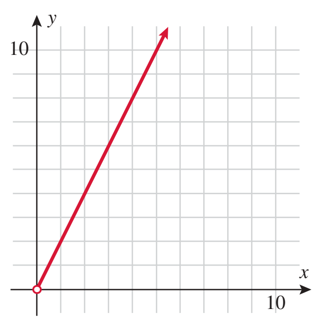
58. Average speed is directly proportional to both distance and time (joint variation).

In Exercises 59 and 60, match the variation with the graph.

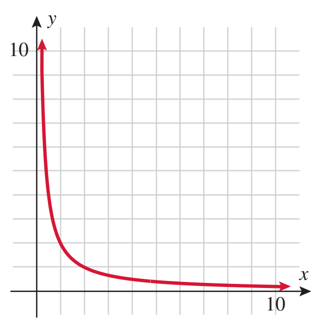
59. Inverse variation

60. Direct variation

a.



b.



CHALLENGE

Exercises 61 and 62 involve the theory governing laser propagation through Earth's atmosphere.

The three parameters that help classify the strength of optical turbulence are the following:

- C_n^2, index of refraction structure parameter
- k, wave number of the laser, which is inversely proportional to the wavelength λ of the laser:

$$k = \frac{2\pi}{\lambda}$$

- L, propagation distance

The variance of the irradiance of a laser, σ^2, is directly proportional to C_n^2, $k^{7/6}$, and $L^{11/16}$.

61. When $C_n^2 = 1.0 \times 10^{-13}\,\text{m}^{-2/3}$, $L = 2$ km, and $\lambda = 1.55\ \mu$m, the variance of irradiance for a plane wave σ_{pl}^2 is 7.1. Find the equation that describes this variation.

62. When $C_n^2 = 1.0 \times 10^{-13}\,\text{m}^{-2/3}$, $L = 2$ km, and $\lambda = 1.55\ \mu$m, the variance of irradiance for a spherical wave σ_{sp}^2 is 2.3. Find the equation that describes this variation.

TECHNOLOGY

For Exercises 63–66, refer to the following:

Data from 1995 to 2006 for oil prices in dollars per barrel, the U.S. Dow Jones Utilities Stock Index, New Privately Owned Housing, and 5-year Treasury Constant Maturity Rate are given in the table below. (These data are from Forecast Center's Historical Economic and Market Home Page at www.neatideas.com/djutil.htm.)

Use the calculator STAT EDIT commands to enter the table with L_1 as the oil price, L_2 as the utilities stock index, L_3 as number of housing units, and L_4 as the 5-year maturity rate.

January of Each Year	Oil Price, \$ per Barrel	U.S. Dow Jones Utilities Stock Index	New, Privately Owned Housing Units	5-year Treasury Constant Maturity Rate
1995	17.99	193.12	1407	7.76
1996	18.88	230.85	1467	5.36
1997	25.17	232.53	1355	6.33
1998	16.71	263.29	1525	5.42
1999	12.47	302.80	1748	4.60
2000	27.18	315.14	1636	6.58
2001	29.58	372.32	1600	4.86
2002	19.67	285.71	1698	4.34
2003	32.94	207.75	1853	3.05
2004	32.27	271.94	1911	3.12
2005	46.84	343.46	2137	3.71
2006	65.51	413.84	2265	4.35

63. An increase in oil price in dollars per barrel will drive the U.S. Dow Jones Utilities Stock Index to soar.

a. Use the calculator commands STAT, linReg $(ax + b)$, and STATPLOT to model the data using the least squares regression. Find the equation of the least squares regression line using x as the oil price in dollars per barrel.

b. If the U.S. Dow Jones Utilities Stock Index varies directly as the oil price in dollars per barrel, then use the calculator commands STAT, PwrReg, and STATPLOT to model the data using the power function. Find the variation constant and equation of variation using x as the oil price in dollars per barrel.

c. Use the equations you found in (a) and (b) to calculate the stock index when the oil price hit \$72.70 per barrel in September 2006. Which answer is closer to the actual stock index of 417? Round all answers to the nearest whole number.

64. An increase in oil price in dollars per barrel will affect the interest rates across the board—in particular, the 5-year Treasury constant maturity rate.

a. Use the calculator commands STAT, linReg $(ax + b)$, and STATPLOT to model the data using the least squares regression. Find the equation of the least squares regression line using x as the oil price in dollars per barrel.

b. If the 5-year Treasury constant maturity rate varies inversely as the oil price in dollars per barrel, then use the calculator commands STAT, PwrReg, and STATPLOT to model the data using the power function. Find the variation constant and equation of variation using x as the oil price in dollars per barrel.

c. Use the equations you found in (a) and (b) to calculate the maturity rate when the oil price hit \$72.70 per barrel in September 2006. Which answer is closer to the actual maturity rate at 5.02%? Round all answers to two decimal places.

65. An increase in interest rates—in particular, the 5-year Treasury constant maturity rate—will affect the number of new, privately owned housing units.

a. Use the calculator commands STAT, linReg $(ax + b)$, and STATPLOT to model the data using the least squares regression. Find the equation of the least squares regression line using x as the 5-year rate.

b. If the number of new privately owned housing units varies inversely as the 5-year Treasury constant maturity rate, then use the calculator commands STAT, PwrReg, and STATPLOT to model the data using the power function. Find the variation constant and equation of variation using x as the 5-year rate.

c. Use the equations you found in (a) and (b) to calculate the number of housing units when the maturity rate was 5.02% in September 2006. Which answer is closer to the actual number of new, privately owned housing units of 1861? Round all answers to the nearest unit.

66. An increase in the number of new, privately owned housing units will affect the U.S. Dow Jones Utilities Stock Index.

a. Use the calculator commands STAT, linReg $(ax + b)$, and STATPLOT to model the data using the least squares regression. Find the equation of the least squares regression line using x as the number of housing units.

b. If the U.S. Dow Jones Utilities Stock Index varies directly as the number of new, privately owned housing units, then use the calculator commands STAT, PwrReg, and STATPLOT to model the data using the power function. Find the variation constant and equation of variation using x as the number of housing units.

c. Use the equations you found in (a) and (b) to find the utilities stock index if there were 1861 new, privately owned housing units in September 2006. Which answer is closer to the actual stock index of 417? Round all answers to the nearest whole number.

For Exercises 67 and 68, refer to the following:

From March 2000 to March 2008, data for retail gasoline price in dollars per gallon are given in the table below. (These data are from Energy Information Administration, Official Energy Statistics from the U.S. Government at http://tonto.eia.doe.gov/oog/info/gdu/gaspump.html.) Use the calculator STAT EDIT command to enter the table below with L_1 as the year ($x = 1$ for year 2000) and L_2 as the gasoline price in dollars per gallon.

March of each year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Retail gasoline price \$ per gallon	1.517	1.409	1.249	1.693	1.736	2.079	2.425	2.563	3.244

67. a. Use the calculator commands STAT LinReg to model the data using the least squares regression. Find the equation of the least squares regression line using x as the year ($x = 1$ for year 2000) and y as the gasoline price in dollars per gallon. Round all answers to three decimal places.

b. Use the equation to determine the gasoline price in March 2006. Round all answers to three decimal places. Is the answer close to the actual price?

c. Use the equation to find the gasoline price in March 2009. Round all answers to three decimal places.

68. a. Use the calculator commands STAT PwrReg to model the data using the power function. Find the variation constant and equation of variation using x as the year ($x = 1$ for year 2000) and y as the gasoline price in dollars per gallon. Round all answers to three decimal places.

b. Use the equation to find the gasoline price in March 2006. Round all answers to three decimal places. Is the answer close to the actual price?

c. Use the equation to determine the gasoline price in March 2009. Round all answers to three decimal places.

SECTION
0.8* LINEAR REGRESSION: BEST FIT

SKILLS OBJECTIVES

- Draw a scatterplot.
- Use linear regression to determine the line of best fit associated with some data.
- Use the line of best fit to predict values of one variable from the values of another.

CONCEPTUAL OBJECTIVES

- Recognize positive or negative association.
- Recognize linear or nonlinear association.
- Understand what "best fit" means.

Scatterplots

An important aspect of applied research across disciplines is to discover and understand relationships between variables, and often how to use such a relationship to predict values of one variable in terms of another. You have likely encountered such issues while watching TV, reading a magazine or newspaper, or simply talking with friends. Some *questions* include

- Is age predictive of texting speed?
- Is the level of pollution in a country related to the prevalence of asthma in that country?
- Do the ratings of car reliability necessarily increase with the price of the car?

In this section we focus on situations involving relationships between two variables x and y, so that the experimental data gathered consists of ordered pairs $(x_1, y_1), \ldots, (x_n, y_n)$.

A first step in understanding a data set of the form $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ is to create a pictorial representation of it. Identifying the first coordinates of these ordered pairs as values of an **independent variable** (or **predictor variable**) x and the second coordinates as the values of a **dependent variable** (or **response variable**) y, we simply plot them all on a single xy-plane. The resulting picture is called a **scatterplot**.

EXAMPLE 1 Drawing a Scatterplot of Olympic Decathlon Data

The 2004 Men's Olympic Decathlon consisted of the following 10 events: 100 meter, long jump, shot put, high jump, 400 meter, 110 meter hurdles, discus, pole vault, javelin throw, and 1500 meter. Actual scores are converted to a point system where points are assigned to each of these events based on performance. Events are equally weighted when converting to points. These points are then summed to obtain total scores, and, in turn, medals are assigned based on these total scores.

It would be interesting to know if certain events are more predictive of the total scores than are others. If someone does exceedingly well in the javelin throw, for example, is that person more likely to do well across all events and therefore obtain a large total points score?

Data from the Men's 2004 Olympic Decathlon are presented on the next page and were retrieved from the following Web source: **http://rss.acs.unt.edu/Rdoc/library/FactoMineR/html/decathlon.html**.

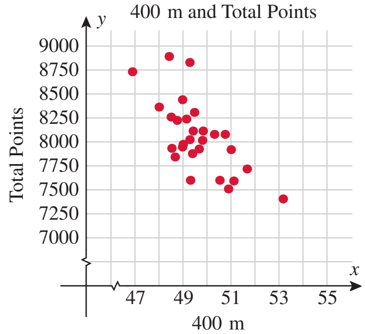
Let's consider the paired data set $\{(x, y)\}$ where x = score on the 400 m and y = total score.

*Optional Technology Required Section.

One scatterplot using the 400 m and total points information from this data set is shown in the following graph.

Several natural questions arise: How are different pairs of these data related? Are there any discernible patterns present, and if so, how strong are they? Is there a single curve that can be used to describe the general trend present in the data? We shall answer these questions one by one in this section.

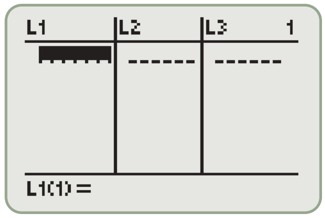
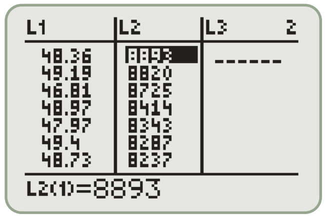
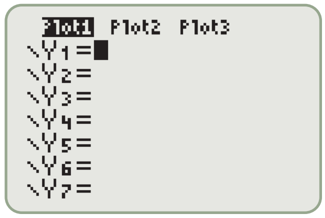
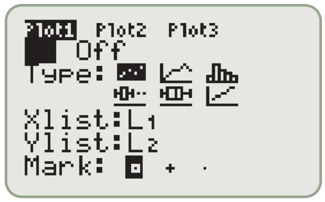
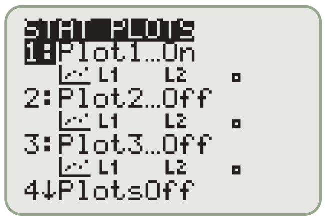
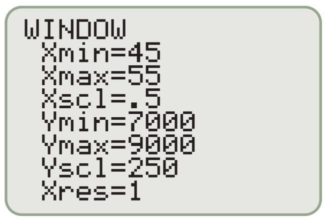
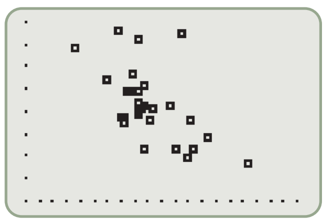
Men's 2004 Olympic Decathlon—
400 m and Total Points



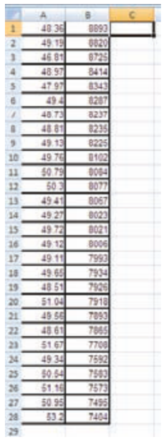
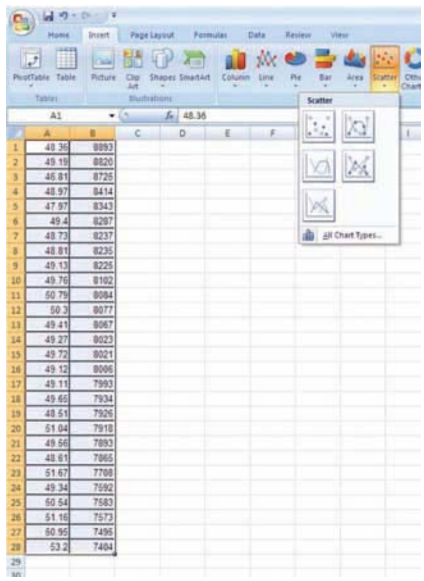
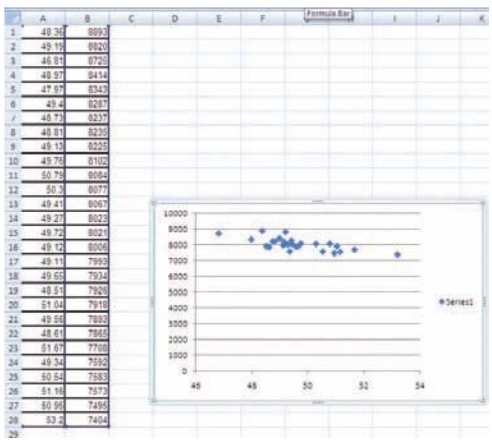
Olympians	X100m	Long Jump	Shot-put	High Jump	X400m	X110m Hurdle	Discus	Pole Vault	Javelin	X1500m	Rank	Total Score
Sebrle	10.85	7.84	16.36	2.12	48.36	14.05	48.72	5.00	70.52	280.01	1	8893
Clay	10.44	7.96	15.23	2.06	49.19	14.13	50.11	4.90	69.71	282.00	2	8820
Karpov	10.50	7.81	15.93	2.09	46.81	13.97	51.65	4.60	55.54	278.11	3	8725
Macey	10.89	7.47	15.73	2.15	48.97	14.56	48.34	4.40	58.46	265.42	4	8414
Warners	10.62	7.74	14.48	1.97	47.97	14.01	43.73	4.90	55.39	278.05	5	8343
Zsivoczky	10.91	7.14	15.31	2.12	49.40	14.95	45.62	4.70	63.45	269.54	6	8287
Hernu	10.97	7.19	14.65	2.03	48.73	14.25	44.72	4.80	57.76	264.35	7	8237
Nool	10.80	7.53	14.26	1.88	48.81	14.80	42.05	5.40	61.33	276.33	8	8235
Bernard	10.69	7.48	14.80	2.12	49.13	14.17	44.75	4.40	55.27	276.31	9	8225
Schwarzl	10.98	7.49	14.01	1.94	49.76	14.25	42.43	5.10	56.32	273.56	10	8102
Pogorelov	10.95	7.31	15.10	2.06	50.79	14.21	44.60	5.00	53.45	287.63	11	8084
Schoenbeck	10.90	7.30	14.77	1.88	50.30	14.34	44.41	5.00	60.89	278.82	12	8077
Barras	11.14	6.99	14.91	1.94	49.41	14.37	44.83	4.60	64.55	267.09	13	8067
Smith	10.85	6.81	15.24	1.91	49.27	14.01	49.02	4.20	61.52	272.74	14	8023
Averyanov	10.55	7.34	14.44	1.94	49.72	14.39	39.88	4.80	54.51	271.02	15	8021
Ojaniemi	10.68	7.50	14.97	1.94	49.12	15.01	40.35	4.60	59.26	275.71	16	8006
Smirnov	10.89	7.07	13.88	1.94	49.11	14.77	42.47	4.70	60.88	263.31	17	7993
Qi	11.06	7.34	13.55	1.97	49.65	14.78	45.13	4.50	60.79	272.63	18	7934
Drews	10.87	7.38	13.07	1.88	48.51	14.01	40.11	5.00	51.53	274.21	19	7926
Parkhomenko	11.14	6.61	15.69	2.03	51.04	14.88	41.90	4.80	65.82	277.94	20	7918
Terek	10.92	6.94	15.15	1.94	49.56	15.12	45.62	5.30	50.62	290.36	21	7893
Gomez	11.08	7.26	14.57	1.85	48.61	14.41	40.95	4.40	60.71	269.70	22	7865
Turi	11.08	6.91	13.62	2.03	51.67	14.26	39.83	4.80	59.34	290.01	23	7708
Lorenzo	11.10	7.03	13.22	1.85	49.34	15.38	40.22	4.50	58.36	263.08	24	7592
Karlivans	11.33	7.26	13.30	1.97	50.54	14.98	43.34	4.50	52.92	278.67	25	7583
Korkizoglou	10.86	7.07	14.81	1.94	51.16	14.96	46.07	4.70	53.05	317.00	26	7573
Uldal	11.23	6.99	13.53	1.85	50.95	15.09	43.01	4.50	60.00	281.70	27	7495
Casarsa	11.36	6.68	14.92	1.94	53.20	15.39	48.66	4.40	58.62	296.12	28	7404

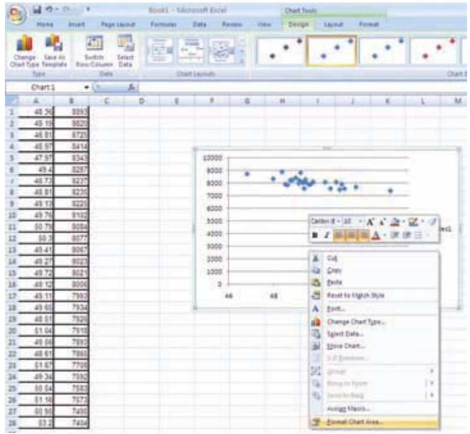
Creating a scatterplot by hand can be tedious, especially for large data sets. You can also very easily lose precision and detail. Using technology to create a scatterplot is very appropriate and quite easy. Below are the procedures for how you would create the scatterplot shown in Example 1 using the TI-83+ (or TI-84) and *Excel* 2007.

Creating a Scatterplot Using the TI-83+ (or TI-84)

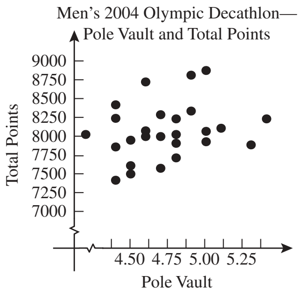
		INSTRUCTION	SCREENSHOT
Entering the Data	**Step 1**	Press **STAT**, followed by **1:Edit**. . . . Clear any data already present in columns L1 and L2 so that the screen looks like the one to the right.	
	Step 2	Input the values of the *x*-variable (first entries in the ordered pairs) in column L1, pressing **ENTER** after each entry. Then, right arrow over to column L2 and input the values of the *y*-variable. The screen (starting from the beginning of the data set) should look like the one to the right when you are done.	
Plotting the Data	**Step 3**	Press **Y=** and then select **Plot1** in the top row of the screen. If either **Plot2** or **Plot3** is darkened, move the cursor onto it and press **ENTER** to undarken it. The screen should look like the one to the right when you are done.	
	Step 4	Press **2nd**, followed by **Y=** (for **StatPlot**). Select 1: and modify the entries to make the screen look like the one to the right.	
	Step 5	Press **2nd**, followed by **Y=** and make certain that both **Plot2** and **Plot3** are OFF. The screen should look like the one to the right.	
	Step 6	Make certain the ranges for the *x* and *y* values are appropriate for the given data set. Here, we use the window shown to the right.	
	Step 7	Press **GRAPH** and you should get the scatterplot shown to the right.	

Creating a Scatterplot Using *Excel* 2007

		Instruction	Screenshot
Entering the Data	**Step 1**	Open a new *Excel* spreadsheet. Input the values of the x-variable (first entries in the ordered pairs) in column A, starting with cell 1A. Then, input the values of the y-variable in column B, starting with cell 1B. The screen should look like the one to the right when you are done.	
Plotting the Data	**Step 2**	Highlight the data. The screen should look like the one to the right when you are done.	**Step 2** 
	Step 3	Go to the **Insert** tab and select the icon labeled **Scatter**. A window list of five possible choices pops up.	**Step 3** 
	Step 4	Select the leftmost choice in the top row. Press **ENTER**. The scatterplot shown to the right should appear on the screen.	

INSTRUCTION	SCREENSHOT
Step 5 You can alter the format of the scatterplot with various bells and whistles by *right-clicking* anywhere near the data points and then selecting **Format Plot Area** at the bottom of the pop-up window.	

■ **Answer:**



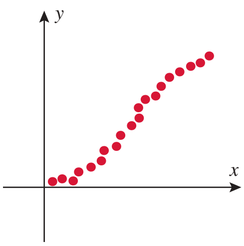
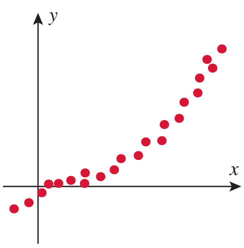
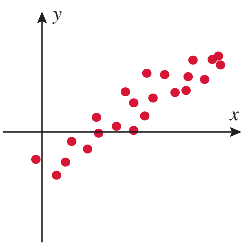
■ **YOUR TURN** Using the data in Example 1, identify x = *score on the pole vault* and z = *total score*. Use technology to create a scatterplot for the data set consisting of the ordered pairs (x, z).

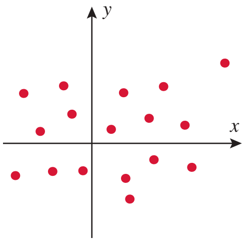
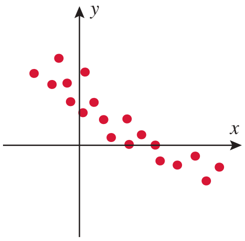
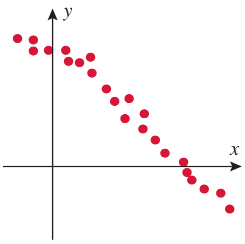
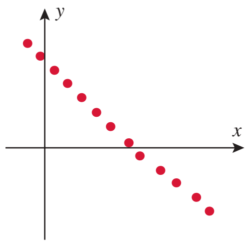
Identifying Patterns

While scatterplots are comprised merely of clusters of ordered pairs, patterns of various types can emerge that can provide insight into how the variables x and y are related.

Direction of Association

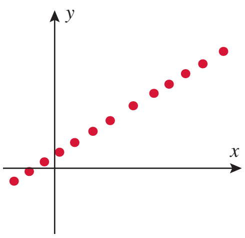
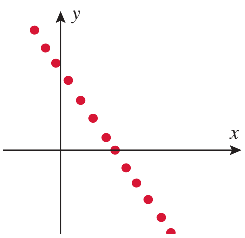
This characteristic is analogous to the concept of slope of a line. If the cluster of points tends to rise from left to right, we say that x and y are **positively associated**, whereas if the cluster of points falls from left to right, we say that x and y are **negatively associated**. Certainly, the more closely packed together the points are to an identifiable curve, the easier it is to make such a determination. Some examples of scatterplots of varying degrees of positive and negative association are shown in the following table.

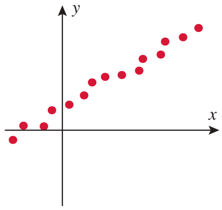
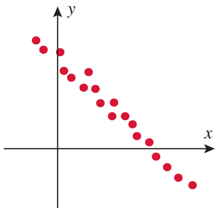
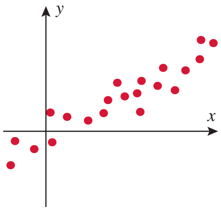
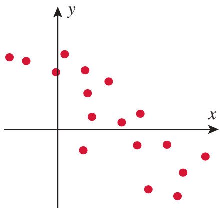
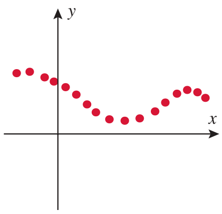
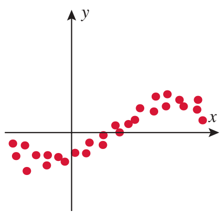
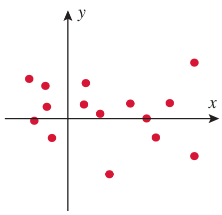
SCATTERPLOT		DIRECTION OF ASSOCIATION	VERBAL DESCRIPTION
		**Positive Association**	A shape that increases from left to right is very discernible in each case.
			The association is a bit loose, but you can still tell the data points tend to rise from left to right.

Scatterplot	Direction of Association	Verbal Description
	**No Association**	Haphazard scattering of points suggests neither positive nor negative association.
	**Negative Association**	The association is a bit loose, but you can still tell the data points tend to fall from left to right.
 	**Negative Association**	A shape that decreases from left to right is very discernible in each case.

Linearity

Depending on the phenomena being studied and the actual sample being used, the data points comprising a scatterplot can conform very closely to an actual curve. If the curve is a line, we say that the relationship between x and y is **linear**; otherwise, we say the relationship is **nonlinear**. Some illustrative examples follow.

Scatterplot	Linearity	Verbal Description
	**Linear**	Perfect linear relationship; positive association
	**Linear**	Perfect linear relationship; negative association

Scatterplot	Linearity	Verbal Description
	**Linear**	Fairly tight linear relationship; positive association
	**Linear**	Fairly tight linear relationship; negative association
	**Linear**	Rather loose, but still somewhat discernible, linear relationship; positive association
	**Linear**	Rather loose, but still somewhat discernible, linear relationship; negative association
	**Nonlinear**	Perfect nonlinear relationship
	**Nonlinear**	Fairly tight nonlinear relationship
	**No specifically identifiable relationship**	No discernible linear relationship or degree of association

EXAMPLE 2 Describing Patterns in a Data Set

Describe the patterns present in the paired data set (x, y) considered in Example 1, where x = *points on the 400 m* and y = *total score*. Is this intuitive?

Solution:

We can surmise that the variable *score on the 400 m* has a relatively strong linear, negative association with the variable *total score*. This means that as the 400 m score decreases, the total score tends to increase. A negative relationship makes sense here in that 400 m scores reflect how much time it took to complete this race. So, lower scores (less time) reflect better performance and therefore more total points.

■ **YOUR TURN** Using the data in Example 1, identify x = *score on the pole vault* and z = *total score*. Comment on the degree of association and linearity of the scatterplot consisting of the ordered pairs (x, z). Is this intuitive?

■ **Answer:** The variable *score the pole vault* has a rather weak linear, positive association with the variable *total score*. This means that as the pole vault score increases, the total score tends to increase. In this case, a positive relationship makes sense in that pole vault scores reflect the height achieved. So, higher scores (greater height) reflect better performance and therefore more total points.

Strength of Linear Relationship

The variability in the data can render it difficult to determine if there is a linear relationship between two variables. As such, it is useful to have a way of measuring how tightly a paired data set conforms to a linear shape. This measure is called the **correlation coefficient**, r, and is defined by the following formula:

DEFINITION

For a paired data set $\{(x_1, y_1), \ldots, (x_n, y_n)\}$, the *correlation coefficient*, r, is defined by

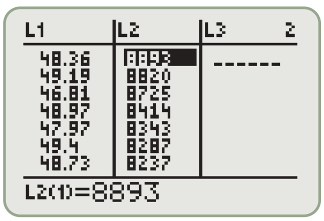
$$r = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\sum x^2 - \left(\sum x\right)^2} \cdot \sqrt{n\sum y^2 - \left(\sum y\right)^2}}$$

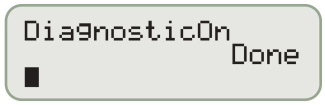
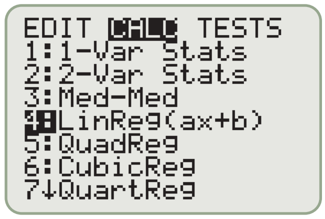
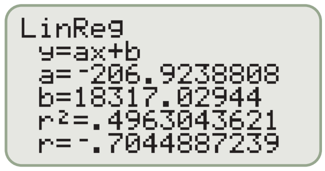
The symbol $\sum z$ is a shorthand way of writing $z_1 + \cdots + z_n$. So, for instance,

$\sum x^2 = x_1^2 + \cdots + x_n^2$.

This is tedious to calculate by hand but is easily computed using technology. Below are the procedures for how you would compute the correlation coefficient for the data set introduced in Example 1 using the TI-83+ (or TI-84) and *Excel* 2007.

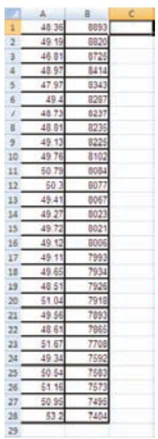
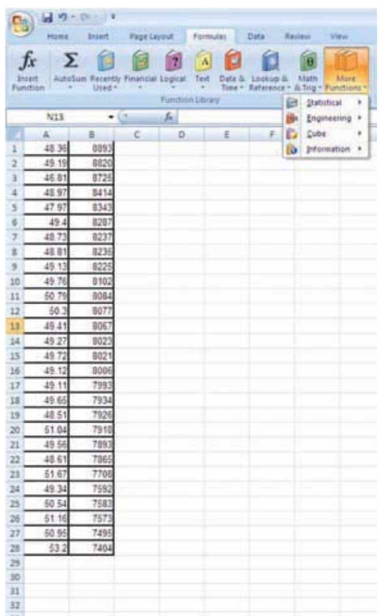
Computing a Correlation Coefficient Using the TI-83+ (or TI-84)

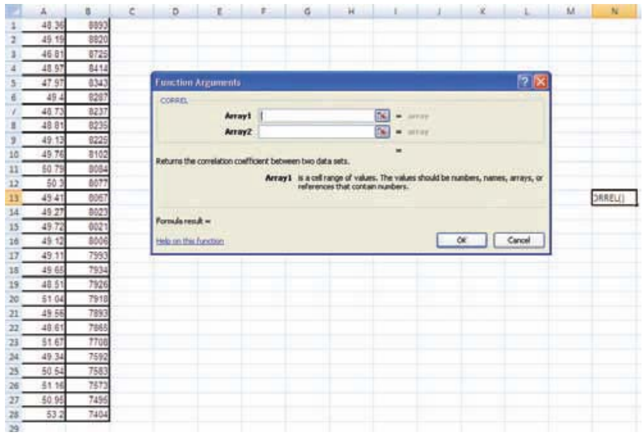
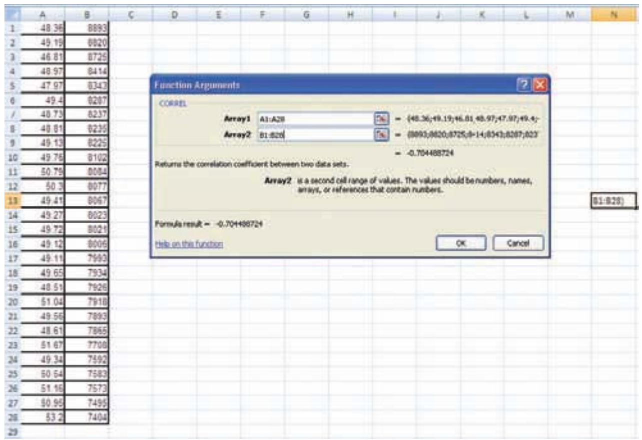
	INSTRUCTION	SCREENSHOT
Step 1	**Enter the data** following the procedure outlined earlier in this section. The screen should look like the one to the right.	

	INSTRUCTION	SCREENSHOT
Step 2	**Set up what will display!** In order for the desired output to display once we execute the commands to follow, we must tell the calculator to do so. As such, do the following: **i.** Press **2nd**, followed by 0 to get **CATALOG**. **ii.** Scroll down until you get to **DiagnosticOn**. Press **ENTER**. Then, this command will appear on the home screen. Press **ENTER** again. The resulting screen should look like the one to the right.	
Step 3	Press **STAT**, followed by **CALC**, and then by **4:LinReg**(ax+b). The resulting screen should look like the one to the right. Press **ENTER**.	
Step 4	Press **ENTER** again. After a brief moment, your screen should look like the one to the right. The value we want is in the bottom row of the screen, about $r = -0.7045$.	

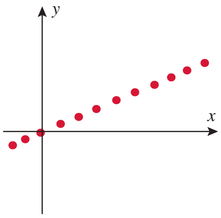
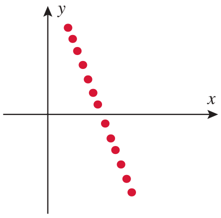
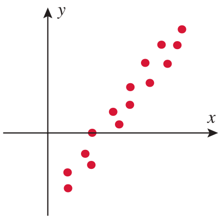
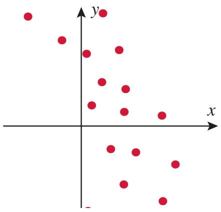
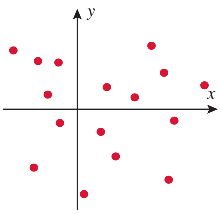
Note: The other information provided will be pertinent once we define the *best fit line* in the next subsection.

Computing a Correlation Coefficient Using *Excel* 2007

	INSTRUCTION	SCREENSHOT
Step 1	**Enter the data** following the procedure outlined earlier in this section. The screen should look like the one to the right.	**Step 1** 
Step 2	Select the **Formulas** tab at the top of the screen and then choose the **More Functions** within the *Function Library* grouping (on the left). The pull-down menu should be as shown to the right.	**Step 2** 

	Instruction	Screenshot
Step 3	From here, select **Statistical**, and then from this list, scroll down and choose **CORREL**. A pop-up window should appear, as shown to the right.	
Step 4	Enter **A1:A28** in Array 1 and **B1:B28** in Array 2, as shown to the right. Press **OK**. You will notice that the correlation coefficient appears directly beneath Array 2. In this case, r is about -0.7045.	

The square of the correlation coefficient is interpreted as a signed percentage of the variability among the y-values that is actually explained by the linear relationship, where the sign corresponds to the direction of the association. For instance, an r-value of $+1$ means that 100% of the variability among the y-values is explained by a line with positive slope; in such a case, all of the points in the data set actually lie on a single line. An r-value of -1 means the same thing, but the line has a negative slope. As the r-values get closer to zero, the more dispersed the points become from a line describing the pattern, so that an r-value very close to zero suggests no linear relationship whatsoever is discernible. The following sample of scatterplots with the associated correlation coefficients should provide you with a feel for the strength of linearity suggested by various values of r.

Scatterplot	Correlation Coefficient r	Strength of Linearity
	$r = 1.0$	Perfect positive linear relationship
	$r = -1.0$	Perfect negative linear relationship
	$r = 0.80$	Reasonably strong, though not perfect, positive linear relationship
	$r = -0.45$	Pretty weak, barely discernible, negative linear relationship
	$r = 0.10$	Essentially no discernible linear relationship whatsoever

EXAMPLE 3 Calculating the Correlation Coefficient Associated with a Data Set

Use technology to calculate the correlation coefficient r for the paired data set (x, y) considered in Example 1, where $x =$ *points on the 400 m* and $y =$ *total score*. Interpret the strength of the linear relationship.

Solution:

We see that using either form of technology yields $r = -0.7045$. This suggests that the data follow a relatively strong negative (i.e., negative slope) linear pattern.

■ **YOUR TURN** Using the data in Example 1, identify $x =$ *score on the pole vault* and $z =$ *total score*, and calculate the correlation coefficient for the data set consisting of the ordered pairs (x, z) using technology. Interpret the strength of the linear relationship.

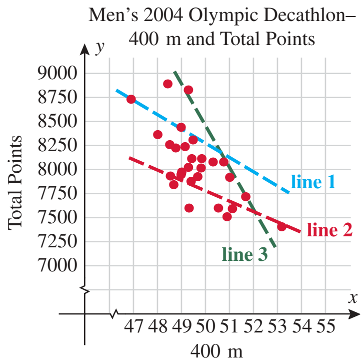
■ **Answer:** The correlation coefficient is approximately $r = 0.28$. This suggests that while the data follow a positive (i.e., positive slope) pattern, the degree to which an actual line describes the trend in the data is rather weak.

Linear Regression

Determining the "Best Fit" Line

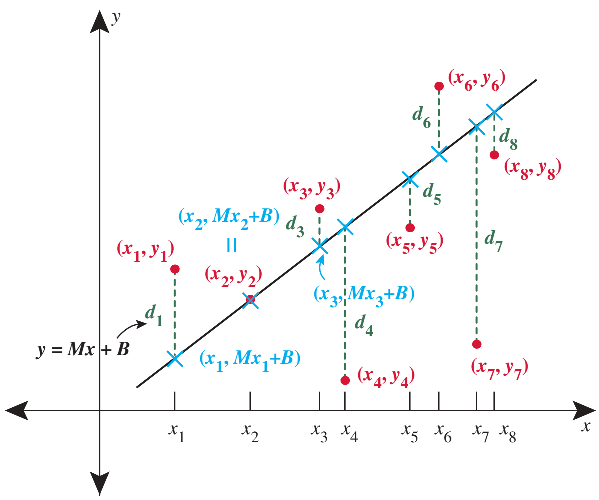
Assuming that a data set follows a reasonably strong linear pattern, it is natural to ask which *single straight line* best describes this pattern. Having such a line would enable us to not only describe the relationship between the two variables x and y precisely, but it would also enable us to predict values of y from values of x not present among the points of the data set.

Consider the paired data set (x, y) from Example 1, where $x =$ *points on the 400 m* and $y =$ *total score*. You learned in Section 0.6 that between any two points there is a unique line whose equation can be determined. Three such lines passing through various pairs of points in the data set are illustrated below.



The unavoidable shortcoming of all of these lines, however, is that not all of the data points lie on a single one of them. Each has a negative slope, which *is* characteristic of the data set, and each of the lines is close to some of the data points, but not close to others. In fact, we could draw infinitely many such lines and make a similar assessment. But which one *best fits* the data?

The answer to this question depends on how you define "best." Reasonably, for the line that *best fits* the data, the error incurred in using it to describe *all* of the points in the data set should be as small as possible. The conventional approach is to define this error by summing the n distances d_i between the y-coordinates of the data points and the corresponding y-value on the line $y = Mx + B$ (that is, the y-value of the point on the line corresponding to the same x-value). These distances are, in effect, the error in making the approximation. This is illustrated below:



Using the distance formula, we find that

$$d_i = \sqrt{(x_i - x_i)^2 + (y_i - (Mx_i + B))^2} = |y_i - (Mx_i + B)|.$$

Note that $d_i = 0$ precisely when the data point (x_i, y_i) lies directly on the line $y = Mx + B$, and that the closer d_i is to 0, the closer the point (x_i, y_i) is to the line $y = Mx + B$. As such, the goal is to determine the values of the slope M and y-intercept B for which the sum $d_1 + \cdots + d_n$ is as small as possible. Then, the resulting straight line $y = Mx + B$ best fits the data set $\{(x_1, y_1), \ldots, (x_n, y_n)\}$.

This is fine, in theory, but it turns out to be inconvenient to work with a sum of absolute value expressions. It is actually much more convenient to work with the *squared distances* d_i^2. The values of M and B that minimize $d_1 + \cdots + d_n$ are precisely the same as those that minimize $d_1^2 + \cdots + d_n^2$. Using calculus, it can be shown that the formulas for M and B are as follows:

$$M = \frac{n\sum xy - \left(\sum x\right)\left(\sum y\right)}{n\sum x^2 - \left(\sum x\right)^2}, \qquad B = \frac{\sum y}{n} - M\frac{\sum x}{n}$$

The resulting line $y = Mx + B$ is called the **best fit least-squares regression line** for the data set $\{(x_1, y_1), \ldots, (x_n, y_n)\}$.

Again, it is tedious to compute these by hand, but their values are actually produced easily using technology.

EXAMPLE 4 Finding the Line of Best Fit by Linear Regression

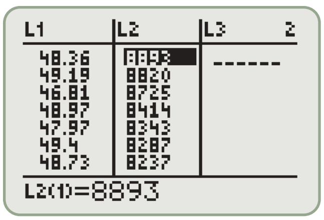
Find the line of best fit (best fit least-squares regression line) for the paired data set (x, y) from Example 1, where x = *points on the 400 m* and y = *total score,* using (a) the TI-83+ (or TI-84) and (b) *Excel* 2007.

Solution (a):

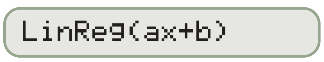
Determining the Best Fit Least-Squares Regression Line Using the TI-83+ (or TI-84).

INSTRUCTION | SCREENSHOT

STEP 1 **Enter the data** following the procedure outlined earlier in this section. The screen should look like the one to the right.



STEP 2 Press **STAT**, followed by **CALC**, and then by **4:LinReg(ax+b)**. The resulting screen should look like the one to the right.



INSTRUCTION	SCREENSHOT

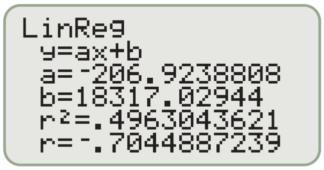
STEP 3 For this example, the data is stored in lists L1 and L2. And, since we will want to graph our best fit line on the scatterplot, it will need to be stored as a function of x, say as Y1. In order to do this, proceed as follows:

Directly next to **LinReg(ax+b)** on the home screen, we need to type the following: **L1, L2, Y1**.

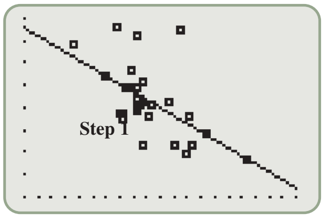
Use the following key strokes: **2nd**, **1**, $\boxed{,}$ **2nd**, **2**, $\boxed{,}$ **VARS**, **Y-VARS**, **1:FUNCTION**, **Y1**
The resulting screen should look like the one to the right.

STEP 4 Press **ENTER**. The equation of the best fit least-squares line with the slope (labeled as a) and the y-intercept (labeled as b) appears on the screen as shown to the right.

So, the equation of the best fit least-squares regression line is approximately
$\mathbf{y = -206.9x + 18317.03.}$



STEP 5 In order to obtain a graph of the scatterplot *with* the best fit line from Step 4 superimposed on it, press **ZOOM**, then **9:ZoomStat**. The resulting screen should look like the one to the right.

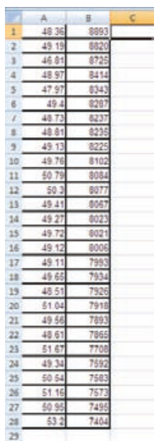


Solution (b):

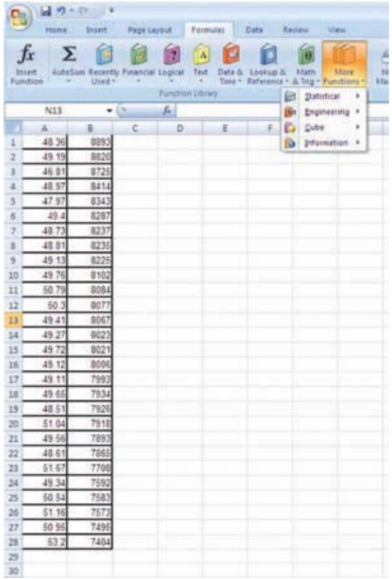
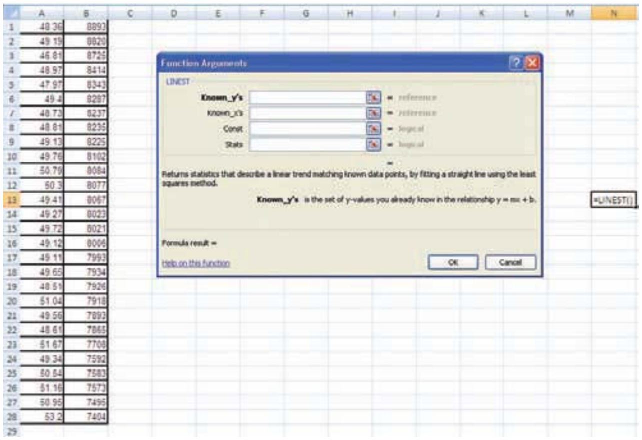
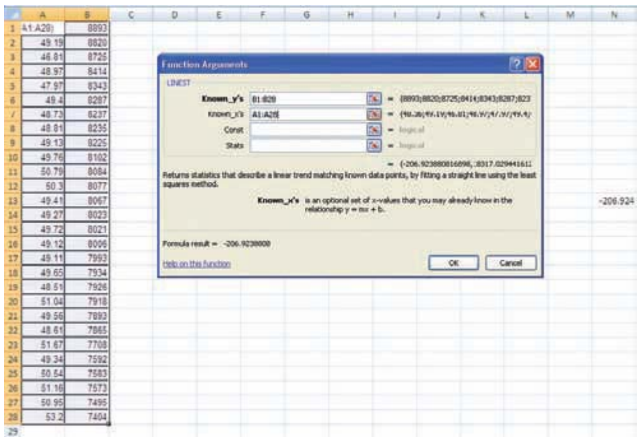
Determining the Best Fit Least-Squares Regression Line Using *Excel* 2007.

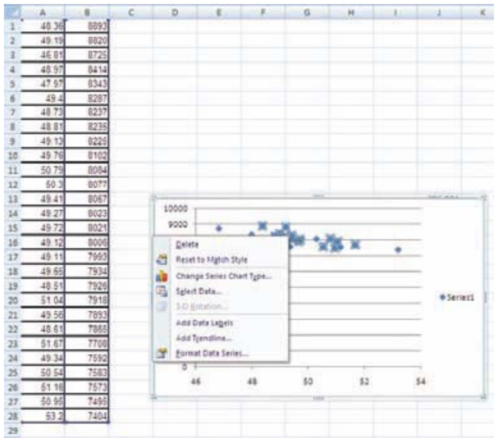
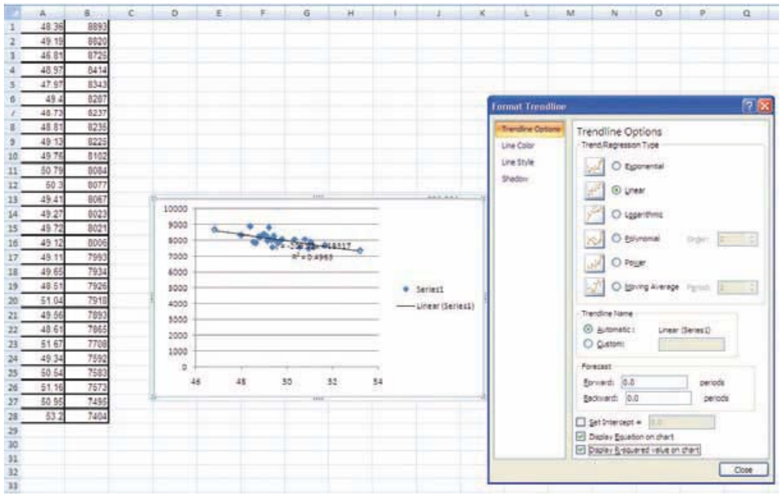
INSTRUCTION	SCREENSHOT

STEP 1 **Enter the data** following the procedure outlined earlier in this section. The screen should look like the one to the right.



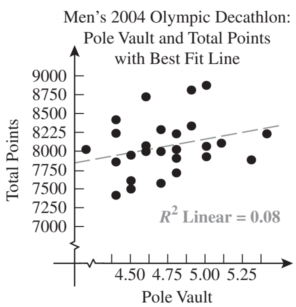
	A	B	C
1	48.36	8893	
2	49.19	8820	
3	46.81	8725	
4	48.97	8414	
5	47.97	8343	
6	49.4	8287	
7	48.73	8237	
8	48.81	8235	
9	49.13	8225	
10	49.76	8102	
11	50.79	8084	
12	50.3	8077	
13	49.41	8067	
14	49.27	8023	
15	49.72	8021	
16	49.12	8006	
17	49.11	7993	
18	49.65	7934	
19	48.51	7926	
20	51.04	7918	
21	49.56	7893	
22	48.61	7865	
23	51.67	7708	
24	49.34	7592	
25	50.54	7583	
26	51.16	7573	
27	50.95	7495	
28	53.2	7404	
29			

INSTRUCTION	SCREENSHOT
STEP 2 Select the **Formulas** tab at the top of the screen and then choose the **More Functions** within the *Function Library* grouping (on the left). The pull-down menu should be as shown to the right.	
STEP 3 From here, select **Statistical**, and then from this list, scroll down and choose **LINEST**. A pop-up window should appear, as shown to the right.	
STEP 4 Enter **B1:B28** in **Known_y's** and **A1:A28** in **Known_x's**, as shown to the right. You will notice that a set of two values occurs directly beneath the entry boxes—the output is about $\{-206.92, 18{,}317.03\}$. The first value is the slope M, and the second value is the y-intercept B of the best fit line. So, the equation of the best fit least-squares regression line is approximately $\boldsymbol{y = -206.92x + 18{,}317.03}$.	

INSTRUCTION	SCREENSHOT
STEP 5 In order to obtain a graph of the scatterplot *with* the best fit line from Step 4 superimposed on it, construct the scatterplot as before, right-click on the scatterplot near the data points, choose **Add Trendline**, and press **ENTER**.	
STEP 6 The best fit line will appear on the scatterplot, along with a pop-up window allowing you to change the format of the line/curve that is displayed.	

■ **YOUR TURN** Using the data in Example 1, identify x = *score on the pole vault* and z = *total score*, and determine the best fit least-squares regression line for the data set consisting of the ordered pairs (x, z). Superimpose the graph of this line on the scatterplot.

■ **Answer:**



Here, the best fit line displays a positive relationship, and its equation is $z = 364.97x + 6324.46$.

It is important to realize that the correlation coefficient is NOT equal, or even related to, the actual slope of the best fit line. In fact, two distinct perfectly linear, positively associated scatterplots will both have $r = 1$, even though the actual lines that fit the data might have slope $M = 15$ and $M = 0.04$.

Using the "Best Fit" Line for Prediction

It is important to realize that for any scatterplot, no matter how haphazardly dispersed the data, a best fit least-squares regression line can be created. This is true even when the relationship between x and y is *non*linear. However, the utility of such a line in these instances is very limited. In fact, a best fit line should only be created when the linear relationship is reasonably strong, which means the correlation coefficient is "reasonably far away from 0." This criterion can be made more precise using statistical methods, but for our present purposes, we shall make the blanket assumption that it makes sense to form the best fit line in all of the scenarios we present.

Once we have the best fit line in hand, we can use it to predict y-values for values of x that do not correspond to any of the points in the data set. For instance, consider the following:

EXAMPLE 5 Making Predictions Using the Line of Best Fit

Consider the data set from Example 1.

a. Use the best fit line to predict the *total score* given that an Olympian scored 50.05 points in the *400 m*. Is it reasonable to use the best fit line to make such a prediction?

b. Use the best fit line to predict the *total score* given that an Olympian scored 40.05 points in the *400 m*. Is it reasonable to use the best fit line to make such a prediction?

Solution:

a. Using the line $y = -206.92x + 18{,}317.03$, we see that y is approximately 7961 when $x = 50.05$. This means that if an Olympian were to score 50.05 on the *400 m*, then his predicted *total score* would be approximately 7961. Using the best fit line to predict the total score in this case is reasonable because the value 50.05 is well within the range of x-values already present in the data set.

b. Using the line $y = -206.92x + 18{,}317.03$, we see that y is approximately 10,030 when $x = 40.05$. This means that if an Olympian were to score 40.05 on the *400 m*, then his predicted *total score* would be approximately 10,030. This prediction is questionable because the x-value at which you are using the best fit line to predict y is sufficiently far away from the rest of the data points that were used to construct the line. As such, there is no reason to believe that the line is valid for such x-values.

■ **YOUR TURN** Using the data in Example 1, identify x = *score on the pole vault* and z = *total score*, and use the best fit line to predict the *total score* given that an Olympian scored 4.65 points on the *pole vault* and then, given that an Olympian scored 5.9 points on the *pole vault*. Comment on the validity of these predictions.

■ **Answer:**

Using the line $z = 364.97x + 6324.46$, we see that z is approximately 8022 when $x = 4.65$. This means that if an Olympian were to score 4.65 on the *pole vault*, then his predicted *total score* would be approximately 8022. Using the best fit line to predict the total score in this case is reasonable because the value 4.65 is well within the range of x-values already present in the data set.

Next, using the same line, we see that z is approximately 8478 when $x = 5.9$. This prediction is less reliable than the former one because 5.9 is outside of the range of x-values corresponding to the rest of the data points used to construct the line. But it isn't too far outside this range, so there is a degree of validity to the prediction.

EXAMPLE 6 The Power of TV Advertisement

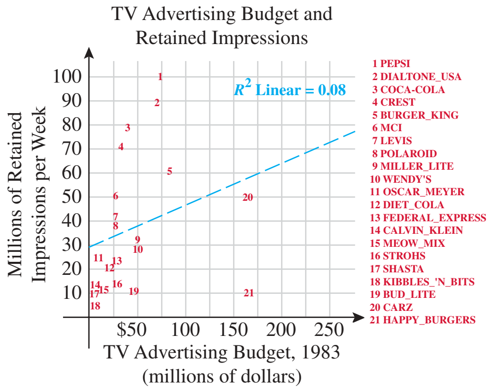
Video Board Tests, Inc., an advertising testing agency, collected data based on 4000 adult participants of a survey. The participants (who were regular product users) were asked to recall a commercial that they had viewed for a given product category in the previous week. The goal was to examine the relationship between retained impressions of commercials and the corresponding TV advertising budget for a given product. The data were published in the *Wall Street Journal* in March 1984.

The following is an adaptation of the original data set (**TV Ad Yields was obtained from the following Web source: http://lib.stat.cmu.edu/DASL/Datafiles/tvadsdat.html**), but contains three modifications made for illustrative purposes. Specifically, ATT/BELL, FORD, and MCDONALD'S have been replaced with DIALTONE USA, CARZ, and HAPPY BURGERS, respectively. These changes have been highlighted in green.

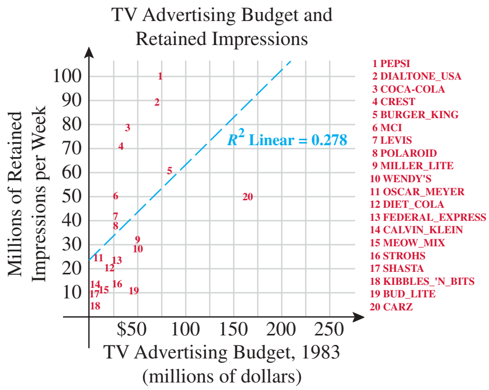
COMPANY	TV ADVERTISING BUDGET, 1983 ($ MILLIONS)	MILLIONS RETAINED IMPRESSIONS PER WEEK
MILLER_LITE	50.1	32.1
PEPSI	74.1	99.6
STROH'S	19.3	11.7
FEDERAL_EXPRESS	22.9	21.9
BURGER_KING	82.4	60.8
COCA-COLA	40.1	78.6
HAPPY_BURGERS	**165.0**	**10.0**
MCI	26.9	50.7
DIET_COLA	20.4	21.4
CARZ	**165.0**	**50.0**
LEVI'S	27.0	40.8
BUD_LITE	45.6	10.4
DIALTONE_USA	**70.0**	**88.9**
CALVIN_KLEIN	5.0	12.0
WENDY'S	49.7	29.2
POLAROID	26.9	38.0
SHASTA	5.7	10.0
MEOW_MIX	7.6	12.3
OSCAR_MEYER	9.2	23.4
CREST	32.4	71.1
KIBBLES_'N_BITS	6.1	4.4

First, a scatterplot for this data set (formed using PASW Statistics 18) is shown below. The approximate regression line is $y = 0.18x + 29.04$ and $r = 0.28$.

It appears that there *might* be a relationship between the budget and the retained impressions. Both CARZ and HAPPY_BURGERS are pretty far away from the bulk of the data and might be skewing an otherwise tighter relationship between x and y; such points are potential *outliers*. What would happen to the regression line if we removed each of these points, one at a time? Would the new line be dramatically different from the original one, or might there be very little change?



Let's start by removing CARZ. The resulting regression line is $y = 0.21x + 27.96$. We observe only a small change in both the slope and intercept. Next, let's put CARZ back into the data set and remove HAPPY_BURGERS instead. In this case, the best fit regression line is $y = 0.40x + 22.61$. This time, we have a slightly larger change in the y-intercept, but more importantly, the slope is more than double the slope of the regression line from the original data set. The resulting best fit regression line is displayed to the right.



Clearly, HAPPY_BURGERS was a very influential data point since removing it dramatically changed the slope between the budget and retained impressions, thereby considerably changing the mathematical description of the relationship between these two variables. But why did this happen?

There was a large distance between it and both the x- and y-directions from the bulk of the data set. When a potential outlier is distant in only one of the two directions (x or y), as was the case with CARZ, it is far less likely to be influential.

SECTION 0.8* SUMMARY

Two variables x and y can be related in different ways. A paired data set $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ obtained experimentally can be illustrated using a *scatterplot*. Patterns concerning the direction of association and linearity can be used to describe the relationship between x and y, and the strength of the linear relationship can be measured using the correlation coefficient r. If r is sufficiently far from 0, a *best fit least-squares regression line* can be formed to precisely describe the linear relationship and used for reasonable prediction purposes.

SECTION 0.8* EXERCISES

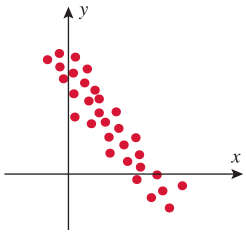
SKILLS

In Exercises 1–4, for each of the following scatterplots, identify the pattern as

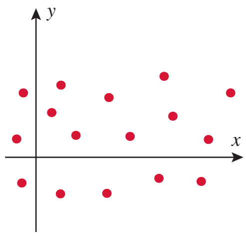
a. having a positive association, negative association, or no identifiable association.

b. being linear or nonlinear.

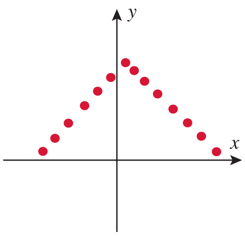
1.



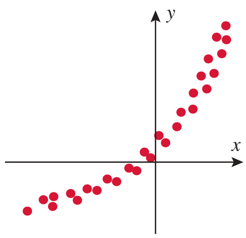
2.



3.



4.



In Exercises 5–8, match the following scatterplots with the following correlation coefficients.

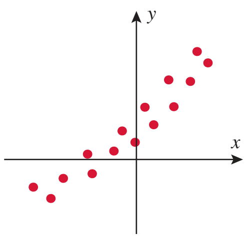
a. $r = -0.90$

b. $r = 0.80$

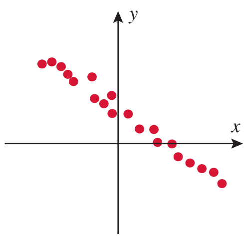
c. $r = -0.68$

d. $r = 0.20$

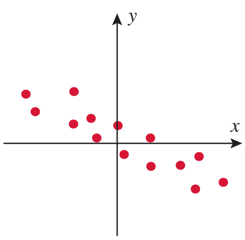
5.



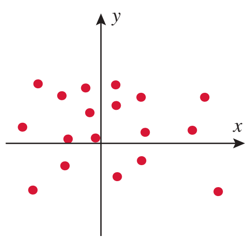
6.



7.



8.



For each of the following data sets,

a. create a scatterplot.

b. guess the value of the correlation coefficient r.

c. use technology to determine the equation of the best fit line and to calculate r.

d. give a verbal description of the relationship between x and y.

9.

x	y
−3	14
−1	8
0	5
1	2
3	−4
5	−10

10.

x	y
−8	−16
−6	−12
−4	−8
−2	−4
1	2
3	6

11.

x	y
−10	1
−6	0
0	−2
8	−10
14	−11
20	−16

12.

x	y
−1	−17
−1/2	−11
−1/4	−5
−1/10	0
0	1
1/10	1
1/5	8
1	12

13.

x	y
−3	−6
−2	3
−1	1
0	1
1	5
2	−1
4	1

14.

x	y
−6	−1
−3	4
−1	3
1	0
2	−6
5	−4
8	1

In Exercises 15–18, for each of the data sets,

a. use technology to create a scatterplot, to determine the best fit line, and to compute r.

b. indicate whether or not the best fit line can be used for predictive purposes for the following x-values. For those for which it can be used, give the predicted value of y:

i. $x = 0$ **iii.** $x = 12$

ii. $x = -6$ **iv.** $x = -15$

c. Using the best fit line, at what x-value would you expect y to be equal to 2?

15.

x	y
−5	−8
−3	0
−2	0
2	1.5
5	4
7	2
10	8

16.

x	y
5	0
5	3
6	3
7	6
7	9
8	9
8	15
9	9
9	15
10	15
10	18

17.

x	y
−20	15
−18	10
−14	3
−14	8
−13	3
−8	0
−8	−3
−5	−6
1	−11
1	−15

18.

x	y
−15	4
−15	12
−15	16
−13	3
−10	4
−10	8
−10	12
−5	4
−2	3
−2	−2
2	3
2	6
4	−1
4	0
4	4
7	−2
7	3
10	−4
10	−2
10	3

For Exercises 19–22,

a. use technology to create a scatterplot, to determine the best fit line, and to compute r for the *entire* data set.

b. repeat (a), but with the data set obtained by removing the *starred* (***) data points.

c. compare the r-values from (a) and (b), as well as the slopes of the best fit lines. Comment on any differences, whether they are substantive, and why this seems reasonable.

19.

x	y
−3	14
−1	8
0	5
1	2
3	−4
*** 5	−10

20.

x	y
−10	1
−6	0
0	−2
8	−10
14	−11
*** 20	−16

21.

x	y
−3	14
−1	8
0	5
1	2
*** 3	−4
5	−15
*** 6	−16

22.

x	y
0	0
1	2
2	4
3	6
4	8
6	12
*** 7	25

CONCEPTUAL

23. Consider the data set from Exercise 17.

a. Reverse the roles of x and y so that now y is the *explanatory* variable and x is the *response* variable. Create a scatterplot for the ordered pairs of the form (y, x) using this data set.

b. Compute r. How does it compare to the r-value from Exercise 17? Why does this make sense?

c. The best fit line for the scatterplot in (a) will be of the form $x = my + b$. Determine this line.

d. Using the line from (c), find the predicted x-value for the following y-values, if appropriate. If it is not appropriate, tell why.

i. $y = 23$ **ii.** $y = 2$ **iii.** $y = -16$

24. Consider the data set from Exercise 16. Redo the parts in Exercise 23.

25. Consider the following data set.

x	y
3	0
3	1
3	−1
3	−2
3	4
3	15
3	−6
3	8
3	10

Guess the values of r and the best fit line. Then, check your answers using technology. What happens? Can you reason why this is the case?

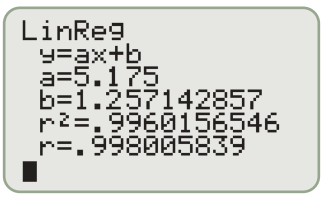
26. Consider the following data set.

x	y
−5	−2
−4	−2
−1	−2
0	−2
1	−2
3	−2
8	−2
17	−2

Guess the values of r and the best fit line. Then, check your answers using technology. What happens? Can you reason why this is the case?

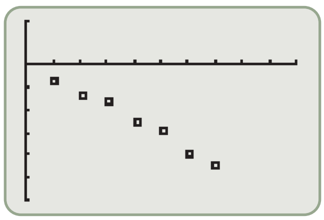
CATCH THE MISTAKE

27. The following screenshot was taken when using the TI-83+ to determine the equation of the best fit line for paired data (x, y):



Using the regression line, we observe that there is a strong positive linear association between x and y, and that for every unit increase in x, the y-value increases by about 1.257 units.

28. The following scatterplot was produced using the TI-83+ for paired data (x, y).



The equation of the best fit line was reported to be $y = -3.207x + 0.971$ with $r^2 = 0.9827$. Thus, the correlation coefficient is given by $r = 0.9913$, which indicates a strong linear association between x and y.

APPLICATIONS

For Exercises 29 and 30, refer to the data set in Example 1.

29. **a.** Examine the relationship between each of the decathlon events and the total points by computing the correlation coefficient in each case.

b. Using the information from part (a), which event has the strongest relationship to the total points?

c. What is the equation of the best fit line that describes the relationship between the event from part (b) and the total points?

d. Using the best fit line, if you had a score of 40 for this event, what would the predicted total points score be?

30. **a.** Using the information from part (a), which event has the second strongest relationship to the total points?

b. What is the equation of the best fit line that describes the relationship between the event in part (b) and the total points?

c. Is it reasonable to expect the best fit line from part (c) to produce accurate predictions of total points using this event?

d. Using the best fit line, if you had a score of 40 for this event, what would the total points score be?

For Exercises 31 and 32, refer to the following scenario: *Texting Speed.*

According to the CTIA—The Wireless Association, as of December 2010, 187.7 billion messages were sent per month or 2.1 trillion messages in that year.[1] According to a 2010 Pew Internet survey, 72% of all teens—or 88% of teen cell phone users—are text-messagers. Teens make and receive far fewer phone calls than text messages on their cell phones.

A number of competitions regarding texting speed have taken place worldwide. According to the Guinness World Records, "The fastest completion of a prescribed 160-character text message is 34.65 seconds and was achieved by Frode Ness (Norway) at the Norwegian SMS championships held at the Oslo City shopping centre in Oslo, Østlandet, Norway, on 13 November 2010."[2]

The data set regarding texting speed on the next page was provided by AP Central. **(http://apcentral.collegeboard.com/apc/public/courses/teachers_corner/195435.html)**

In the data given, the ***A total score*** is the amount of time (in seconds) it took to text the following message, "Statistics students are above average." The ***B total score*** is the amount of time (in seconds) to type, "Meet me at my car after school today." The ***Total both scores*** is the sum of the ***A total score*** and ***B total score***.

What influences texting speed in this group? Let's consider thumb length.

[1] http://www.cita.org/advocacy/research/index.cfm/aid/10323

[2] http://www.guinnessworldrecords.com/Search/Details/Fastest-text-message/57979.htm

31. What is the relationship between the variables ***left thumb length*** and ***total both scores***?

a. Create a scatterplot to show the relationship between ***left thumb length*** and ***total both scores***.

b. What is the correlation coefficient between ***left thumb length*** and ***total both scores***?

c. Describe the strength of the relationship between ***left thumb length*** and ***total both scores***.

d. What is the equation of the best fit line that describes the relationship between ***left thumb length*** and ***total both scores***?

e. Could you use the best fit line to produce accurate predictions of ***total both scores*** using ***left thumb length***?

32. Repeat Exercise 31 for ***right thumb length*** and ***total both scores***.

Gender	Texting Style	Left Thumb Length	Right Thumb Length	A Total Score	B Total Score	Total Both Scores	A Minus B	Avg Thumb	Diff Thumb
Male	Char	6.5	6.5	35	25	60	10	6.5	0
Female	Char	5	5	61	57	118	4	5	0
Female	Word	6	6	24	20	44	4	6	0
Male	Word	7	7	43	60	103	−17	7	0
Female	Word	6	6	14	19	33	−5	6	0
Male	Word	7	6	15	18	33	−3	6.5	−1
Female	Word	6	6	13	14	27	−1	6	0
Female	Word	6	6	22	10	22	12	6	0
Male	Word	6.5	6	13	15	28	−2	6.25	−0.5
Female	Word	5.5	5.5	16	16	32	0	5.5	0
Male	Char	6	5	85	78	163	7	5.5	−1
Male	Char	6	6	126	120	246	6	6	0
Male	Word	7.5	6.5	67	69	136	−2	7	−1
Female	Char	5.5	5.5	11	7	18	4	5.5	0
Female	Word	5.5	5.7	14	17	31	−3	5.6	0.2
Female	Word	6	6	17	14	31	3	6	0
Female	Word	5	5	20	15	35	5	5	0
Male	Word	6.5	6.5	15	13	28	2	6.5	0
Male	Word	7	7	30	31	61	−1	7	0
Male	Word	6	6.1	120	117	237	3	6.05	0.1
Male	Word	6	6	74	25	99	49	6	0
Male	Word	6.3	6	23	21	44	2	6.15	−0.3
Male	Char	6	5.9	45	50	95	−5	5.95	−0.1
Male	Word	6	6.1	86	100	186	−14	6.05	0.1
Male	Char	6	6	25	23	48	2	6	0
Male	Char	6.3	6.3	81	57	138	24	6.3	0
Female	Char	5.5	5.5	88	66	154	22	5.5	0
Female	Word	7	6.8	10	9	19	1	6.9	−0.2
Male	Char	6.5	7	21	18	39	3	6.75	0.5
Female	Char	5.4	5.2	72	48	121	24	5.3	−0.2
Female	Char	8	8	36	23	59	13	8	0
Male	Char	7	6.5	46	45	91	1	6.75	−0.5
Female	Char	6	6.8	48	39	87	9	6.4	0.8
Female	Char	7.1	7.1	84	57	141	27	7.1	0
Female	Word	5.9	5.5	25	23	48	2	5.7	−0.4
Female	Char	7.6	7.2	32	45	77	−13	7.4	−0.4
Male	Word	6.9	7	23	28	51	−5	6.95	0.1
Female	Char	7.7	7.5	18	15	33	3	7.6	−0.2
Male	Char	8.6	(cast)	22	20	42	2	N/A	N/A
Male	Char	7.3	7.1	54	50	104	4	7.2	−0.2

For Exercises 33 and 34, refer to the following data set: *Herd Immunity.*

According to the U.S. Department of Health and Human Services, herd immunity is defined as "a concept of protecting a community against certain diseases by having a high percentage of the community's population immunized. Even if a few members of the community are unable to be immunized, the entire community will be indirectly protected because the disease has little opportunity for an outbreak. However, with a low percentage of population immunity, the disease would have great opportunity for an outbreak."[3]

Suppose a study is conducted in the year 2016 looking at the outbreak of *Haemophilus influenzae type b* in the winter of 2015 across 22 nursing homes. We might look at the percentage of residents in each of the nursing homes that were immunized and the percentage of residents who were infected with this type of influenza.

The fictional data set is as follows.

Nursing Home	% Residents Immunized	% Residents with Influenza
1	70	11
2	68	9
3	80	8
4	10	34
5	12	30
6	18	31
7	27	22
8	64	18
9	73	6
10	9	31
11	35	19
12	56	16
13	57	22
14	83	10
15	74	13
16	64	15
17	16	28
18	23	25
19	29	24
20	33	20
21	82	28
22	67	9

33. What is the relationship between the variables ***% residents immunized*** and ***% residents with influenza***?

a. Create a scatterplot to illustrate the relationship between ***% residents immunized*** and ***% residents with influenza***.

b. What is the correlation coefficient between ***% residents immunized*** and ***% residents with influenza***?

c. Describe the strength of the relationship between ***% residents immunized*** and ***% residents with influenza***.

d. What is the equation of the best fit line that describes the relationship between ***% residents immunized*** and ***% residents with influenza***?

e. Could you use the best fit line to produce accurate predictions of ***% residents with influenza*** using ***% residents immunized***?

34. What is the impact of the outlier(s) on this data set?

a. Identify the outlier in this data set. What is the nursing home number for this outlier?

b. Remove the outlier and re-create the scatterplot to show the relationship between ***% residents immunized*** and ***% residents with influenza***.

c. What is the revised correlation coefficient between ***% residents immunized*** and ***% residents with influenza***?

d. By removing the outlier is the strength of the relationship between ***% residents immunized*** and ***% residents with influenza*** increased or decreased?

e. What is the revised equation of the best fit line that describes the relationship between ***% residents immunized*** and ***% residents with influenza***?

For Exercises 35–38, refer to the following data set: *Amusement Park Rides*.

According to the International Association of Amusement Parks and Attractions (IAAPA), "There are more than 400 amusement parks and traditional attractions in the United States alone. In 2008, amusement parks in the United States entertained 300 million visitors who safely enjoyed more than 1.7 billion rides."[4] Despite the popularity of amusement parks, the wait times, especially for the most popular rides, are not so highly regarded. There are different approaches and tactics that people take to get the most rides in their visit to the park. Now, there are even apps for the iPhone and Android to track waiting times at various amusement parks.

One might ask, "Are the wait times worth it? Are the rides with the longest wait times, the most enjoyable?"

Consider the following fictional data.

[3] http://www.hhs.gov/nvpo/glossary1.htm

[4] http://www.iaapa.org/pressroom/Amusement_Park_Industry_Statistics.asp

Ride ID	Ride Name	Avg Wait Time	Avg Enjoyment Rating	Park	Park Location
1	Xoom	45	58	1	Florida
2	Accentuator	35	40	1	Florida
3	Wobbler	15	15	1	Florida
4	Arctic_Attack	75	75	1	Florida
5	Gusher	60	70	1	Florida
6	Alley_Cats	5	60	1	Florida
7	Moon_Swing	10	15	1	Florida
8	Speedster	70	50	1	Florida
9	Hailstorm	80	90	1	Florida
10	DragonFire	70	88	1	Florida
1	Xoom	50	10	2	California
2	Accentuator	35	40	2	California
3	Wobbler	20	75	2	California
4	Arctic_Attack	70	60	2	California
5	Gusher	70	80	2	California
6	Alley_Cats	10	18	2	California
7	Moon_Swing	15	80	2	California
8	Speedster	80	35	2	California
9	Hailstorm	95	40	2	California
10	DragonFire	55	60	2	California

The data shows 10 popular rides in two sister parks located in Florida and California. For each ride in each park, average wait times (in minutes) in the summer of 2010 and the average rating of ride enjoyment (on a scale of 1–100) are provided.

35. What is the relationship between the variables ***average wait times*** and ***average rating of enjoyment***?

a. Create a scatterplot to show the relationship between ***average wait times*** and ***average rating of enjoyment***.

b. What is the correlation coefficient between ***average wait times*** and ***average rating of enjoyment***?

c. Describe the strength of the relationship between ***average wait times*** and ***average rating of enjoyment***.

d. What is the equation of the best fit line that describes the relationship between ***average wait times*** and ***average rating of enjoyment***?

e. Could you use the best fit line to produce accurate predictions of ***average wait times*** using ***average rating of enjoyment***?

36. Examine the relationship between ***average wait times*** and ***average rating of enjoyment*** for ***Park 1*** in ***Florida*** by repeating Exercise 35 for only ***Park 1***.

37. Examine the relationship between ***average wait times*** and ***average rating of enjoyment*** for ***Park 2*** in ***California*** repeating Exercise 35 for only ***Park 2***.

38. Compare the relationship between ***average wait times*** and ***average rating of enjoyment*** for ***Park 1 in Florida*** versus ***Park 2 in California***.

CHALLENGE

For Exercises 39–42, refer to the following:

Exploring other types of best-fit curves

When describing the patterns that emerge in paired data sets, there are many more possibilities other than best fit *lines*. Indeed, once you have drawn a scatterplot and are ready to identify the curve that *best fits* the data, there is a substantive collection of other curves that might more accurately describe the data. The following are listed among those in **STATS/CALC** on the TI-83+, along with some comments:

Name of Regression Curve	Form of the Curve	Comments
5: QuadReg	$y = ax^2 + bx + c$	The data set must have at least 3 points to be able to select this option.
6: CubicReg	$y = ax^3 + bx^2 + cx + d$	The data set must have at least 4 points to be able to select this option.
7: QuartReg	$y = ax^4 + bx^3 + cx^2 + dx + e$	The data set must have at least 5 points to be able to select this option.
9: LnReg	$y = a + b \ln x$	The data set must have at least 2 points to be able to select this option, and x cannot take on negative values.
0: ExpReg	$y = a * b^x$	The data set must have at least 2 points to be able to select this option, and y cannot take on the value of 0.
A: PwrReg	$y = a * x^b$	The data set must have at least 2 points to be able to select this option.

For each of the following data sets,

a. Create a scatterplot.

b. Use **LinReg(ax+b)** to determine the best fit *line* and r. Does the line seem to accurately describe the pattern in the data?

c. For each of the different choices listed in the above chart, find the equation of the best fit curve and its associated r^2 value. Of all of the curves, which seems to provide the best fit?

Note: The r^2-value reported in each case is NOT the *linear* correlation coefficient reported when running LinReg(ax+B). Rather, the value will typically change depending on the curve. The reason why is that each time, the r^2-value is measuring how accurate the fit is between the data and *that type of curve*. A value of r^2 close to 1 still corresponds to a good fit with whichever curve you are fitting to the data.

39.

x	y
1	16.2
2	21
3	23.7
4	24.8
5	23.9
6	20.7
7	15.8
8	9.1
9	0.3

40.

x	y
0.5	1.20
1.0	0.760
1.5	0.412
2.1	0.196
2.9	0.131
3.3	0.071

41.

x	y
1	0.2
1.5	0.93
2	1.46
3	2.25
10	4.51
15	5.50

42.

x	y
1	32.3
2	8.12
3	−16.89
5	−45.2
6	0.89
8	62.1

CHAPTER 0 INQUIRY-BASED LEARNING PROJECT (1)

Equivalent Equations and Extraneous Solutions

A general strategy for solving all the various types of equations you encountered in this chapter can be summarized as follows: From a given equation, perform algebraic operations on both sides in order to generate equivalent equations. Remember, *equivalent equations* have the same solution set.

1. Consider first a linear equation: $3x - 1 = 5$.

a. Use a graphing utility to show $y_1 = 3x - 1$ and $y_2 = 5$ and determine the point of intersection. Make a sketch and label it.

b. How does the graph in part (a) relate to the solution set of the equation $3x - 1 = 5$?

c. To solve the equation $3x - 1 = 5$ algebraically, the first step is to add 1 to both sides of the equation, as follows:

$$\begin{array}{rcr} 3x - 1 & = & 5 \\ +1 & & +1 \\ \hline 3x \quad & = & 6 \end{array}$$

Use a graphing utility to show $y_1 = 3x$ and $y_2 = 6$, and determine the point of intersection. Make a sketch and label it. How does this graph relate to the equation $3x = 6$?

d. The final algebraic step to solve the equation is to divide both sides of the equation by 3.

$$\frac{3x}{3} = \frac{6}{3}$$
$$x = 2$$

Use a graphing utility to show $y_1 = x$ and $y_2 = 2$, and determine the point of intersection. Make a sketch and label it. How does this graph relate to the equation $x = 2$?

e. The algebraic steps to solve the equation $3x - 1 = 5$ produce two equations: $3x = 6$ and $x = 2$. How do the graphs you sketched above represent the fact that these equations are equivalent to the original?

2. Next consider the equation $x - 2 = \sqrt{4 - x}$

a. Use a graphing utility to show $y_1 = x - 2$ and $y_2 = \sqrt{4 - x}$ and determine any points of intersection. What do you learn about the solution set of the equation $x - 2 = \sqrt{4 - x}$? Make a sketch to explain.

b. The algebraic steps to solve this equation are as follows:

$x - 2 = \sqrt{4 - x}$	
$(x - 2)^2 = (\sqrt{4 - x})^2$	Square both sides.
$x^2 - 4x + 4 = 4 - x$	Simplify.
$x^2 - 3x = 0$	Write the quadratic equation in standard form.
$x(x - 3) = 0$	Factor.
$x = 0 \quad \text{or} \quad x = 3$	Use the zero product property.

Use a graphing utility to show the first step above: $y_1 = (x - 2)^2$ and $y_2 = (\sqrt{4 - x})^2$. What do you learn about the solution set of $(x - 2)^2 = (\sqrt{4 - x})^2$?

c. Discuss whether $x - 2 = \sqrt{4 - x}$ and $(x - 2)^2 = (\sqrt{4 - x})^2$ are equivalent equations.

d. The algebraic process of squaring both sides introduced an *extraneous solution*. What do you think that means?

e. Why is it important to always check the solutions you obtain when solving equations?

3. Consider $x^4 - x^2 = 0$

a. A fellow student suggests dividing both sides of the equation by x^2. What will be the resulting equation?

b. Is the equation you wrote in part (a) equivalent to the original equation? How can you use a graphing utility to illustrate this?

c. Show the algebraic steps you should take to solve $x^4 - x^2 = 0$.

CHAPTER 0 INQUIRY-BASED LEARNING PROJECT (2)

The following table shows U.S. population estimates for the years 1991 to 2002, along with the number of tons of municipal waste generated (in 100 million tons) and the percentage that was recycled in the United States during those years.

Year	U.S. Population Estimate (100 million people)	Municipal Waste Generated (100 million tons)	Percentage Recycled
1991	2.46	2.69	8%
1992	2.49	2.93	11.5%
1993	2.52	2.80	14%
1994	2.55	2.91	17%
1995	2.58	3.06	19%
1996	2.60	3.22	23%
1997	2.63	3.26	27%
1998	2.65	3.27	28%
1999	2.68	3.40	30%
2000	2.73	3.74	31.5%
2001	2.80	3.82	33%
2002	2.86	4.09	32%

Adapted from: A Yearly Snapshot of U.S. (Municipal) Waste and Recycling (Data Source: BIOCYCLE/Table & Conversion: ZWA http://www.zerowasteamerica.org/statistics.htm)

1. For parts (a)–(f), consider the columns for U.S. Population Estimate, x, and Municipal Waste Generated, y.

a. Write the equation (in slope–intercept form) of the line that passes through the points (2.60, 3.22) and (2.80, 3.82).

b. How can you interpret the slope of the line in part (a)?

c. Now choose two other data points and write the equation of the line that passes through your chosen points. How can you interpret the slope of this line?

d. How does the slope of this line compare to the line in part (a)?

e. Sketch a graph of the two lines. What do you notice about their y-intercepts?

f. Finally, plot all the other ordered points (Population, Waste Generated).

The graph of all the data points is called a **scatterplot.** Since the data fall in approximately a straight line, each of the lines you graphed above is an approximation of the data. Section 0.8 presented methods for finding the line that *best* fits a set of data points, called the least-squares regression line.

2. For parts (a)–(c) below, consider the columns U.S. Population Estimate, x, and Percentage Recycled, y.

a. Graph the scatterplot for this data.

b. Sketch the graph of a line that contains two data points. Choose a line you think fits the data well.

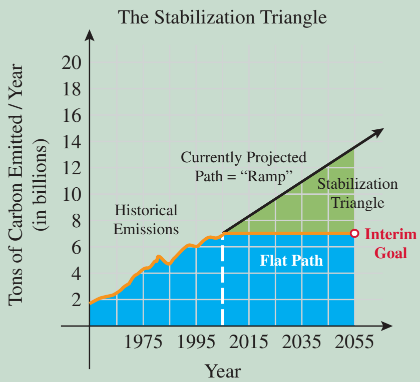
c. Write the equation of the line.

d. How can you interpret the slope of this line?

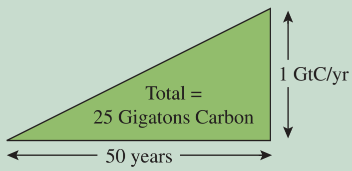
MODELING OUR WORLD

The Intergovernmental Panel on Climate Change (IPCC) claims that carbon dioxide (CO_2) production from industrial activity (such as fossil fuel burning and other human activities) has increased the CO_2 concentrations in the atmosphere. Because it is a greenhouse gas, elevated CO_2 levels will increase global mean (average) temperature. In this section, we will examine the increasing rate of carbon emissions on Earth.

In 1955 there were (globally) 2 billion tons of carbon emitted per year. In 2005 the carbon emissions more than tripled to reach approximately 7 billion tons of carbon emitted per year. Currently, we are on the path to doubling our current carbon emissions in the next 50 years.



Two Princeton professors* (Stephen Pacala and Rob Socolow) introduced the Climate Carbon Wedge concept. A "wedge" is a strategy to reduce carbon emissions over a 50-year time period from zero to 1.0 GtC/yr (gigatons of carbon per year).



1. Draw the Cartesian plane. Label the vertical axis C, where C represents the number of gigatons (billions of tons) of carbon emitted, and label the horizontal axis t, where t is the number of years. Let $t = 0$ correspond to 2005.

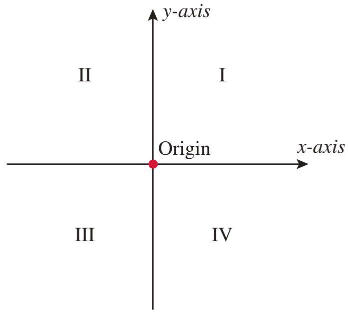
*S. Pacala and R. Socolow, "Stabilization Wedges: Solving the Climate Problem for the Next 50 Years with Current Technologies," *Science*, Vol. 305 (2004).

2. Find the equations of the flat path and the seven lines corresponding to the seven wedges.
   - **a.** Flat path (no increase) over 50 years (2005 to 2055)
   - **b.** Increase of 1 GtC over 50 years (2005 to 2055)
   - **c.** Increase of 2 GtC over 50 years (2005 to 2055)
   - **d.** Increase of 3 GtC over 50 years (2005 to 2055)
   - **e.** Increase of 4 GtC over 50 years (2005 to 2055)
   - **f.** Increase of 5 GtC over 50 years (2005 to 2055)
   - **g.** Increase of 6 GtC over 50 years (2005 to 2055)
   - **h.** Increase of 7 GtC over 50 years (2005 to 2055) [projected path]

3. For each of the seven wedges and the flat path, determine how many **total** gigatons of carbon will be reduced over a 50-year period. In other words, how many gigatons of carbon would the world have to reduce in each of the eight cases?
   - **a.** Flat path
   - **b.** Increase of 1 GtC over 50 years
   - **c.** Increase of 2 GtC over 50 years
   - **d.** Increase of 3 GtC over 50 years
   - **e.** Increase of 4 GtC over 50 years
   - **f.** Increase of 5 GtC over 50 years
   - **g.** Increase of 6 GtC over 50 years
   - **h.** Increase of 7 GtC over 50 years (projected path)

4. Research the "climate carbon wedge" concept and discuss the types of changes (transportation efficiency, transportation conservation, building efficiency, efficiency in electricity production, alternate energies, etc.) the world would have to make that would correspond to each of the seven wedges and the flat path.
   - **a.** Flat path
   - **b.** Wedge 1
   - **c.** Wedge 2
   - **d.** Wedge 3
   - **e.** Wedge 4
   - **f.** Wedge 5
   - **g.** Wedge 6

SECTION	CONCEPT	KEY IDEAS/FORMULAS
0.1	**Linear equations**	$ax + b = 0$
	Solving linear equations in one variable	Isolate variable on one side and constants on the other side.
	Applications involving linear equations	Six-step procedure: Step 1: Identify the question. Step 4: Set up an equation. Step 2: Make notes. Step 5: Solve the equation. Step 3: Assign a variable. Step 6: Check the solution. **Geometry problems:** Formulas for rectangles, triangles, and circles **Interest problems:** Simple interest: $I = Prt$ **Mixture problems:** Whenever two *distinct* quantities are mixed, the result is a mixture. **Distance–rate–time problems:** $d = r \cdot t$
0.2	**Quadratic equations**	$ax^2 + bx + c = 0$
	Factoring	If $(x - h)(x - k) = 0$, then $x = h$ or $x = k$.
	Square root method	If $x^2 = P$, then $x = \pm\sqrt{P}$.
	Completing the square	Find half of b; square that quantity; add the result to both sides.
	The quadratic formula	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
0.3	**Other types of equations**	
	Rational equations	Eliminate any values that make the denominator equal to 0.
	Radical equations	Check solutions to avoid extraneous solutions.
	Equations quadratic in form: u-substitution	Use a u-substitution to write the equation in quadratic form.
	Factorable equations	Extract common factor or factor by grouping.
	Absolute value equations	If $\|x\| = a$, then $x = -a$ or $x = a$.
0.4	**Inequalities**	Solutions are a range of real numbers.
	Graphing inequalities and interval notation	■ $a < x < b$ is equivalent to (a, b). ■ $x \le a$ is equivalent to $(-\infty, a]$. ■ $x > a$ is equivalent to (a, ∞).
	Linear inequalities	If an inequality is multiplied or divided by a *negative* number, the inequality sign must be reversed.
	Polynomial inequalities	Zeros are values that make the polynomial equal to 0.
	Rational inequalities	The number line is divided into intervals. The endpoints of these intervals are values that make either the numerator or denominator equal to 0. Always exclude values that make the denominator equal to 0.
	Absolute value inequalities	■ $\|x\| \le a$ is equivalent to $-a \le x \le a$. ■ $\|x\| > a$ is equivalent to $x < -a$ or $x > a$.

Section	Concept	Key Ideas/Formulas
0.5	**Graphing equations**	
	Cartesian plane	
	The distance and midpoint formulas	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
	Point-plotting	List a table with several coordinates that are solutions to the equation; plot and connect.
	Using intercepts and symmetry as graphing aids	If (a, b) is on the graph of the equation, then $(-a, b)$ is on the graph if symmetric about the y-axis, $(a, -b)$ is on the graph if symmetric about the x-axis, and $(-a, -b)$ is on the graph if symmetric about the origin. **Intercepts:** x-intercept: let $y = 0$. y-intercept: let $x = 0$. **Symmetry:** The graph of an equation can be symmetric about the x-axis, y-axis, or origin.
	Circles	Standard equation of a circle with center (h, k) and radius r. $(x - h)^2 + (y - k)^2 = r^2$ General form: $x^2 + y^2 + ax + by + c = 0$ Transform equations of circles to the standard form by completing the square
0.6	**Lines**	General form: $Ax + By = C$
	Graphing a line	Vertical: $x = a$ Slant: $Ax + By = C$, where $A \neq 0$ and $B \neq 0$ Horizontal: $y = b$
	Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}$, where $x_1 \neq x_2$ $\frac{\text{“rise”}}{\text{“run”}}$
	Equations of lines	**Slope–intercept form:** $y = mx + b$ m is the slope and b is the y-intercept. **Point–slope form:** $y - y_1 = m(x - x_1)$
	Parallel and perpendicular lines	$L_1 \parallel L_2$ if and only if $m_1 = m_2$ (slopes are equal). $L_1 \perp L_2$ if and only if $m_1 = -\frac{1}{m_2}$ $\begin{cases} m_1 \neq 0 \\ m_2 \neq 0 \end{cases}$ (slopes are negative reciprocals).

Section	Concept	Key Ideas/Formulas
0.7	**Modeling variation**	
	Direct variation	$y = kx$
	Inverse variation	$y = \frac{k}{x}$
	Joint variation and combined variation	Joint: One quantity is directly proportional to the product of two or more other quantities. Combined: Direct variation and inverse variation occur at the same time.
0.8*	**Linear regression: Best fit**	Fitting data with a line
	Scatterplots	Creating a scatterplot: ■ Using Microsoft Excel ■ Using a Graphing Calculator
	Identifying patterns	Association: Positive or Negative Linearity: Linear or Nonlinear Correlation coefficient, r
	Linear regression	■ Determine the "best fit" line ■ Making predictions

0.1 Linear Equations

Solve for the variable.

1. $7x - 4 = 12$

2. $13d + 12 = 7d + 6$

3. $20p + 14 = 6 - 5p$

4. $4(x - 7) - 4 = 4$

5. $3(x + 7) - 2 = 4(x - 2)$

6. $7c + 3(c - 5) = 2(c + 3) - 14$

7. $14 - [-3(y - 4) + 9] = [4(2y + 3) - 6] + 4$

8. $[6 - 4x + 2(x - 7)] - 52 = 3(2x - 4) + 6[3(2x - 3) + 6]$

9. $\frac{12}{b} - 3 = \frac{6}{b} + 4$

10. $\frac{g}{3} + g = \frac{7}{9}$

11. $\frac{13x}{7} - x = \frac{x}{4} - \frac{3}{14}$

12. $5b + \frac{b}{6} = \frac{b}{3} - \frac{29}{6}$

13. Investments. You win \$25,000 and you decide to invest the money in two different investments: one paying 20% and the other paying 8%. A year later you have \$27,600 total. How much did you originally invest in each account?

14. Investments. A college student on summer vacation was able to make \$5000 by working a full-time job every summer. He invested half the money in a mutual fund and half the money in a stock that yielded four times as much interest as the mutual fund. After a year he earned \$250 in interest. What were the interest rates of the mutual fund and the stock?

15. Chemistry. For an experiment, a student requires 150 milliliters of a solution that is 8% NaCl (sodium chloride). The storeroom has only solutions that are 10% NaCl and 5% NaCl. How many milliliters of each available solution should be mixed to get 150 milliliters of 8% NaCl?

16. Chemistry. A mixture containing 8% salt is to be mixed with 4 ounces of a mixture that is 20% salt, in order to obtain a solution that is 12% salt. How much of the first solution must be used?

0.2 Quadratic Equations

Solve by factoring.

17. $b^2 = 4b + 21$

18. $x(x - 3) = 54$

19. $x^2 = 8x$

20. $6y^2 - 7y - 5 = 0$

Solve by the square root method.

21. $q^2 - 169 = 0$

22. $c^2 + 36 = 0$

23. $(2x - 4)^2 = -64$

24. $(d + 7)^2 - 4 = 0$

Solve by completing the square.

25. $x^2 - 4x - 12 = 0$

26. $2x^2 - 5x - 7 = 0$

27. $\frac{x^2}{2} = 4 + \frac{x}{2}$

28. $8m = m^2 + 15$

Solve by the quadratic formula.

29. $3t^2 - 4t = 7$

30. $4x^2 + 5x + 7 = 0$

31. $8f^2 - \frac{1}{3}f = \frac{7}{6}$

32. $x^2 = -6x + 6$

Solve by any method.

33. $5q^2 - 3q - 3 = 0$

34. $(x - 7)^2 = -12$

35. $2x^2 - 3x - 5 = 0$

36. $(g - 2)(g + 5) = -7$

37. $7x^2 = -19x + 6$

38. $7 = \left(2b^2 + 1\right)$

39. Geometry. Find the base and height of a triangle with an area of 2 square feet if its base is 3 feet longer than its height.

40. Falling Objects. A man is standing on top of a building 500 feet tall. If he drops a penny off the roof, the height of the penny is given by $h = -16t^2 + 500$, where t is in seconds. Determine how many seconds it takes until the penny hits the ground.

0.3 Other Types of Equations

Specify any values that must be excluded from the solution set and then solve the rational equation.

41. $\frac{1}{x} - 4 = 3(x - 7) + 5$

42. $\frac{4}{x + 1} - \frac{8}{x - 1} = 3$

43. $\frac{2}{t + 4} - \frac{7}{t} = \frac{6}{t(t + 4)}$

44. $\frac{3}{2x - 7} = \frac{-2}{3x + 1}$

45. $\frac{3}{2x} - \frac{6}{x} = 9$

46. $\frac{3 - 5/m}{2 + 5/m} = 1$

Solve the radical equation for the given variable.

47. $\sqrt[3]{2x - 4} = 2$

48. $\sqrt{x - 2} = -4$

49. $(2x - 7)^{1/5} = 3$

50. $x = \sqrt{7x - 10}$

51. $x - 4 = \sqrt{x^2 + 5x + 6}$

52. $\sqrt{2x - 7} = \sqrt{x + 3}$

53. $\sqrt{x + 3} = 2 - \sqrt{3x + 2}$

54. $4 + \sqrt{x - 3} = \sqrt{x - 5}$

Solve the equation by introducing a substitution that transforms the equation to quadratic form.

55. $y^{-2} - 5y^{-1} + 4 = 0$

56. $p^{-2} + 4p^{-1} = 12$

57. $3x^{1/3} + 2x^{2/3} = 5$

58. $2x^{2/3} - 3x^{1/3} - 5 = 0$

59. $x^{-2/3} + 3x^{-1/3} + 2 = 0$

60. $y^{-1/2} - 2y^{-1/4} + 1 = 0$

61. $x^4 + 5x^2 = 36$

62. $3 - 4x^{-1/2} + x^{-1} = 0$

Solve the equation by factoring.

63. $x^3 + 4x^2 - 32x = 0$

64. $9t^3 - 25t = 0$

65. $p^3 - 3p^2 - 4p + 12 = 0$

66. $4x^3 - 9x^2 + 4x - 9 = 0$

67. $p(2p - 5)^2 - 3(2p - 5) = 0$

68. $2(t^2 - 9)^3 - 20(t^2 - 9)^2 = 0$

69. $y - 81y^{-1} = 0$

70. $9x^{3/2} - 37x^{1/2} + 4x^{-1/2} = 0$

Solve the absolute value equation.

71. $|x - 3| = -4$

72. $|2 + x| = 5$

73. $|3x - 4| = 1.1$

74. $|x^2 - 6| = 3$

0.4 Inequalities

Graph the indicated set and write as a single interval, if possible.

75. $(4, 6] \cup [5, \infty)$

76. $(-\infty, -3) \cup [-7, 2]$

77. $(3, 12] \cap [8, \infty)$

78. $(-\infty, -2) \cap [-2, 9)$

Solve the linear inequality and express the solution set in interval notation.

79. $2x < 5 - x$

80. $6x + 4 \le 2$

81. $4(x - 1) > 2x - 7$

82. $\dfrac{x + 3}{3} \ge 6$

83. $6 < 2 + x \le 11$

84. $-6 \le 1 - 4(x + 2) \le 16$

85. $\dfrac{2}{3} \le \dfrac{1 + x}{6} \le \dfrac{3}{4}$

86. $\dfrac{x}{3} + \dfrac{x + 4}{9} > \dfrac{x}{6} - \dfrac{1}{3}$

Solve the polynomial inequality and express the solution set using interval notation.

87. $x^2 \le 36$

88. $6x^2 - 7x < 20$

89. $4x \le x^2$

90. $-x^2 \ge 9x + 14$

91. $4x^2 - 12 > 13x$

92. $3x \le x^2 + 2$

Solve the rational inequality and express the solution set using interval notation.

93. $\dfrac{x}{x - 3} < 0$

94. $\dfrac{x - 1}{x - 4} > 0$

95. $\dfrac{x^2 - 3x}{3} \ge 18$

96. $\dfrac{x^2 - 49}{x - 7} \ge 0$

97. $\dfrac{3}{x - 2} - \dfrac{1}{x - 4} \le 0$

98. $\dfrac{4}{x - 1} \le \dfrac{2}{x + 3}$

Solve the absolute value inequality and express the solution set using interval notation.

99. $|x + 4| > 7$

100. $|-7 + y| \le 4$

101. $|2x| > 6$

102. $\left|\dfrac{4 + 2x}{3}\right| \ge \dfrac{1}{7}$

103. $|2 + 5x| \ge 0$

104. $|1 - 2x| \le 4$

0.5 Graphing Equations

Calculate the distance between the two points.

105. $(-2, 0)$ and $(4, 3)$

106. $(1, 4)$ and $(4, 4)$

107. $(-4, -6)$ and $(2, 7)$

108. $\left(\frac{1}{4}, \frac{1}{12}\right)$ and $\left(\frac{1}{3}, -\frac{7}{3}\right)$

Calculate the midpoint of the segment joining the two points.

109. $(2, 4)$ and $(3, 8)$

110. $(-2, 6)$ and $(5, 7)$

111. $(2.3, 3.4)$ and $(5.4, 7.2)$

112. $(-a, 2)$ and $(a, 4)$

Find the x-intercept(s) and y-intercept(s) if any.

113. $x^2 + 4y^2 = 4$

114. $y = x^2 - x + 2$

115. $y = \sqrt{x^2 - 9}$

116. $y = \dfrac{x^2 - x - 12}{x - 12}$

Use algebraic tests to determine symmetry with respect to the x-axis, y-axis, or origin.

117. $x^2 + y^3 = 4$

118. $y = x^2 - 2$

119. $xy = 4$

120. $y^2 = 5 + x$

Use symmetry as a graphing aid and point-plot the given equations.

121. $y = x^2 - 3$

122. $y = |x| - 4$

123. $y = \sqrt[3]{x}$

124. $x = y^2 - 2$

125. $y = x\sqrt{9 - x^2}$

126. $x^2 + y^2 = 36$

Find the center and the radius of the circle given by the equation.

127. $(x + 2)^2 + (y + 3)^2 = 81$

128. $(x - 4)^2 + (y + 2)^2 = 32$

129. $x^2 + y^2 + 2y - 4x + 11 = 0$

130. $3x^2 + 3y^2 - 6x - 7 = 0$

0.6 Lines

Write an equation of the line, given the slope and a point that lies on the line.

131. $m = -2 \quad (-3, 4)$

132. $m = \frac{3}{4} \quad (2, 16)$

133. $m = 0 \quad (-4, 6)$

134. m is undefined $\quad (2, -5)$

Write the equation of the line that passes through the given points. Express the equation in slope–intercept form or in the form of $x = a$ or $y = b$.

135. $(-4, -2)$ and $(2, 3)$

136. $(-1, 4)$ and $(-2, 5)$

137. $\left(-\frac{3}{4}, \frac{1}{2}\right)$ and $\left(-\frac{7}{4}, \frac{5}{2}\right)$

138. $(3, -2)$ and $(-9, 2)$

Find the equation of the line that passes through the given point and also satisfies the additional piece of information.

139. $(-2, -1)$ parallel to the line $2x - 3y = 6$

140. $(5, 6)$ perpendicular to the line $5x - 3y = 0$

0.7 Modeling Variation

Write an equation that describes each variation.

141. C is directly proportional to r; $C = 2\pi$ when $r = 1$.

142. V is directly proportional to both l and w; $V = 12h$ when $w = 6$ and $l = 2$.

143. A varies directly with the square of r; $A = 25\pi$ when $r = 5$.

144. F varies inversely with both λ and L; $F = 20\pi$ when $\lambda = 10$ μm and $L = 10$ km.

Solve the equation.

1. $4p - 7 = 6p - 1$
2. $-2(z - 1) + 3 = -3z + 3(z - 1)$
3. $3t = t^2 - 28$
4. $8x^2 - 13x = 6$
5. $6x^2 - 13x = 8$
6. $\dfrac{3}{x - 1} = \dfrac{5}{x + 2}$
7. $\dfrac{5}{y - 3} + 1 = \dfrac{30}{y^2 - 9}$
8. $x^4 - 5x^2 - 36 = 0$
9. $\sqrt{2x + 1} + x = 7$
10. $2x^{2/3} + 3x^{1/3} - 2 = 0$
11. $\sqrt{3y - 2} = 3 - \sqrt{3y + 1}$
12. $x(3x - 5)^3 - 2(3x - 5)^2 = 0$
13. $x^{7/3} - 8x^{4/3} + 12x^{1/3} = 0$
14. Solve for x: $\left|\frac{1}{5}x + \frac{2}{3}\right| = \frac{7}{15}$.

Solve the inequality and express the solution in interval notation.

15. $3x + 19 \geq 5(x - 3)$
16. $-1 \leq 3x + 5 < 26$
17. $\dfrac{2}{5} < \dfrac{x + 8}{4} \leq \dfrac{1}{2}$
18. $3x \geq 2x^2$
19. $3p^2 \geq p + 4$
20. $|5 - 2x| > 1$
21. $\dfrac{x - 3}{2x + 1} \leq 0$
22. $\dfrac{x + 4}{x^2 - 9} \geq 0$
23. Find the distance between the points $(-7, -3)$ and $(2, -2)$.
24. Find the midpoint between $(-3, 5)$ and $(5, -1)$.

In Exercises 25 and 26, graph the equations.

25. $2x^2 + y^2 = 8$
26. $y = \dfrac{4}{x^2 + 1}$
27. Find the x-intercept and the y-intercept of the line $x - 3y = 6$.
28. Find the x-intercept(s) and the y-intercept(s), if any: $4x^2 - 9y^2 = 36$.
29. Express the line in slope–intercept form: $\frac{2}{3}x - \frac{1}{4}y = 2$.
30. Express the line in slope–intercept form: $4x - 6y = 12$.

Find the equation of the line that is characterized by the given information. Graph the line.

31. Passes through the points $(-3, 2)$ and $(4, 9)$
32. Parallel to the line $y = 4x + 3$ and passes through the point $(1, 7)$
33. Perpendicular to the line $2x - 4y = 5$ and passes through the point $(1, 1)$
34. Determine the center and radius of the circle $x^2 + y^2 - 10x + 6y + 22 = 0$.

In Exercises 35 and 36, use variation to find a model for the given problem.

35. F varies directly with m and inversely with p; $F = 20$ when $m = 2$ and $p = 3$.
36. y varies directly with the square of x; $y = 8$ when $x = 5$.

1

Functions and Their Graphs

You are buying a pair of running shoes. Their original price was $100, but they have been discounted 30% as part of a weekend sale. Because you arrived early, you can take advantage of door-buster savings: an additional 20% off the sale price. Naïve shoppers might be lured into thinking these shoes will cost $50 because they add the 20% and 30% to get 50% off, but they will end up paying more than that. Experienced shoppers know that the store will first take 30% off of $100, which results in a price of $70, and then it will take an additional 20% off of the sale price, $70, which results in a final discounted price of $56. Experienced shoppers already understand the concept of a function—taking an input (original price) and mapping it to an output (sale price).

A composition of functions can be thought of as a function of a function. One function takes an input (original price, $100) and maps it to an output (sale price, $70), and then another function takes that output as its input (sale price, $70) and maps that to an output (checkout price, $56).

IN THIS CHAPTER we will establish what a relation is, and then we will determine whether a relation is a function. We will discuss common functions, the domain and range of functions, and graphs of functions. We will determine whether a function is increasing, decreasing, or constant on an interval and calculate the average rate of change of a function. We will perform operations on functions and the composition of functions. Lastly, we will discuss one-to-one functions and inverse functions.

FUNCTIONS AND THEIR GRAPHS

1.1 Functions
- Definition of a Function
- Functions Defined by Equations
- Function Notation
- Domain of a Function

1.2 Graphs of Functions
- Recognizing and Classifying Functions
- Increasing and Decreasing Functions
- Average Rate of Change
- Piecewise-Defined Functions

1.3 Graphing Techniques: Transformations
- Horizontal and Vertical Shifts
- Reflection About the Axes
- Stretching and Compressing

1.4 Combining Functions
- Adding, Subtracting, Multiplying, and Dividing Functions
- Composition of Functions

1.5 One-to-One Functions and Inverse Functions
- One-to-One Functions
- Inverse Functions
- Graphical Interpretation of Inverse Functions
- Finding the Inverse Function

LEARNING OBJECTIVES

- Evaluate a function for any argument using placeholder notation.
- Determine characteristics of graphs: even or odd, increasing or decreasing, and the average rate of change.
- Graph functions that are transformations of common functions.
- Find composite functions and their domains.
- Find inverse functions and their domains and ranges.

SECTION 1.1 FUNCTIONS

SKILLS OBJECTIVES

- Determine whether a relation is a function.
- Determine whether an equation represents a function.
- Use function notation.
- Find the value of a function.
- Determine the domain and range of a function.

CONCEPTUAL OBJECTIVES

- Think of function notation as a placeholder or mapping.
- Understand that all functions are relations but not all relations are functions.

Definition of a Function

What do the following pairs have in common?

- Every person has a blood type.
- Temperature is some typical value at a particular time of day.
- Every working household phone in the United States has a 10-digit phone number.
- First-class postage rates correspond to the weight of a letter.
- Certain times of the day are start times for sporting events at a university.

They all describe a particular correspondence between two groups. A **relation** is a correspondence between two sets. The first set is called the **domain** and the corresponding second set is called the **range**. Members of these sets are called **elements**.

DEFINITION Relation

A **relation** is a correspondence between two sets where each element in the first set, called the **domain**, corresponds to *at least* one element in the second set, called the **range**.

A relation is a set of ordered pairs. The domain is the set of all the first components of the ordered pairs, and the range is the set of all the second components of the ordered pairs.

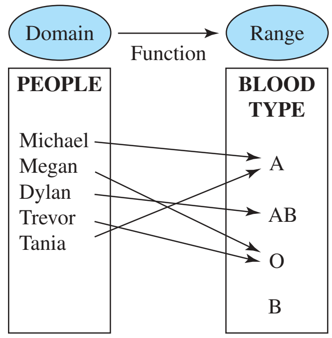
PERSON	BLOOD TYPE	ORDERED PAIR
Michael	A	(Michael, A)
Tania	A	(Tania, A)
Dylan	AB	(Dylan, AB)
Trevor	O	(Trevor, O)
Megan	O	(Megan, O)

WORDS	MATH
The domain is the set of all the first components.	{Michael, Tania, Dylan, Trevor, Megan}
The range is the set of all the second components.	{A, AB, O}

A relation in which each element in the domain corresponds to exactly one element in the range is a **function**.

DEFINITION Function

A **function** is a correspondence between two sets where each element in the first set, called the **domain**, corresponds to *exactly* one element in the second set, called the **range**.

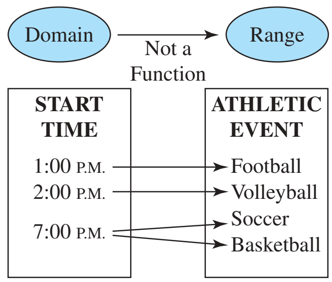


Note that the definition of a function is more restrictive than the definition of a relation. For a relation, each input corresponds to *at least* one output, whereas, for a function, each input corresponds to *exactly* one output. The blood-type example given is both a relation and a function.

Also note that the range (set of values to which the elements of the domain correspond) is a subset of the set of all blood types. Although all functions are relations, not all relations are functions.

For example, at a university, four primary sports typically overlap in the late fall: football, volleyball, soccer, and basketball. On a given Saturday, the table to the right indicates the start times for the competitions.

Time of Day	Competition
1:00 P.M.	Football
2:00 P.M.	Volleyball
7:00 P.M.	Soccer
7:00 P.M.	Basketball



Words	Math
The 1:00 start time corresponds to exactly one event, Football.	(1:00 P.M., Football)
The 2:00 start time corresponds to exactly one event, Volleyball.	(2:00 P.M., Volleyball)
The 7:00 start time corresponds to two events, Soccer and Basketball.	(7:00 P.M., Soccer) (7:00 P.M., Basketball)

Because an element in the domain, 7:00 P.M., corresponds to more than one element in the range, Soccer and Basketball, this is not a function. It is, however, a relation.

Study Tip

All functions are relations but not all relations are functions.

EXAMPLE 1 Determining Whether a Relation Is a Function

Determine whether the following relations are functions:

a. $\{(-3, 4), (2, 4), (3, 5), (6, 4)\}$
b. $\{(-3, 4), (2, 4), (3, 5), (2, 2)\}$
c. Domain = Set of all items for sale in a grocery store; Range = Price

Solution:

a. No x-value is repeated. Therefore, each x-value corresponds to exactly one y-value.
$\boxed{\text{This relation is a function.}}$

b. The value $x = 2$ corresponds to *both* $y = 2$ and $y = 4$. $\boxed{\text{This relation is not a function.}}$

c. Each item in the grocery store corresponds to exactly one price. $\boxed{\text{This relation is a function.}}$

■ **YOUR TURN** Determine whether the following relations are functions:

a. $\{(1, 2), (3, 2), (5, 6), (7, 6)\}$
b. $\{(1, 2), (1, 3), (5, 6), (7, 8)\}$
c. $\{(11{:}00 \text{ A.M.}, 83°\text{F}), (2{:}00 \text{ P.M.}, 89°\text{F}), (6{:}00 \text{ P.M.}, 85°\text{F})\}$

■ **Answer:** **a.** function
b. not a function
c. function

All of the examples we have discussed thus far are **discrete** sets in that they represent a countable set of distinct pairs of (x, y). A function can also be defined algebraically by an equation.

Functions Defined by Equations

Let's start with the equation $y = x^2 - 3x$, where x can be any real number. This equation assigns to each x-value exactly one corresponding y-value. For example,

x	$y = x^2 - 3x$	y
1	$y = (1)^2 - 3(1)$	-2
5	$y = (5)^2 - 3(5)$	10
$-\frac{2}{3}$	$y = \left(-\frac{2}{3}\right)^2 - 3\left(-\frac{2}{3}\right)$	$\frac{22}{9}$
1.2	$y = (1.2)^2 - 3(1.2)$	-2.16

Since the variable y *depends* on what value of x is selected, we denote y as the **dependent variable**. The variable x can be any number in the domain; therefore, we denote x as the **independent variable**.

Although functions are defined by equations, it is important to recognize that *not all equations define functions*. The requirement for an equation to define a function is that each element in the domain corresponds to exactly one element in the range. Throughout the ensuing discussion, we assume x to be the independent variable and y to be the dependent variable.

CAUTION

Not all equations are functions.

Equations that represent functions of x:	$y = x^2$	$y = \lvert x\rvert$	$y = x^3$
Equations that do not represent functions of x:	$x = y^2$	$x^2 + y^2 = 1$	$x = \lvert y\rvert$

Study Tip

We say that $x = y^2$ is not a function of x. However, if we reverse the independent and dependent variables, then $x = y^2$ is a function of y.

In the "equations that represent functions of x," every x-value corresponds to exactly one y-value. Some ordered pairs that correspond to these functions are

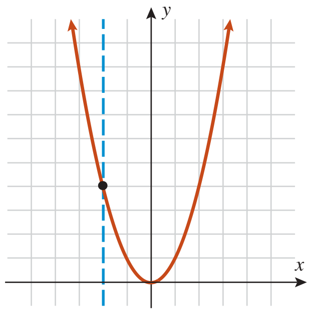
$$y = x^2: \quad (-1, 1)\ (0, 0)\ (1, 1)$$
$$y = |x|: \quad (-1, 1)\ (0, 0)\ (1, 1)$$
$$y = x^3: \quad (-1, -1)\ (0, 0)\ (1, 1)$$

The fact that $x = -1$ and $x = 1$ both correspond to $y = 1$ in the first two examples does not violate the definition of a function.

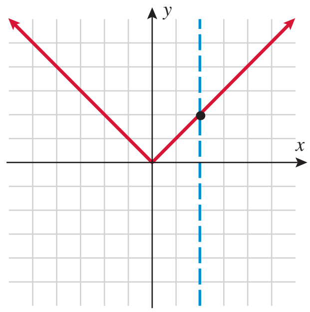
In the "equations that do not represent functions of x," some x-values correspond to *more than one* y-value. Some ordered pairs that correspond to these equations are given in the two right-hand columns of the table below.

RELATION	SOLVE RELATION FOR y	POINTS THAT LIE ON GRAPH	
$x = y^2$	$y = \pm\sqrt{x}$	$(1, -1)\ (0, 0)\ (1, 1)$	$x = 1$ maps to **both** $y = -1$ and $y = 1$
$x^2 + y^2 = 1$	$y = \pm\sqrt{1 - x^2}$	$(0, -1)\ (0, 1)\ (-1, 0)\ (1, 0)$	$x = 0$ maps to **both** $y = -1$ and $y = 1$
$x = \lvert y\rvert$	$y = \pm x$	$(1, -1)\ (0, 0)\ (1, 1)$	$x = 1$ maps to **both** $y = -1$ and $y = 1$

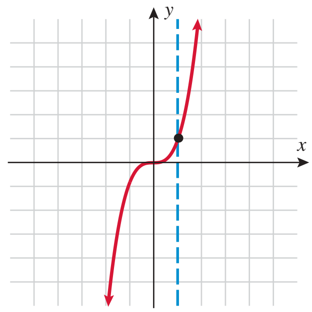
Let's look at the graphs of the three **functions of *x***:



$y = x^2$

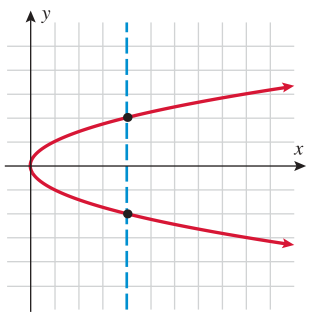


$y = |x|$

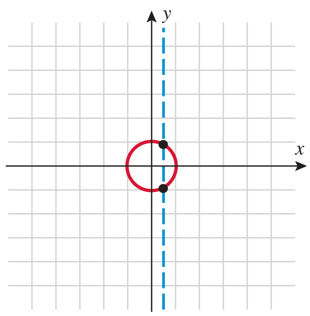


$y = x^3$

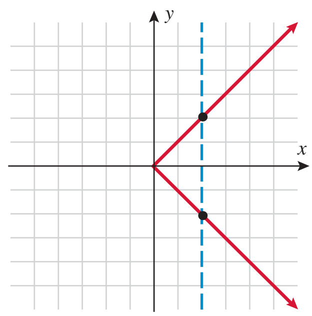
Let's take any value for x, say, $x = a$. The graph of $x = a$ corresponds to a vertical line. A function of x maps each x-value to exactly one y-value; therefore, there should be at most one point of intersection with any vertical line. We see in the three graphs of the functions above that if a vertical line is drawn at any value of x on any of the three graphs, the vertical line only intersects the graph in one place. Look at the graphs of the three equations that do **not** represent **functions of *x***.



$x = y^2$



$x^2 + y^2 = 1$



$x = |y|$

A vertical line can be drawn on any of the three graphs such that the vertical line will intersect each of these graphs at two points. Thus, there is more than one y-value that corresponds to some x-value in the domain, which is why these equations do not define y as functions of x.

DEFINITION **Vertical Line Test**

Given the graph of an equation, if any vertical line that can be drawn intersects the graph at no more than one point, the equation defines y as a function of x. This test is called the **vertical line test**.

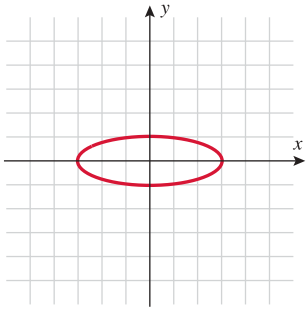
Study Tip

If any x-value corresponds to more than one y-value, then y is **not** a function of x.

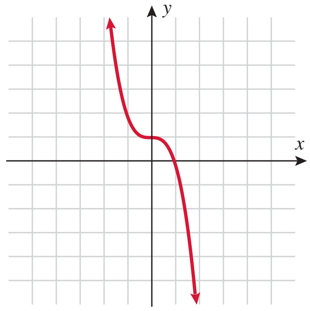
EXAMPLE 2 Using the Vertical Line Test

Use the vertical line test to determine whether the graphs of equations define functions of x.

a.



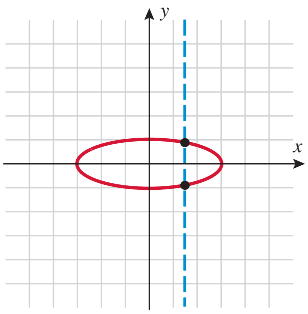
b.



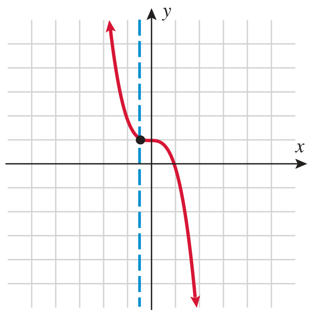
Solution:

Apply the vertical line test.

a.



b.



a. Because the vertical line intersects the graph of the equation at two points, this equation $\boxed{\text{does not represent a function}}$.

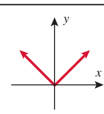
b. Because any vertical line will intersect the graph of this equation at no more than one point, this equation $\boxed{\text{represents a function}}$.

■ **YOUR TURN** Determine whether the equation $(x - 3)^2 + (y + 2)^2 = 16$ is a function of x.

■ **Answer:** The graph of the equation is a circle, which does not pass the vertical line test. Therefore, the equation does not define a function.

To recap, a function can be expressed one of four ways: verbally, numerically, algebraically, and graphically. This is sometimes called the Rule of 4.

Expressing a Function

VERBALLY	NUMERICALLY	ALGEBRAICALLY	GRAPHICALLY
Every real number has a corresponding absolute value.	$\{(-3, 3), (-1, 1), (0, 0), (1, 1), (5, 5)\}$	$y = \lvert x \rvert$	

Function Notation

We know that the equation $y = 2x + 5$ defines y as a function of x because its graph is a nonvertical line and thus passes the vertical line test. We can select x-values (input) and determine unique corresponding y-values (output). The output is found by taking 2 times the input and then adding 5. If we give the function a name, say, "f", then we can use **function notation**:

$$f(x) = 2x + 5$$

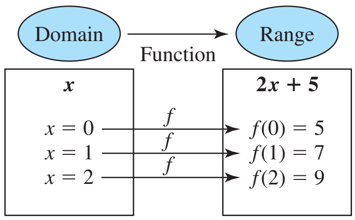
DEFINITION **Function Notation**

The symbol $f(x)$ is read "f evaluated at x" or "f of x" and represents the y-value that corresponds to a particular x-value. In other words, $y = f(x)$.

INPUT	FUNCTION	OUTPUT	EQUATION
x	f	$f(x)$	$f(x) = 2x + 5$
Independent variable	Mapping	Dependent variable	Mathematical rule

It is important to note that f is the function name, whereas $f(x)$ is the value of the function. In other words, the function f maps some value x in the domain to some value $f(x)$ in the range.

x	$f(x) = 2x + 5$	$f(x)$
0	$f(0) = 2(0) + 5$	$f(0) = 5$
1	$f(1) = 2(1) + 5$	$f(1) = 7$
2	$f(2) = 2(2) + 5$	$f(2) = 9$



The independent variable is also referred to as the **argument** of a function. To evaluate functions, it is often useful to think of the independent variable or argument as a placeholder. For example, $f(x) = x^2 - 3x$ can be thought of as

$$f(\square) = (\square)^2 - 3(\square)$$

In other words, "f of the argument is equal to the argument squared minus 3 times the argument." Any expression can be substituted for the argument:

$$f(1) = (1)^2 - 3(1)$$
$$f(x + 1) = (x + 1)^2 - 3(x + 1)$$
$$f(-x) = (-x)^2 - 3(-x)$$

It is important to note:

- $f(x)$ does *not* mean f times x.
- The most common function names are f and F since the word function begins with an "f". Other common function names are g and G, but any letter can be used.
- The letter most commonly used for the independent variable is x. The letter t is also common because in real-world applications it represents time, but any letter can be used.
- Although we can think of y and $f(x)$ as interchangeable, the function notation is useful when we want to consider two or more functions of the same independent variable or when we want to evaluate a function at more than one argument.

Study Tip

It is important to note that $f(x)$ does not mean f times x.

EXAMPLE 3 Evaluating Functions by Substitution

Given the function $f(x) = 2x^3 - 3x^2 + 6$, find $f(-1)$.

Solution:

Consider the independent variable x to be a placeholder. $\quad f(\square) = 2(\square)^3 - 3(\square)^2 + 6$

To find $f(-1)$, substitute $x = -1$ into the function. $\quad f(-1) = 2(-1)^3 - 3(-1)^2 + 6$

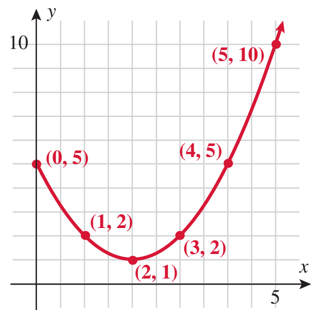
Evaluate the right side. $\quad f(-1) = -2 - 3 + 6$

Simplify. $\quad \boxed{f(-1) = 1}$

EXAMPLE 4 Finding Function Values from the Graph of a Function

The graph of f is given on the right.

a. Find $f(0)$.
b. Find $f(1)$.
c. Find $f(2)$.
d. Find $4f(3)$.
e. Find x such that $f(x) = 10$.
f. Find x such that $f(x) = 2$.



Solution (a): The value $x = 0$ corresponds to the value $y = 5$. $\boxed{f(0) = 5}$

Solution (b): The value $x = 1$ corresponds to the value $y = 2$. $\boxed{f(1) = 2}$

Solution (c): The value $x = 2$ corresponds to the value $y = 1$. $\boxed{f(2) = 1}$

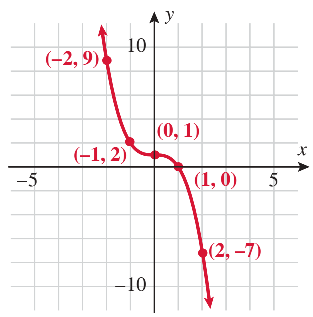
Solution (d): The value $x = 3$ corresponds to the value $y = 2$. $\quad 4f(3) = 4 \cdot 2 = \boxed{8}$

Solution (e): The value $y = 10$ corresponds to the value $\boxed{x = 5}$.

Solution (f): The value $y = 2$ corresponds to the values $\boxed{x = 1}$ and $\boxed{x = 3}$.

■ **YOUR TURN** For the following graph of a function, find

a. $f(-1)$ **b.** $f(0)$ **c.** $3f(2)$
d. the value of x that corresponds to $f(x) = 0$



■ **Answer:** **a.** $f(-1) = 2$
b. $f(0) = 1$
c. $3f(2) = -21$
d. $x = 1$

EXAMPLE 5 Evaluating Functions with Variable Arguments (Inputs)

For the given function $f(x) = x^2 - 3x$, evaluate $f(x + 1)$ and simplify if possible.

COMMON MISTAKE

A common misunderstanding is to interpret the notation $f(x + 1)$ as a sum: $f(x + 1) \neq f(x) + f(1)$.

CORRECT

Write the original function.

$$f(x) = x^2 - 3x$$

Replace the argument x with a placeholder.

$$f(\square) = (\square)^2 - 3(\square)$$

Substitute $x + 1$ for the argument.

$$f(x + 1) = (x + 1)^2 - 3(x + 1)$$

Eliminate the parentheses.

$$f(x + 1) = x^2 + 2x + 1 - 3x - 3$$

Combine like terms.

$$\boxed{f(x + 1) = x^2 - x - 2}$$

INCORRECT

The **ERROR** is in interpreting the notation as a sum.

$$f(x + 1) \neq f(x) + f(1)$$
$$f(x + 1) \neq x^2 - 3x - 2$$

CAUTION

$f(x + 1) \neq f(x) + f(1)$

YOUR TURN For the given function $g(x) = x^2 - 2x + 3$, evaluate $g(x - 1)$.

Answer: $g(x - 1) = x^2 - 4x + 6$

EXAMPLE 6 Evaluating Functions: Sums

For the given function $H(x) = x^2 + 2x$, evaluate

a. $H(x + 1)$ **b.** $H(x) + H(1)$

Solution (a):

Write the function H in placeholder notation. $H(\square) = (\square)^2 + 2(\square)$

Substitute $x + 1$ for the argument of H. $H(x + 1) = (x + 1)^2 + 2(x + 1)$

Eliminate the parentheses on the right side. $H(x + 1) = x^2 + 2x + 1 + 2x + 2$

Combine like terms on the right side. $\boxed{H(x + 1) = x^2 + 4x + 3}$

Solution (b):

Write $H(x)$. $H(x) = x^2 + 2x$

Evaluate H at $x = 1$. $H(1) = (1)^2 + 2(1) = 3$

Evaluate the sum $H(x) + H(1)$. $H(x) + H(1) = x^2 + 2x + 3$

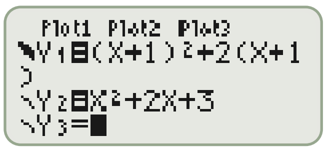
$$\boxed{H(x) + H(1) = x^2 + 2x + 3}$$

Note: Comparing the results of part (a) and part (b), we see that

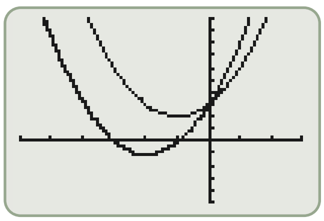
$$H(x + 1) \neq H(x) + H(1).$$

Technology Tip

Use a graphing utility to display graphs of $y_1 = H(x + 1) = (x + 1)^2 + 2(x + 1)$ and $y_2 = H(x) + H(1) = x^2 + 2x + 3$.



The graphs are not the same.



EXAMPLE 7 Evaluating Functions: Negatives

For the given function $G(t) = t^2 - t$, evaluate

a. $G(-t)$ **b.** $-G(t)$

Solution (a):

Write the function G in placeholder notation. $G(\square) = (\square)^2 - (\square)$

Substitute $-t$ for the argument of G. $G(-t) = (-t)^2 - (-t)$

Eliminate the parentheses on the right side. $\boxed{G(-t) = t^2 + t}$

Solution (b):

Write $G(t)$. $G(t) = t^2 - t$

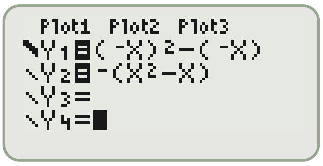
Multiply by -1. $-G(t) = -(t^2 - t)$

Eliminate the parentheses on the right side. $\boxed{-G(t) = -t^2 + t}$

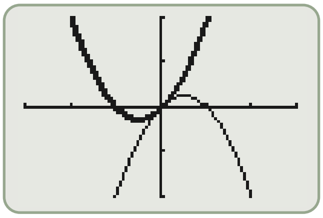
Note: Comparing the results of part (a) and part (b), we see that $\mathbf{G(-t) \neq -G(t)}$. If $G(t)$ was an odd function, then $G(-t) = -G(t)$, but in general this is not true.

Technology Tip

Use a graphing utility to display graphs of $y_1 = G(-x) = (-x)^2 - (-x)$ and $y_2 = -G(x) = -(x^2 - x)$.



The graphs are not the same.



EXAMPLE 8 Evaluating Functions: Quotients

For the given function $F(x) = 3x + 5$, evaluate

a. $F\left(\frac{1}{2}\right)$ **b.** $\frac{F(1)}{F(2)}$

Solution (a):

Write F in placeholder notation. $F(\square) = 3(\square) + 5$

Replace the argument with $\frac{1}{2}$. $F\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right) + 5$

Simplify the right side. $\boxed{F\left(\frac{1}{2}\right) = \frac{13}{2}}$

Solution (b):

Evaluate $F(1)$. $F(1) = 3(1) + 5 = 8$

Evaluate $F(2)$. $F(2) = 3(2) + 5 = 11$

Divide $F(1)$ by $F(2)$. $\boxed{\frac{F(1)}{F(2)} = \frac{8}{11}}$

Note: Comparing the results of part (a) and part (b), we see that $\mathbf{F\left(\frac{1}{2}\right) \neq \frac{F(1)}{F(2)}}$.

CAUTION

$$f\left(\frac{a}{b}\right) \neq \frac{f(a)}{f(b)}$$

YOUR TURN Given the function $G(t) = 3t - 4$, evaluate

a. $G(t - 2)$ **b.** $G(t) - G(2)$ **c.** $\frac{G(1)}{G(3)}$ **d.** $G\left(\frac{1}{3}\right)$

Answer: **a.** $G(t - 2) = 3t - 10$
b. $G(t) - G(2) = 3t - 6$
c. $\frac{G(1)}{G(3)} = -\frac{1}{5}$
d. $G\left(\frac{1}{3}\right) = -3$

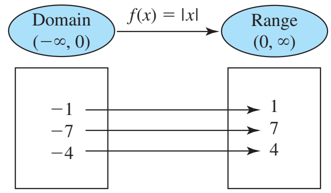
Examples 6–8 illustrate the following in general:

$$f(a + b) \neq f(a) + f(b) \qquad f(-t) \neq -f(t) \qquad f\left(\frac{a}{b}\right) \neq \frac{f(a)}{f(b)}$$

Domain of a Function

Sometimes the domain of a function is stated *explicitly*. For example,

$$f(x) = |x| \qquad \underbrace{x < 0}_{\text{domain}}$$



Here, the **explicit domain** is the set of all negative real numbers, $(-\infty, 0)$. Every negative real number in the domain is mapped to a positive real number in the range through the absolute value function.

If the expression that defines the function is given but the domain is not stated explicitly, then the domain is implied. The **implicit domain** is the largest set of real numbers for which the function is defined and the output value $f(x)$ is a real number. For example,

$$f(x) = \sqrt{x}$$

does not have the domain explicitly stated. There is, however, an implicit domain. Note that if the argument is negative, that is, if $x < 0$, then the result is an imaginary number. In order for the output of the function, $f(x)$, to be a real number, we must restrict the domain to nonnegative numbers, that is, if $x \geq 0$.

Function	Implicit Domain
$f(x) = \sqrt{x}$	$[0, \infty)$

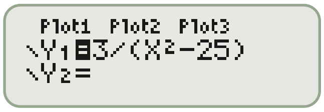
In general, we ask the question, "What can x be?" The implicit domain of a function excludes values that cause a function to be undefined or have outputs that are not real numbers.

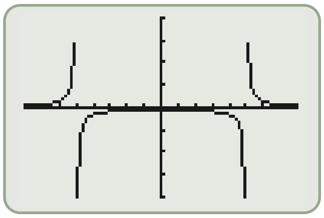
Expression That Defines the Function	Excluded x-Values	Example	Implicit Domain
Polynomial	None	$f(x) = x^3 - 4x^2$	All real numbers
Rational	x-values that make the denominator equal to 0	$g(x) = \dfrac{2}{x^2 - 9}$	$x \neq \pm 3$ or $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Radical	x-values that result in a square (even) root of a negative number	$h(x) = \sqrt{x - 5}$	$x \geq 5$ or $[5, \infty)$

Technology Tip

To visualize the domain of each function, ask the question: What are the excluded x-values in the graph?

Graph of $F(x) = \dfrac{3}{x^2 - 25}$ is shown.





The excluded x-values are -5 and 5.

EXAMPLE 9 Determining the Domain of a Function

State the domain of the given functions.

a. $F(x) = \dfrac{3}{x^2 - 25}$ **b.** $H(x) = \sqrt[4]{9 - 2x}$ **c.** $G(x) = \sqrt[3]{x - 1}$

Solution (a):

Write the original equation.	$F(x) = \dfrac{3}{x^2 - 25}$
Determine any restrictions on the values of x.	$x^2 - 25 \neq 0$
Solve the restriction equation.	$x^2 \neq 25$ or $x \neq \pm\sqrt{25} = \pm 5$
State the domain restrictions.	$x \neq \pm 5$
Write the domain in interval notation.	$\boxed{(-\infty, -5) \cup (-5, 5) \cup (5, \infty)}$

Solution (b):

Write the original equation.	$H(x) = \sqrt[4]{9 - 2x}$
Determine any restrictions on the values of x.	$9 - 2x \geq 0$
Solve the restriction equation.	$9 \geq 2x$
State the domain restrictions.	$x \leq \dfrac{9}{2}$
Write the domain in interval notation.	$\boxed{\left(-\infty, \dfrac{9}{2}\right]}$

Solution (c):

Write the original equation.	$G(x) = \sqrt[3]{x - 1}$
Determine any restrictions on the values of x.	no restrictions
State the domain.	$\mathbb{R}$
Write the domain in interval notation.	$\boxed{(-\infty, \infty)}$

YOUR TURN State the domain of the given functions.

a. $f(x) = \sqrt{x - 3}$ **b.** $g(x) = \dfrac{1}{x^2 - 4}$

Answer:
a. $x \geq 3$ or $[3, \infty)$
b. $x \neq \pm 2$ or $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Applications

Functions that are used in applications often have restrictions on the domains due to physical constraints. For example, the volume of a cube is given by the function $V(x) = x^3$, where x is the length of a side. The function $f(x) = x^3$ has no restrictions on x, and therefore, the domain is the set of all real numbers. However, the volume of any cube has the restriction that the length of a side can never be negative or zero.

EXAMPLE 10 The Dimensions of a Pool

Express the volume of a 30 foot by 10 foot rectangular swimming pool as a function of its depth.

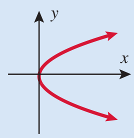
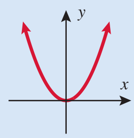
Solution:

The volume of any rectangular box is $V = lwh$, where V is the volume, l is the length, w is the width, and h is the height. In this example, the length is 30 feet, the width is 10 feet, and the height represents the depth d of the pool.

Write the volume as a function of depth d.	$V(d) = (30)(10)d$
Simplify.	$\boxed{V(d) = 300d}$
Determine any restrictions on the domain.	$d > 0$

SECTION 1.1 SUMMARY

Relations and Functions (Let *x* represent the independent variable and *y* the dependent variable.)

TYPE	MAPPING/CORRESPONDENCE	EQUATION	GRAPH
Relation	Every x-value in the domain maps to **at least one** y-value in the range.	$x = y^2$	
Function	Every x-value in the domain maps to **exactly one** y-value in the range.	$y = x^2$	 Passes vertical line test

All functions are relations, but not all relations are functions. Functions can be represented by equations. In the following table, each column illustrates an alternative notation.

INPUT	CORRESPONDENCE	OUTPUT	EQUATION
x	Function	y	$y = 2x + 5$
Independent variable	Mapping	Dependent variable	Mathematical rule
Argument	f	$f(x)$	$f(x) = 2x + 5$

The **domain** is the set of all inputs (x-values) and the **range** is the set of all corresponding outputs (y-values). Placeholder notation is useful when evaluating functions.

$$f(x) = 3x^2 + 2x$$

$$f(\square) = 3(\square)^2 + 2(\square)$$

An explicit domain is stated, whereas an **implicit domain** is found by *excluding* x-values that

- make the function undefined (denominator = 0).
- result in a nonreal output (even roots of negative real numbers).

SECTION 1.1 EXERCISES

SKILLS

In Exercises 1–18, determine whether each relation is a function. Assume that the coordinate pair (x, y) represents the independent variable x and the dependent variable y.

1. $\{(0, -3), (0, 3), (-3, 0), (3, 0)\}$

2. $\{(2, -2), (2, 2), (5, -5), (5, 5)\}$

3. $\{(0, 0), (9, -3), (4, -2), (4, 2), (9, 3)\}$

4. $\{(0, 0), (-1, -1), (-2, -8), (1, 1), (2, 8)\}$

5. $\{(0, 1), (1, 0), (2, 1), (-2, 1), (5, 4), (-3, 4)\}$

6. $\{(0, 1), (1, 1), (2, 1), (3, 1)\}$

7. $x^2 + y^2 = 9$

8. $x = |y|$

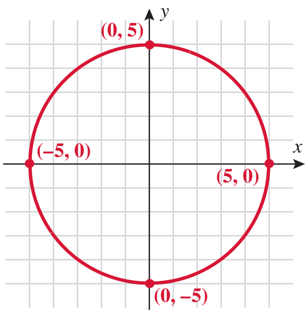
9. $x = y^2$

10. $y = x^3$

11. $y = |x - 1|$

12. $y = 3$

13.



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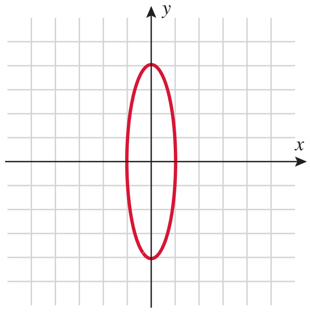

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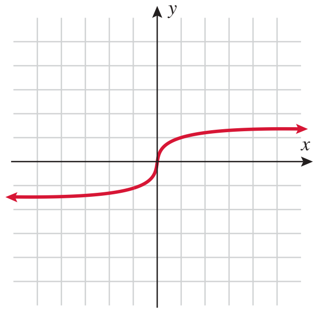

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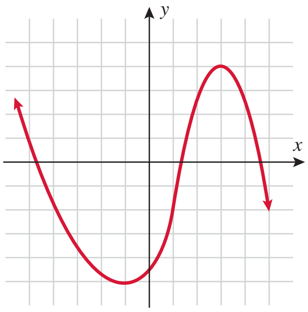

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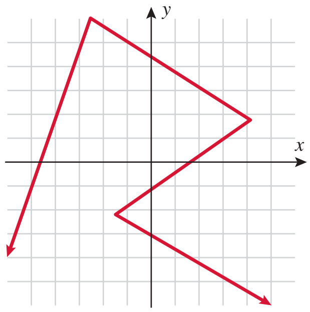
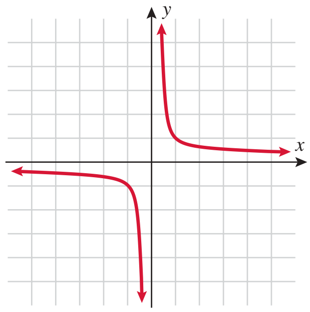
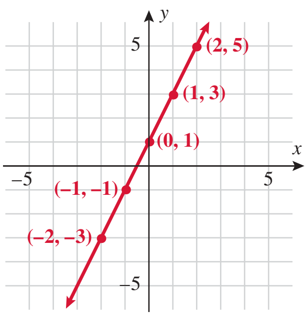


18.



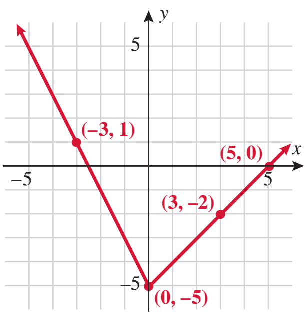
In Exercises 19–26, use the given graphs to evaluate the functions.

19. $y = f(x)$



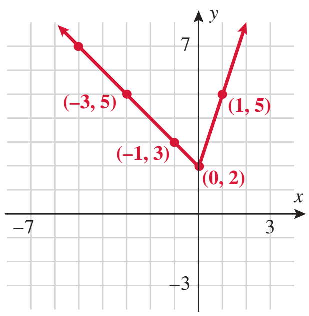
a. $f(2)$ **b.** $f(0)$ **c.** $f(-2)$

20. $y = g(x)$



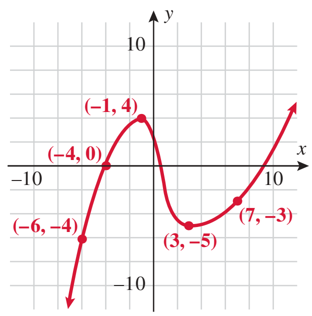
a. $g(-3)$ **b.** $g(0)$ **c.** $g(5)$

21. $y = p(x)$



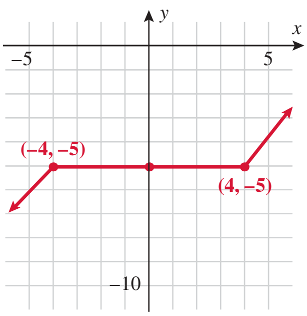
a. $p(-1)$ **b.** $p(0)$ **c.** $p(1)$

22. $y = r(x)$



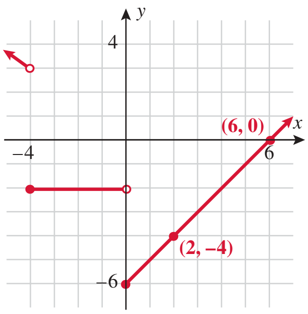
a. $r(-4)$ **b.** $r(-1)$ **c.** $r(3)$

23. $y = C(x)$



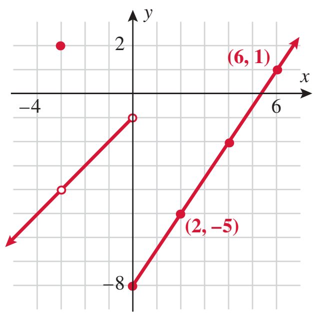
a. $C(2)$ **b.** $C(0)$ **c.** $C(-2)$

24. $y = q(x)$



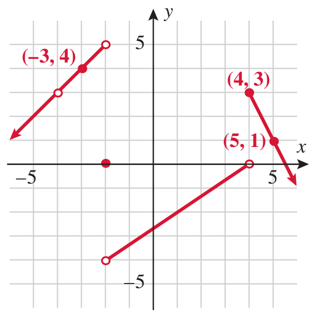
a. $q(-4)$ **b.** $q(0)$ **c.** $q(2)$

25. $y = S(x)$



a. $S(-3)$ **b.** $S(0)$ **c.** $S(2)$

26. $y = T(x)$



a. $T(-5)$ **b.** $T(-2)$ **c.** $T(4)$

27. Find x if $f(x) = 3$ in Exercise 19.

28. Find x if $g(x) = -2$ in Exercise 20.

29. Find x if $p(x) = 5$ in Exercise 21.

30. Find x if $C(x) = -7$ in Exercise 22.

31. Find x if $C(x) = -5$ in Exercise 23.

32. Find x if $q(x) = -2$ in Exercise 24.

33. Find x if $S(x) = 1$ in Exercise 25.

34. Find x if $T(x) = 4$ in Exercise 26.

In Exercises 35–50, evaluate the given quantities applying the following four functions:

$f(x) = 2x - 3 \qquad F(t) = 4 - t^2 \qquad g(t) = 5 + t \qquad G(x) = x^2 + 2x - 7$

35. $f(-2)$

36. $G(-3)$

37. $g(1)$

38. $F(-1)$

39. $f(-2) + g(1)$

40. $G(-3) - F(-1)$

41. $3f(-2) - 2g(1)$

42. $2F(-1) - 2G(-3)$

43. $\dfrac{f(-2)}{g(1)}$

44. $\dfrac{G(-3)}{F(-1)}$

45. $\dfrac{f(0) - f(-2)}{g(1)}$

46. $\dfrac{G(0) - G(-3)}{F(-1)}$

47. $f(x + 1) - f(x - 1)$

48. $F(t + 1) - F(t - 1)$

49. $g(x + a) - f(x + a)$

50. $G(x + b) + F(b)$

In Exercises 51–82, find the domain of the given function. Express the domain in interval notation.

51. $f(x) = 2x - 5$

52. $f(x) = -2x - 5$

53. $g(t) = t^2 + 3t$

54. $h(x) = 3x^4 - 1$

55. $P(x) = \dfrac{x + 5}{x - 5}$

56. $Q(t) = \dfrac{2 - t^2}{t + 3}$

57. $T(x) = \dfrac{2}{x^2 - 4}$

58. $R(x) = \dfrac{1}{x^2 - 1}$

59. $F(x) = \dfrac{1}{x^2 + 1}$

60. $G(t) = \dfrac{2}{t^2 + 4}$

61. $q(x) = \sqrt{7 - x}$

62. $k(t) = \sqrt{t - 7}$

63. $f(x) = \sqrt{2x + 5}$

64. $g(x) = \sqrt{5 - 2x}$

65. $G(t) = \sqrt{t^2 - 4}$

66. $F(x) = \sqrt{x^2 - 25}$

67. $F(x) = \dfrac{1}{\sqrt{x - 3}}$

68. $G(x) = \dfrac{2}{\sqrt{5 - x}}$

69. $f(x) = \sqrt[3]{1 - 2x}$

70. $g(x) = \sqrt[5]{7 - 5x}$

71. $P(x) = \dfrac{1}{\sqrt[5]{x + 4}}$

72. $Q(x) = \dfrac{x}{\sqrt[3]{x^2 - 9}}$

73. $R(x) = \dfrac{x + 1}{\sqrt[4]{3 - 2x}}$

74. $p(x) = \dfrac{x^2}{\sqrt{25 - x^2}}$

75. $H(t) = \dfrac{t}{\sqrt{t^2 - t - 6}}$

76. $f(t) = \dfrac{t - 3}{\sqrt[4]{t^2 + 9}}$

77. $f(x) = (x^2 - 16)^{1/2}$

78. $g(x) = (2x - 5)^{1/3}$

79. $r(x) = x^2(3 - 2x)^{-1/2}$

80. $p(x) = (x - 1)^2\,(x^2 - 9)^{-3/5}$

81. $f(x) = \frac{2}{5}x - \frac{2}{4}$

82. $g(x) = \frac{2}{3}x^2 - \frac{1}{6}x - \frac{3}{4}$

83. Let $g(x) = x^2 - 2x - 5$ and find the values of x that correspond to $g(x) = 3$.

84. Let $g(x) = \frac{5}{6}x - \frac{3}{4}$ and find the value of x that corresponds to $g(x) = \frac{2}{3}$.

85. Let $f(x) = 2x(x - 5)^3 - 12(x - 5)^2$ and find the values of x that correspond to $f(x) = 0$.

86. Let $f(x) = 3x(x + 3)^2 - 6(x + 3)^3$ and find the values of x that correspond to $f(x) = 0$.

APPLICATIONS

87. Temperature. The average temperature in Tampa, Florida in the springtime is given by the function $T(x) = -0.7x^2 + 16.8x - 10.8$, where T is the temperature in degrees Fahrenheit and x is the time of day in military time and is restricted to $6 \le x \le 18$ (sunrise to sunset). What is the temperature at 6 A.M.? What is the temperature at noon?

88. Temperature. The average temperature in Orlando, Florida in the summertime is given by the function $T(x) = -0.5x^2 + 14.2x - 2.8$, where T is the temperature in degrees Fahrenheit and x is the time of the day in military time and is restricted to $7 \le x \le 20$ (sunrise to sunset). What is the temperature at 9 A.M.? What is the temperature at 3 P.M.?

89. Falling Objects: Baseballs. A baseball is hit and its height is a function of time, $h(t) = -16t^2 + 45t + 1$, where h is the height in feet and t is the time in seconds, with $t = 0$ corresponding to the instant the ball is hit. What is the height after 2 seconds? What is the domain of this function?

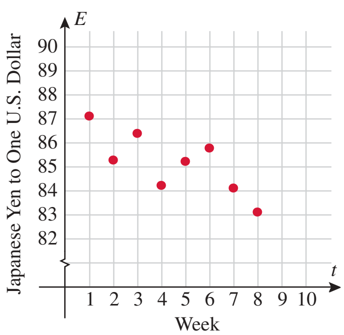
90. Falling Objects: Firecrackers. A firecracker is launched straight up, and its height is a function of time, $h(t) = -16t^2 + 128t$, where h is the height in feet and t is the time in seconds, with $t = 0$ corresponding to the instant it launches. What is the height 4 seconds after launch? What is the domain of this function?

91. Volume. An open box is constructed from a square 10-inch piece of cardboard by cutting squares of length x inches out of each corner and folding the sides up. Express the volume of the box as a function of x, and state the domain.

92. Volume. A cylindrical water basin will be built to harvest rainwater. The basin is limited in that the largest radius it can have is 10 feet. Write a function representing the volume of water V as a function of height h. How many additional gallons of water will be collected if you increase the height by 2 feet? *Hint:* 1 cu ft = 7.48 gal.

For Exercises 93–94, refer to the following:

The weekly exchange rate of the U.S. dollar to the Japanese yen is shown in the graph as varying over an 8-week period. Assume the exchange rate $E(t)$ is a function of time (week); let $E(1)$ be the exchange rate during Week 1.



93. Economics. Approximate the exchange rates of the U.S. dollar to the nearest yen during Weeks 4, 7, and 8.

94. Economics. Find the increase or decrease in the number of Japanese yen to the U.S. dollar exchange rate, to the nearest yen, from (a) Week 2 to Week 3 and (b) Week 6 to Week 7.

For Exercises 95–96, refer to the following:

An epidemiological study of the spread of malaria in a rural area finds that the total number P of people who contracted malaria t days into an outbreak is modeled by the function

$$P(t) = -\frac{1}{4}t^2 + 7t + 180 \qquad 1 \le t \le 14$$

95. Medicine/Health. How many people have contracted malaria 14 days into the outbreak?

96. Medicine/Health. How many people have contracted malaria 6 days into the outbreak?

97. Environment: Tossing the Envelopes. The average American adult receives 24 pieces of mail per week, usually of some combination of ads and envelopes with windows. Suppose each of these adults throws away a dozen envelopes per week.

a. The width of the window of an envelope is 3.375 inches less than its length x. Create the function $A(x)$ that represents the area of the window in square inches. Simplify, if possible.
b. Evaluate $A(4.5)$ and explain what this value represents.
c. Assume the dimensions of the envelope are 8 inches by 4 inches. Evaluate $A(8.5)$. Is this possible for this particular envelope? Explain.

98. Environment: Tossing the Envelopes. Each month, Jack receives his bank statement in a 9.5 inch by 6 inch envelope. Each month, he throws away the envelope after removing the statement.

a. The width of the window of the envelope is 2.875 inches less than its length x. Create the function $A(x)$ that represents the area of the window in square inches. Simplify, if possible.
b. Evaluate $A(5.25)$ and explain what this value represents.
c. Evaluate $A(10)$. Is this possible for this particular envelope? Explain.

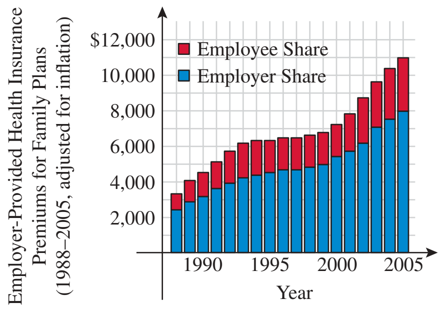
For Exercises 99 and 100, refer to the table below. It illustrates the average federal funds rate for the month of January (2000 to 2008).

Year	Fed. Rate
2000	5.45
2001	5.98
2002	1.73
2003	1.24
2004	1.00
2005	2.25
2006	4.50
2007	5.25
2008	3.50

99. Finance. Is the relation whose domain is the year and whose range is the average federal funds rate for the month of January a function? Explain.

100. Finance. Write five ordered pairs whose domain is the set of even years from 2000 to 2008 and whose range is the set of corresponding average federal funds rate for the month of January.

For Exercises 101 and 102, use the following figure:



Source: Kaiser Family Foundation Health Research and Education Trust. *Note:* The following years were interpolated: 1989–1992; 1994–1995; 1997–1998.

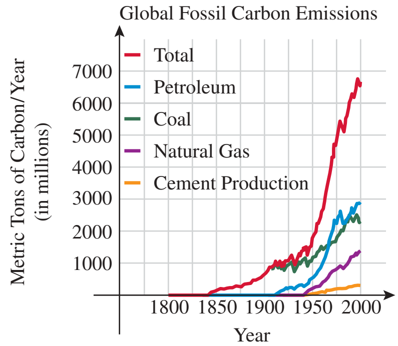
101. Health-Care Costs: Fill in the following table. Round dollars to the nearest $1000.

Year	Total Health-Care Cost for Family Plans
1989	
1993	
1997	
2001	
2005	

Write the five ordered pairs resulting from the table.

102. Health-Care Costs. Using the table found in Exercise 101, let the years correspond to the domain and the total costs correspond to the range. Is this relation a function? Explain.

For Exercises 103 and 104, use the following information:



Source: http:/www.naftc.wvu.edu

Let the functions f, F, g, G, and H represent the number of tons of carbon emitted per year as a function of year corresponding to cement production, natural gas, coal, petroleum, and the total amount, respectively. Let t represent the year, with $t = 0$ corresponding to 1900.

103. Environment: Global Climate Change. Estimate (to the nearest thousand) the value of

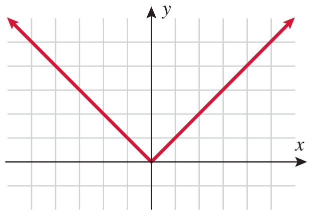
a. $F(50)$ **b.** $g(50)$ **c.** $H(50)$

104. Environment: Global Climate Change. Explain what the sum $F(100) + g(100) + G(100)$ represents.

CATCH THE MISTAKE

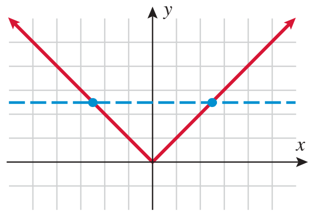
In Exercises 105–110, explain the mistake that is made.

105. Determine whether the relationship is a function.



Solution:

Apply the horizontal line test.



Because the horizontal line intersects the graph in two places, this is not a function.

This is incorrect. What mistake was made?

106. Given the function $H(x) = 3x - 2$, evaluate the quantity $H(3) - H(-1)$.

Solution: $H(3) - H(-1) = H(3) + H(1) = 7 + 1 = 8$

This is incorrect. What mistake was made?

107. Given the function $f(x) = x^2 - x$, evaluate the quantity $f(x + 1)$.

Solution: $f(x + 1) = f(x) + f(1) = x^2 - x + 0$

$f(x + 1) = x^2 - x$

This is incorrect. What mistake was made?

108. Determine the domain of the function $g(t) = \sqrt{3 - t}$ and express it in interval notation.

Solution:

What can t be? Any nonnegative real number.

$3 - t > 0$

$3 > t$ or $t < 3$

Domain: $(-\infty, 3)$

This is incorrect. What mistake was made?

109. Given the function $G(x) = x^2$, evaluate $\dfrac{G(-1 + h) - G(-1)}{h}$.

Solution:

$$\frac{G(-1 + h) - G(-1)}{h} = \frac{G(-1) + G(h) - G(-1)}{h}$$

$$= \frac{G(h)}{h} = \frac{h^2}{h} = h$$

This is incorrect. What mistake was made?

110. Given the functions $f(x) = |x - A| - 1$ and $f(1) = -1$, find A.

Solution:

Since $f(1) = -1$, the point $(-1, 1)$ must satisfy the function. $\quad -1 = |-1 - A| - 1$

Add 1 to both sides of the equation. $\quad |-1 - A| = 0$

The absolute value of zero is zero, so there is no need for the absolute value signs: $-1 - A = 0 \Rightarrow A = -1$.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 111–116, determine whether each statement is true or false.

111. If a vertical line does not intersect the graph of an equation, then that equation does not represent a function.

112. If a horizontal line intersects a graph of an equation more than once, the equation does not represent a function.

113. For $x = y^2$, x is a function of y.

114. For $y = x^2$, y is a function of x.

115. If $f(x) = Ax^2 - 3x$ and $f(1) = -1$, find A.

116. If $g(x) = \dfrac{1}{b - x}$ and $g(3)$ is undefined, find b.

CHALLENGE

117. If $F(x) = \dfrac{C - x}{D - x}$, $F(-2)$ is undefined, and $F(-1) = 4$, find C and D.

118. Construct a function that is undefined at $x = 5$ and whose graph passes through the point $(1, -1)$.

In Exercises 119 and 120, find the domain of each function, where *a* is any positive real number.

119. $f(x) = \dfrac{-100}{x^2 - a^2}$

120. $f(x) = -5\sqrt{x^2 - a^2}$

TECHNOLOGY

121. Using a graphing utility, graph the temperature function in Exercise 87. What time of day is it the warmest? What is the temperature? Looking at this function, explain why this model for Tampa, Florida, is valid only from sunrise to sunset (6 to 18).

122. Using a graphing utility, graph the height of the firecracker in Exercise 90. How long after liftoff is the firecracker airborne? What is the maximum height that the firecracker attains? Explain why this height model is valid only for the first 8 seconds.

123. Let $f(x) = x^2 + 1$. Graph $y_1 = f(x)$ and $y_2 = f(x - 2)$ in the same viewing window. Describe how the graph of y_2 can be obtained from the graph of y_1.

124. Let $f(x) = 4 - x^2$. Graph $y_1 = f(x)$ and $y_2 = f(x + 2)$ in the same viewing window. Describe how the graph of y_2 can be obtained from the graph of y_1.

PREVIEW TO CALCULUS

For Exercises 125–128, refer to the following:

In calculus, the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ of a function f is used to find a new function f', called the *derivative of f*. To find f', we let h approach 0, $h \to 0$, in the difference quotient. For example, if $f(x) = x^2$, $\dfrac{f(x+h) - f(x)}{h} = 2x + h$, and allowing $h = 0$, we have $f'(x) = 2x$.

125. Given $f(x) = x^3 + x$, find $f'(x)$.

126. Given $f(x) = 6x + \sqrt{x}$, find $f'(x)$.

127. Given $f(x) = \dfrac{x - 5}{x + 3}$, find $f'(x)$.

128. Given $f(x) = \sqrt{\dfrac{x + 7}{5 - x}}$, find $f'(x)$.

SECTION 1.2 GRAPHS OF FUNCTIONS

SKILLS OBJECTIVES

- Classify functions as even, odd, or neither.
- Determine whether functions are increasing, decreasing, or constant.
- Calculate the average rate of change of a function.
- Evaluate the difference quotient for a function.
- Graph piecewise-defined functions.

CONCEPTUAL OBJECTIVES

- Identify common functions.
- Develop and graph piecewise-defined functions.
  - Identify and graph points of discontinuity.
  - State the domain and range.
- Understand that even functions have graphs that are symmetric about the y-axis and odd functions have graphs that are symmetric about the origin.
- Understand that the difference quotient is an average rate of change.

Recognizing and Classifying Functions

Common Functions

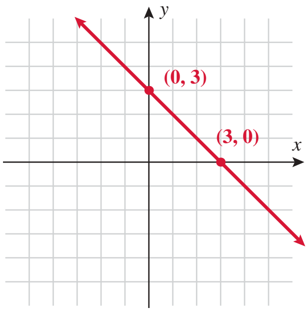
The nine main functions you will read about in this section will constitute a "library" of functions that you should commit to memory. We will draw on this library of functions in the next section when graphing transformations are discussed.

In Section 0.6, we discussed equations and graphs of lines. All lines (with the exception of vertical lines) pass the vertical line test, and hence are classified as functions. Instead of the traditional notation of a line, $y = mx + b$, we use function notation and classify a function whose graph is a *line* as a *linear* function.

LINEAR FUNCTION

$f(x) = mx + b$ $\qquad$ m and b are real numbers.

Linear Function
Domain: $(-\infty, \infty)$



The domain of a linear function $f(x) = mx + b$ is the set of all real numbers $\mathbb{R}$. The graph of this function has slope m and y-intercept b.

LINEAR FUNCTION: $f(x) = mx + b$	SLOPE: m	y-INTERCEPT: b
$f(x) = 2x - 7$	$m = 2$	$b = -7$
$f(x) = -x + 3$	$m = -1$	$b = 3$
$f(x) = x$	$m = 1$	$b = 0$
$f(x) = 5$	$m = 0$	$b = 5$

One special case of the linear function is the *constant function* ($m = 0$).

CONSTANT FUNCTION

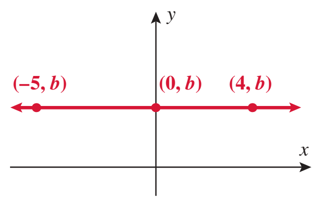
$f(x) = b$ $\qquad$ b is any real number.

The graph of a constant function $f(x) = b$ is a horizontal line. The y-intercept corresponds to the point $(0, b)$. The domain of a constant function is the set of all real numbers $\mathbb{R}$. The range, however, is a single value b. In other words, all x-values correspond to a single y-value.

Points that lie on the graph of a constant function $f(x) = b$ are

$(-5, b)$
$(-1, b)$
$(0, b)$
$(2, b)$
$(4, b)$
. . .
(x, b)

Domain: $(-\infty, \infty)$ $\quad$ Range: $[b, b]$ or $\{b\}$

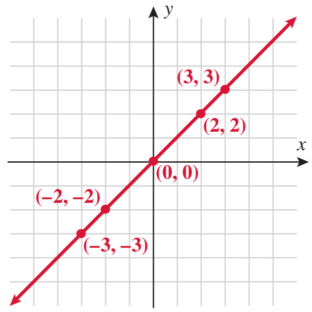


Another specific example of a linear function is the function having a slope of one ($m = 1$) and a y-intercept of zero ($b = 0$). This special case is called the *identity function.*

IDENTITY FUNCTION

$$f(x) = x$$

Identity Function
Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$



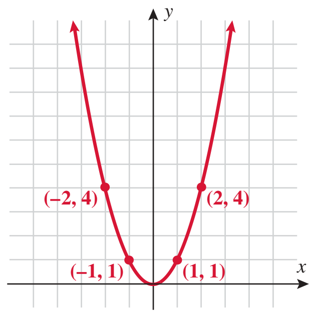
The graph of the identity function has the following properties: it passes through the origin, and every point that lies on the line has equal x- and y-coordinates. Both the domain and the range of the identity function are the set of all real numbers $\mathbb{R}$.

A function that squares the input is called the *square function*.

SQUARE FUNCTION

$$f(x) = x^2$$

Square Function
Domain: $(-\infty, \infty)$ Range: $[0, \infty)$



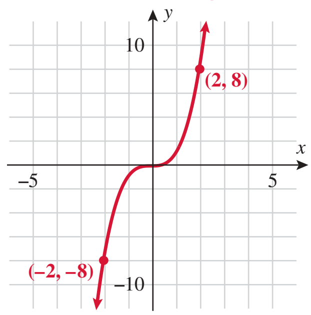
The graph of the square function is called a parabola and will be discussed in further detail in Chapter 9. The domain of the square function is the set of all real numbers $\mathbb{R}$. Because squaring a real number always yields a positive number or zero, the range of the square function is the set of all nonnegative numbers. Note that the only intercept is the origin and the square function is symmetric about the y-axis. This graph is contained in quadrants I and II.

A function that cubes the input is called the *cube function*.

CUBE FUNCTION

$$f(x) = x^3$$

Cube Function
Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$



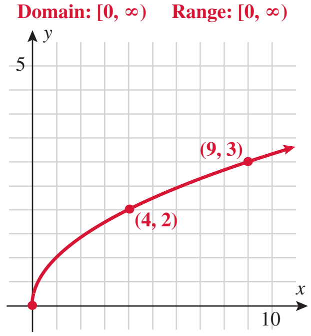
The domain of the cube function is the set of all real numbers $\mathbb{R}$. Because cubing a negative number yields a negative number, cubing a positive number yields a positive number, and cubing 0 yields 0, the range of the cube function is also the set of all real numbers $\mathbb{R}$. Note that the only intercept is the origin and the cube function is symmetric about the origin. This graph extends only into quadrants I and III.

The next two functions are counterparts of the previous two functions: square root and cube root. When a function takes the square root of the input or the cube root of the input, the function is called the *square root function* or the *cube root function*, respectively.

SQUARE ROOT FUNCTION

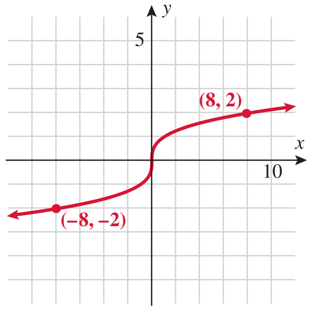
$$f(x) = \sqrt{x} \quad \text{or} \quad f(x) = x^{1/2}$$

Square Root Function
Domain: $[0, \infty)$ Range: $[0, \infty)$

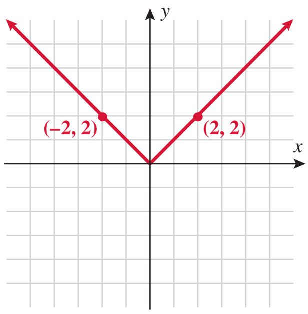


In Section 1.1, we found the domain to be $[0, \infty)$. The output of the function will be all real numbers greater than or equal to zero. Therefore, the range of the square root function is $[0, \infty)$. The graph of this function will be contained in quadrant I.

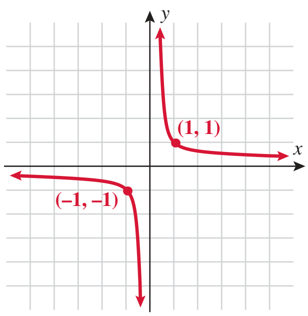
Cube Root Function
Domain: (−∞, ∞) Range: (−∞, ∞)



Absolute Value Function
Domain: (−∞, ∞) Range: [0, ∞)



Reciprocal Function
Domain: (−∞, 0) ∪ (0, ∞)
Range: (−∞, 0) ∪ (0, ∞)



CUBE ROOT FUNCTION

$$f(x) = \sqrt[3]{x} \quad \text{or} \quad f(x) = x^{1/3}$$

In Section 1.1, we stated the domain of the cube root function to be $(-\infty, \infty)$. We see by the graph that the range is also $(-\infty, \infty)$. This graph is contained in quadrants I and III and passes through the origin. This function is symmetric about the origin.

In Sections 0.3 and 0.4, absolute value equations and inequalities were reviewed. Now we shift our focus to the graph of the *absolute value function.*

ABSOLUTE VALUE FUNCTION

$$f(x) = |x|$$

Some points that are on the graph of the absolute value function are $(-1, 1)$, $(0, 0)$, and $(1, 1)$. The domain of the absolute value function is the set of all real numbers $\mathbb{R}$, yet the range is the set of nonnegative real numbers. The graph of this function is symmetric with respect to the y-axis and is contained in quadrants I and II.

A function whose output is the reciprocal of its input is called the *reciprocal function.*

RECIPROCAL FUNCTION

$$f(x) = \frac{1}{x} \qquad x \neq 0$$

The only restriction on the domain of the reciprocal function is that $x \neq 0$. Therefore, we say the domain is the set of all real numbers excluding zero. The graph of the reciprocal function illustrates that its range is also the set of all real numbers except zero. Note that the reciprocal function is symmetric with respect to the origin and is contained in quadrants I and III.

Even and Odd Functions

Of the nine functions discussed above, several have similar properties of symmetry. The constant function, square function, and absolute value function are all symmetric with respect to the y-axis. The identity function, cube function, cube root function, and reciprocal function are all symmetric with respect to the origin. The term **even** is used to describe functions that are symmetric with respect to the y-axis, or vertical axis, and the term **odd** is used to describe functions that are symmetric with respect to the origin. Recall from Section 0.5 that symmetry can be determined both graphically and algebraically. The box below summarizes the graphic and algebraic characteristics of even and odd functions.

EVEN AND ODD FUNCTIONS

Function	Symmetric with Respect to	On Replacing x with $-x$
Even	y-axis or vertical axis	$f(-x) = f(x)$
Odd	origin	$f(-x) = -f(x)$

The algebraic method for determining symmetry with respect to the y-axis, or vertical axis, is to substitute in $-x$ for x. If the result is an equivalent equation, the function is symmetric with respect to the y-axis. Some examples of even functions are $f(x) = b$, $f(x) = x^2$, $f(x) = x^4$, and $f(x) = |x|$. In any of these equations, if $-x$ is substituted for x, the result is the same, that is, $f(-x) = f(x)$. Also note that, with the exception of the absolute value function, these examples are all even-degree polynomial equations. All constant functions are degree zero and are even functions.

The algebraic method for determining symmetry with respect to the origin is to substitute $-x$ for x. If the result is the negative of the original function, that is, if $f(-x) = -f(x)$, then the function is symmetric with respect to the origin and, hence, classified as an odd function. Examples of odd functions are $f(x) = x$, $f(x) = x^3$, $f(x) = x^5$, and $f(x) = x^{1/3}$. In any of these functions, if $-x$ is substituted for x, the result is the negative of the original function. Note that with the exception of the cube root function, these equations are odd-degree polynomials.

Be careful, though, because functions that are combinations of even- and odd-degree polynomials can turn out to be neither even nor odd, as we will see in Example 1.

EXAMPLE 1 Determining Whether a Function Is Even, Odd, or Neither

Determine whether the functions are even, odd, or neither.

a. $f(x) = x^2 - 3$ **b.** $g(x) = x^5 + x^3$ **c.** $h(x) = x^2 - x$

Solution (a):

Original function. $f(x) = x^2 - 3$

Replace x with $-x$. $f(-x) = (-x)^2 - 3$

Simplify. $f(-x) = x^2 - 3 = f(x)$

Because $f(-x) = f(x)$, we say that $\boxed{f(x) \text{ is an } \textit{even} \text{ function}}$.

Solution (b):

Original function. $g(x) = x^5 + x^3$

Replace x with $-x$. $g(-x) = (-x)^5 + (-x)^3$

Simplify. $g(-x) = -x^5 - x^3 = -(x^5 + x^3) = -g(x)$

Because $g(-x) = -g(x)$, we say that $\boxed{g(x) \text{ is an } \textit{odd} \text{ function}}$.

Solution (c):

Original function. $h(x) = x^2 - x$

Replace x with $-x$. $h(-x) = (-x)^2 - (-x)$

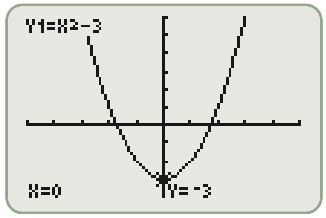
Simplify. $h(-x) = x^2 + x$

$h(-x)$ **is neither** $-h(x)$ **nor** $h(x)$; therefore, the function $h(x)$ is $\boxed{\text{neither even nor odd}}$.

In parts (a), (b), and (c), we classified these functions as either even, odd, or neither, using the algebraic test. Look back at them now and reflect on whether these classifications agree with your intuition. In part (a), we combined two functions: the square function and the constant function. Both of these functions are even, and adding even functions yields another even function. In part (b), we combined two odd functions: the fifth-power function and the cube function. Both of these functions are odd, and adding two odd functions yields another odd function. In part (c), we combined two functions: the square function and the identity function. The square function is even, and the identity function is odd. In this part, combining an even function with an odd function yields a function that is neither even nor odd and, hence, has no symmetry with respect to the vertical axis or the origin.

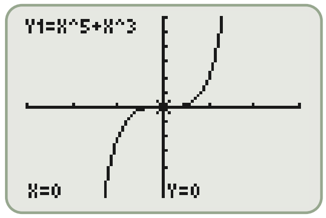
Technology Tip

a. Graph $y_1 = f(x) = x^2 - 3$.



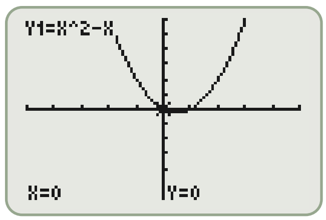
Even; symmetric with respect to the y-axis.

b. Graph $y_1 = g(x) = x^5 + x^3$.



Odd; symmetric with respect to origin.

c. Graph $y_1 = h(x) = x^2 - x$.



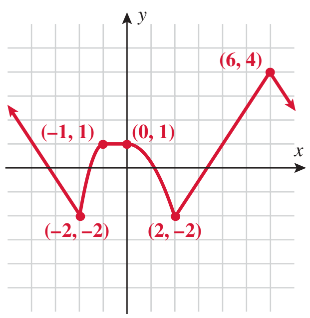
No symmetry with respect to y-axis or origin.

■ **YOUR TURN** Classify the functions as even, odd, or neither.

a. $f(x) = |x| + 4$ **b.** $f(x) = x^3 - 1$

■ **Answer:** **a.** even **b.** neither

Increasing and Decreasing Functions



Look at the figure on the left. Graphs are read from *left to right*. If we start at the left side of the graph and trace the red curve, we see that the function values (values in the vertical direction) are decreasing until arriving at the point $(-2, -2)$. Then, the function values increase until arriving at the point $(-1, 1)$. The values then remain constant $(y = 1)$ between the points $(-1, 1)$ and $(0, 1)$. Proceeding beyond the point $(0, 1)$, the function values decrease again until the point $(2, -2)$. Beyond the point $(2, -2)$, the function values increase again until the point $(6, 4)$. Finally, the function values decrease and continue to do so.

When specifying a function as increasing, decreasing, or constant, the *intervals are classified according to the x-coordinate*. For instance, in this graph, we say the function is increasing when x is between $x = -2$ and $x = -1$ and again when x is between $x = 2$ and $x = 6$. The graph is classified as decreasing when x is less than -2 and again when x is between 0 and 2 and again when x is greater than 6. The graph is classified as constant when x is between -1 and 0. In interval notation, this is summarized as

Decreasing	Increasing	Constant
$(-\infty, -2) \cup (0, 2) \cup (6, \infty)$	$(-2, -1) \cup (2, 6)$	$(-1, 0)$

An algebraic test for determining whether a function is increasing, decreasing, or constant is to compare the value $f(x)$ of the function for particular points in the intervals.

Study Tip

- Graphs are read from left to right.
- Intervals correspond to the x-coordinates.

Study Tip

Increasing: Graph of function rises from left to right.
Decreasing: Graph of function falls from left to right.
Constant: Graph of function does not change height from left to right.

Increasing, Decreasing, and Constant Functions

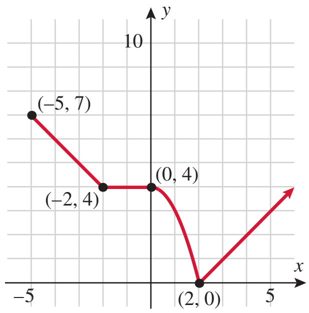
1. A function f is **increasing** on an open interval I if for any x_1 and x_2 in I, where $x_1 < x_2$, then $f(x_1) < f(x_2)$.
2. A function f is **decreasing** on an open interval I if for any x_1 and x_2 in I, where $x_1 < x_2$, then $f(x_1) > f(x_2)$.
3. A function f is **constant** on an open interval I if for any x_1 and x_2 in I, then $f(x_1) = f(x_2)$.

In addition to classifying a function as increasing, decreasing, or constant, we can also determine the domain and range of a function by inspecting its graph from left to right:

- The domain is the set of all x-values where the function is defined.
- The range is the set of all y-values that the graph of the function corresponds to.
- A solid dot on the left or right end of a graph indicates that the graph terminates there and the point is included in the graph.
- An open dot indicates that the graph terminates there and the point is not included in the graph.
- Unless a dot is present, it is assumed that a graph continues indefinitely in the same direction. (An arrow is used in some books to indicate direction.)

EXAMPLE 2 **Finding Intervals When a Function Is Increasing or Decreasing**

Given the graph of a function:



a. State the domain and range of the function.

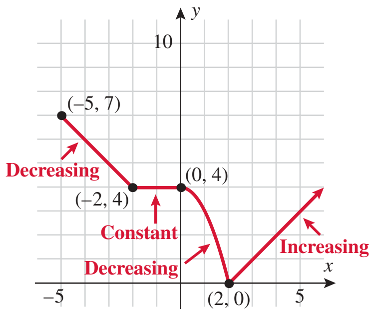
b. Find the intervals when the function is increasing, decreasing, or constant.

Solution:

Domain: $[-5, \infty)$

Range: $[0, \infty)$

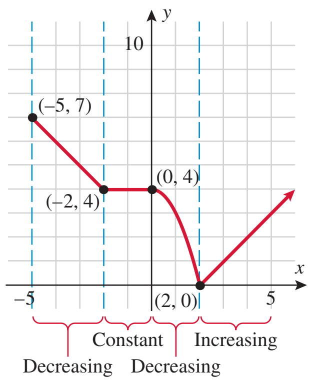
Reading the graph from **left to right**, we see that the graph



- decreases from the point $(-5, 7)$ to the point $(-2, 4)$.
- is constant from the point $(-2, 4)$ to the point $(0, 4)$.
- decreases from the point $(0, 4)$ to the point $(2, 0)$.
- increases from the point $(2, 0)$ on.

The intervals of increasing and decreasing correspond to the ***x*-coordinates**.

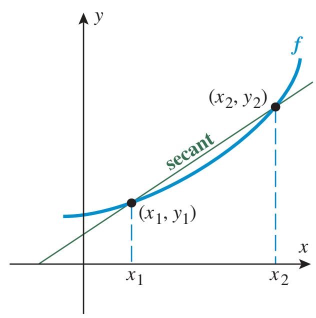
We say that this function is



- increasing on the interval $(2, \infty)$.
- decreasing on the interval $(-5, -2) \cup (0, 2)$.
- constant on the interval $(-2, 0)$.

Note: The intervals of increasing or decreasing are defined on *open* intervals. This should not be confused with the domain. For example, the point $x = -5$ is included in the domain of the function but not in the interval where the function is classified as decreasing.

Average Rate of Change



How do we know *how much* a function is increasing or decreasing? For example, is the price of a stock slightly increasing or is it doubling every week? One way we determine how much a function is increasing or decreasing is by calculating its *average rate of change*.

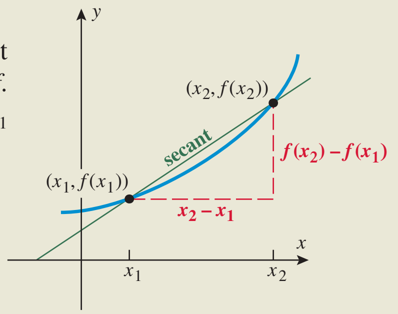
Let (x_1, y_1) and (x_2, y_2) be two points that lie on the graph of a function f. Draw the line that passes through these two points (x_1, y_1) and (x_2, y_2). This line is called a **secant line**.

Note that the slope of the secant line is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, and recall that the slope of a line is the rate of change of that line. The **slope of the secant line** is used to represent the *average rate of change* of the function.

AVERAGE RATE OF CHANGE

Let $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two distinct points, $(x_1 \neq x_2)$, on the graph of the function f. The **average rate of change** of f between x_1 and x_2 is given by

$$\textbf{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



EXAMPLE 3 Average Rate of Change

Find the average rate of change of $f(x) = x^4$ from

a. $x = -1$ to $x = 0$ **b.** $x = 0$ to $x = 1$ **c.** $x = 1$ to $x = 2$

Solution (a):

Write the average rate of change formula. $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Let $x_1 = -1$ and $x_2 = 0$. $= \dfrac{f(0) - f(-1)}{0 - (-1)}$

Substitute $f(-1) = (-1)^4 = 1$ and $f(0) = 0^4 = 0$. $= \dfrac{0 - 1}{0 - (-1)}$

Simplify. $= \boxed{-1}$

Solution (b):

Write the average rate of change formula. $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Let $x_1 = 0$ and $x_2 = 1$. $= \dfrac{f(1) - f(0)}{1 - 0}$

Substitute $f(0) = 0^4 = 0$ and $f(1) = (1)^4 = 1$. $= \dfrac{1 - 0}{1 - 0}$

Simplify. $= \boxed{1}$

Solution (c):

Write the average rate of change formula. $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

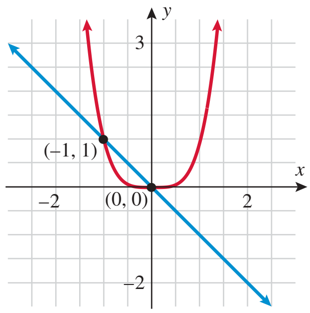
Let $x_1 = 1$ and $x_2 = 2$. $= \dfrac{f(2) - f(1)}{2 - 1}$

Substitute $f(1) = 1^4 = 1$ and $f(2) = (2)^4 = 16$. $= \dfrac{16 - 1}{2 - 1}$

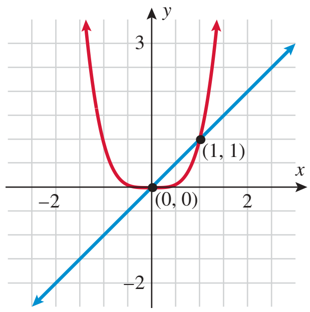
Simplify. $= \boxed{15}$

Graphical Interpretation: Slope of the Secant Line

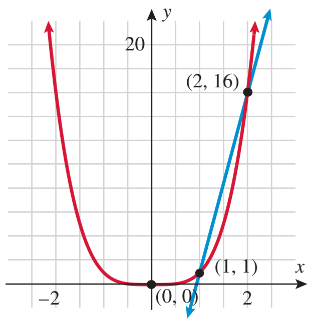
a. Between $(-1, 1)$ and $(0, 0)$, this function is decreasing at a rate of 1.



b. Between $(0, 0)$ and $(1, 1)$, this function is increasing at a rate of 1.



c. Between $(1, 1)$ and $(2, 16)$, this function is increasing at a rate of 15.



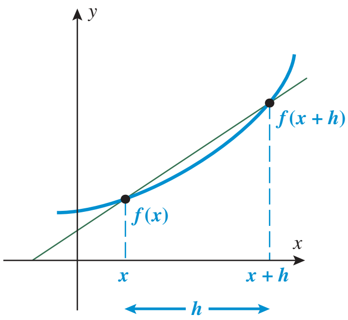
■ **YOUR TURN** Find the average rate of change of $f(x) = x^2$ from

a. $x = -2$ to $x = 0$ **b.** $x = 0$ to $x = 2$

■ **Answer: a.** -2 **b.** 2

The average rate of change can also be written in terms of the *difference quotient.*

WORDS	MATH
Let the distance between x_1 and x_2 be h.	$x_2 - x_1 = h$
Solve for x_2.	$x_2 = x_1 + h$
Substitute $x_2 - x_1 = h$ into the denominator and $x_2 = x_1 + h$ into the numerator of the average rate of change.	Average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$
	$= \dfrac{f(x_1 + h) - f(x_1)}{h}$
Let $x_1 = x$.	$= \boxed{\dfrac{f(x + h) - f(x)}{h}}$



When written in this form, the average rate of change is called the **difference quotient**.

DEFINITION **Difference Quotient**

The expression $\dfrac{f(x+h)-f(x)}{h}$, where $h \neq 0$, is called the **difference quotient**.

The difference quotient is more meaningful when h is small. In calculus the difference quotient is used to define a *derivative*.

Study Tip

Use brackets or parentheses around $f(x)$ to avoid forgetting to distribute the negative sign:

$$\frac{f(x+h)-[f(x)]}{h}$$

EXAMPLE 4 Calculating the Difference Quotient

Calculate the difference quotient for the function $f(x) = 2x^2 + 1$.

Solution:

Find $f(x+h)$.
$$f(x+h) = 2(x+h)^2 + 1$$
$$= 2(x^2 + 2xh + h^2) + 1$$
$$= 2x^2 + 4xh + 2h^2 + 1$$

Find the difference quotient.
$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{2x^2 + 4xh + 2h^2 + 1}^{f(x+h)} - \overbrace{(2x^2+1)}^{f(x)}}{h}$$

Simplify.
$$= \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+1} \cancel{-2x^2} \cancel{-1}}{h}$$
$$= \frac{4xh + 2h^2}{h}$$

Factor the numerator.
$$= \frac{h(4x+2h)}{h}$$

Cancel (divide out) the common h.
$$= \boxed{4x + 2h} \qquad h \neq 0$$

YOUR TURN Calculate the difference quotient for the function $f(x) = -x^2 + 2$.

Answer:
$$\frac{f(x+h)-f(x)}{h} = -2x - h$$

EXAMPLE 5 Evaluating the Difference Quotient

For the function $f(x) = x^2 - x$, find $\dfrac{f(x+h)-f(x)}{h}$.

Solution:

Use placeholder notation for the function $f(x) = x^2 - x$.
$$f(\square) = (\square)^2 - (\square)$$

Calculate $f(x+h)$.
$$f(x+h) = (x+h)^2 - (x+h)$$

Write the difference quotient.
$$\frac{f(x+h)-f(x)}{h}$$

Let $f(x+h) = (x+h)^2 - (x+h)$ and $f(x) = x^2 - x$.

$$\frac{f(x+h)-f(x)}{h} = \frac{\overbrace{\left[(x+h)^2 - (x+h)\right]}^{f(x+h)} - \overbrace{\left[x^2 - x\right]}^{f(x)}}{h} \qquad h \neq 0$$

Eliminate the parentheses inside the first set of brackets. $$= \frac{[x^2 + 2xh + h^2 - x - h] - [x^2 - x]}{h}$$

Eliminate the brackets in the numerator. $$= \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h}$$

Combine like terms. $$= \frac{2xh + h^2 - h}{h}$$

Factor the numerator. $$= \frac{h(2x + h - 1)}{h}$$

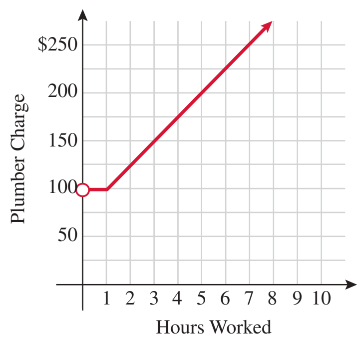
Divide out the common factor, h. $$= \boxed{2x + h - 1} \qquad h \neq 0$$

■ **YOUR TURN** Evaluate the difference quotient for $f(x) = x^2 - 1$.

■ **Answer:** $2x + h$

Piecewise-Defined Functions

Most of the functions that we have seen in this text are functions defined by polynomials. Sometimes the need arises to define functions in terms of *pieces*. For example, most plumbers charge a flat fee for a house call and then an additional hourly rate for the job. For instance, if a particular plumber charges \$100 to drive out to your house and work for 1 hour and then \$25 an hour for every additional hour he or she works on your job, we would define this function in pieces. If we let h be the number of hours worked, then the charge is defined as



$$\text{Plumbing charge} = \begin{cases} 100 & 0 < h \leq 1 \\ 100 + 25(h - 1) & h > 1 \end{cases}$$

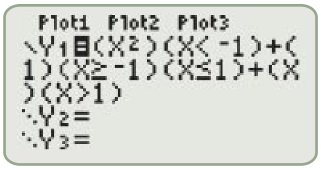
We can see in the graph of this function that there is 1 hour that is constant and after that the function continually increases.

The next example is a piecewise-defined function given in terms of pieces of functions from our "library of functions." Because the function is defined in terms of pieces of other functions, we draw the graph of each individual function and, then, for each function darken the piece corresponding to its part of the domain.

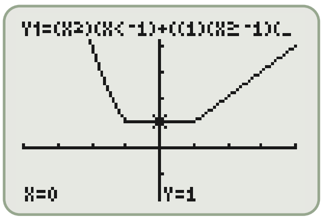
Technology Tip

Plot a piecewise-defined function using the TEST menu operations to define the inequalities in the function. Press:

Y= (x, T, θ, n x²) (x, T, θ, n 2nd TEST 5:< ENTER (−) 1) + (1) (x, T, θ, n 2nd TEST 4:≥ ENTER (−) 1) (x, T, θ, n 2nd TEST 6:≤ ENTER 1) + (x, T, θ, n) (x, T, θ, n 2nd TEST 3:> ENTER 1)



Set the viewing rectangle as $[-4, 4]$ by $[-2, 5]$; then press GRAPH.



EXAMPLE 6 Graphing Piecewise-Defined Functions

Graph the piecewise-defined function, and state the domain, range, and intervals when the function is increasing, decreasing, or constant.

$$G(x) = \begin{cases} x^2 & x < -1 \\ 1 & -1 \le x \le 1 \\ x & x > 1 \end{cases}$$

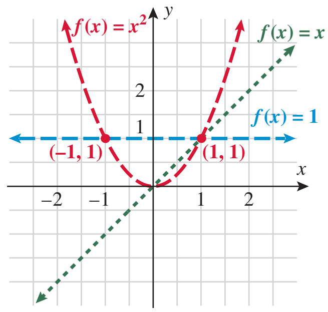
Solution:

Graph each of the functions on the same plane.

Square function:
$f(x) = x^2$

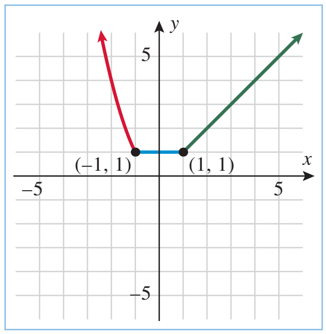
Constant function:
$f(x) = 1$

Identity function:
$f(x) = x$



The points to focus on in particular are the x-values where the pieces change over—that is, $x = -1$ and $x = 1$.

Let's now investigate each piece. When $x < -1$, this function is defined by the square function, $f(x) = x^2$, so darken that particular function to the left of $x = -1$. When $-1 \le x \le 1$, the function is defined by the constant function, $f(x) = 1$, so darken that particular function between the x-values of -1 and 1. When $x > 1$, the function is defined by the identity function, $f(x) = x$, so darken that function to the right of $x = 1$. Erase everything that is not darkened, and the resulting graph of the piecewise-defined function is given on the right.



This function is defined for all real values of x, so the domain of this function is the set of all real numbers. The values that this function yields in the vertical direction are all real numbers greater than or equal to 1. Hence, the range of this function is $[1, \infty)$. The intervals of increasing, decreasing, and constant are as follows:

$$\text{Decreasing: } (-\infty, -1)$$

$$\text{Constant: } (-1, 1)$$

$$\text{Increasing: } (1, \infty)$$

The term **continuous** implies that there are no holes or jumps and that the graph can be drawn without picking up your pencil. A function that does have holes or jumps and cannot be drawn in one motion without picking up your pencil is classified as **discontinuous**, and the points where the holes or jumps occur are called *points of discontinuity*.

The previous example illustrates a *continuous* piecewise-defined function. At the $x = -1$ junction, the square function and constant function both pass through the point $(-1, 1)$. At the $x = 1$ junction, the constant function and the identity function both pass through the point $(1, 1)$. Since the graph of this piecewise-defined function has no holes or jumps, we classify it as a continuous function.

The next example illustrates a *discontinuous* piecewise-defined function.

EXAMPLE 7 Graphing a Discontinuous Piecewise-Defined Function

Graph the piecewise-defined function, and state the intervals where the function is increasing, decreasing, or constant, along with the domain and range.

$$f(x) = \begin{cases} 1 - x & x < 0 \\ x & 0 \le x < 2 \\ -1 & x > 2 \end{cases}$$

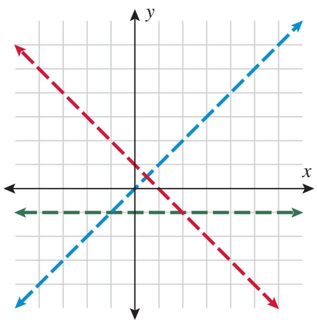
Solution:

Graph these functions on the same plane.

Linear function:
$f(x) = 1 - x$

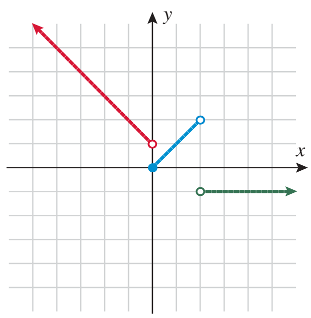
Identity function:
$f(x) = x$

Constant function:
$f(x) = -1$



Darken the piecewise-defined function on the graph. For all values less than zero **$(x < 0)$**, the function is defined by the **linear function**. Note the use of an open circle, indicating up to but not including $x = 0$. For values **$0 \le x < 2$**, the function is defined by the **identity function**.

The circle is filled in at the left endpoint, $x = 0$. An open circle is used at $x = 2$. For all values greater than 2, **$x > 2$**, the function is defined by the **constant function**. Because this interval does not include the point $x = 2$, an open circle is used.



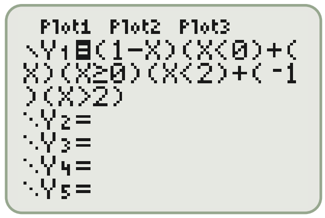
At what intervals is the function increasing, decreasing, or constant? Remember that the intervals correspond to the x-values.

Decreasing: **$(-\infty, 0)$** Increasing: **$(0, 2)$** Constant: **$(2, \infty)$**

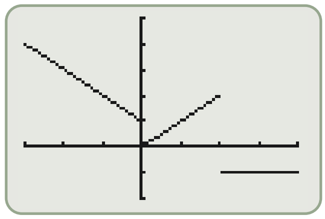
Technology Tip

Plot a piecewise-defined function using the $\boxed{\text{TEST}}$ menu operations to define the inequalities in the function. Press:

$\boxed{\text{Y=}}\ \boxed{(}\ \boxed{1}\ \boxed{-}\ \boxed{\text{X, T, θ, n}}\ \boxed{)}\ \boxed{(}\ \boxed{\text{X, T, θ, n}}\ \boxed{\text{2nd}}\ \boxed{\text{MATH}}\ \boxed{5}\ \boxed{<}\ \boxed{0}\ \boxed{)}\ \boxed{+}\ \boxed{(}\ \boxed{\text{X, T, θ, n}}\ \boxed{)}\ \boxed{(}\ \boxed{\text{X, T, θ, n}}\ \boxed{\text{2nd}}\ \boxed{\text{MATH}}\ \boxed{4}\ \boxed{\ge}\ \boxed{0}\ \boxed{)}\ \boxed{(}\ \boxed{\text{X, T, θ, n}}\ \boxed{\text{2nd}}\ \boxed{\text{MATH}}\ \boxed{5}\ \boxed{<}\ \boxed{2}\ \boxed{)}\ \boxed{+}\ \boxed{(}\ \boxed{(-)}\ \boxed{1}\ \boxed{)}\ \boxed{(}\ \boxed{\text{X, T, θ, n}}\ \boxed{\text{2nd}}\ \boxed{\text{MATH}}\ \boxed{3}\ \boxed{>}\ \boxed{2}\ \boxed{)}$

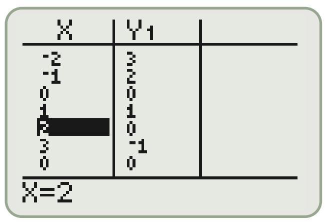


To avoid connecting graphs of the pieces, press $\boxed{\text{MODE}}$ and $\boxed{\text{Dot}}$. Set the viewing rectangle as $[-3, 4]$ by $[-2, 5]$; then press $\boxed{\text{GRAPH}}$.



Be sure to include the open circle and closed circle at the appropriate endpoints of each piece in the function.

The table of values supports the graph, except at $x = 2$. The function is not defined at $x = 2$.



The function is defined for all values of x except $x = 2$.

$$\text{Domain:} \quad (-\infty, 2) \cup (2, \infty)$$

The output of this function (vertical direction) takes on the y-values $y \geq 0$ and the additional single value $y = -1$.

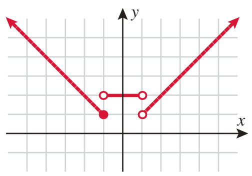
$$\text{Range:} \quad [-1, -1] \cup [0, \infty) \text{ or } \{-1\} \cup [0, \infty)$$

We mentioned earlier that a discontinuous function has a graph that exhibits holes or jumps. In Example 6, the point $x = 0$ corresponds to a jump, because you would have to pick up your pencil to continue drawing the graph. The point $x = 2$ corresponds to both a hole and a jump. The hole indicates that the function is not defined at that point, and there is still a jump because the identity function and the constant function do not meet at the same y-value at $x = 2$.

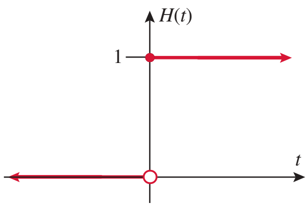
YOUR TURN Graph the piecewise-defined function, and state the intervals where the function is increasing, decreasing, or constant, along with the domain and range.

$$f(x) = \begin{cases} -x & x \leq -1 \\ 2 & -1 < x < 1 \\ x & x > 1 \end{cases}$$

Answer: Increasing: $(1, \infty)$
Decreasing: $(-\infty, -1)$
Constant: $(-1, 1)$
Domain: $(-\infty, 1) \cup (1, \infty)$
Range: $[1, \infty)$



Piecewise-defined functions whose "pieces" are constants are called **step functions**. The reason for this name is that the graph of a step function looks like steps of a staircase. A common step function used in engineering is the **Heaviside step function** (also called the **unit step function**):



$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

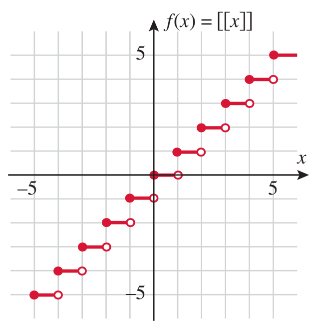
This function is used in signal processing to represent a signal that turns on at some time and stays on indefinitely.

A common step function used in business applications is the *greatest integer function.*

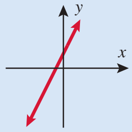
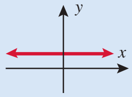
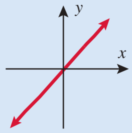
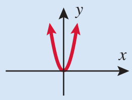
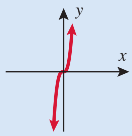
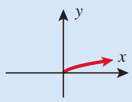
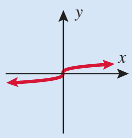
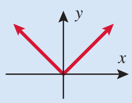
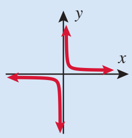
GREATEST INTEGER FUNCTION

$$f(x) = [[x]] = \text{greatest integer less than or equal to } x$$

x	1.0	1.3	1.5	1.7	1.9	2.0
$f(x) = [[x]]$	1	1	1	1	1	2



SECTION 1.2 SUMMARY

NAME	FUNCTION	DOMAIN	RANGE	GRAPH	EVEN/ODD
Linear	$f(x) = mx + b,\ m \neq 0$	$(-\infty, \infty)$	$(-\infty, \infty)$		Neither (unless $y = x$)
Constant	$f(x) = c$	$(-\infty, \infty)$	$[c, c]$ or $\{c\}$		Even
Identity	$f(x) = x$	$(-\infty, \infty)$	$(-\infty, \infty)$		Odd
Square	$f(x) = x^2$	$(-\infty, \infty)$	$[0, \infty)$		Even
Cube	$f(x) = x^3$	$(-\infty, \infty)$	$(-\infty, \infty)$		Odd
Square Root	$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$		Neither
Cube Root	$f(x) = \sqrt[3]{x}$	$(-\infty, \infty)$	$(-\infty, \infty)$		Odd
Absolute Value	$f(x) = \lvert x \rvert$	$(-\infty, \infty)$	$[0, \infty)$		Even
Reciprocal	$f(x) = \dfrac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$		Odd

Domain and Range of a Function

- **Implied Domain:** Exclude any values that lead to the function being undefined (dividing by zero) or imaginary outputs (square root of a negative real number).
- Inspect the graph to determine the set of all inputs (domain) and the set of all outputs (range).

Finding Intervals Where a Function Is Increasing, Decreasing, or Constant

- **Increasing:** Graph of function rises from left to right.
- **Decreasing:** Graph of function falls from left to right.
- **Constant:** Graph of function does not change height from left to right.

Average Rate of Change $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \quad x_1 \neq x_2$

Difference Quotient $\dfrac{f(x + h) - f(x)}{h} \quad h \neq 0$

Piecewise-Defined Functions

- **Continuous:** You can draw the graph of a function without picking up the pencil.
- **Discontinuous:** Graph has holes and/or jumps.

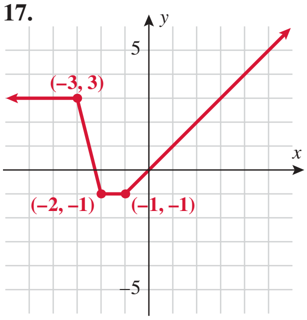
SECTION 1.2 EXERCISES

SKILLS

In Exercises 1–16, determine whether the function is even, odd, or neither.

1. $h(x) = x^2 + 2x$
2. $G(x) = 2x^4 + 3x^3$
3. $h(x) = x^{1/3} - x$
4. $g(x) = x^{-1} + x$
5. $f(x) = |x| + 5$
6. $f(x) = |x| + x^2$
7. $f(x) = |x|$
8. $f(x) = |x^3|$
9. $G(t) = |t - 3|$
10. $g(t) = |t + 2|$
11. $G(t) = \sqrt{t - 3}$
12. $f(x) = \sqrt{2 - x}$
13. $g(x) = \sqrt{x^2 + x}$
14. $f(x) = \sqrt{x^2 + 2}$
15. $h(x) = \dfrac{1}{x} + 3$
16. $h(x) = \dfrac{1}{x} - 2x$

In Exercises 17–28, state the (a) domain, (b) range, and (c) x-interval(s) where the function is increasing, decreasing, or constant. Find the values of (d) $f(0)$, (e) $f(-2)$, and (f) $f(2)$.

17.


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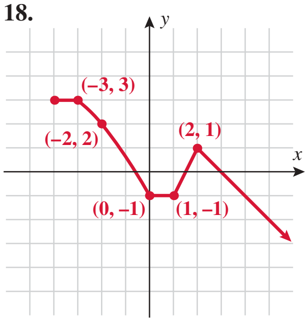

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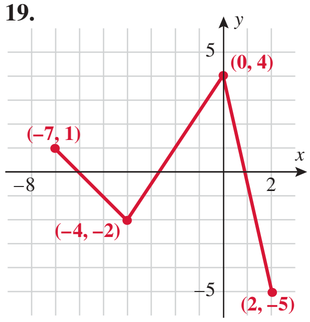

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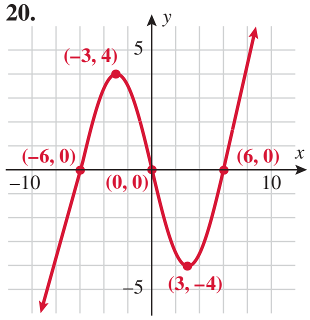

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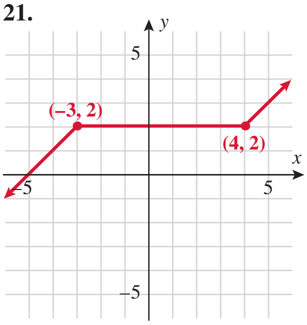

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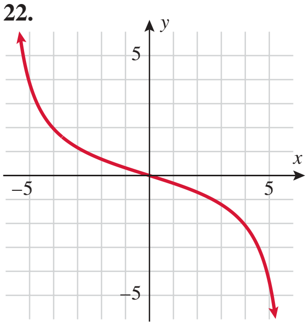

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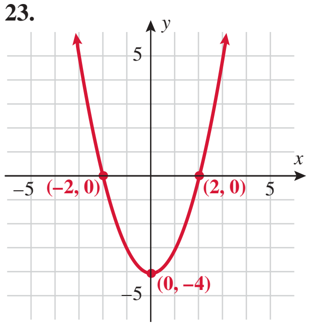

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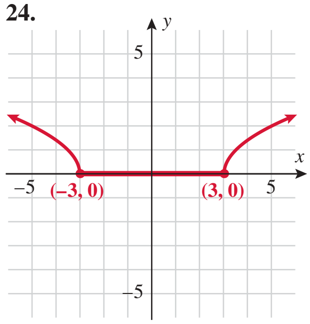

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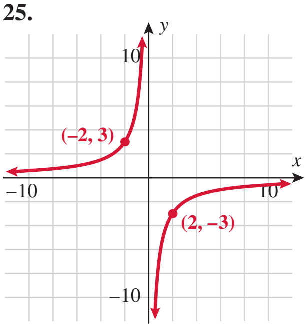

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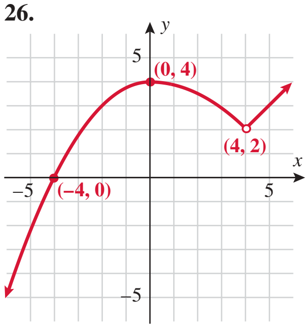

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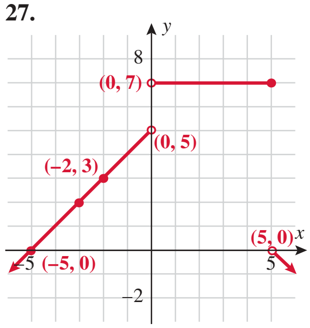

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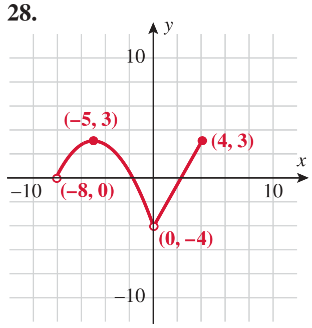


In Exercises 29–44, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for each function.

29. $f(x) = x^2 - x$ **30.** $f(x) = x^2 + 2x$ **31.** $f(x) = 3x + x^2$ **32.** $f(x) = 5x - x^2$

33. $f(x) = x^2 - 3x + 2$ **34.** $f(x) = x^2 - 2x + 5$ **35.** $f(x) = -3x^2 + 5x - 4$ **36.** $f(x) = -4x^2 + 2x - 3$

37. $f(x) = x^3 + x^2$ **38.** $f(x) = (x-1)^4$ **39.** $f(x) = \dfrac{2}{x-2}$ **40.** $f(x) = \dfrac{x+5}{x-7}$

41. $f(x) = \sqrt{1-2x}$ **42.** $f(x) = \sqrt{x^2+x+1}$ **43.** $f(x) = \dfrac{4}{\sqrt{x}}$ **44.** $f(x) = \sqrt{\dfrac{x}{x+1}}$

In Exercises 45–52, find the average rate of change of the function from $x = 1$ to $x = 3$.

45. $f(x) = x^3$ **46.** $f(x) = \dfrac{1}{x}$ **47.** $f(x) = |x|$ **48.** $f(x) = 2x$

49. $f(x) = 1 - 2x$ **50.** $f(x) = 9 - x^2$ **51.** $f(x) = |5 - 2x|$ **52.** $f(x) = \sqrt{x^2-1}$

In Exercises 53–78, graph the piecewise-defined functions. State the domain and range in interval notation. Determine the intervals where the function is increasing, decreasing, or constant.

53. $f(x) = \begin{cases} x & x < 2 \\ 2 & x \geq 2 \end{cases}$ **54.** $f(x) = \begin{cases} -x & x < -1 \\ -1 & x \geq -1 \end{cases}$ **55.** $f(x) = \begin{cases} 1 & x < -1 \\ x^2 & x \geq -1 \end{cases}$

56. $f(x) = \begin{cases} x^2 & x < 2 \\ 4 & x \geq 2 \end{cases}$ **57.** $f(x) = \begin{cases} x & x < 0 \\ x^2 & x \geq 0 \end{cases}$ **58.** $f(x) = \begin{cases} -x & x \leq 0 \\ x^2 & x > 0 \end{cases}$

59. $f(x) = \begin{cases} -x+2 & x < 1 \\ x^2 & x \geq 1 \end{cases}$ **60.** $f(x) = \begin{cases} 2+x & x \leq -1 \\ x^2 & x > -1 \end{cases}$ **61.** $f(x) = \begin{cases} 5 - 2x & x < 2 \\ 3x - 2 & x > 2 \end{cases}$

62. $f(x) = \begin{cases} 3 - \frac{1}{2}x & x < -2 \\ 4 + \frac{3}{2}x & x > -2 \end{cases}$ **63.** $G(x) = \begin{cases} -1 & x < -1 \\ x & -1 \leq x \leq 3 \\ 3 & x > 3 \end{cases}$ **64.** $G(x) = \begin{cases} -1 & x < -1 \\ x & -1 < x < 3 \\ 3 & x > 3 \end{cases}$

65. $G(t) = \begin{cases} 1 & t < 1 \\ t^2 & 1 \leq t \leq 2 \\ 4 & t > 2 \end{cases}$ **66.** $G(t) = \begin{cases} 1 & t < 1 \\ t^2 & 1 < t < 2 \\ 4 & t > 2 \end{cases}$ **67.** $f(x) = \begin{cases} -x-1 & x < -2 \\ x+1 & -2 < x < 1 \\ -x+1 & x \geq 1 \end{cases}$

68. $f(x) = \begin{cases} -x-1 & x \leq -2 \\ x+1 & -2 < x < 1 \\ -x+1 & x > 1 \end{cases}$ **69.** $G(x) = \begin{cases} 0 & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$ **70.** $G(x) = \begin{cases} 1 & x < 1 \\ \sqrt[3]{x} & x > 1 \end{cases}$

71. $G(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$ **72.** $G(x) = \begin{cases} 0 & x = 0 \\ -\frac{1}{x} & x \neq 0 \end{cases}$ **73.** $G(x) = \begin{cases} -\sqrt[3]{x} & x \leq -1 \\ x & -1 < x < 1 \\ -\sqrt{x} & x > 1 \end{cases}$

74. $G(x) = \begin{cases} -\sqrt[3]{x} & x < -1 \\ x & -1 \le x < 1 \\ \sqrt{x} & x > 1 \end{cases}$

75. $f(x) = \begin{cases} x + 3 & x \le -2 \\ |x| & -2 < x < 2 \\ x^2 & x \ge 2 \end{cases}$

76. $f(x) = \begin{cases} |x| & x < -1 \\ 1 & -1 < x < 1 \\ |x| & x > 1 \end{cases}$

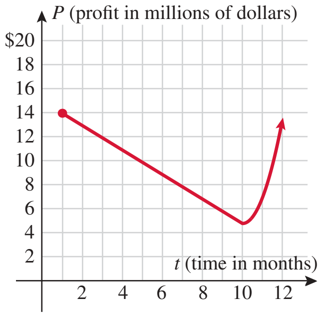
77. $f(x) = \begin{cases} x & x \le -1 \\ x^3 & -1 < x < 1 \\ x^2 & x > 1 \end{cases}$

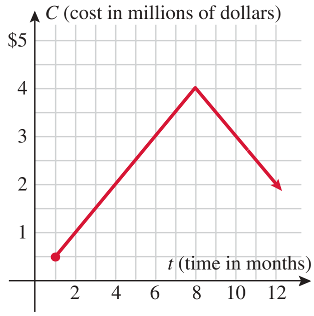
78. $f(x) = \begin{cases} x^2 & x \le -1 \\ x^3 & -1 < x < 1 \\ x & x \ge 1 \end{cases}$

APPLICATIONS

For Exercises 79 and 80, refer to the following:

A manufacturer determines that his *profit* and *cost* functions over one year are represented by the following graphs.





79. Business. Find the intervals on which profit is increasing, decreasing, and constant.

80. Business. Find the intervals on which cost is increasing, decreasing, and constant.

81. Budget: Costs. The Kappa Kappa Gamma sorority decides to order custom-made T-shirts for its *Kappa Krush* mixer with the Sigma Alpha Epsilon fraternity. If the sorority orders 50 or fewer T-shirts, the cost is \$10 per shirt. If it orders more than 50 but fewer than 100, the cost is \$9 per shirt. If it orders 100 or more the cost is \$8 per shirt. Find the cost function $C(x)$ as a function of the number of T-shirts x ordered.

82. Budget: Costs. The marching band at a university is ordering some additional uniforms to replace existing uniforms that are worn out. If the band orders 50 or fewer, the cost is \$176.12 per uniform. If it orders more than 50 but fewer than 100, the cost is \$159.73 per uniform. Find the cost function $C(x)$ as a function of the number of new uniforms x ordered.

83. Budget: Costs. The Richmond rowing club is planning to enter the *Head of the Charles* race in Boston and is trying to figure out how much money to raise. The entry fee is \$250 per boat for the first 10 boats and \$175 for each additional boat. Find the cost function $C(x)$ as a function of the number of boats x the club enters.

84. Phone Cost: Long-Distance Calling. A phone company charges \$0.39 per minute for the first 10 minutes of an international long-distance phone call and \$0.12 per minute every minute after that. Find the cost function $C(x)$ as a function of the length of the phone call x in minutes.

85. Event Planning. A young couple are planning their wedding reception at a yacht club. The yacht club charges a flat rate of \$1000 to reserve the dining room for a private party. The cost of food is \$35 per person for the first 100 people and \$25 per person for every additional person beyond the first 100. Write the cost function $C(x)$ as a function of the number of people x attending the reception.

86. Home Improvement. An irrigation company gives you an estimate for an eight-zone sprinkler system. The parts are \$1400, and the labor is \$25 per hour. Write a function $C(x)$ that determines the cost of a new sprinkler system if you choose this irrigation company.

87. Sales. A famous author negotiates with her publisher the monies she will receive for her next suspense novel. She will receive \$50,000 up front and a 15% royalty rate on the first 100,000 books sold, and 20% on any books sold beyond that. If the book sells for \$20 and royalties are based on the selling price, write a royalties function $R(x)$ as a function of total number x of books sold.

88. Sales. Rework Exercise 87 if the author receives \$35,000 up front, 15% for the first 100,000 books sold, and 25% on any books sold beyond that.

89. Profit. A group of artists are trying to decide whether they will make a profit if they set up a Web-based business to market and sell stained glass that they make. The costs associated with this business are \$100 per month for the website and \$700 per month for the studio they rent. The materials cost \$35 for each work in stained glass, and the artists charge \$100 for each unit they sell. Write the monthly profit as a function of the number of stained-glass units they sell.

90. Profit. Philip decides to host a shrimp boil at his house as a fund-raiser for his daughter's AAU basketball team. He orders gulf shrimp to be flown in from New Orleans. The shrimp costs \$5 per pound. The shipping costs \$30. If he charges \$10 per person, write a function $F(x)$ that represents either his loss or profit as a function of the number of people x that attend. Assume that each person will eat 1 pound of shrimp.

91. Postage Rates. The following table corresponds to first-class postage rates for the U.S. Postal Service. Write a piecewise-defined function in terms of the greatest integer function that models this cost of mailing flat envelopes first class.

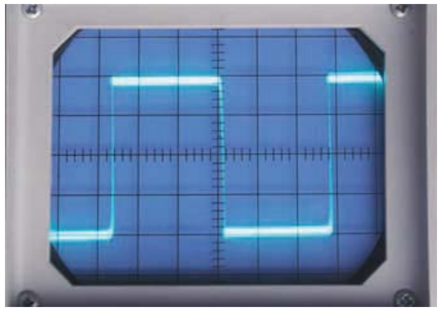
Weight Less Than (ounces)	First-Class Rate (Large Envelopes)
1	$0.88
2	$1.05
3	$1.22
4	$1.39
5	$1.56
6	$1.73
7	$1.90
8	$2.07
9	$2.24
10	$2.41
11	$2.58
12	$2.75
13	$2.92

92. Postage Rates. The following table corresponds to first-class postage rates for the U.S. Postal Service. Write a piecewise-defined function in terms of the greatest integer function that models this cost of mailing parcels first class.

Weight Less Than (ounces)	First-Class Rate (Packages)
1	$1.22
2	$1.39
3	$1.56
4	$1.73
5	$1.90
6	$2.07
7	$2.24
8	$2.41
9	$2.58
10	$2.75
11	$2.92
12	$3.09
13	$3.26

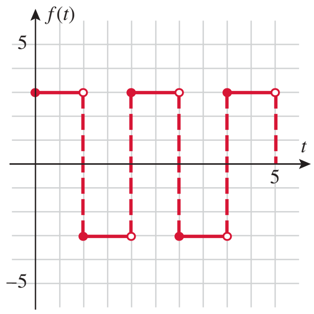
For Exercises 93 and 94, refer to the following:

A square wave is a waveform used in electronic circuit testing and signal processing. A square wave alternates regularly and instantaneously between two levels.

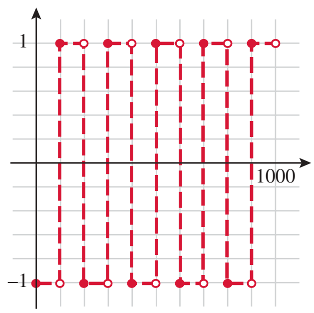


sciencephotos/Alamy

93. Electronics: Signals. Write a step function $f(t)$ that represents the following square wave:



94. Electronics: Signals. Write a step function $f(x)$ that represents the following square wave, where x represents frequency in Hz:



For Exercises 95 and 96, refer to the following table:

Global Carbon Emissions from Fossil Fuel Burning

Year	Millions of Tons of Carbon
1900	500
1925	1000
1950	1500
1975	5000
2000	7000

95. Climate Change: Global Warming. What is the average rate of change in global carbon emissions from fossil fuel burning from

a. 1900 to 1950? **b.** 1950 to 2000?

96. Climate Change: Global Warming. What is the average rate of change in global carbon emissions from fossil fuel burning from

a. 1950 to 1975? **b.** 1975 to 2000?

For Exercises 97 and 98, use the following information:

The height (in feet) of a falling object with an initial velocity of 48 feet per second launched straight upward from the ground is given by $h(t) = -16t^2 + 48t$, where t is time (in seconds).

97. **Falling Objects.** What is the average rate of change of the height as a function of time from $t = 1$ to $t = 2$?

98. **Falling Objects.** What is the average rate of change of the height as a function of time from $t = 1$ to $t = 3$?

For Exercises 99 and 100, refer to the following:

An analysis of sales indicates that demand for a product during a calendar year (no leap year) is modeled by

$$d(t) = 3\sqrt{t^2 + 1} - 2.75t$$

where d is demand in thousands of units and t is the day of the year and $t = 1$ represents January 1.

99. **Economics.** Find the average rate of change of the demand of the product over the first quarter.

100. **Economics.** Find the average rate of change of the demand of the product over the fourth quarter.

CATCH THE MISTAKE

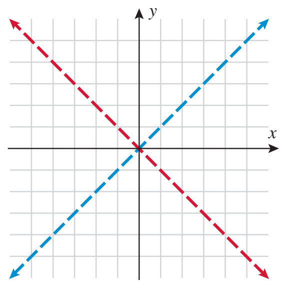
In Exercises 101–104, explain the mistake that is made.

101. Graph the piecewise-defined function. State the domain and range.

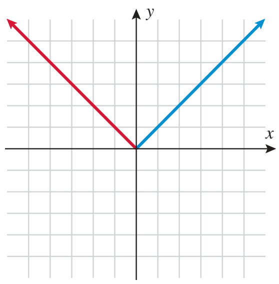
$$f(x) = \begin{cases} -x & x < 0 \\ x & x > 0 \end{cases}$$

Solution:

Draw the graphs of $f(x) = -x$ and $f(x) = x$.



Darken the graph of $f(x) = -x$ when $x < 0$ and the graph of $f(x) = x$ when $x > 0$. This gives us the familiar absolute value graph.



Domain: $(-\infty, \infty)$ or $\mathbb{R}$
Range: $[0, \infty)$

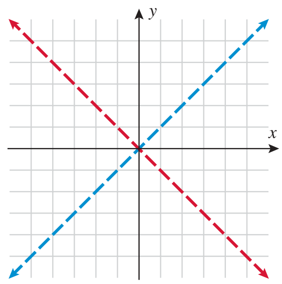
This is incorrect. What mistake was made?

102. Graph the piecewise-defined function. State the domain and range.

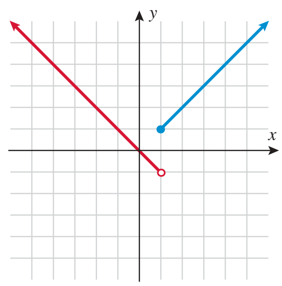
$$f(x) = \begin{cases} -x & x \le 1 \\ x & x > 1 \end{cases}$$

Solution:

Draw the graphs of $f(x) = -x$ and $f(x) = x$.



Darken the graph of $f(x) = -x$ when $x < 1$ and the graph of $f(x) = x$ when $x > 1$.



The resulting graph is as shown.

Domain: $(-\infty, \infty)$ or $\mathbb{R}$
Range: $(-1, \infty)$

This is incorrect. What mistake was made?

103. The cost of airport Internet access is \$15 for the first 30 minutes and \$1 per minute for each additional minute. Write a function describing the cost of the service as a function of minutes used online.

Solution: $C(x) = \begin{cases} 15 & x \le 30 \\ 15 + x & x > 30 \end{cases}$

This is incorrect. What mistake was made?

104. Most money market accounts pay a higher interest with a higher principal. If the credit union is offering 2% on accounts with less than or equal to \$10,000 and 4% on the additional money over \$10,000, write the interest function $I(x)$ that represents the interest earned on an account as a function of dollars in the account.

Solution: $I(x) = \begin{cases} 0.02x & x \le 10{,}000 \\ 0.02(10{,}000) + 0.04x & x > 10{,}000 \end{cases}$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 105 and 106, determine whether each statement is true or false.

105. If an odd function has an interval where the function is increasing, then it also has to have an interval where the function is decreasing.

106. If an even function has an interval where the function is increasing, then it also has to have an interval where the function is decreasing.

In Exercises 107 and 108, for a and b real numbers, can the function given ever be a continuous function? If so, specify the value for a and b that would make it so.

107. $f(x) = \begin{cases} ax & x \leq 2 \\ bx^2 & x > 2 \end{cases}$

108. $f(x) = \begin{cases} -\dfrac{1}{x} & x < a \\ \dfrac{1}{x} & x \geq a \end{cases}$

CHALLENGE

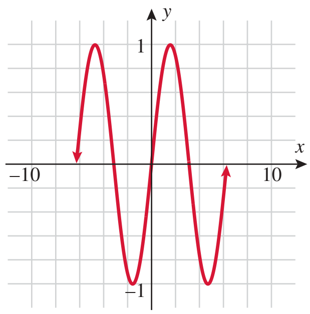
In Exercises 109 and 110, find the values of a and b that make f continuous.

109. $f(x) = \begin{cases} -x^2 - 10x - 13 & x \leq -2 \\ ax + b & -2 < x < 1 \\ \sqrt{x - 1} - 9 & x \geq 1 \end{cases}$

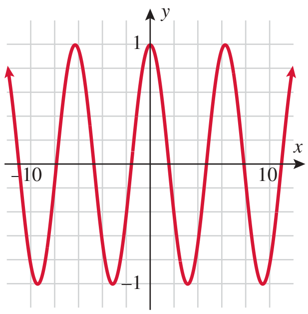
110. $f(x) = \begin{cases} -2x - a + 2b & x \leq -2 \\ \sqrt{x + a} & -2 < x \leq 2 \\ x^2 - 4x + a + 4 & x > 2 \end{cases}$

TECHNOLOGY

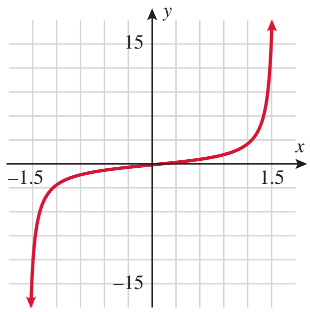
111. In trigonometry you will learn about the sine function, $\sin x$. Plot the function $f(x) = \sin x$, using a graphing utility. It should look like the graph on the right. Is the sine function even, odd, or neither?



112. In trigonometry you will learn about the cosine function, $\cos x$. Plot the function $f(x) = \cos x$, using a graphing utility. It should look like the graph on the right. Is the cosine function even, odd, or neither?



113. In trigonometry you will learn about the tangent function, $\tan x$. Plot the function $f(x) = \tan x$, using a graphing utility. If you restrict the values of x so that $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$, the graph should resemble the graph below. Is the tangent function even, odd, or neither?



114. Plot the function $f(x) = \dfrac{\sin x}{\cos x}$. What function is this?

115. Graph the function $f(x) = [[3x]]$ using a graphing utility. State the domain and range.

116. Graph the function $f(x) = \left[\left[\frac{1}{3}x\right]\right]$ using a graphing utility. State the domain and range.

PREVIEW TO CALCULUS

For Exercises 117–120, refer to the following:

In calculus, the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ of a function f is used to find the derivative f' of f, by allowing h to approach zero, $h \rightarrow 0$. Find the derivative of the following functions.

117. $f(x) = k$, where k is a constant

118. $f(x) = mx + b$, where m and b are constants, $m \neq 0$

119. $f(x) = ax^2 + bx + c$, where a, b, and c are constants, $a \neq 0$

120. $f(x) = \begin{cases} 7 & x < 0 \\ 2 - 3x & 0 < x < 4 \\ x^2 + 4x - 6 & x > 4 \end{cases}$

SECTION
1.3 GRAPHING TECHNIQUES: TRANSFORMATIONS

SKILLS OBJECTIVES

- Sketch the graph of a function using horizontal and vertical shifting of common functions.
- Sketch the graph of a function by reflecting a common function about the x-axis or y-axis.
- Sketch the graph of a function by stretching or compressing a common function.
- Sketch the graph of a function using a sequence of transformations.

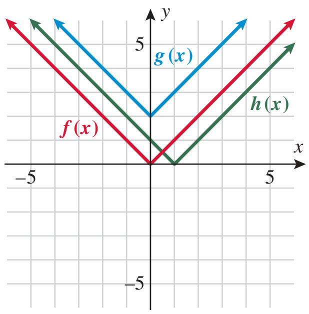
CONCEPTUAL OBJECTIVES

- Identify the common functions by their graphs.
- Apply multiple transformations of common functions to obtain graphs of functions.
- Understand that domain and range also are transformed.

Horizontal and Vertical Shifts

The focus of the previous section was to learn the graphs that correspond to particular functions such as identity, square, cube, square root, cube root, absolute value, and reciprocal. Therefore, at this point, you should be able to recognize and generate the graphs of $y = x$, $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = |x|$, and $y = \frac{1}{x}$. In this section, we will discuss how to sketch the graphs of functions that are very simple modifications of these functions. For instance, a common function may be shifted (horizontally or vertically), reflected, or stretched (or compressed). Collectively, these techniques are called **transformations**.

Let's take the absolute value function as an example. The graphs of $f(x) = |x|$, $g(x) = |x| + 2$, and $h(x) = |x - 1|$ are shown below.



x	$f(x)$
-2	2
-1	1
0	0
1	1
2	2

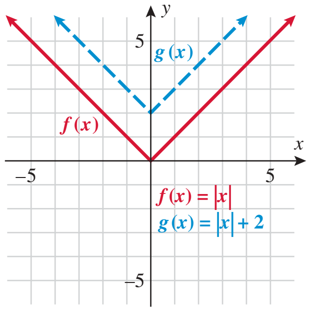
x	$g(x)$
-2	4
-1	3
0	2
1	3
2	4

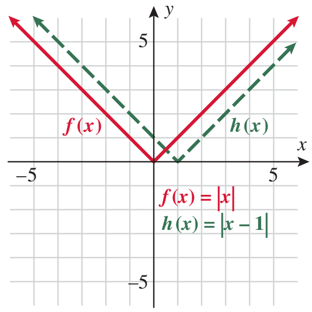
x	$h(x)$
-2	3
-1	2
0	1
1	0
2	1

Notice that the graph of $g(x) = |x| + 2$ is the graph of $f(x) = |x|$ shifted *up* two units. Similarly, the graph of $h(x) = |x - 1|$ is the graph of $f(x) = |x|$ shifted to the *right* one unit. In both cases, the base or starting function is $f(x) = |x|$.

Note that we could rewrite the functions $g(x)$ and $h(x)$ in terms of $f(x)$:

$$g(x) = |x| + 2 = f(x) + 2$$
$$h(x) = |x - 1| = f(x - 1)$$





In the case of $g(x)$, the shift $(+2)$ occurs "outside" the function—that is, outside the parentheses showing the argument. Therefore, the output for $g(x)$ is 2 more than the typical output for $f(x)$. Because the output corresponds to the vertical axis, this results in a shift *upward* of two units. In general, shifts that occur *outside* the function correspond to a *vertical* shift corresponding to the sign of the shift. For instance, had the function been $G(x) = |x| - 2$, this graph would have started with the graph of the function $f(x)$ and shifted down two units.

In the case of $h(x)$, the shift occurs "inside" the function—that is, inside the parentheses showing the argument. Note that the point $(0, 0)$ that lies on the graph of $f(x)$ was shifted to the point $(1, 0)$ on the graph of the function $h(x)$. The y-value remained the same, but the x-value shifted to the right one unit. Similarly, the points $(-1, 1)$ and $(1, 1)$ were shifted to the points $(0, 1)$ and $(2, 1)$, respectively. In general, shifts that occur *inside* the function correspond to a *horizontal* shift opposite the sign. In this case, the graph of the function $h(x) = |x - 1|$ shifted the graph of the function $f(x)$ to the right one unit. If, instead, we had the function $H(x) = |x + 1|$, this graph would have started with the graph of the function $f(x)$ and shifted to the left one unit.

VERTICAL SHIFTS

Assuming that c is a positive constant,

To Graph	Shift the Graph of $f(x)$
$f(x) + c$	c units upward
$f(x) - c$	c units downward

Adding or subtracting a constant **outside** the function corresponds to a **vertical** shift that goes **with the sign**.

HORIZONTAL SHIFTS

Assuming that c is a positive constant,

To Graph	Shift the Graph of $f(x)$
$f(x + c)$	c units to the left
$f(x - c)$	c units to the right

Adding or subtracting a constant **inside** the function corresponds to a **horizontal** shift that goes **opposite the sign**.

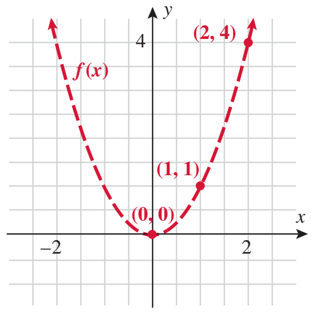
Study Tip

- Shifts *outside* the function are *vertical* shifts with the sign.
- Shifts *inside* the function are *horizontal* shifts opposite the sign.

EXAMPLE 1 Horizontal and Vertical Shifts

Sketch the graphs of the given functions using horizontal and vertical shifts.

a. $g(x) = x^2 - 1$
b. $H(x) = (x + 1)^2$

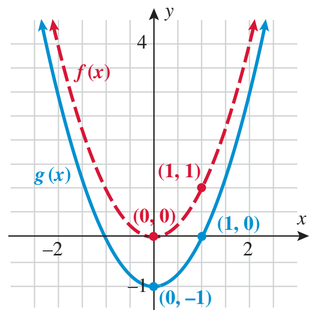


Solution:

In both cases, the function to start with is $f(x) = x^2$.

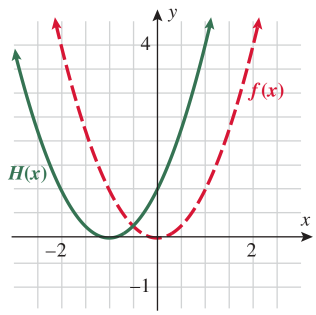
a. $g(x) = x^2 - 1$ can be rewritten as $g(x) = f(x) - 1$.

1. The shift (one unit) occurs *outside* of the function. Therefore, we expect a vertical shift that goes with the sign.
2. Since the sign is *negative*, this corresponds to a *downward* shift.
3. Shifting the graph of the function $f(x) = x^2$ down one unit yields the graph of $g(x) = x^2 - 1$.



b. $H(x) = (x + 1)^2$ can be rewritten as $H(x) = f(x + 1)$.

1. The shift (one unit) occurs *inside* of the function. Therefore, we expect a horizontal shift that goes *opposite* the sign.
2. Since the sign is *positive*, this corresponds to a shift to the *left*.
3. Shifting the graph of the function $f(x) = x^2$ to the left one unit yields the graph of $H(x) = (x + 1)^2$.

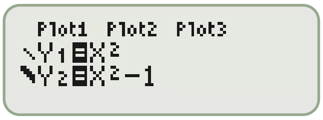


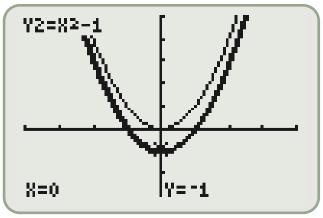
■ **YOUR TURN** Sketch the graphs of the given functions using horizontal and vertical shifts.

a. $g(x) = x^2 + 1$ **b.** $H(x) = (x - 1)^2$

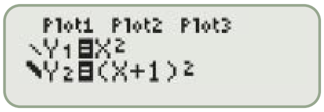
Technology Tip

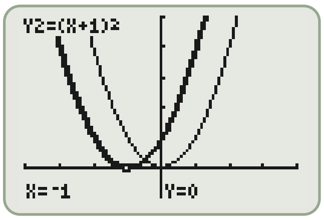
a. Graphs of $y_1 = x^2$ and $y_2 = g(x) = x^2 - 1$ are shown.





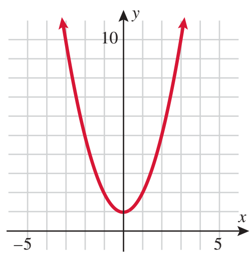
b. Graphs of $y_1 = x^2$ and $y_2 = H(x) = (x + 1)^2$ are shown.



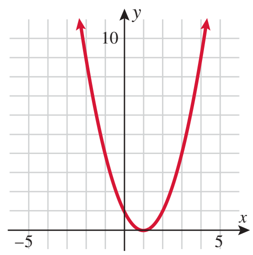


■ **Answer:**

a.



b.



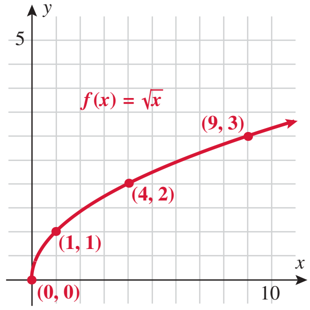
It is important to note that the domain and range of the resulting function can be thought of as also being shifted. Shifts in the domain correspond to horizontal shifts, and shifts in the range correspond to vertical shifts.

EXAMPLE 2 Horizontal and Vertical Shifts and Changes in the Domain and Range

Graph the functions using translations and state the domain and range of each function.

a. $g(x) = \sqrt{x+1}$

b. $G(x) = \sqrt{x} - 2$



Solution:

In both cases the function to start with is $f(x) = \sqrt{x}$.

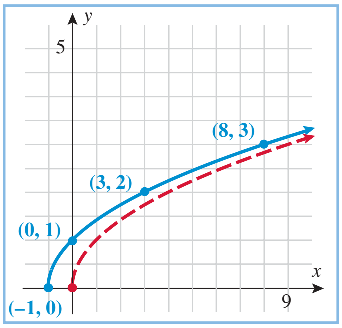
Domain: $[0, \infty)$

Range: $[0, \infty)$

a. $g(x) = \sqrt{x+1}$ can be rewritten as $g(x) = f(x+1)$.

1. The shift (one unit) is *inside* the function, which corresponds to a *horizontal* shift *opposite the sign.*

2. Shifting the graph of $f(x) = \sqrt{x}$ to the *left* one unit yields the graph of $g(x) = \sqrt{x+1}$. Notice that the point $(0, 0)$, which lies on the graph of $f(x)$, gets shifted to the point $(-1, 0)$ on the graph of $g(x)$.



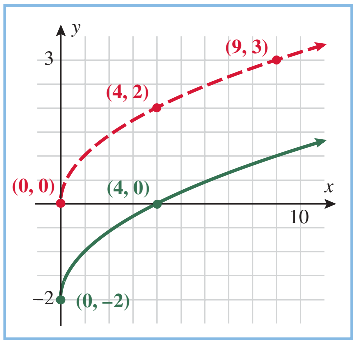
Although the original function $f(x) = \sqrt{x}$ had an implicit restriction on the domain $[0, \infty)$, the function $g(x) = \sqrt{x+1}$ has the implicit restriction that $x \geq -1$. We see that the output or range of $g(x)$ is the same as the output of the original function $f(x)$.

Domain: $[-1, \infty)$ **Range:** $[0, \infty)$

b. $G(x) = \sqrt{x} - 2$ can be rewritten as $G(x) = f(x) - 2$.

1. The shift (two units) is *outside* the function, which corresponds to a *vertical* shift *with the sign.*

2. The graph of $G(x) = \sqrt{x} - 2$ is found by shifting $f(x) = \sqrt{x}$ down two units. Note that the point $(0, 0)$, which lies on the graph of $f(x)$, gets shifted to the point $(0, -2)$ on the graph of $G(x)$.



The original function $f(x) = \sqrt{x}$ has an implicit restriction on the domain: $[0, \infty)$. The function $G(x) = \sqrt{x} - 2$ also has the implicit restriction that $x \geq 0$. The output or range of $G(x)$ is always two units less than the output of the original function $f(x)$.

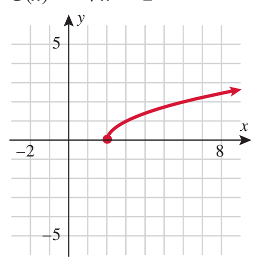
Domain: $[0, \infty)$ **Range:** $[-2, \infty)$

YOUR TURN Sketch the graph of the functions using shifts and state the domain and range.

a. $G(x) = \sqrt{x-2}$ **b.** $h(x) = |x| + 1$

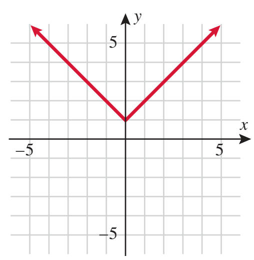
Answer:

a. $G(x) = \sqrt{x-2}$



Domain: $[2, \infty)$ Range: $[0, \infty)$

b. $h(x) = |x| + 1$



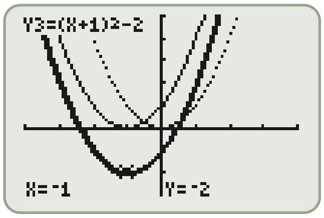
Domain: $(-\infty, \infty)$ Range: $[1, \infty)$

The previous examples have involved graphing functions by shifting a known function either in the horizontal or vertical direction. Let us now look at combinations of horizontal and vertical shifts.

Technology Tip

Graphs of $y_1 = x^2$, $y_2 = (x + 1)^2$, and $y_3 = F(x) = (x + 1)^2 - 2$ are shown.

```
Plot1 Plot2 Plot3
\Y1=X²
\Y2=(X+1)²
\Y3=(X+1)²-2
```



EXAMPLE 3 Combining Horizontal and Vertical Shifts

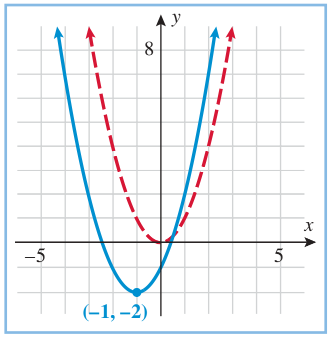
Sketch the graph of the function $F(x) = (x + 1)^2 - 2$. State the domain and range of F.

Solution:

The base function is $y = x^2$.

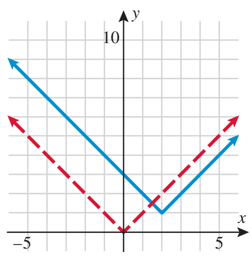
1. The shift (one unit) is *inside* the function, so it represents a *horizontal* shift *opposite the sign.*
2. The -2 shift is *outside* the function, which represents a *vertical* shift *with the sign.*
3. Therefore, we shift the graph of $y = x^2$ to the left one unit and down two units. For instance, the point **(0, 0)** on the graph of $y = x^2$ shifts to the point **(−1, −2)** on the graph of $F(x) = (x + 1)^2 - 2$.

Domain: $(-\infty, \infty)$ **Range:** $[-2, \infty)$



YOUR TURN Sketch the graph of the function $f(x) = |x - 2| + 1$. State the domain and range of f.

Answer:
$f(x) = |x - 2| + 1$
$f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



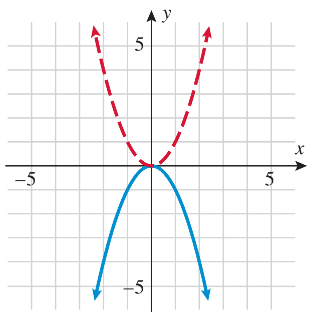
All of the previous transformation examples involve starting with a common function and shifting the function in either the horizontal or the vertical direction (or a combination of both). Now, let's investigate *reflections* of functions about the x-axis or y-axis.

Reflection About the Axes

To sketch the graphs of $f(x) = x^2$ and $g(x) = -x^2$, start by first listing points that are on each of the graphs and then connecting the points with smooth curves.

x	$f(x)$
−2	4
−1	1
0	0
1	1
2	4

x	$g(x)$
−2	−4
−1	−1
0	0
1	−1
2	−4

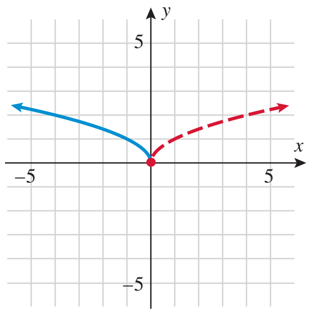


Note that if the graph of $f(x) = x^2$ is reflected about the x-axis, the result is the graph of $g(x) = -x^2$. Also note that the function $g(x)$ can be written as the negative of the function $f(x)$; that is $g(x) = -f(x)$. In general, **reflection about the x-axis** is produced by multiplying a function by -1.

Let's now investigate reflection about the y-axis. To sketch the graphs of $f(x) = \sqrt{x}$ and $g(x) = \sqrt{-x}$, start by listing points that are on each of the graphs and then connecting the points with smooth curves.

x	$f(x)$
0	0
1	1
4	2
9	3

x	$g(x)$
−9	3
−4	2
−1	1
0	0



Note that if the graph of $f(x) = \sqrt{x}$ is reflected about the y-axis, the result is the graph of $g(x) = \sqrt{-x}$. Also note that the function $g(x)$ can be written as $g(x) = f(-x)$. In general, **reflection about the y-axis** is produced by replacing x with $-x$ in the function. Notice that the domain of f is $[0, \infty)$, whereas the domain of g is $(-\infty, 0]$.

Reflection about the Axes

The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ about the **x-axis**.
The graph of $f(-x)$ is obtained by reflecting the graph of $f(x)$ about the **y-axis**.

EXAMPLE 4 Sketching the Graph of a Function Using Both Shifts and Reflections

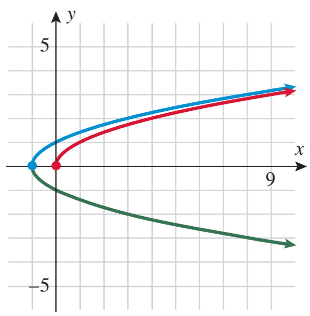
Sketch the graph of the function $G(x) = -\sqrt{x+1}$.

Solution:

Start with the square root function. $\quad f(x) = \sqrt{x}$

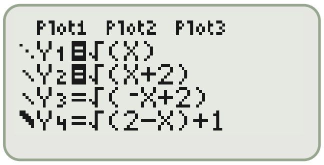
Shift the graph of $f(x)$ to the left one unit to arrive at the graph of $f(x+1)$. $\quad f(x+1) = \sqrt{x+1}$

Reflect the graph of $f(x+1)$ about the x-axis to arrive at the graph of $-f(x+1)$. $\quad -f(x+1) = -\sqrt{x+1}$

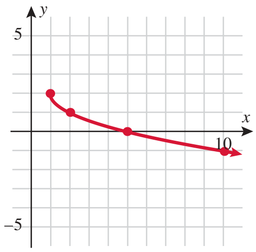


Technology Tip

Graphs of $y_1 = \sqrt{x}$, $y_2 = \sqrt{x+2}$, $y_3 = \sqrt{-x+2}$, and $y_4 = f(x) = \sqrt{2-x} + 1$ are shown.



■ **Answer:**
Domain: $[1, \infty)$
Range: $(-\infty, 2]$

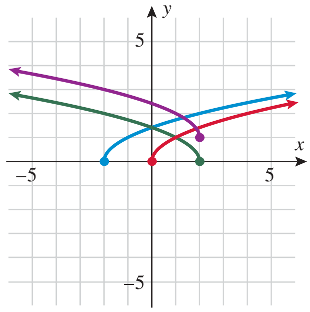


EXAMPLE 5 Sketching the Graph of a Function Using Both Shifts and Reflections

Sketch the graph of the function $f(x) = \sqrt{2-x} + 1$.

Solution:

Start with the square root function.	$g(x) = \sqrt{x}$
Shift the graph of $g(x)$ to the left two units to arrive at the graph of $g(x+2)$.	$g(x+2) = \sqrt{x+2}$
Reflect the graph of $g(x+2)$ about the y-axis to arrive at the graph of $g(-x+2)$.	$g(-x+2) = \sqrt{-x+2}$
Shift the graph $g(-x+2)$ up one unit to arrive at the graph of $g(-x+2)+1$.	$g(-x+2) + 1 = \sqrt{2-x} + 1$

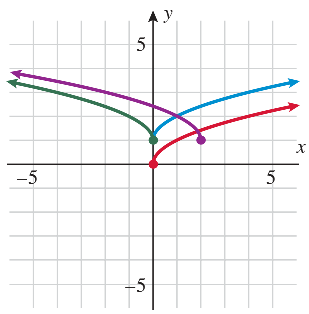


■ **YOUR TURN** Use shifts and reflections to sketch the graph of the function $f(x) = -\sqrt{x-1} + 2$. State the domain and range of $f(x)$.

Look back at the order in which transformations were performed in Example 5: horizontal shift, reflection, and then vertical shift. Let us consider an alternate order of transformations.

Words	Math
Start with the square root function.	$g(x) = \sqrt{x}$
Shift the graph of $g(x)$ up one unit to arrive at the graph of $g(x) + 1$.	$g(x) + 1 = \sqrt{x} + 1$
Reflect the graph of $g(x) + 1$ about the y-axis to arrive at the graph of $g(-x) + 1$.	$g(-x) + 1 = \sqrt{-x} + 1$
Replace x with $x - 2$, which corresponds to a shift of the graph of $g(-x) + 1$ to the right two units to arrive at the graph of $g[-(x-2)] + 1$.	$g(-x+2) + 1 = \sqrt{2-x} + 1$

In the last step we replaced x with $x - 2$, which required us to think ahead, knowing the desired result was $2 - x$ inside the radical. To avoid any possible confusion, follow this order of transformations:



1. Horizontal shifts: $f(x \pm c)$
2. Reflection: $f(-x)$ and/or $-f(x)$
3. Vertical shifts: $f(x) \pm c$

Stretching and Compressing

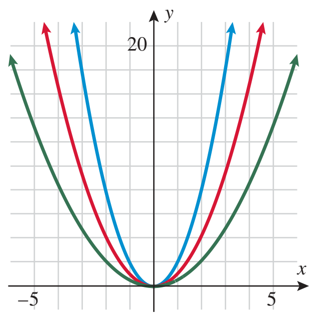
Horizontal shifts, vertical shifts, and reflections change only the position of the graph in the Cartesian plane, leaving the basic shape of the graph unchanged. These transformations (shifts and reflections) are called **rigid transformations** because they alter only the *position*. **Nonrigid transformations**, on the other hand, distort the *shape* of the original graph. We now consider *stretching* and *compressing* of graphs in both the vertical and the horizontal direction.

A vertical stretch or compression of a graph occurs when the function is multiplied by a positive constant. For example, the graphs of the functions $f(x) = x^2$, $g(x) = 2f(x) = 2x^2$, and $h(x) = \frac{1}{2}f(x) = \frac{1}{2}x^2$ are illustrated below. Depending on if the constant is larger than 1 or smaller than 1 will determine whether it corresponds to a stretch (expansion) or compression (contraction) in the vertical direction.

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

x	$g(x)$
-2	8
-1	2
0	0
1	2
2	8

x	$h(x)$
-2	2
-1	$\frac{1}{2}$
0	0
1	$\frac{1}{2}$
2	2



Note that when the function $f(x) = x^2$ is multiplied by 2, so that $g(x) = 2f(x) = 2x^2$, the result is a graph stretched in the vertical direction. When the function $f(x) = x^2$ is multiplied by $\frac{1}{2}$, so that $h(x) = \frac{1}{2}f(x) = \frac{1}{2}x^2$, the result is a graph that is compressed in the vertical direction.

Vertical stretching and vertical compressing of graphs

The graph of $cf(x)$ is found by:

- **Vertically stretching** the graph of $f(x)$ if $c > 1$
- **Vertically compressing** the graph of $f(x)$ if $0 < c < 1$

Note: c is any positive real number.

EXAMPLE 6 Vertically Stretching and Compressing Graphs

Graph the function $h(x) = \frac{1}{4}x^3$.

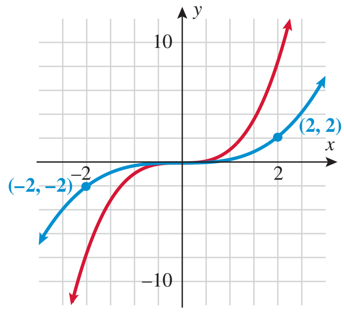
Solution:

1. Start with the cube function. $f(x) = x^3$

2. Vertical compression is expected because $\frac{1}{4}$ is less than 1. $h(x) = \frac{1}{4}x^3$

3. Determine a few points that lie on the graph of h.

$(0, 0)$ $(2, 2)$ $(-2, -2)$



Conversely, if the argument x of a function f is multiplied by a positive real number c, then the result is a *horizontal* stretch of the graph of f if $0 < c < 1$. If $c > 1$, then the result is a *horizontal* compression of the graph of f.

HORIZONTAL STRETCHING AND HORIZONTAL COMPRESSING OF GRAPHS

The graph of $f(cx)$ is found by:

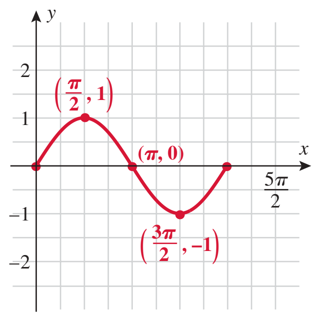
- **Horizontally stretching** the graph of $f(x)$ if $0 < c < 1$
- **Horizontally compressing** the graph of $f(x)$ if $c > 1$

Note: c is any positive real number.

EXAMPLE 7 Vertically Stretching and Horizontally Compressing Graphs

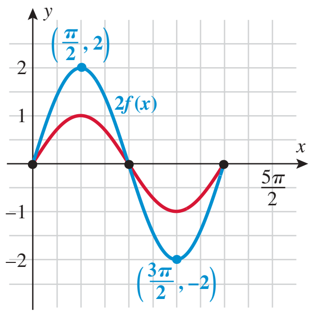
Given the graph of $f(x)$, graph

a. $2f(x)$ **b.** $f(2x)$



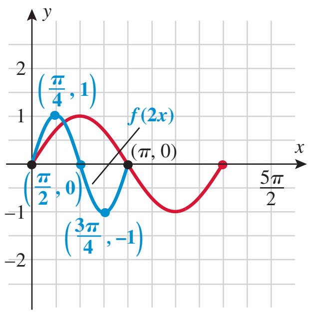
Solution (a):

Since the function is multiplied (on the outside) by 2, the result is that each ***y*-value** of $f(x)$ is ***multiplied* by 2**, which corresponds to vertical stretching.



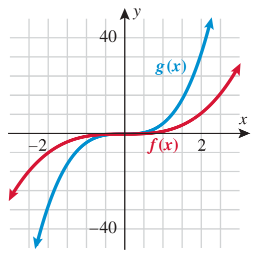
Solution (b):

Since the argument of the function is multiplied (on the inside) by 2, the result is that each ***x*-value** of $f(x)$ is **divided by 2**, which corresponds to horizontal compression.



■ **YOUR TURN** Graph the function $g(x) = 4x^3$.

■ **Answer:** Vertical stretch of the graph $f(x) = x^3$.



EXAMPLE 8 Sketching the Graph of a Function Using Multiple Transformations

Sketch the graph of the function $H(x) = -2(x - 3)^2$.

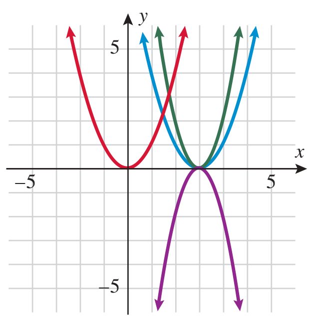
Solution:

Start with the square function. $\quad f(x) = x^2$

Shift the graph of $f(x)$ to the right three units to arrive at the graph of $f(x - 3)$. $\quad f(x - 3) = (x - 3)^2$

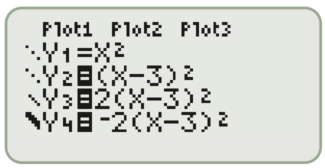
Vertically stretch the graph of $f(x - 3)$ by a factor of 2 to arrive at the graph of $2f(x - 3)$. $\quad 2f(x - 3) = 2(x - 3)^2$

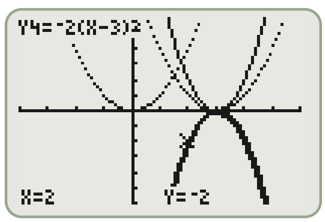
Reflect the graph $2f(x - 3)$ about the x-axis to arrive at the graph of $-2f(x - 3)$. $\quad -2f(x - 3) = -2(x - 3)^2$



Technology Tip

Graphs of $y_1 = x^2$, $y_2 = (x - 3)^2$, $y_3 = 2(x - 3)^2$, and $y_4 = H(x) = -2(x - 3)^2$ are shown.





In Example 8 we followed the same "inside out" approach with the functions to determine the order for the transformations: horizontal shift, vertical stretch, and reflection.

SECTION 1.3 SUMMARY

Transformation	To graph the function...	Draw the graph of f and then...	Description
Horizontal shifts ($c > 0$)	$f(x + c)$ $f(x - c)$	Shift the graph of f to the left c units. Shift the graph of f to the right c units.	Replace x by $x + c$. Replace x by $x - c$.
Vertical shifts ($c > 0$)	$f(x) + c$ $f(x) - c$	Shift the graph of f up c units. Shift the graph of f down c units.	Add c to $f(x)$. Subtract c from $f(x)$.
Reflection about the x-axis	$-f(x)$	Reflect the graph of f about the x-axis.	Multiply $f(x)$ by -1.
Reflection about the y-axis	$f(-x)$	Reflect the graph of f about the y-axis.	Replace x by $-x$.
Vertical stretch	$cf(x)$, where $c > 1$	Vertically stretch the graph of f.	Multiply $f(x)$ by c.
Vertical compression	$cf(x)$, where $0 < c < 1$	Vertically compress the graph of f.	Multiply $f(x)$ by c.
Horizontal stretch	$f(cx)$, where $0 < c < 1$	Horizontally stretch the graph of f.	Replace x by cx.
Horizontal compression	$f(cx)$, where $c > 1$	Horizontally compress the graph of f.	Replace x by cx.

SECTION 1.3 EXERCISES

SKILLS

In Exercises 1–6, write the function whose graph is the graph of $y = |x|$, but is transformed accordingly.

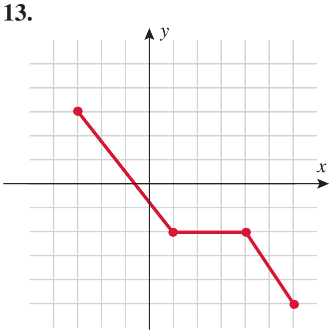
1. Shifted up three units
2. Shifted to the left four units
3. Reflected about the y-axis
4. Reflected about the x-axis
5. Vertically stretched by a factor of 3
6. Vertically compressed by a factor of 3

In Exercises 7–12, write the function whose graph is the graph of $y = x^3$, but is transformed accordingly.

7. Shifted down four units
8. Shifted to the right three units
9. Shifted up three units and to the left one unit
10. Reflected about the x-axis
11. Reflected about the y-axis
12. Reflected about both the x-axis and the y-axis

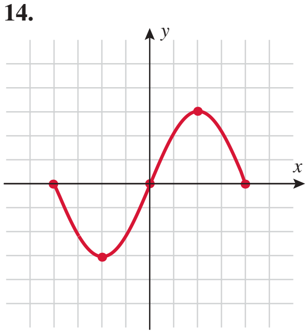
In Exercises 13–36, use the given graph to sketch the graph of the indicated functions.

13.



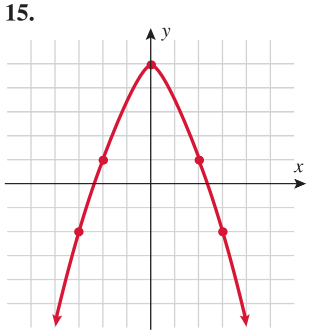
a. $y = f(x - 2)$
b. $y = f(x) - 2$

14.



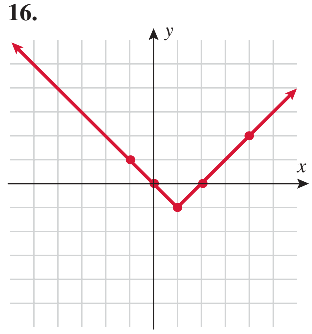
a. $y = f(x + 2)$
b. $y = f(x) + 2$

15.



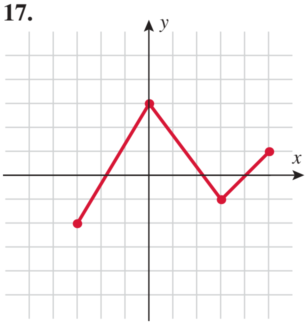
a. $y = f(x) - 3$
b. $y = f(x - 3)$

16.



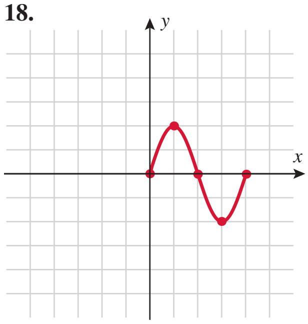
a. $y = f(x) + 3$
b. $y = f(x + 3)$

17.



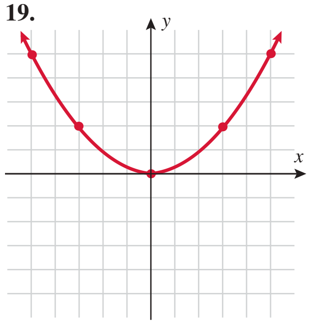
a. $y = -f(x)$
b. $y = f(-x)$

18.



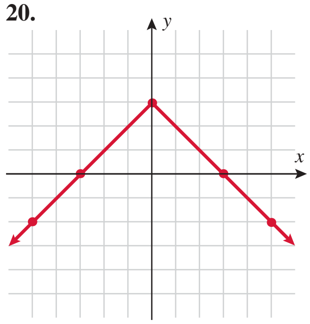
a. $y = -f(x)$
b. $y = f(-x)$

19.



a. $y = 2f(x)$
b. $y = f(2x)$

20.



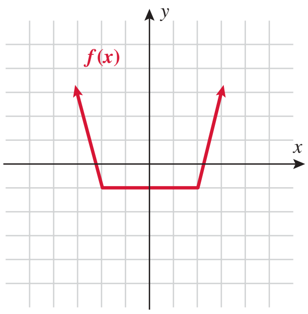
a. $y = 2f(x)$
b. $y = f(2x)$

21. $y = f(x - 2) - 3$

22. $y = f(x + 1) - 2$

23. $y = -f(x - 1) + 2$

24. $y = -2f(x) + 1$

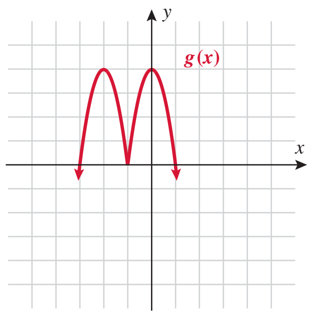


25. $y = -\frac{1}{2}g(x)$

26. $y = \frac{1}{4}g(-x)$

27. $y = -g(2x)$

28. $y = g\left(\frac{1}{2}x\right)$

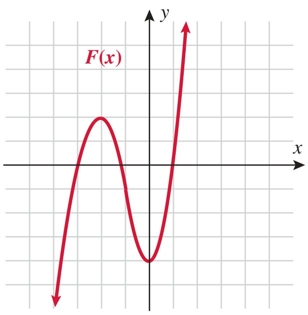


29. $y = \frac{1}{2}F(x - 1) + 2$

30. $y = \frac{1}{2}F(-x)$

31. $y = -F(1 - x)$

32. $y = -F(x - 2) - 1$

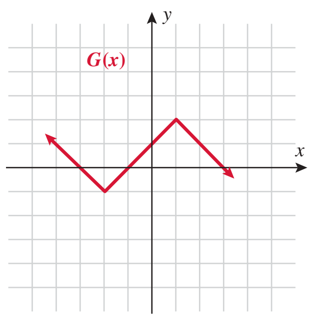


33. $y = 2G(x + 1) - 4$

34. $y = 2G(-x) + 1$

35. $y = -2G(x - 1) + 3$

36. $y = -G(x - 2) - 1$



In Exercises 37–62, graph the function using transformations.

37. $y = x^2 - 2$

38. $y = x^2 + 3$

39. $y = (x + 1)^2$

40. $y = (x - 2)^2$

41. $y = (x - 3)^2 + 2$

42. $y = (x + 2)^2 + 1$

43. $y = -(1 - x)^2$

44. $y = -(x + 2)^2$

45. $y = |-x|$

46. $y = -|x|$

47. $y = -|x + 2| - 1$

48. $y = |1 - x| + 2$

49. $y = 2x^2 + 1$

50. $y = 2|x| + 1$

51. $y = -\sqrt{x - 2}$

52. $y = \sqrt{2 - x}$

53. $y = -\sqrt{2 + x} - 1$

54. $y = \sqrt{2 - x} + 3$

55. $y = \sqrt[3]{x - 1} + 2$

56. $y = \sqrt[3]{x + 2} - 1$

57. $y = \dfrac{1}{x + 3} + 2$

58. $y = \dfrac{1}{3 - x}$

59. $y = 2 - \dfrac{1}{x + 2}$

60. $y = 2 - \dfrac{1}{1 - x}$

61. $y = 5\sqrt{-x}$

62. $y = -\frac{1}{5}\sqrt{x}$

In Exercises 63–68, transform the function into the form $f(x) = c(x - h)^2 + k$, where c, k, and h are constants, by completing the square. Use graph-shifting techniques to graph the function.

63. $y = x^2 - 6x + 11$

64. $f(x) = x^2 + 2x - 2$

65. $f(x) = -x^2 - 2x$

66. $f(x) = -x^2 + 6x - 7$

67. $f(x) = 2x^2 - 8x + 3$

68. $f(x) = 3x^2 - 6x + 5$

APPLICATIONS

69. Salary. A manager hires an employee at a rate of $10 per hour. Write the function that describes the current salary of the employee as a function of the number of hours worked per week, x. After a year, the manager decides to award the employee a raise equivalent to paying him for an additional 5 hours per week. Write a function that describes the salary of the employee after the raise.

70. Profit. The profit associated with St. Augustine sod in Florida is typically $P(x) = -x^2 + 14{,}000x - 48{,}700{,}000$, where x is the number of pallets sold per year in a normal year. In rainy years Sod King gives away 10 free pallets per year. Write the function that describes the profit of x pallets of sod in rainy years.

71. Taxes. Every year in the United States each working American typically pays in taxes a percentage of his or her earnings (minus the standard deduction). Karen's 2005 taxes were calculated based on the formula $T(x) = 0.22(x - 6500)$. That year the standard deduction was $6500 and her tax bracket paid 22% in taxes. Write the function that will determine her 2006 taxes, assuming she receives a raise that places her in the 33% bracket.

72. Medication. The amount of medication that an infant requires is typically a function of the baby's weight. The number of milliliters of an antiseizure medication A is given by $A(x) = \sqrt{x} + 2$, where x is the weight of the infant in ounces. In emergencies there is often not enough time to weigh the infant, so nurses have to estimate the baby's weight. What is the function that represents the actual amount of medication the infant is given if his weight is overestimated by 3 ounces?

73. Taxi Rates. Victoria lives in a condo on Peachtree Street in downtown Atlanta and works at the Federal Reserve Bank of Atlanta, which is 1 mile north of her condo on Peachtree Street. She often eats lunch at Nava Restaurant in Buckhead that is x miles north of the Federal Reserve Bank on Peachtree Street. A taxi in downtown Atlanta costs $7.00 for the first mile and $0.30 for every mile after that. Write a function that shows the cost of traveling from Victoria's office to Nava for lunch. Then rewrite the same function to show the cost of the taxi on days when Victoria walks home first to let her dog out and then takes the taxi from her condo to the Nava Restaurant.

74. Taxi Rates. Victoria (in Exercise 73) also likes to eat lunch at a sushi bar x miles south of the Federal Reserve Bank on Peachtree Street. Write a function that shows the cost of traveling from Victoria's office to the sushi bar for lunch. Then rewrite the same function to show the cost of the taxi on days when Victoria walks home first to let her dog out and then takes the taxi from her condo to the sushi bar.

75. Profit. A company that started in 1900 has made a profit corresponding to $P(t) = t^3 - t^2 + t - 1$, where P is the profit in dollars and t is the year (with $t = 0$ corresponding to 1950). Write the profit function with $t = 0$ corresponding to the year 2000.

76. Profit. For the company in Exercise 75, write the profit function with $t = 0$ corresponding to the year 2010.

For Exercises 77 and 78, refer to the following:

Body Surface Area (BSA) is used in physiology and medicine for many clinical purposes. BSA can be modeled by the function

$$BSA = \sqrt{\frac{wh}{3600}}$$

where w is weight in kilograms and h is height in centimeters. Since BSA depends on weight and height, it is often thought of as a function of both weight and height. However, for an individual adult height is generally considered constant; thus BSA can be thought of as a function of weight alone.

77. Health/Medicine. (a) If an adult female is 162 centimeters tall, find her BSA as a function of weight. (b) If she loses 3 kilograms, find a function that represents her new BSA.

78. Health/Medicine. (a) If an adult male is 180 centimeters tall, find his BSA as a function of weight. (b) If he gains 5 kilograms, find a function that represents his new BSA.

CATCH THE MISTAKE

In Exercises 79–82, explain the mistake that is made.

79. Describe a procedure for graphing the function $f(x) = \sqrt{x - 3} + 2$.

Solution:

a. Start with the function $f(x) = \sqrt{x}$.
b. Shift the function to the left three units.
c. Shift the function up two units.

This is incorrect. What mistake was made?

80. Describe a procedure for graphing the function $f(x) = -\sqrt{x + 2} - 3$.

Solution:

a. Start with the function $f(x) = \sqrt{x}$.
b. Shift the function to the left two units.
c. Reflect the function about the y-axis.
d. Shift the function down three units.

This is incorrect. What mistake was made?

81. Describe a procedure for graphing the function $f(x) = |3 - x| + 1$.

Solution:

a. Start with the function $f(x) = |x|$.
b. Reflect the function about the y-axis.
c. Shift the function to the left three units.
d. Shift the function up one unit.

This is incorrect. What mistake was made?

82. Describe a procedure for graphing the function $f(x) = -2x^2 + 1$.

Solution:

a. Start with the function $f(x) = x^2$.
b. Reflect the function about the y-axis.
c. Shift the function up one unit.
d. Expand in the vertical direction by a factor of 2.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 83–88, determine whether each statement is true or false.

83. The graph of $y = |-x|$ is the same as the graph of $y = |x|$.

84. The graph of $y = \sqrt{-x}$ is the same as the graph of $y = \sqrt{x}$.

85. If the graph of an odd function is reflected around the x-axis and then the y-axis, the result is the graph of the original odd function.

86. If the graph of $y = \frac{1}{x}$ is reflected around the x-axis, it produces the same graph as if it had been reflected about the y-axis.

87. If f is a function and $c > 1$ is a constant, then the graph of $-cf$ is a reflection about the x-axis of a vertical stretch of the graph of f.

88. If a and b are positive constants and f is a function, then the graph of $f(x + a) + b$ is obtained by shifting the graph of f to the right a units and then shifting this graph up b units.

CHALLENGE

89. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $f(x - 3) + 2$?

90. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $-f(-x) + 1$?

91. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $2f(x + 1) - 1$?

92. The point (a, b) lies on the graph of the function $y = f(x)$. What point is guaranteed to lie on the graph of $-2f(x - 3) + 4$?

TECHNOLOGY

93. Use a graphing utility to graph

a. $y = x^2 - 2$ and $y = |x^2 - 2|$
b. $y = x^3 - 1$ and $y = |x^3 - 1|$

What is the relationship between $f(x)$ and $|f(x)|$?

94. Use a graphing utility to graph

a. $y = x^2 - 2$ and $y = |x|^2 - 2$
b. $y = x^3 + 1$ and $y = |x|^3 + 1$

What is the relationship between $f(x)$ and $f(|x|)$?

95. Use a graphing utility to graph

a. $y = \sqrt{x}$ and $y = \sqrt{0.1x}$
b. $y = \sqrt{x}$ and $y = \sqrt{10x}$

What is the relationship between $f(x)$ and $f(ax)$, assuming that a is positive?

96. Use a graphing utility to graph

a. $y = \sqrt{x}$ and $y = 0.1\sqrt{x}$
b. $y = \sqrt{x}$ and $y = 10\sqrt{x}$

What is the relationship between $f(x)$ and $af(x)$, assuming that a is positive?

97. Use a graphing utility to graph $y = f(x) = [[0.5x]] + 1$. Use transformations to describe the relationship between $f(x)$ and $y = [[x]]$.

98. Use a graphing utility to graph $y = g(x) = 0.5\,[[x]] + 1$. Use transformations to describe the relationship between $g(x)$ and $y = [[x]]$.

PREVIEW TO CALCULUS

For Exercises 99–102, refer to the following:

In calculus, the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ of a function f is used to find the derivative f' of f, by letting h approach 0, $h \to 0$. Find the derivatives of f and g.

99. Horizontal Shift. $f(x) = x^2$, $g(x) = (x - 1)^2$. How are the graphs of g' and f' related?

100. Horizontal Shift. $f(x) = \sqrt{x}$, $g(x) = \sqrt{x + 5}$. How are the graphs of g' and f' related?

101. Vertical Shift. $f(x) = 2x$, $g(x) = 2x + 7$. How are the graphs of g' and f' related?

102. Vertical Shift. $f(x) = x^3$, $g(x) = x^3 - 4$. How are the graphs of g' and f' related?

SECTION 1.4 COMBINING FUNCTIONS

SKILLS OBJECTIVES

- Add, subtract, multiply, and divide functions.
- Evaluate composite functions.
- Determine domain of functions resulting from operations on and composition of functions.

CONCEPTUAL OBJECTIVES

- Understand domain restrictions when dividing functions.
- Realize that the domain of a composition of functions excludes values that are not in the domain of the inside function.

Adding, Subtracting, Multiplying, and Dividing Functions

Two functions can be added, subtracted, and multiplied. The domain of the resulting function is the intersection of the domains of the two functions. However, for division, any value of x (input) that makes the denominator equal to zero must be eliminated from the domain.

Function	Notation	Domain
Sum	$(f + g)(x) = f(x) + g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Difference	$(f - g)(x) = f(x) - g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$	$\{\text{domain of } f\} \cap \{\text{domain of } g\}$
Quotient	$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	$\{\text{domain of } f\} \cap \{\text{domain of } g\} \cap \{g(x) \neq 0\}$

We can think of this in the following way: Any number that is in the domain of *both* the functions is in the domain of the combined function. The exception to this is the quotient function, which also eliminates values that make the denominator equal to zero.

EXAMPLE 1 Operations on Functions: Determining Domains of New Functions

For the functions $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{4-x}$, determine the sum function, difference function, product function, and quotient function. State the domain of these four new functions.

Solution:

Sum function: $f(x) + g(x) = \sqrt{x-1} + \sqrt{4-x}$

Difference function: $f(x) - g(x) = \sqrt{x-1} - \sqrt{4-x}$

Product function:
$$f(x) \cdot g(x) = \sqrt{x-1} \cdot \sqrt{4-x} = \sqrt{(x-1)(4-x)} = \sqrt{-x^2 + 5x - 4}$$

Quotient function:
$$\frac{f(x)}{g(x)} = \frac{\sqrt{x-1}}{\sqrt{4-x}} = \sqrt{\frac{x-1}{4-x}}$$

The domain of the square root function is determined by setting the argument under the radical greater than or equal to zero.

Domain of $f(x)$: $[1, \infty)$

Domain of $g(x)$: $(-\infty, 4]$

The domain of the sum, difference, and product functions is

$$[1, \infty) \cap (-\infty, 4] = [1, 4]$$

The quotient function has the additional constraint that the denominator cannot be zero. This implies that $x \neq 4$, so the domain of the quotient function is $[1, 4)$.

YOUR TURN Given the function $f(x) = \sqrt{x+3}$ and $g(x) = \sqrt{1-x}$, find $(f+g)(x)$ and state its domain.

Answer:
$(f+g)(x) = \sqrt{x+3} + \sqrt{1-x}$
Domain: $[-3, 1]$

EXAMPLE 2 Quotient Function and Domain Restrictions

Given the functions $F(x) = \sqrt{x}$ and $G(x) = |x-3|$, find the quotient function, $\left(\frac{F}{G}\right)(x)$, and state its domain.

Solution:

The quotient function is written as

$$\boxed{\left(\frac{F}{G}\right)(x) = \frac{F(x)}{G(x)} = \frac{\sqrt{x}}{|x-3|}}$$

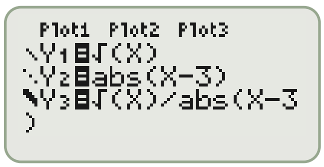
Domain of $F(x)$: $[0, \infty)$ Domain of $G(x)$: $(-\infty, \infty)$

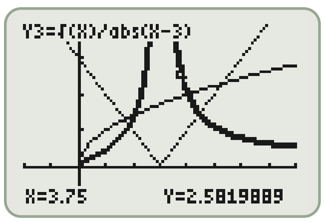
The real numbers that are in both the domain for $F(x)$ and the domain for $G(x)$ are represented by the intersection $[0, \infty) \cap (-\infty, \infty) = [0, \infty)$. Also, the denominator of the quotient function is equal to zero when $x = 3$, so we must eliminate this value from the domain.

$$\boxed{\text{Domain of } \left(\frac{F}{G}\right)(x)\text{: } [0, 3) \cup (3, \infty)}$$

Technology Tip

The graphs of $y_1 = F(x) = \sqrt{x}$, $y_2 = G(x) = |x-3|$, and $y_3 = \frac{F(x)}{G(x)} = \frac{\sqrt{x}}{|x-3|}$ are shown.



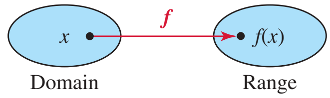


YOUR TURN For the functions given in Example 2, determine the quotient function $\left(\frac{G}{F}\right)(x)$, and state its domain.

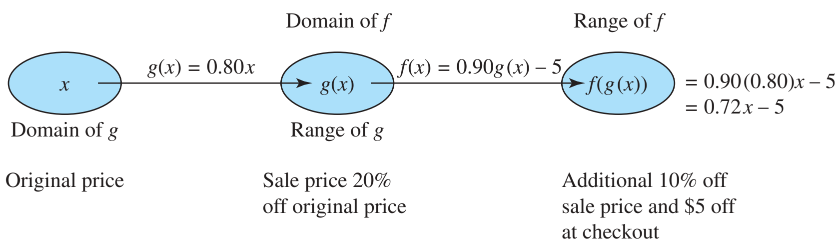
Answer:
$\left(\frac{G}{F}\right)(x) = \frac{G(x)}{F(x)} = \frac{|x-3|}{\sqrt{x}}$
Domain: $(0, \infty)$

Composition of Functions

Recall that a function maps every element in the domain to exactly one corresponding element in the range as shown in the figure below.



Suppose there is a sales rack of clothes in a department store. Let x correspond to the original price of each item on the rack. These clothes have recently been marked down 20%. Therefore, the function $g(x) = 0.80x$ represents the current sale price of each item. You have been invited to a special sale that lets you take 10% off the current sale price and an additional \$5 off every item at checkout. The function $f(g(x)) = 0.90g(x) - 5$ determines the checkout price. Note that the output of the function g is the input of the function f as shown in the figure below.



The "checkout" price is found by taking 28% off the original price and subtracting an additional \$5.

This is an example of a **composition of functions**, when the output of one function is the input of another function. It is commonly referred to as a function of a function.

An algebraic example of this is the function $y = \sqrt{x^2 - 2}$. Suppose we let $g(x) = x^2 - 2$ and $f(x) = \sqrt{x}$. Recall that the independent variable in function notation is a placeholder. Since $f(\square) = \sqrt{(\square)}$, then $f(g(x)) = \sqrt{(g(x))}$. Substituting the expression for $g(x)$, we find $f(g(x)) = \sqrt{x^2 - 2}$. The function $y = \sqrt{x^2 - 2}$ is said to be a composite function, $y = f(g(x))$.

Note that the domain of $g(x)$ is the set of all real numbers, and the domain of $f(x)$ is the set of all nonnegative numbers. The domain of a composite function is the set of all x such that $g(x)$ is in the domain of f. For instance, in the composite function $y = f(g(x))$, we know that the allowable inputs into f are all numbers greater than or equal to zero. Therefore, we restrict the outputs of $g(x) \geq 0$ and find the corresponding x-values. Those x-values are the only allowable inputs and constitute the domain of the composite function $y = f(g(x))$.

CAUTION

$f \circ g \neq f \cdot g$

The symbol that represents composition of functions is a small open circle; thus $(f \circ g)(x) = f(g(x))$ and is read aloud as "f of g." It is important not to confuse this with the multiplication sign: $(f \cdot g)(x) = f(x)g(x)$.

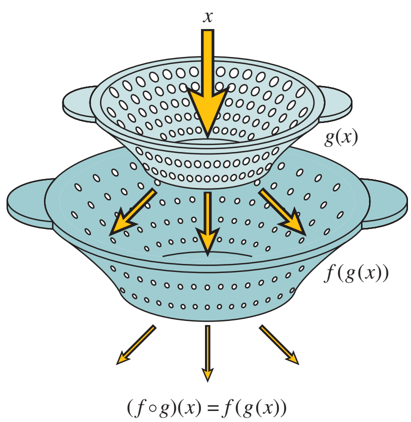
Composition of Functions

Given two functions f and g, there are two **composite functions** that can be formed.

Notation	Words	Definition	Domain
$f \circ g$	f composed with g	$f(g(x))$	The set of all real numbers x in the domain of g such that $g(x)$ is also in the domain of f.
$g \circ f$	g composed with f	$g(f(x))$	The set of all real numbers x in the domain of f such that $f(x)$ is also in the domain of g.

It is important to realize that there are two "filters" that allow certain values of x into the domain. The first filter is $g(x)$. If x is not in the domain of $g(x)$, it cannot be in the domain of $(f \circ g)(x) = f(g(x))$. Of those values for x that are in the domain of $g(x)$, only some pass through, because we restrict the output of $g(x)$ to values that are allowable as input into f. This adds an additional filter.

The domain of $f \circ g$ is always a subset of the domain of g, and the range of $f \circ g$ is always a subset of the range of f.



Study Tip

Order is important:

$$(f \circ g)(x) = f(g(x))$$
$$(g \circ f)(x) = g(f(x))$$

Study Tip

The domain of $f \circ g$ is always a subset of the domain of g, and the range of $f \circ g$ is always a subset of the range of f.

Example 3 Finding a Composite Function

Given the functions $f(x) = x^2 + 1$ and $g(x) = x - 3$, find $(f \circ g)(x)$.

Solution:

Write $f(x)$ using placeholder notation. $\quad f(\square) = (\square)^2 + 1$

Express the composite function $f \circ g$. $\quad f(g(x)) = (g(x))^2 + 1$

Substitute $g(x) = x - 3$ into f. $\quad f(g(x)) = (x - 3)^2 + 1$

Eliminate the parentheses on the right side. $\quad f(g(x)) = x^2 - 6x + 10$

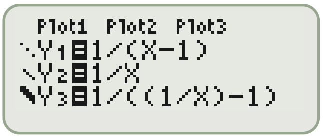
$$\boxed{(f \circ g)(x) = f(g(x)) = x^2 - 6x + 10}$$

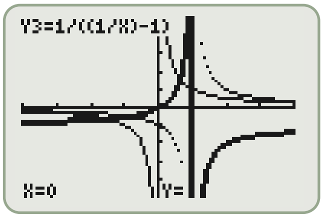
Your Turn Given the functions in Example 3, find $(g \circ f)(x)$.

Answer: $g \circ f = g(f(x)) = x^2 - 2$

Technology Tip

The graphs of $y_1 = f(x) = \dfrac{1}{x-1}$, $y_2 = g(x) = \dfrac{1}{x}$, and $y_3 = (f \circ g)(x) = \dfrac{1}{\frac{1}{x} - 1} = \dfrac{x}{1-x}$ are shown.





■ **Answer:** $g(f(x)) = x - 1$. Domain of $g \circ f$ is $x \neq 1$, or in interval notation, $(-\infty, 1) \cup (1, \infty)$.

CAUTION

The domain of the composite function cannot always be determined by examining the final form of $f \circ g$.

EXAMPLE 4 Determining the Domain of a Composite Function

Given the functions $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{x}$, determine $f \circ g$, and state its domain.

Solution:

Write $f(x)$ using placeholder notation. $\quad f(\square) = \dfrac{1}{(\square) - 1}$

Express the composite function $f \circ g$. $\quad f(g(x)) = \dfrac{1}{g(x) - 1}$

Substitute $g(x) = \dfrac{1}{x}$ into f. $\quad f(g(x)) = \dfrac{1}{\frac{1}{x} - 1}$

Multiply the right side by $\dfrac{x}{x}$. $\quad f(g(x)) = \dfrac{1}{\frac{1}{x} - 1} \cdot \dfrac{x}{x} = \dfrac{x}{1-x}$

$$\boxed{(f \circ g) = f(g(x)) = \frac{x}{1-x}}$$

What is the domain of $(f \circ g)(x) = f(g(x))$? By inspecting the final result of $f(g(x))$, we see that the denominator is zero when $x = 1$. Therefore, $x \neq 1$. Are there any other values for x that are not allowed? The function $g(x)$ has the domain $x \neq 0$; therefore, we must also exclude zero.

The domain of $(f \circ g)(x) = f(g(x))$ excludes $x = 0$ and $x = 1$ or, in interval notation,

$$\boxed{(-\infty, 0) \cup (0, 1) \cup (1, \infty)}$$

■ **YOUR TURN** For the functions f and g given in Example 4, determine the composite function $g \circ f$ and state its domain.

The domain of the composite function cannot always be determined by examining the final form of $f \circ g$, as illustrated in Example 4.

EXAMPLE 5 Determining the Domain of a Composite Function (Without Finding the Composite Function)

Let $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+3}$. Find the domain of $f(g(x))$. Do not find the composite function.

Solution:

Find the domain of g. $\quad [-3, \infty)$

Find the range of g. $\quad [0, \infty)$

In $f(g(x))$, the output of g becomes the input for f. Since the domain of f is the set of all real numbers except 2, we eliminate any values of x in the domain of g that correspond to $g(x) = 2$.

Let $g(x) = 2$. $\quad \sqrt{x+3} = 2$

Square both sides. $\quad x + 3 = 4$

Solve for x. $\quad x = 1$

Eliminate $x = 1$ from the domain of g, $[-3, \infty)$.

State the domain of $f(g(x))$. $\quad \boxed{[-3, 1) \cup (1, \infty)}$

EXAMPLE 6 Evaluating a Composite Function

Given the functions $f(x) = x^2 - 7$ and $g(x) = 5 - x^2$, evaluate

a. $f(g(1))$ **b.** $f(g(-2))$ **c.** $g(f(3))$ **d.** $g(f(-4))$

Solution:

One way of evaluating these composite functions is to calculate the two individual composites in terms of x: $f(g(x))$ and $g(f(x))$. Once those functions are known, the values can be substituted for x and evaluated.

Another way of proceeding is as follows:

a. Write the desired quantity. $f(g(1))$
Find the value of the inner function g. $g(1) = 5 - 1^2 = 4$
Substitute $g(1) = 4$ into f. $f(g(1)) = f(4)$
Evaluate $f(4)$. $f(4) = 4^2 - 7 = 9$

$$\boxed{f(g(1)) = 9}$$

b. Write the desired quantity. $f(g(-2))$
Find the value of the inner function g. $g(-2) = 5 - (-2)^2 = 1$
Substitute $g(-2) = 1$ into f. $f(g(-2)) = f(1)$
Evaluate $f(1)$. $f(1) = 1^2 - 7 = -6$

$$\boxed{f(g(-2)) = -6}$$

c. Write the desired quantity. $g(f(3))$
Find the value of the inner function f. $f(3) = 3^2 - 7 = 2$
Substitute $f(3) = 2$ into g. $g(f(3)) = g(2)$
Evaluate $g(2)$. $g(2) = 5 - 2^2 = 1$

$$\boxed{g(f(3)) = 1}$$

d. Write the desired quantity. $g(f(-4))$
Find the value of the inner function f. $f(-4) = (-4)^2 - 7 = 9$
Substitute $f(-4) = 9$ into g. $g(f(-4)) = g(9)$
Evaluate $g(9)$. $g(9) = 5 - 9^2 = -76$

$$\boxed{g(f(-4)) = -76}$$

■ **YOUR TURN** Given the functions $f(x) = x^3 - 3$ and $g(x) = 1 + x^3$, evaluate $f(g(1))$ and $g(f(1))$.

■ **Answer:** $f(g(1)) = 5$
$g(f(1)) = -7$

Application Problems

Recall the example at the beginning of this section regarding the clothes that are on sale. Often, real-world applications are modeled with composite functions. In the clothes example, x is the original price of each item. The first function maps its input (original price) to an output (sale price). The second function maps its input (sale price) to an output (checkout price). Example 7 is another real-world application of composite functions.

Three temperature scales are commonly used:

- The degree Celsius (°C) scale
  - This scale was devised by dividing the range between the freezing (0°C) and boiling (100°C) points of pure water at sea level into 100 equal parts. This scale is used in science and is one of the standards of the "metric" (SI) system of measurements.

- The Kelvin (K) temperature scale
  - This scale shifts the Celsius scale down so that the zero point is equal to absolute zero (about $-273.15°C$), a hypothetical temperature at which there is a complete absence of heat energy.
  - Temperatures on this scale are called **kelvins**, *not* degrees kelvin, and kelvin is not capitalized. The symbol for the kelvin is K.
- The degree Fahrenheit (°F) scale
  - This scale evolved over time and is still widely used mainly in the United States, although Celsius is the preferred "metric" scale.
  - With respect to pure water at sea level, the **degrees Fahrenheit** are gauged by the spread from 32°F (freezing) to 212°F (boiling).

The equations that relate these temperature scales are

$$F = \frac{9}{5}C + 32 \qquad C = K - 273.15$$

EXAMPLE 7 Applications Involving Composite Functions

Determine degrees Fahrenheit as a function of kelvins.

Solution:

Degrees Fahrenheit is a function of degrees Celsius. $F = \frac{9}{5}C + 32$

Now substitute $C = K - 273.15$ into the equation for F. $F = \frac{9}{5}(K - 273.15) + 32$

Simplify. $F = \frac{9}{5}K - 491.67 + 32$

$$\boxed{F = \frac{9}{5}K - 459.67}$$

SECTION 1.4 SUMMARY

Operations on Functions

Function	Notation
Sum	$(f + g)(x) = f(x) + g(x)$
Difference	$(f - g)(x) = f(x) - g(x)$
Product	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$

The domain of the sum, difference, and product functions is the intersection of the domains, or common domain shared by both f and g. The domain of the quotient function is also the intersection of the domain shared by both f and g with an additional restriction that $g(x) \neq 0$.

Composition of Functions

$$(f \circ g)(x) = f(g(x))$$

The domain restrictions cannot always be determined simply by inspecting the final form of $f(g(x))$. Rather, the domain of the composite function is a subset of the domain of $g(x)$. Values of x must be eliminated if their corresponding values of $g(x)$ are not in the domain of f.

SECTION 1.4 EXERCISES

SKILLS

In Exercises 1–10, given the functions f and g, find $f+g$, $f-g$, $f\cdot g$, and $\frac{f}{g}$, and state the domain of each.

1. $f(x) = 2x + 1$, $g(x) = 1 - x$
2. $f(x) = 3x + 2$, $g(x) = 2x - 4$
3. $f(x) = 2x^2 - x$, $g(x) = x^2 - 4$
4. $f(x) = 3x + 2$, $g(x) = x^2 - 25$
5. $f(x) = \dfrac{1}{x}$, $g(x) = x$
6. $f(x) = \dfrac{2x+3}{x-4}$, $g(x) = \dfrac{x-4}{3x+2}$
7. $f(x) = \sqrt{x}$, $g(x) = 2\sqrt{x}$
8. $f(x) = \sqrt{x-1}$, $g(x) = 2x^2$
9. $f(x) = \sqrt{4-x}$, $g(x) = \sqrt{x+3}$
10. $f(x) = \sqrt{1-2x}$, $g(x) = \dfrac{1}{x}$

In Exercises 11–20, for the given functions f and g, find the composite functions $f \circ g$ and $g \circ f$, and state their domains.

11. $f(x) = 2x + 1$, $g(x) = x^2 - 3$
12. $f(x) = x^2 - 1$, $g(x) = 2 - x$
13. $f(x) = \dfrac{1}{x-1}$, $g(x) = x + 2$
14. $f(x) = \dfrac{2}{x-3}$, $g(x) = 2 + x$
15. $f(x) = |x|$, $g(x) = \dfrac{1}{x-1}$
16. $f(x) = |x-1|$, $g(x) = \dfrac{1}{x}$
17. $f(x) = \sqrt{x-1}$, $g(x) = x + 5$
18. $f(x) = \sqrt{2-x}$, $g(x) = x^2 + 2$
19. $f(x) = x^3 + 4$, $g(x) = (x-4)^{1/3}$
20. $f(x) = \sqrt[3]{x^2 - 1}$, $g(x) = x^{2/3} + 1$

In Exercises 21–38, evaluate the functions for the specified values, if possible.

$$f(x) = x^2 + 10 \qquad g(x) = \sqrt{x-1}$$

21. $(f+g)(2)$
22. $(f+g)(10)$
23. $(f-g)(2)$
24. $(f-g)(5)$
25. $(f\cdot g)(4)$
26. $(f\cdot g)(5)$
27. $\left(\dfrac{f}{g}\right)(10)$
28. $\left(\dfrac{f}{g}\right)(2)$
29. $f(g(2))$
30. $f(g(1))$
31. $g(f(-3))$
32. $g(f(4))$
33. $f(g(0))$
34. $g(f(0))$
35. $f(g(-3))$
36. $g(f(\sqrt{7}))$
37. $(f \circ g)(4)$
38. $(g \circ f)(-3)$

In Exercises 39–50, evaluate $f(g(1))$ and $g(f(2))$, if possible.

39. $f(x) = \dfrac{1}{x}$, $g(x) = 2x + 1$
40. $f(x) = x^2 + 1$, $g(x) = \dfrac{1}{2-x}$
41. $f(x) = \sqrt{1-x}$, $g(x) = x^2 + 2$
42. $f(x) = \sqrt{3-x}$, $g(x) = x^2 + 1$
43. $f(x) = \dfrac{1}{|x-1|}$, $g(x) = x + 3$
44. $f(x) = \dfrac{1}{x}$, $g(x) = |2x - 3|$
45. $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 5$
46. $f(x) = \sqrt[3]{x-3}$, $g(x) = \dfrac{1}{x-3}$
47. $f(x) = \dfrac{1}{x^2-3}$, $g(x) = \sqrt{x-3}$
48. $f(x) = \dfrac{x}{2-x}$, $g(x) = 4 - x^2$
49. $f(x) = (x-1)^{1/3}$, $g(x) = x^2 + 2x + 1$
50. $f(x) = (1-x^2)^{1/2}$, $g(x) = (x-3)^{1/3}$

In Exercises 51–60, show that $f(g(x)) = x$ and $g(f(x)) = x$.

51. $f(x) = 2x + 1, \quad g(x) = \dfrac{x - 1}{2}$

52. $f(x) = \dfrac{x - 2}{3}, \quad g(x) = 3x + 2$

53. $f(x) = \sqrt{x - 1}, \quad g(x) = x^2 + 1$ for $x \geq 1$

54. $f(x) = 2 - x^2, \quad g(x) = \sqrt{2 - x}$ for $x \leq 2$

55. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$ for $x \neq 0$

56. $f(x) = (5 - x)^{1/3}, \quad g(x) = 5 - x^3$

57. $f(x) = 4x^2 - 9, \quad g(x) = \dfrac{\sqrt{x + 9}}{2}$ for $x \geq 0$

58. $f(x) = \sqrt[3]{8x - 1}, \quad g(x) = \dfrac{x^3 + 1}{8}$

59. $f(x) = \dfrac{1}{x - 1}, \quad g(x) = \dfrac{x + 1}{x}$ for $x \neq 0, x \neq 1$

60. $f(x) = \sqrt{25 - x^2}, \quad g(x) = \sqrt{25 - x^2}$ for $0 \leq x \leq 5$

In Exercises 61–66, write the function as a composite of two functions f and g. (More than one answer is correct.)

61. $f(g(x)) = 2(3x - 1)^2 + 5(3x - 1)$

62. $f(g(x)) = \dfrac{1}{1 + x^2}$

63. $f(g(x)) = \dfrac{2}{|x - 3|}$

64. $f(g(x)) = \sqrt{1 - x^2}$

65. $f(g(x)) = \dfrac{3}{\sqrt{x + 1} - 2}$

66. $f(g(x)) = \dfrac{\sqrt{x}}{3\sqrt{x} + 2}$

APPLICATIONS

Exercises 67 and 68 depend on the relationship between degrees Fahrenheit, degrees Celsius, and kelvins:

$$F = \frac{9}{5}C + 32 \qquad C = K - 273.15$$

67. Temperature. Write a composite function that converts kelvins into degrees Fahrenheit.

68. Temperature. Convert the following degrees Fahrenheit to kelvins: 32°F and 212°F.

69. Dog Run. Suppose that you want to build a *square* fenced-in area for your dog. Fencing is purchased in linear feet.

a. Write a composite function that determines the area of your dog pen as a function of how many linear feet are purchased.

b. If you purchase 100 linear feet, what is the area of your dog pen?

c. If you purchase 200 linear feet, what is the area of your dog pen?

70. Dog Run. Suppose that you want to build a *circular* fenced-in area for your dog. Fencing is purchased in linear feet.

a. Write a composite function that determines the area of your dog pen as a function of how many linear feet are purchased.

b. If you purchase 100 linear feet, what is the area of your dog pen?

c. If you purchase 200 linear feet, what is the area of your dog pen?

71. Market Price. Typical supply and demand relationships state that as the number of units for sale increases, the market price decreases. Assume that the market price p and the number of units for sale x are related by the demand equation:

$$p = 3000 - \frac{1}{2}x$$

Assume that the cost $C(x)$ of producing x items is governed by the equation

$$C(x) = 2000 + 10x$$

and the revenue $R(x)$ generated by selling x units is governed by

$$R(x) = 100x$$

a. Write the cost as a function of price p.

b. Write the revenue as a function of price p.

c. Write the profit as a function of price p.

72. Market Price. Typical supply and demand relationships state that as the number of units for sale increases, the market price decreases. Assume that the market price p and the number of units for sale x are related by the demand equation:

$$p = 10{,}000 - \frac{1}{4}x$$

Assume that the cost $C(x)$ of producing x items is governed by the equation

$$C(x) = 30{,}000 + 5x$$

and the revenue $R(x)$ generated by selling x units is governed by

$$R(x) = 1000x$$

a. Write the cost as a function of price p.

b. Write the revenue as a function of price p.

c. Write the profit as a function of price p.

In Exercises 73 and 74, refer to the following:

The cost of manufacturing a product is a function of the number of hours t the assembly line is running per day. The number of products manufactured n is a function of the number of hours t the assembly line is operating and is given by the function $n(t)$. The cost of manufacturing the product C measured in thousands of dollars is a function of the quantity manufactured, that is, the function $C(n)$.

73. Business. If the quantity of a product manufactured during a day is given by

$$n(t) = 50t - t^2$$

and the cost of manufacturing the product is given by

$$C(n) = 10n + 1375$$

a. Find a function that gives the cost of manufacturing the product in terms of the number of hours t the assembly line was functioning, $C(n(t))$.
b. Find the cost of production on a day when the assembly line was running for 16 hours. Interpret your answer.

74. Business. If the quantity of a product manufactured during a day is given by

$$n(t) = 100t - 4t^2$$

and the cost of manufacturing the product is given by

$$C(n) = 8n + 2375$$

a. Find a function that gives the cost of manufacturing the product in terms of the number of hours t the assembly line was functioning, $C(n(t))$.
b. Find the cost of production on a day when the assembly line was running for 24 hours. Interpret your answer.

In Exercises 75 and 76, refer to the following:

Surveys performed immediately following an accidental oil spill at sea indicate the oil moved outward from the source of the spill in a nearly circular pattern. The radius of the oil spill r measured in miles is a function of time t measured in days from the start of the spill, while the area of the oil spill is a function of radius, that is, the function $A(r)$.

75. Environment: Oil Spill. If the radius of the oil spill is given by

$$r(t) = 10t - 0.2t^2$$

and the area of the oil spill is given by

$$A(r) = \pi r^2$$

a. Find a function that gives the area of the oil spill in terms of the number of days since the start of the spill, $A(r(t))$.
b. Find the area of the oil spill to the nearest square mile 7 days after the start of the spill.

76. Environment: Oil Spill. If the radius of the oil spill is given by

$$r(t) = 8t - 0.1t^2$$

and the area of the oil spill is given by

$$A(r) = \pi r^2$$

a. Find a function that gives the area of the oil spill in terms of the number of days since the start of the spill, $A(r(t))$.
b. Find the area of the oil spill to the nearest square mile 5 days after the start of the spill.

77. Environment: Oil Spill. An oil spill makes a circular pattern around a ship such that the radius in feet grows as a function of time in hours $r(t) = 150\sqrt{t}$. Find the area of the spill as a function of time.

78. Pool Volume. A 20 foot by 10 foot rectangular pool has been built. If 50 cubic feet of water is pumped into the pool per hour, write the water-level height (feet) as a function of time (hours).

79. Fireworks. A family is watching a fireworks display. If the family is 2 miles from where the fireworks are being launched and the fireworks travel vertically, what is the distance between the family and the fireworks as a function of height above ground?

80. Real Estate. A couple are about to put their house up for sale. They bought the house for $172,000 a few years ago; if they list it with a realtor, they will pay a 6% commission. Write a function that represents the amount of money they will make on their home as a function of the asking price p.

CATCH THE MISTAKE

In Exercises 81–85, for the functions $f(x) = x + 2$ and $g(x) = x^2 - 4$, find the indicated function and state its domain. Explain the mistake that is made in each problem.

81. $\dfrac{g}{f}$

Solution:

$$\frac{g(x)}{f(x)} = \frac{x^2 - 4}{x + 2} = \frac{(x - 2)(x + 2)}{x + 2} = x - 2$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

82. $\dfrac{f}{g}$

Solution:

$$\frac{f(x)}{g(x)} = \frac{x + 2}{x^2 - 4} = \frac{x + 2}{(x - 2)(x + 2)} = \frac{1}{x - 2} = \frac{1}{x - 2}$$

Domain: $(-\infty, 2) \cup (2, \infty)$

This is incorrect. What mistake was made?

83. $f \circ g$

Solution: $$\begin{aligned} f \circ g &= f(x)g(x) \\ &= (x + 2)(x^2 - 4) \\ &= x^3 + 2x^2 - 4x - 8 \end{aligned}$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

84. $$\begin{aligned} f(x) - g(x) &= x + 2 - x^2 - 4 \\ &= -x^2 + x - 2 \end{aligned}$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

85. $$\begin{aligned} (f + g)(2) &= (x + 2 + x^2 - 4)(2) \\ &= (x^2 + x - 2)(2) \\ &= 2x^2 + 2x - 4 \end{aligned}$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

86. Given the function $f(x) = x^2 + 7$ and $g(x) = \sqrt{x - 3}$, find $f \circ g$, and state the domain.

Solution: $$\begin{aligned} f \circ g &= f(g(x)) = (\sqrt{x - 3})^2 + 7 \\ &= f(g(x)) = x - 3 + 7 \\ &= x - 4 \end{aligned}$$

Domain: $(-\infty, \infty)$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 87–90, determine whether each statement is true or false.

87. When adding, subtracting, multiplying, or dividing two functions, the domain of the resulting function is the union of the domains of the individual functions.

88. For any functions f and g, $f(g(x)) = g(f(x))$ for all values of x that are in the domain of both f and g.

89. For any functions f and g, $(f \circ g)(x)$ exists for all values of x that are in the domain of $g(x)$, provided the range of g is a subset of the domain of f.

90. The domain of a composite function can be found by inspection, without knowledge of the domain of the individual functions.

CHALLENGE

91. For the functions $f(x) = x + a$ and $g(x) = \dfrac{1}{x - a}$, find $g \circ f$ and state its domain.

92. For the functions $f(x) = ax^2 + bx + c$ and $g(x) = \dfrac{1}{x - c}$, find $g \circ f$ and state its domain.

93. For the functions $f(x) = \sqrt{x + a}$ and $g(x) = x^2 - a$ find $g \circ f$ and state its domain.

94. For the functions $f(x) = \dfrac{1}{x^a}$ and $g(x) = \dfrac{1}{x^b}$, find $g \circ f$ and state its domain. Assume $a > 1$ and $b > 1$.

TECHNOLOGY

95. Using a graphing utility, plot $y_1 = \sqrt{x + 7}$ and $y_2 = \sqrt{9 - x}$. Plot $y_3 = y_1 + y_2$. What is the domain of y_3?

96. Using a graphing utility, plot $y_1 = \sqrt[3]{x + 5}$, $y_2 = \dfrac{1}{\sqrt{3 - x}}$, and $y_3 = \dfrac{y_1}{y_2}$. What is the domain of y_3?

97. Using a graphing utility, plot $y_1 = \sqrt{x^2 - 3x - 4}$, $y_2 = \dfrac{1}{x^2 - 14}$, and $y_3 = \dfrac{1}{y_1^2 - 14}$. If y_1 represents a function f and y_2 represents a function g, then y_3 represents the composite function $g \circ f$. The graph of y_3 is only defined for the domain of $g \circ f$. State the domain of $g \circ f$.

98. Using a graphing utility, plot $y_1 = \sqrt{1 - x}$, $y_2 = x^2 + 2$, and $y_3 = y_1^2 + 2$. If y_1 represents a function f and y_2 represents a function g, then y_3 represents the composite function $g \circ f$. The graph of y_3 is only defined for the domain of $g \circ f$. State the domain of $g \circ f$.

PREVIEW TO CALCULUS

For Exercises 99–102, refer to the following:

In calculus, the difference quotient $\frac{f(x+h)-f(x)}{h}$ of a function f is used to find the derivative f' of f by letting h approach 0, $h \to 0$.

99. Addition. Find the derivatives of $F(x) = x$, $G(x) = x^2$, and $H(x) = (F+G)(x) = x + x^2$. What do you observe?

100. Subtraction. Find the derivatives of $F(x) = \sqrt{x}$, $G(x) = x^3 + 1$, and $H(x) = (F - G)(x) = \sqrt{x} - x^3 - 1$. What do you observe?

101. Multiplication. Find the derivatives of $F(x) = 5$, $G(x) = \sqrt{x-1}$, and $H(x) = (FG)(x) = 5\sqrt{x-1}$. What do you observe?

102. Division. Find the derivatives of $F(x) = x$, $G(x) = \sqrt{x+1}$, and $H(x) = \left(\frac{F}{G}\right)(x) = \frac{x}{\sqrt{x+1}}$. What do you observe?

SECTION 1.5 ONE-TO-ONE FUNCTIONS AND INVERSE FUNCTIONS

SKILLS OBJECTIVES

- Determine algebraically and graphically whether a function is a one-to-one function.
- Verify that two functions are inverses of one another.
- Graph the inverse function given the graph of the function.
- Find the inverse of a function.

CONCEPTUAL OBJECTIVES

- Visualize the relationships between domain and range of a function and the domain and range of its inverse.
- Understand why functions and their inverses are symmetric about $y = x$.

One-to-One Functions

Every human being has a blood type, and every human being has a DNA sequence. These are examples of functions, where a person is the input and the output is blood type or DNA sequence. These relationships are classified as functions because each person can have one and only one blood type or DNA strand. The difference between these functions is that many people have the same blood type, but DNA is unique to each individual. Can we map backwards? For instance, if you know the blood type, do you know specifically which person it came from? No, but, if you know the DNA sequence, you know that the sequence belongs to only one person. When a function has a one-to-one correspondence, like the DNA example, then mapping backwards is possible. The map back is called the *inverse function.*

In Section 1.1, we defined a function as a relationship that maps an input (contained in the domain) to exactly one output (found in the range). Algebraically, each value for x can correspond to only a single value for y. Recall the square, identity, absolute value, and reciprocal functions from our library of functions in Section 1.3.

All of the graphs of these functions satisfy the vertical line test. Although the square function and the absolute value function map each value of x to exactly one value for y, these two functions map two values of x to the same value for y. For example, $(-1, 1)$ and $(1, 1)$ lie on both graphs. The identity and reciprocal functions, on the other hand, map each x to a single value for y, and no two x-values map to the same y-value. These two functions are examples of *one-to-one functions.*

DEFINITION **One-to-One Function**

A function $f(x)$ is **one-to-one** if no two elements in the domain correspond to the same element in the range; that is,

$$\text{if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

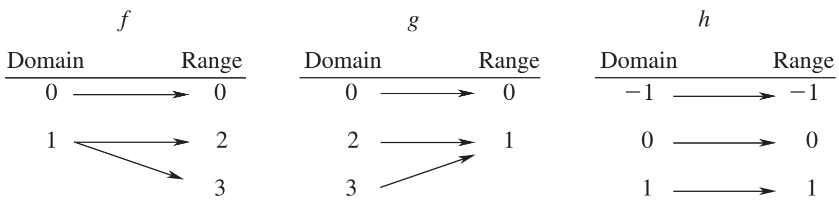
In other words, it is one-to-one if no two inputs map to the same output.

EXAMPLE 1 **Determining Whether a Function Defined as a Set of Points Is a One-to-One Function**

For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function.

$$f = \{(0, 0), (1, 2), (1, 3)\}$$
$$g = \{(2, 1), (0, 0), (3, 1)\}$$
$$h = \{(-1, -1), (0, 0), (1, 1)\}$$

Solution:



f is not a function.

g is a function, but not one-to-one.

h is a one-to-one function.

Just as there is a graphical test for functions, the vertical line test, there is a graphical test for one-to-one functions, the *horizontal line test.* Note that a horizontal line can be drawn on the square and absolute value functions so that it intersects the graph of each function at two points. The identity and reciprocal functions, however, will intersect a horizontal line in at most only one point. This leads us to the horizontal line test for one-to-one functions.

DEFINITION **Horizontal Line Test**

If every horizontal line intersects the graph of a function in at most one point, then the function is classified as a one-to-one function.

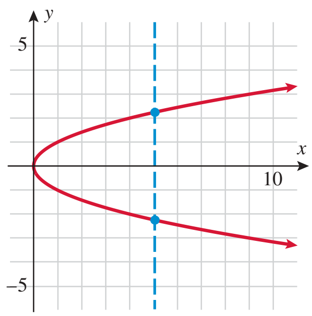
EXAMPLE 2 Using the Horizontal Line Test to Determine Whether a Function Is One-to-One

For each of the three relations, determine whether the relation is a function. If it is a function, determine whether it is a one-to-one function. Assume that x is the independent variable and y is the dependent variable.

$$x = y^2 \qquad y = x^2 \qquad y = x^3$$

Solution:

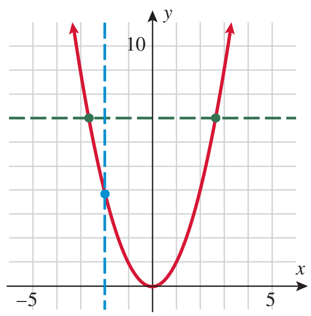
$x = y^2$



Not a function

(fails vertical line test)

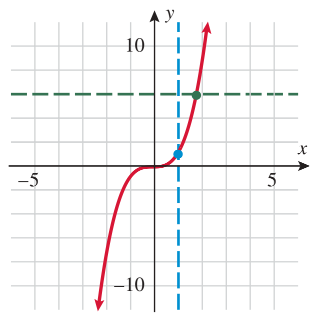
$y = x^2$



Function, but not one-to-one

(passes vertical line test, but fails horizontal line test)

$y = x^3$



One-to-one function

(passes both vertical and horizontal line tests)

YOUR TURN Determine whether each of the functions is a one-to-one function.

a. $f(x) = x + 2$ **b.** $f(x) = x^2 + 1$

Answer:
a. yes **b.** no

Another way of writing the definition of a one-to-one function is

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

In the Your Turn following Example 2, we found (using the horizontal line test) that $f(x) = x + 2$ is a one-to-one function, but that $f(x) = x^2 + 1$ is not a one-to-one function. We can also use this alternative definition to determine algebraically whether a function is one-to-one.

WORDS	MATH
State the function.	$f(x) = x + 2$
Let there be two real numbers, x_1 and x_2, such that $f(x_1) = f(x_2)$.	$x_1 + 2 = x_2 + 2$
Subtract 2 from both sides of the equation.	$x_1 = x_2$

$f(x) = x + 2$ is a one-to-one function.

WORDS	MATH
State the function.	$f(x) = x^2 + 1$
Let there be two real numbers, x_1 and x_2, such that $f(x_1) = f(x_2)$.	$x_1^2 + 1 = x_2^2 + 1$
Subtract 1 from both sides of the equation.	$x_1^2 = x_2^2$
Solve for x_1.	$x_1 = \pm x_2$

$f(x) = x^2 + 2$ is *not* a one-to-one function.

EXAMPLE 3 Determining Algebraically Whether a Function Is One-to-One

Determine algebraically whether the functions are one-to-one.

a. $f(x) = 5x^3 - 2$ **b.** $f(x) = |x + 1|$

Solution (a):

Find $f(x_1)$ and $f(x_2)$.	$f(x_1) = 5x_1^3 - 2$ and $f(x_2) = 5x_2^3 - 2$
Let $f(x_1) = f(x_2)$.	$5x_1^3 - 2 = 5x_2^3 - 2$
Add 2 to both sides of the equation.	$5x_1^3 = 5x_2^3$
Divide both sides of the equation by 5.	$x_1^3 = x_2^3$
Take the cube root of both sides of the equation.	$(x_1^3)^{1/3} = (x_2^3)^{1/3}$
Simplify.	$x_1 = x_2$

$\boxed{f(x) = 5x^3 - 2 \text{ is a one-to-one function.}}$

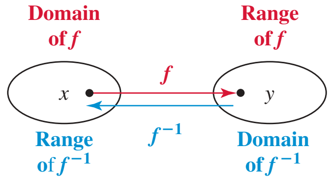
Solution (b):

Find $f(x_1)$ and $f(x_2)$.	$f(x_1) = \lvert x_1 + 1\rvert$ and $f(x_2) = \lvert x_2 + 1\rvert$
Let $f(x_1) = f(x_2)$.	$\lvert x_1 + 1\rvert = \lvert x_2 + 1\rvert$
Solve the absolute value equation.	$(x_1 + 1) = (x_2 + 1)$ or $(x_1 + 1) = -(x_2 + 1)$
	$x_1 = x_2$ or $x_1 = -x_2 - 2$

$\boxed{f(x) = |x + 1| \text{ is } \textbf{not} \text{ a one-to-one function.}}$

Inverse Functions

If a function is one-to-one, then the function maps each x to exactly one y, and no two x-values map to the same y-value. This implies that there is a one-to-one correspondence between the inputs (domain) and outputs (range) of a one-to-one function $f(x)$. In the special case of a one-to-one function, it would be possible to map from the output (range of f) back to the input (domain of f), and this mapping would also be a function. The function that maps the output back to the input of a function f is called the **inverse function** and is denoted $f^{-1}(x)$.



A one-to-one function f maps every x in the domain to a unique and distinct corresponding y in the range. Therefore, the inverse function f^{-1} maps every y back to a unique and distinct x.

The function notations $f(x) = y$ and $f^{-1}(y) = x$ indicate that if the point (x, y) satisfies the function, then the point (y, x) satisfies the inverse function.

For example, let the function $h(x) = \{(-1, 0), (1, 2), (3, 4)\}$.

$$h = \{(-1, 0), (1, 2), (3, 4)\}$$

Domain **Range**

$-1 \rightleftarrows 0$

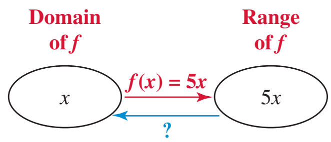
$1 \rightleftarrows 2$ h is a one-to-one function

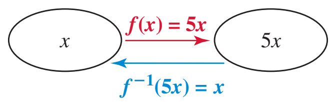
$3 \rightleftarrows 4$

Range **Domain**

$$h^{-1} = \{(0, -1), (2, 1), (4, 3)\}$$

The inverse function undoes whatever the function does. For example, if $f(x) = 5x$, then the function f maps any value x in the domain to a value $5x$ in the range. If we want to map backwards or undo the $5x$, we develop a function called the inverse function that takes $5x$ as input and maps back to x as output. The inverse function is $f^{-1}(x) = \frac{1}{5}x$. Note that if we input $5x$ into the inverse function, the output is x: $f^{-1}(5x) = \frac{1}{5}(5x) = x$.





DEFINITION **Inverse Function**

If f and g denote two one-to-one functions such that

$$f(g(x)) = x \text{ for every } x \text{ in the domain of } g$$
and
$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f,$$

then g is the **inverse** of the function f. The function g is denoted by f^{-1} (read "f-inverse").

CAUTION

$$f^{-1} \neq \frac{1}{f}$$

Note: f^{-1} is used to denote the inverse of f. The superscript -1 is not used as an exponent and, therefore, does not represent the reciprocal of f: $\dfrac{1}{f}$.

Two properties hold true relating one-to-one functions to their inverses: (1) The range of the function is the domain of the inverse, and the range of the inverse is the domain of the function, and (2) the composite function that results with a function and its inverse (and vice versa) is the identity function x.

Domain of f = range of f^{-1} and range of f = domain of f^{-1}

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

EXAMPLE 4 Verifying Inverse Functions

Verify that $f^{-1}(x) = \frac{1}{2}x - 2$ is the inverse of $f(x) = 2x + 4$.

Solution:

Show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Write f^{-1} using placeholder notation. $\quad f^{-1}(\square) = \frac{1}{2}(\square) - 2$

Substitute $f(x) = 2x + 4$ into f^{-1}. $\quad f^{-1}(f(x)) = \frac{1}{2}(2x + 4) - 2$

Simplify. $\quad f^{-1}(f(x)) = x + 2 - 2 = x$

$$f^{-1}(f(x)) = x$$

Write f using placeholder notation. $\quad f(\square) = 2(\square) + 4$

Substitute $f^{-1}(x) = \frac{1}{2}x - 2$ into f. $\quad f(f^{-1}(x)) = 2\left(\frac{1}{2}x - 2\right) + 4$

Simplify. $\quad f(f^{-1}(x)) = x - 4 + 4 = x$

$$f(f^{-1}(x)) = x$$

Note the relationship between the domain and range of f and f^{-1}.

	DOMAIN	RANGE
$f(x) = 2x + 4$	$(-\infty, \infty)$	$(-\infty, \infty)$
$f^{-1}(x) = \frac{1}{2}x - 2$	$(-\infty, \infty)$	$(-\infty, \infty)$

EXAMPLE 5 Verifying Inverse Functions with Domain Restrictions

Verify that $f^{-1}(x) = x^2$, for $x \geq 0$, is the inverse of $f(x) = \sqrt{x}$.

Solution:

Show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Write f^{-1} using placeholder notation. $\quad f^{-1}(\square) = (\square)^2$

Substitute $f(x) = \sqrt{x}$ into f^{-1}. $\quad f^{-1}(f(x)) = \left(\sqrt{x}\right)^2 = x$

$$f^{-1}(f(x)) = x \text{ for } x \geq 0$$

Write f using placeholder notation. $\quad f(\square) = \sqrt{(\square)}$

Substitute $f^{-1}(x) = x^2, x \geq 0$ into f. $\quad f(f^{-1}(x)) = \sqrt{x^2} = x, x \geq 0$

$$f(f^{-1}(x)) = x \text{ for } x \geq 0$$

	DOMAIN	RANGE
$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$f^{-1}(x) = x^2, x \geq 0$	$[0, \infty)$	$[0, \infty)$

Graphical Interpretation of Inverse Functions

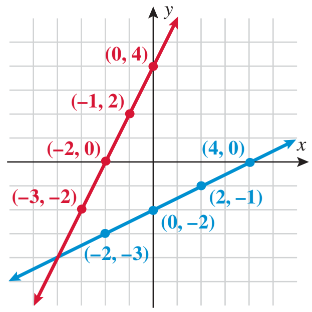
In Example 4, we showed that $f^{-1}(x) = \frac{1}{2}x - 2$ is the inverse of $f(x) = 2x + 4$. Let's now investigate the graphs that correspond to the function f and its inverse f^{-1}.

$f(x)$

x	y
-3	-2
-2	0
-1	2
0	4

$f^{-1}(x)$

x	y
-2	-3
0	-2
2	-1
4	0



Note that the point $(-3, -2)$ lies on the function and the point $(-2, -3)$ lies on the inverse. In fact, every point (a, b) that lies on the function corresponds to a point (b, a) that lies on the inverse.

Draw the line $y = x$ on the graph. In general, the point (b, a) on the inverse $f^{-1}(x)$ is the reflection (about $y = x$) of the point (a, b) on the function $f(x)$.

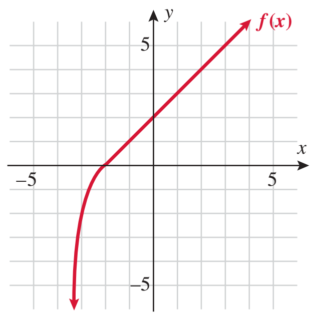
In general, if the point (a, b) is on the graph of a function, then the point (b, a) is on the graph of its inverse.

Study Tip

If the point (a, b) is on the function, then the point (b, a) is on the inverse. Notice the interchanging of the x- and y-coordinates.

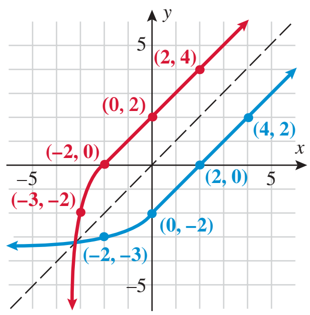
EXAMPLE 6 Graphing the Inverse Function

Given the graph of the function $f(x)$, plot the graph of its inverse $f^{-1}(x)$.

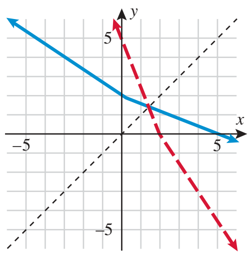


Solution:

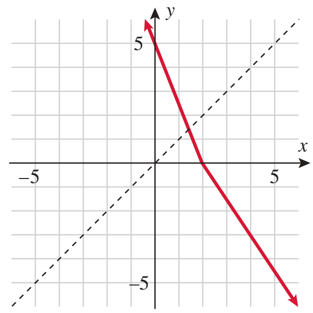
Because the points $(-3, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, 4)$ lie on the graph of f, then the points $(-2, -3)$, $(0, -2)$, $(2, 0)$, and $(4, 2)$ lie on the graph of f^{-1}.



■ **Answer:**

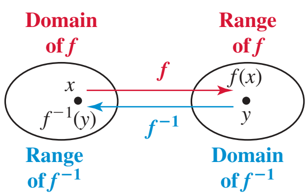


■ **YOUR TURN** Given the graph of a function f, plot the inverse function.



We have developed the definition of an inverse function, and properties of inverses. At this point, you should be able to determine whether two functions are inverses of one another. Let's turn our attention to another problem: How do you find the inverse of a function?

Finding the Inverse Function



If the point (a, b) lies on the graph of a function, then the point (b, a) lies on the graph of the inverse function. The symmetry about the line $y = x$ tells us that the roles of x and y interchange. Therefore, if we start with every point (x, y) that lies on the graph of a function, then every point (y, x) lies on the graph of its inverse. Algebraically, this corresponds to interchanging x and y. Finding the inverse of a finite set of ordered pairs is easy: Simply interchange the x- and y-coordinates. Earlier, we found that if $h(x) = \{(-1, 0), (1, 2), (3, 4)\}$, then $h^{-1}(x) = \{(0, -1), (2, 1), (4, 3)\}$. But how do we find the inverse of a function defined by an equation?

Recall the mapping relationship if f is a one-to-one function. This relationship implies that $\mathbf{f(x) = y}$ and $\mathbf{f^{-1}(y) = x}$. Let's use these two identities to find the inverse. Now consider the function defined by $f(x) = 3x - 1$. To find f^{-1}, we let $\mathbf{f(x) = y}$, which yields $y = 3x - 1$. Solve for the variable x: $x = \frac{1}{3}y + \frac{1}{3}$.

Recall that $\mathbf{f^{-1}(y) = x}$, so we have found the inverse to be $f^{-1}(y) = \frac{1}{3}y + \frac{1}{3}$. It is customary to write the independent variable as x, so we write the inverse as $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$. Now that we have found the inverse, let's confirm that the properties $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$ hold.

$$f\left(f^{-1}(x)\right) = 3\left(\frac{1}{3}x + \frac{1}{3}\right) - 1 = x + 1 - 1 = x$$

$$f^{-1}(f(x)) = \frac{1}{3}(3x - 1) + \frac{1}{3} = x - \frac{1}{3} + \frac{1}{3} = x$$

Finding the inverse of a function

Let f be a one-to-one function, then the following procedure can be used to find the inverse function f^{-1} if the inverse exists.

Step	Procedure	Example
1	Let $y = f(x)$.	$f(x) = -3x + 5$ $y = -3x + 5$
2	Solve the resulting equation for x in terms of y (if possible).	$3x = -y + 5$ $x = -\frac{1}{3}y + \frac{5}{3}$
3	Let $x = f^{-1}(y)$.	$f^{-1}(y) = -\frac{1}{3}y + \frac{5}{3}$
4	Let $y = x$ (interchange x and y).	$f^{-1}(x) = -\frac{1}{3}x + \frac{5}{3}$

The same result is found if we first interchange x and y and then solve for y in terms of x.

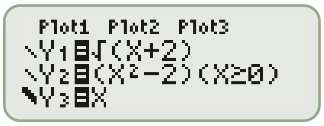
Step	Procedure	Example
1	Let $y = f(x)$.	$f(x) = -3x + 5$ $y = -3x + 5$
2	Interchange x and y.	$x = -3y + 5$
3	Solve for y in terms of x.	$3y = -x + 5$ $y = -\frac{1}{3}x + \frac{5}{3}$
4	Let $y = f^{-1}(x)$	$f^{-1}(x) = -\frac{1}{3}x + \frac{5}{3}$

Note the following:

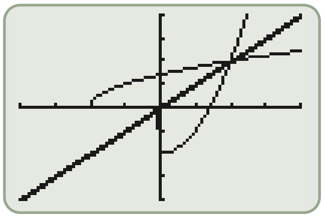
- Verify first that a function is one-to-one prior to finding an inverse (if it is not one-to-one, then the inverse does not exist).
- State the domain restrictions on the inverse function. The domain of f is the range of f^{-1} and vice versa.
- To verify that you have found the inverse, show that $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.

Technology Tip

Using a graphing utility, plot $y_1 = f(x) = \sqrt{x+2}$, $y_2 = f^{-1}(x) = x^2 - 2$ for $x \geq 0$, and $y_3 = x$.



Note that the function $f(x)$ and its inverse $f^{-1}(x)$ are symmetric about the line $y = x$.



Study Tip

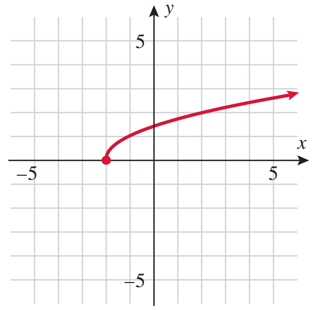
Had we ignored the domain and range in Example 7, we would have found the inverse function to be the square function $f(x) = x^2 - 2$, which is not a one-to-one function. It is only when we restrict the domain of the square function that we get a one-to-one function.

EXAMPLE 7 The Inverse of a Square Root Function

Find the inverse of the function $f(x) = \sqrt{x+2}$ and state the domain and range of both f and f^{-1}.

Solution:

$f(x)$ is a one-to-one function because it passes the horizontal line test.



STEP 1 Let $y = f(x)$. $\qquad y = \sqrt{x+2}$

STEP 2 Interchange x and y. $\qquad x = \sqrt{y+2}$

STEP 3 Solve for y.

Square both sides of the equation. $\qquad x^2 = y + 2$

Subtract 2 from both sides. $\qquad x^2 - 2 = y$ or $y = x^2 - 2$

STEP 4 Let $y = f^{-1}(x)$. $\qquad f^{-1}(x) = x^2 - 2$

Note any domain restrictions. (State the domain and range of both f and f^{-1}.)

f:	Domain: $[-2, \infty)$	Range: $[0, \infty)$
f^{-1}:	Domain: $[0, \infty)$	Range: $[-2, \infty)$

The inverse of $f(x) = \sqrt{x+2}$ is $\boxed{f^{-1}(x) = x^2 - 2 \text{ for } x \geq 0}$.

Check.

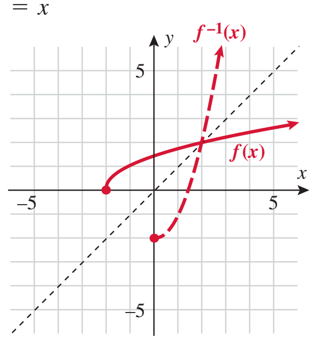
$f^{-1}(f(x)) = x$ for all x in the domain of f.

$$\begin{aligned} f^{-1}(f(x)) &= \left(\sqrt{x+2}\right)^2 - 2 \\ &= x + 2 - 2 \text{ for } x \geq -2 \\ &= x \end{aligned}$$

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

$$\begin{aligned} f(f^{-1}(x)) &= \sqrt{(x^2 - 2) + 2} \\ &= \sqrt{x^2} \text{ for } x \geq 0 \\ &= x \end{aligned}$$

Note that the function $f(x) = \sqrt{x+2}$ and its inverse $f^{-1}(x) = x^2 - 2$ for $x \geq 0$ are symmetric about the line $y = x$.



YOUR TURN Find the inverse of the given function and state the domain and range of the inverse function.

a. $f(x) = 7x - 3$ $\qquad$ **b.** $g(x) = \sqrt{x-1}$

Answers:

a. $f^{-1}(x) = \dfrac{x+3}{7}$, Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$

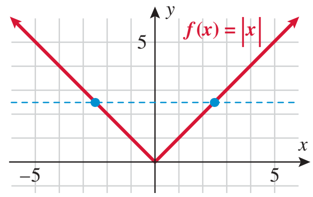
b. $g^{-1}(x) = x^2 + 1$, Domain: $[0, \infty)$, Range: $[1, \infty)$

EXAMPLE 8 A Function That Does Not Have an Inverse Function

Find the inverse of the function $f(x) = |x|$ if it exists.

Solution:

The function $f(x) = |x|$ fails the horizontal line test and therefore is not a one-to-one function. Because f is not a one-to-one function, its inverse function does not exist.



EXAMPLE 9 Finding the Inverse Function

The function $f(x) = \frac{2}{x+3}$, $x \neq -3$, is a one-to-one function. Find its inverse.

Solution:

STEP 1 Let $y = f(x)$. $\quad y = \frac{2}{x+3}$

STEP 2 Interchange x and y. $\quad x = \frac{2}{y+3}$

STEP 3 Solve for y.

Multiply the equation by $(y + 3)$. $\quad x(y+3) = 2$

Eliminate the parentheses. $\quad xy + 3x = 2$

Subtract $3x$ from both sides. $\quad xy = -3x + 2$

Divide the equation by x. $\quad y = \frac{-3x+2}{x} = -3 + \frac{2}{x}$

STEP 4 Let $y = f^{-1}(x)$. $\quad f^{-1}(x) = -3 + \frac{2}{x}$

Note any domain restrictions on $f^{-1}(x)$. $\quad x \neq 0$

The inverse of the function $f(x) = \frac{2}{x+3}$, $x \neq -3$, is $\boxed{f^{-1}(x) = -3 + \frac{2}{x},\ x \neq 0}$.

Check.

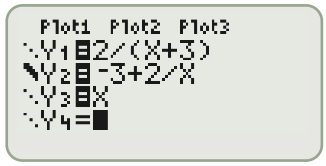
$$f^{-1}(f(x)) = -3 + \frac{2}{\left(\frac{2}{x+3}\right)} = -3 + (x+3) = x,\ x \neq -3$$

$$f\left(f^{-1}(x)\right) = \frac{2}{\left(-3 + \frac{2}{x}\right) + 3} = \frac{2}{\left(\frac{2}{x}\right)} = x,\ x \neq 0$$

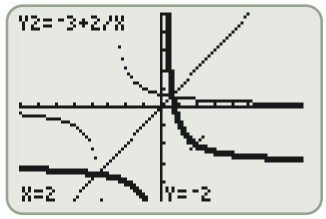
■ **YOUR TURN** The function $f(x) = \frac{4}{x-1}$, $x \neq 1$, is a one-to-one function. Find its inverse.

Technology Tip

The graphs of $y_1 = f(x) = \frac{2}{x+3}$, $x \neq -3$, and $y_2 = f^{-1}(x) = -3 + \frac{2}{x}$, $x \neq 0$, are shown.



Note that the graphs of the function $f(x)$ and its inverse $f^{-1}(x)$ are symmetric about the line $y = x$.



Study Tip

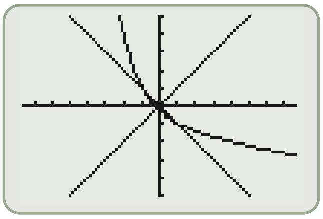
The range of the function is equal to the domain of its inverse function.

■ **Answer:** $f^{-1}(x) = 1 + \frac{4}{x}$, $x \neq 0$

Note in Example 9 that the domain of f is $(-\infty, -3) \cup (-3, \infty)$ and the domain of f^{-1} is $(-\infty, 0) \cup (0, \infty)$. Therefore, we know that the range of f is $(-\infty, 0) \cup (0, \infty)$, and the range of f^{-1} is $(-\infty, -3) \cup (-3, \infty)$.

Technology Tip

The graphs of f and f^{-1} are symmetric about the line $y = x$.



EXAMPLE 10 Finding the Inverse of a Piecewise-Defined Function

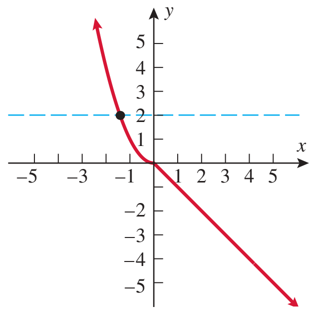
Determine whether the function $f(x) = \begin{cases} x^2 & x < 0 \\ -x & x \geq 0 \end{cases}$ is a one-to-one function. If it is a one-to-one function, find its inverse.

Solution:

The graph of the function f passes the horizontal line test and therefore f is a one-to-one function.

STEP 1 Let $y = f(x)$.

Let $y_1 = x_1^2$ for $x_1 < 0$ and $y_2 = -x_2$ for $x_2 \geq 0$ represent the two pieces of f. Note the domain and range for each piece.



EQUATION	DOMAIN	RANGE
$y_1 = x_1^2$	$x_1 < 0$	$y_1 > 0$
$y_2 = -x_2$	$x_2 \geq 0$	$y_2 \leq 0$

STEP 2 Solve for x in terms of y.

Solve for x_1. $\quad x_1 = \pm\sqrt{y_1}$

Select the negative root since $x_1 < 0$. $\quad x_1 = -\sqrt{y_1}$

Solve for x_2. $\quad x_2 = -y_2$

STEP 3 Let $x = f^{-1}(y)$.

$x_1 = f^{-1}(y_1) = -\sqrt{y_1}$	$x_1 < 0$	$y_1 > 0$
$x_2 = f^{-1}(y_2) = -y_2$	$x_2 \geq 0$	$y_2 \leq 0$

Express the two "pieces" in terms of a piecewise-defined function. $\quad f^{-1}(y) = \begin{cases} -\sqrt{y} & y > 0 \\ -y & y \leq 0 \end{cases}$

STEP 4 Let $y = x$ (interchange x and y). $\quad f^{-1}(x) = \begin{cases} -\sqrt{x} & x > 0 \\ -x & x \leq 0 \end{cases}$

SECTION 1.5 SUMMARY

One-to-One Functions

Each input in the domain corresponds to exactly one output in the range, and no two inputs map to the same output. There are three ways to test a function to determine whether it is a one-to-one function.

1. **Discrete points:** For the set of all points (a, b), verify that no y-values are repeated.
2. **Algebraic equations:** Let $f(x_1) = f(x_2)$; if it can be shown that $x_1 = x_2$, then the function is one-to-one.
3. **Graphs:** Use the horizontal line test; if any horizontal line intersects the graph of the function in more than one point, then the function is not one-to-one.

Properties of Inverse Functions

1. If f is a one-to-one function, then f^{-1} exists.
2. Domain and range
   - Domain of f = range of f^{-1}
   - Domain of f^{-1} = range of f
3. Composition of inverse functions
   - $f^{-1}(f(x)) = x$ for all x in the domain of f
   - $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}
4. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

Procedure for Finding the Inverse of a Function

1. Let $y = f(x)$.
2. Interchange x and y.
3. Solve for y.
4. Let $y = f^{-1}(x)$.

SECTION 1.5 EXERCISES

SKILLS

In Exercises 1–10, determine whether the given relation is a function. If it is a function, determine whether it is a one-to-one function.

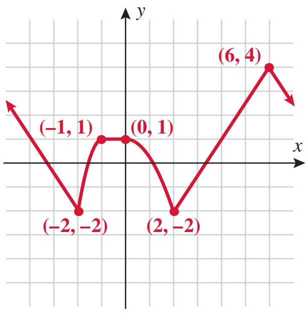
1. $\{(0, 0), (9, -3), (4, -2), (4, 2), (9, 3)\}$

2. $\{(0, 1), (1, 1), (2, 1), (3, 1)\}$

3. $\{(0, 1), (1, 0), (2, 1), (-2, 1), (5, 4), (-3, 4)\}$

4. $\{(0, 0), (-1, -1), (-2, -8), (1, 1), (2, 8)\}$

5.



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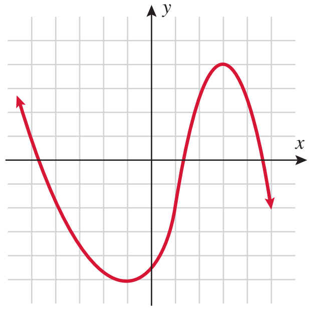

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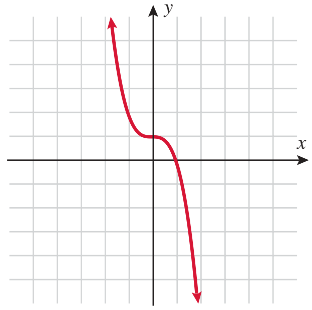

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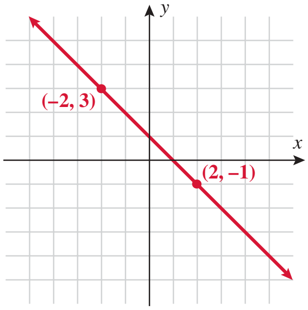

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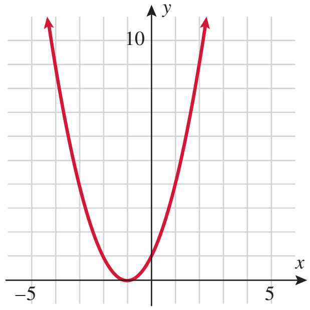

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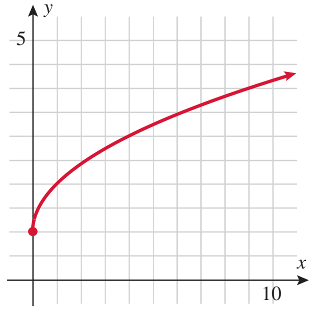

In Exercises 11–18, determine algebraically and graphically whether the function is one-to-one.

11. $f(x) = |x - 3|$

12. $f(x) = (x - 2)^2 + 1$

13. $f(x) = \dfrac{1}{x - 1}$

14. $f(x) = \sqrt[3]{x}$

15. $f(x) = x^2 - 4$

16. $f(x) = \sqrt{x + 1}$

17. $f(x) = x^3 - 1$

18. $f(x) = \dfrac{1}{x + 2}$

In Exercises 19–28, verify that the function $f^{-1}(x)$ is the inverse of $f(x)$ by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Graph $f(x)$ and $f^{-1}(x)$ on the same axes to show the symmetry about the line $y = x$.

19. $f(x) = 2x + 1$; $f^{-1}(x) = \dfrac{x - 1}{2}$

20. $f(x) = \dfrac{x - 2}{3}$; $f^{-1}(x) = 3x + 2$

21. $f(x) = \sqrt{x - 1}, x \geq 1$; $f^{-1}(x) = x^2 + 1, x \geq 0$

22. $f(x) = 2 - x^2, x \geq 0$; $f^{-1}(x) = \sqrt{2 - x}, x \leq 2$

23. $f(x) = \dfrac{1}{x}$; $f^{-1}(x) = \dfrac{1}{x}, x \neq 0$

24. $f(x) = (5 - x)^{1/3}$; $f^{-1}(x) = 5 - x^3$

25. $f(x) = \dfrac{1}{2x + 6}, x \neq -3$; $f^{-1}(x) = \dfrac{1}{2x} - 3, x \neq 0$

26. $f(x) = \dfrac{3}{4 - x}, x \neq 4$; $f^{-1}(x) = 4 - \dfrac{3}{x}, x \neq 0$

27. $f(x) = \dfrac{x + 3}{x + 4}, x \neq -4$; $f^{-1}(x) = \dfrac{3 - 4x}{x - 1}, x \neq 1$

28. $f(x) = \dfrac{x - 5}{3 - x}, x \neq 3$; $f^{-1}(x) = \dfrac{3x + 5}{x + 1}, x \neq -1$

In Exercises 29–36, graph the inverse of the one-to-one function that is given.

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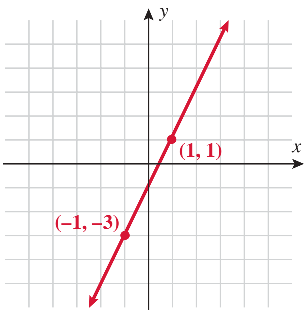

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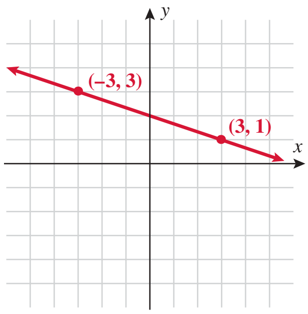

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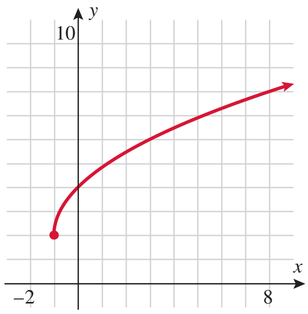

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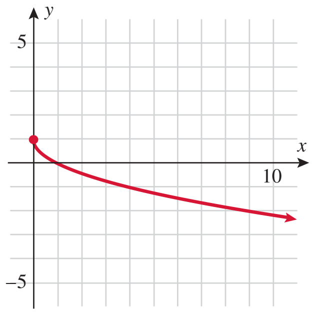

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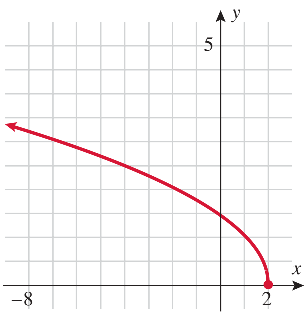

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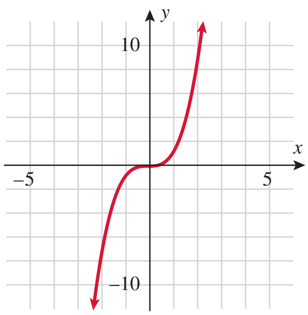

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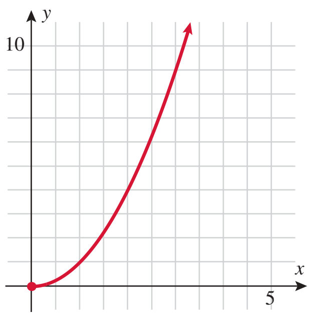
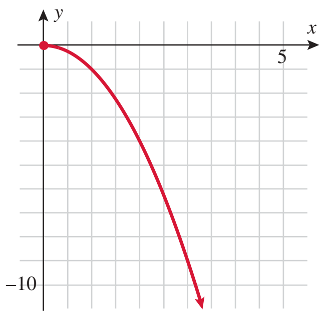


36.



In Exercises 37–56, the function f is one-to-one. Find its inverse, and check your answer. State the domain and range of both f and f^{-1}.

37. $f(x) = -3x + 2$

38. $f(x) = 2x + 3$

39. $f(x) = x^3 + 1$

40. $f(x) = x^3 - 1$

41. $f(x) = \sqrt{x - 3}$

42. $f(x) = \sqrt{3 - x}$

43. $f(x) = x^2 - 1, x \geq 0$

44. $f(x) = 2x^2 + 1, x \geq 0$

45. $f(x) = (x + 2)^2 - 3, x \geq -2$

46. $f(x) = (x - 3)^2 - 2, x \geq 3$

47. $f(x) = \dfrac{2}{x}$

48. $f(x) = -\dfrac{3}{x}$

49. $f(x) = \dfrac{2}{3 - x}$

50. $f(x) = \dfrac{7}{x + 2}$

51. $f(x) = \dfrac{7x + 1}{5 - x}$

52. $f(x) = \dfrac{2x + 5}{7 + x}$

53. $f(x) = \dfrac{1}{\sqrt{x}}$

54. $f(x) = \dfrac{x}{\sqrt{x + 1}}$

55. $f(x) = \sqrt{\dfrac{x + 1}{x - 2}}$

56. $f(x) = \sqrt{x^2 - 1}, x \geq 1$

In Exercises 57–62, graph the piecewise-defined function to determine whether it is a one-to-one function. If it is a one-to-one function, find its inverse.

57. $G(x) = \begin{cases} 0 & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$

58. $G(x) = \begin{cases} \dfrac{1}{x} & x < 0 \\ \sqrt{x} & x \geq 0 \end{cases}$

59. $f(x) = \begin{cases} \sqrt[3]{x} & x \leq -1 \\ x^2 + 2x & -1 < x \leq 1 \\ \sqrt{x} + 2 & x > 1 \end{cases}$

60. $f(x) = \begin{cases} -x & x < -2 \\ \sqrt{4 - x^2} & -2 \leq x \leq 0 \\ -\dfrac{1}{x} & x > 0 \end{cases}$

61. $f(x) = \begin{cases} x & x \leq -1 \\ x^3 & -1 < x < 1 \\ x & x \geq 1 \end{cases}$

62. $f(x) = \begin{cases} x + 3 & x \leq -2 \\ |x| & -2 < x < 2 \\ x^2 & x \geq 2 \end{cases}$

APPLICATIONS

63. Temperature. The equation used to convert from degrees Celsius to degrees Fahrenheit is $f(x) = \frac{9}{5}x + 32$. Determine the inverse function $f^{-1}(x)$. What does the inverse function represent?

64. Temperature. The equation used to convert from degrees Fahrenheit to degrees Celsius is $C(x) = \frac{5}{9}(x - 32)$. Determine the inverse function $C^{-1}(x)$. What does the inverse function represent?

65. Budget. The Richmond rowing club is planning to enter the Head of the Charles race in Boston and is trying to figure out how much money to raise. The entry fee is \$250 per boat for the first 10 boats and \$175 for each additional boat. Find the cost function $C(x)$ as a function of the number of boats x the club enters. Find the inverse function that will yield how many boats the club can enter as a function of how much money it will raise.

66. Long-Distance Calling Plans. A phone company charges \$0.39 per minute for the first 10 minutes of a long-distance phone call and \$0.12 per minute every minute after that. Find the cost function $C(x)$ as a function of length x of the phone call in minutes. Suppose you buy a "prepaid" phone card that is planned for a single call. Find the inverse function that determines how many minutes you can talk as a function of how much you prepaid.

67. Salary. A student works at Target making \$7 per hour and the weekly number of hours worked per week x varies. If Target withholds 25% of his earnings for taxes and Social Security, write a function $E(x)$ that expresses the student's take-home pay each week. Find the inverse function $E^{-1}(x)$. What does the inverse function tell you?

68. Salary. A grocery store pays you \$8 per hour for the first 40 hours per week and time and a half for overtime. Write a piecewise-defined function that represents your weekly earnings $E(x)$ as a function of the number of hours worked x. Find the inverse function $E^{-1}(x)$. What does the inverse function tell you?

In Exercises 69–72, refer to the following:

By analyzing available empirical data it was determined that during an illness a patient's body temperature fluctuated during one 24-hour period according to the function

$$T(t) = 0.0003(t - 24)^3 + 101.70$$

where T represents that patient's temperature in degrees Fahrenheit and t represents the time of day in hours measured from 12:00 A.M. (midnight).

69. Health/Medicine. Find the domain and range of the function $T(t)$.

70. Health/Medicine. Find time as a function of temperature, that is, the inverse function $t(T)$.

71. Health/Medicine. Find the domain and range of the function $t(T)$ found in Exercise 70.

72. Health/Medicine. At what time, to the nearest hour, was the patient's temperature 99.5°F?

73. ATM Charges. A bank charges \$0.60 each time that a client uses an ATM machine for the first 15 transactions of the month, and \$0.90 for every additional transaction. Find a function $M(x)$ as a function of the number of monthly transactions that describes the amount charged by the bank. Find the inverse function that will yield how many transactions per month a client can do as a function of the client's budget.

74. Truck Renting. For renting a truck, a company charges \$19.95 per day plus \$0.80 per mile plus 10% in taxes. Find the cost of renting a truck for two days as a function of the miles driven x. Find the inverse function that determines how many miles you can drive as a function of the money you have available.

75. Depreciation. The value of a family car decreases \$600 per year for the first 5 years; after that it depreciates \$900 per year. Find the resale value function $V(x)$, where x is the number of years the family has owned the car whose original price was \$20,000. Find the inverse function $V^{-1}(x)$. What does the inverse function tell you?

76. Production. The number of pounds of strawberries produced per square yard in a strawberry field depends on the number of plants per square yard. When $x < 25$ plants are planted per square yard, each square yard produces $\frac{50 - x}{5}$ pounds. Find a function P as a function of the number of plants x per square yard. Find the inverse function P^{-1}. What does the inverse function tell you?

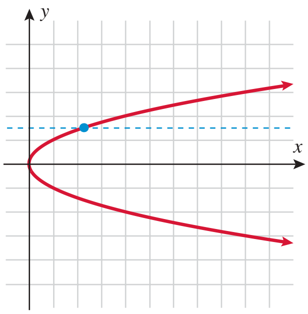
CATCH THE MISTAKE

In Exercises 77–80, explain the mistake that is made.

77. Is $x = y^2$ a one-to-one function?

Solution:

Yes, this graph represents a one-to-one function because it passes the horizontal line test.

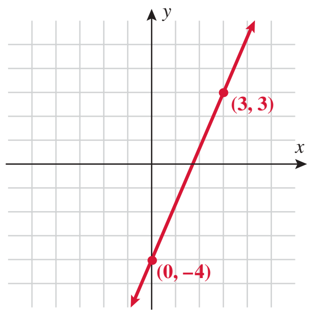


This is incorrect. What mistake was made?

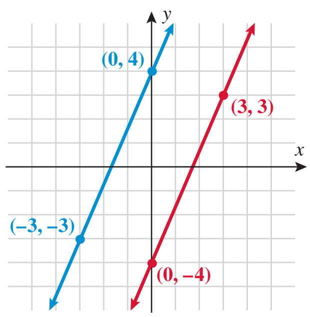
78. A linear one-to-one function is graphed below. Draw its inverse.

Solution:

Note that the points **(3, 3)** and **(0, −4)** lie on the graph of the function.



By symmetry, the points **(−3, −3)** and **(0, 4)** lie on the graph of the inverse.



This is incorrect. What mistake was made?

79. Given the function $f(x) = x^2$, find the inverse function $f^{-1}(x)$.

Solution:

Step 1: Let $y = f(x)$. $\quad y = x^2$

Step 2: Solve for x. $\quad x = \sqrt{y}$

Step 3: Interchange x and y. $\quad y = \sqrt{x}$

Step 4: Let $y = f^{-1}(x)$. $\quad f^{-1}(x) = \sqrt{x}$

Check: $f\left(f^{-1}(x)\right) = \left(\sqrt{x}\right)^2 = x$ and $f^{-1}(f(x)) = \sqrt{x^2} = x$.

The inverse of $f(x) = x^2$ is $f^{-1}(x) = \sqrt{x}$.

This is incorrect. What mistake was made?

80. Given the function $f(x) = \sqrt{x - 2}$, find the inverse function $f^{-1}(x)$, and state the domain restrictions on $f^{-1}(x)$.

Solution:

Step 1: Let $y = f(x)$. $\quad y = \sqrt{x - 2}$

Step 2: Interchange x and y. $\quad x = \sqrt{y - 2}$

Step 3: Solve for y. $\quad y = x^2 + 2$

Step 4: Let $f^{-1}(x) = y$. $\quad f^{-1}(x) = x^2 + 2$

Step 5: Domain restrictions $f(x) = \sqrt{x - 2}$ has the domain restriction that $x \geq 2$.

The inverse of $f(x) = \sqrt{x - 2}$ is $f^{-1}(x) = x^2 + 2$.

The domain of $f^{-1}(x)$ is $x \geq 2$.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 81–84, determine whether each statement is true or false.

81. Every even function is a one-to-one function.

82. Every odd function is a one-to-one function.

83. It is not possible that $f = f^{-1}$.

84. A function f has an inverse. If the function lies in quadrant II, then its inverse lies in quadrant IV.

85. If $(0, b)$ is the y-intercept of a one-to-one function f, what is the x-intercept of the inverse f^{-1}?

86. If $(a, 0)$ is the x-intercept of a one-to-one function f, what is the y-intercept of the inverse f^{-1}?

CHALLENGE

87. The unit circle is not a function. If we restrict ourselves to the semicircle that lies in quadrants I and II, the graph represents a function, but it is not a one-to-one function. If we further restrict ourselves to the quarter circle lying in quadrant I, the graph does represent a one-to-one function. Determine the equations of both the one-to-one function and its inverse. State the domain and range of both.

88. Find the inverse of $f(x) = \dfrac{c}{x}$, $c \neq 0$.

89. Under what conditions is the linear function $f(x) = mx + b$ a one-to-one function?

90. Assuming that the conditions found in Exercise 89 are met, determine the inverse of the linear function.

91. Determine the value of a that makes $f(x) = \dfrac{x-2}{x^2 - a}$ a one-to-one function. Determine $f^{-1}(x)$ and its domain.

92. The point (a, b) lies on the graph of the one-to-one function $y = f(x)$. What other points are guaranteed to lie on the graph of $y = f^{-1}(x)$?

TECHNOLOGY

In Exercises 93–96, graph the following functions and determine whether they are one-to-one.

93. $f(x) = |4 - x^2|$

94. $f(x) = \dfrac{3}{x^3 + 2}$

95. $f(x) = x^{1/3} - x^5$

96. $f(x) = \dfrac{1}{x^{1/2}}$

In Exercises 97–100, graph the functions f and g and the line $y = x$ in the same screen. Do the two functions appear to be inverses of each other?

97. $f(x) = \sqrt{3x - 5}$; $g(x) = \dfrac{x^2}{3} + \dfrac{5}{3}$

98. $f(x) = \sqrt{4 - 3x}$; $g(x) = \dfrac{4}{3} - \dfrac{x^2}{3}$, $x \geq 0$

99. $f(x) = (x - 7)^{1/3} + 2$; $g(x) = x^3 - 6x^2 + 12x - 1$

100. $f(x) = \sqrt[3]{x + 3} - 2$; $g(x) = x^3 + 6x^2 + 12x + 6$

PREVIEW TO CALCULUS

For Exercises 101–104, refer to the following:

In calculus, the difference quotient $\dfrac{f(x+h) - f(x)}{h}$ of a function f is used to find the derivative f' of f, by allowing h to approach zero, $h \to 0$. The derivative of the inverse function $(f^{-1})'$ can be found using the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0 and both f and f^{-1} are differentiable. For the following one-to-one function, find (a) f^{-1}, (b) f', (c) $(f^{-1})'$, and (d) verify the formula above. For (b) and (c), use the difference quotient.

101. $f(x) = 2x + 1$

102. $f(x) = x^2$, $x > 0$

103. $f(x) = \sqrt{x + 2}$, $x > -2$

104. $f(x) = \dfrac{1}{x + 1}$, $x > -1$

CHAPTER 1 INQUIRY-BASED LEARNING PROJECT

Transformations of Functions

Being a creature of habit, Dylan usually sets out each morning at 7 A.M. from his house for a jog. Figure 1 shows the graph of a function, $y = d(t)$, that represents Dylan's jog on Friday.

a. Use the graph in Figure 1 to fill in the table below.

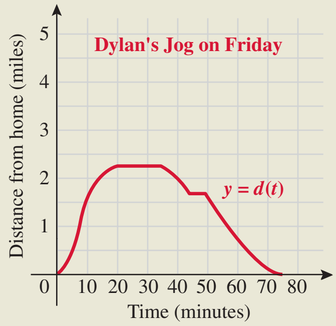


Figure 1

t											
$y = d(t)$											

Describe a jogging scenario that fits the graph and table above.

b. The graph shown in Figure 2 represents Dylan's jog on Saturday. It is a transformation of the function $y = d(t)$ shown in Figure 1.

Complete the table of values below for this transformation. You may find it helpful to refer to the table in part (a).

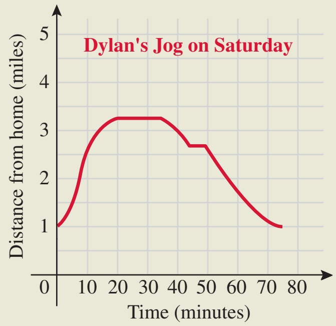


Figure 2

t											
y											

What is the real-world meaning of this transformation? How is Dylan's jog on Saturday different from his usual jog? How is it the same?

The original function (in Figure 1) is represented by the equation $y = d(t)$. Write an equation, in terms of $d(t)$, that represents the function graphed in Figure 2. Explain.

c. The graph shown in Figure 3 represents Dylan's jog on Sunday. It is a transformation of the function $y = d(t)$ shown in Figure 1.

Complete the table of values below for this transformation.

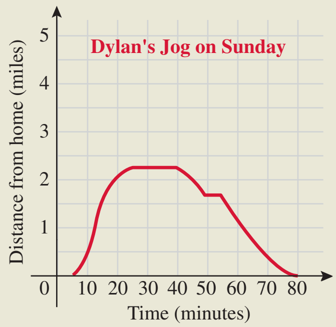


Figure 3

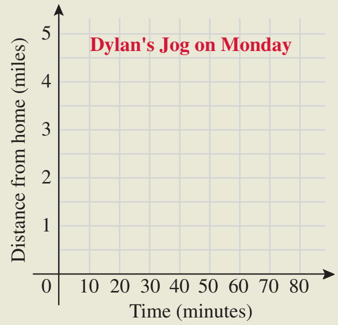
t											
y											

What is the real-world meaning of this transformation? How is Dylan's jog on Sunday different from his usual jog?

The original function (in Figure 1) is represented by the equation $y = d(t)$. Use function notation to represent the function graphed in Figure 3. Explain.

d. Suppose Dylan's jog on Monday can be represented by the equation $y = \frac{1}{2}d(t)$.

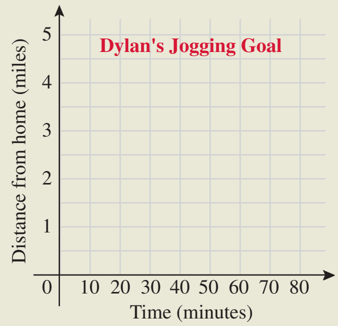
Complete the table of values below and sketch a graph at the right for this transformation.



t											
y											

What is the real-world meaning of this transformation? How does Dylan's jog on Monday differ from his usual jog? How is it the same?

e. Suppose Dylan has a goal of cutting his usual jogging time in half, while covering the same distance. Represent this scenario as a transformation of $y = d(t)$ shown in Figure 1. Complete the table, sketch a graph, and write an equation in function notation. Explain why your equation makes sense. Finally, discuss whether you think Dylan's goal is realistic.



$y =$ ____________

t											
y											

MODELING OUR WORLD

The U.S. National Oceanic and Atmospheric Association (NOAA) monitors temperature and carbon emissions at its observatory in Mauna Loa, Hawaii. NOAA's goal is to help foster an informed society that uses a comprehensive understanding of the role of the oceans, coasts, and atmosphere in the global ecosystem to make the best social and economic decisions. The data presented in this chapter are from the Mauna Loa Observatory, where historical atmospheric measurements have been recorded for the last 50 years. You will develop linear models based on this data to predict temperature and carbon emissions in the future.

The following table summarizes average yearly temperature in degrees Fahrenheit °F and carbon dioxide emissions in parts per million (ppm) for **Mauna Loa, Hawaii**.

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Temperature (°F)	44.45	43.29	43.61	43.35	46.66	45.71	45.53	47.53	45.86	46.23
CO_2 Emissions (ppm)	316.9	320.0	325.7	331.1	338.7	345.9	354.2	360.6	369.4	379.7

1. Plot the temperature data with time on the horizontal axis and temperature on the vertical axis. Let $t = 0$ correspond to 1960.
2. Find a *linear function* that models the temperature in Mauna Loa.
   a. Use data from 1965 and 1995.
   b. Use data from 1960 and 1990.
   c. Use linear regression and all data given.
3. Predict what the temperature will be in Mauna Loa in 2020.
   a. Apply the line found in Exercise 2(a).
   b. Apply the line found in Exercise 2(b).
   c. Apply the line found in Exercise 2(c).
4. Predict what the temperature will be in Mauna Loa in 2100.
   a. Apply the line found in Exercise 2(a).
   b. Apply the line found in Exercise 2(b).
   c. Apply the line found in Exercise 2(c).
5. Do you think your models support the claim of "global warming"? Explain.
6. Plot the carbon dioxide emissions data with time on the horizontal axis and carbon dioxide emissions on the vertical axis. Let $t = 0$ correspond to 1960.
7. Find a *linear function* that models the CO_2 emissions (ppm) in Mauna Loa.
   a. Use data from 1965 and 1995.
   b. Use data from 1960 and 1990.
   c. Use linear regression and all data given.

8. Predict the expected CO_2 emissions in Mauna Loa in 2020.

a. Apply the line found in Exercise 7(a).
b. Apply the line found in Exercise 7(b).
c. Apply the line found in Exercise 7(c).

9. Predict the expected CO_2 emissions in Mauna Loa in 2100.

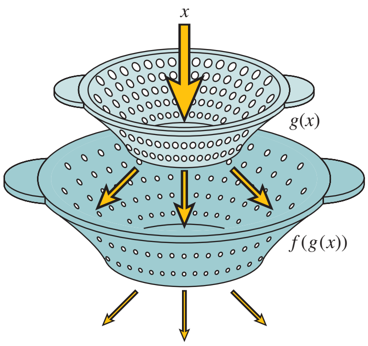
a. Apply the line found in Exercise 7(a).
b. Apply the line found in Exercise 7(b).
c. Apply the line found in Exercise 7(c).

10. Do you think your models support the claim of the "greenhouse effect"? Explain.

CHAPTER 1 REVIEW

SECTION	CONCEPT	KEY IDEAS/FORMULAS
1.1	**Functions**	
	Definition of a function	All functions are relations but not all relations are functions.
	Functions defined by equations	A vertical line can intersect a function in at most one point.
	Function notation	Placeholder notation Difference quotient: $\dfrac{f(x+h)-f(x)}{h}$, $h \neq 0$
	Domain of a function	Are there any restrictions on x?
1.2	**Graphs of functions**	
	Recognizing and classifying functions	**Common functions** $f(x) = mx + b, f(x) = x, f(x) = x^2,$ $f(x) = x^3,\ f(x) = \sqrt{x},\ f(x) = \sqrt[3]{x},$ $f(x) = \lvert x \rvert,\ f(x) = \dfrac{1}{x}$ **Even and odd functions:** *Even:* $f(-x) = f(x)$ Symmetry about y-axis *Odd:* $f(-x) = -f(x)$ Symmetry about origin
	Increasing and decreasing functions	■ Increasing: rises (left to right) ■ Decreasing: falls (left to right)
	Average rate of change	$\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$ $\quad x_1 \neq x_2$
	Piecewise-defined functions	Points of discontinuity
1.3	**Graphing techniques: Transformations**	Shift the graph of $f(x)$.
	Horizontal and vertical shifts	$f(x+c)$ c units to the left where $c > 0$ $f(x-c)$ c units to the right where $c > 0$ $f(x)+c$ c units upward where $c > 0$ $f(x)-c$ c units downward where $c > 0$
	Reflection about the axes	$-f(x)$ Reflection about the x-axis $f(-x)$ Reflection about the y-axis
	Stretching and compressing	$cf(x)$ if $c > 1$ stretch vertically $cf(x)$ if $0 < c < 1$ compress vertically $f(cx)$ if $c > 1$ compress horizontally $f(cx)$ if $0 < c < 1$ stretch horizontally

Section	Concept	Key Ideas/Formulas
1.4	**Combining functions**	
	Adding, subtracting, multiplying, and dividing functions	$(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$ $(f \cdot g)(x) = f(x) \cdot g(x)$ Domain of the resulting function is the intersection of the individual domains. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$ Domain of the quotient is the intersection of the domains of f and g, and any points when $g(x) = 0$ must be eliminated.
	Composition of functions	$(f \circ g)(x) = f(g(x))$ The domain of the composite function is a subset of the domain of $g(x)$. Values for x must be eliminated if their corresponding values $g(x)$ are not in the domain of f.



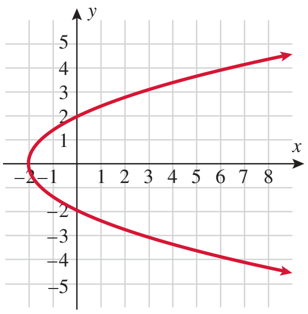
$(f \circ g)(x) = f(g(x))$

Section	Concept	Key Ideas/Formulas
1.5	**One-to-one functions and inverse functions**	
	One-to-one functions	■ No two x-values map to the same y-value. If $f(x_1) = f(x_2)$, then $x_1 = x_2$. ■ A horizontal line may intersect a one-to-one function in at most one point.
	Inverse functions	■ Only one-to-one functions have inverses. ■ $f^{-1}(f(x)) = x$ for all x in the domain of f. ■ $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}. ■ Domain of f = range of f^{-1}. Range of f = domain of f^{-1}.
	Graphical interpretation of inverse functions	■ The graph of a function and its inverse are symmetric about the line $y = x$. ■ If the point (a, b) lies on the graph of a function, then the point (b, a) lies on the graph of its inverse.
	Finding the inverse function	1. Let $y = f(x)$. 2. Interchange x and y. 3. Solve for y. 4. Let $y = f^{-1}(x)$.

1.1 Functions

Determine whether each relation is a function. Assume that the coordinate pair (x, y) represents independent variable x and dependent variable y.

1. $\{(-2, 3), (1, -3), (0, 4), (2, 6)\}$
2. $\{(4, 7), (2, 6), (3, 8), (1, 7)\}$
3. $x^2 + y^2 = 36$
4. $x = 4$
5. $y = |x + 2|$
6. $y = \sqrt{x}$
7.



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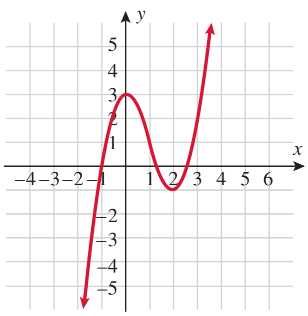
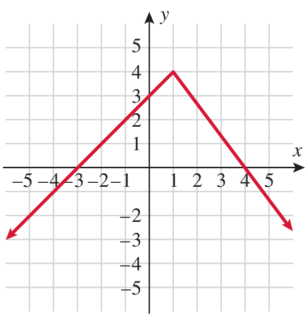


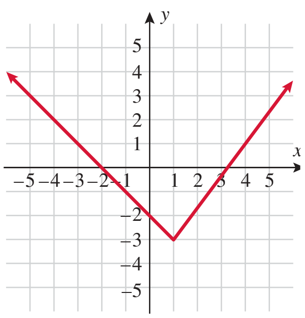
Use the graphs of the functions to find:

9.



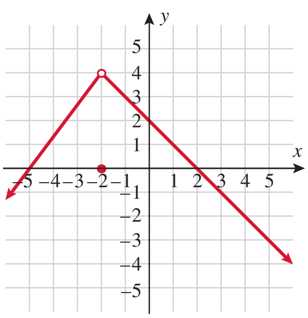
a. $f(-1)$ **b.** $f(1)$
c. x, where $f(x) = 0$

10.



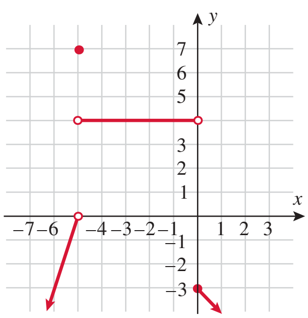
a. $f(-4)$ **b.** $f(0)$
c. x, where $f(x) = 0$

11.



a. $f(-2)$ **b.** $f(4)$
c. x, where $f(x) = 0$

12.



a. $f(-5)$ **b.** $f(0)$
c. x, where $f(x) = 0$

Evaluate the given quantities using the following three functions:

$f(x) = 4x - 7 \qquad F(t) = t^2 + 4t - 3 \qquad g(x) = |x^2 + 2x + 4|$

13. $f(3)$
14. $F(4)$
15. $f(-7) \cdot g(3)$
16. $\dfrac{F(0)}{g(0)}$
17. $\dfrac{f(2) - F(2)}{g(0)}$
18. $f(3 + h)$
19. $\dfrac{f(3 + h) - f(3)}{h}$
20. $\dfrac{F(t + h) - F(t)}{h}$

Find the domain of the given function. Express the domain in interval notation.

21. $f(x) = -3x - 4$
22. $g(x) = x^2 - 2x + 6$
23. $h(x) = \dfrac{1}{x + 4}$
24. $F(x) = \dfrac{7}{x^2 + 3}$
25. $G(x) = \sqrt{x - 4}$
26. $H(x) = \dfrac{1}{\sqrt{2x - 6}}$

Challenge

27. If $f(x) = \dfrac{D}{x^2 - 16}$, $f(4)$ and $f(-4)$ are undefined, and $f(5) = 2$, find D.
28. Construct a function that is undefined at $x = -3$ and $x = 2$ such that the point $(0, -4)$ lies on the graph of the function.

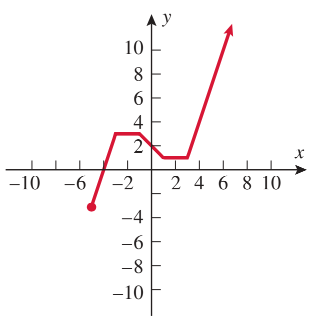
1.2 Graphs of Functions

Determine whether the function is even, odd, or neither.

29. $h(x) = x^3 - 7x$
30. $f(x) = x^4 + 3x^2$
31. $f(x) = \dfrac{1}{x^3} + 3x$
32. $f(x) = \dfrac{1}{x^2} + 3x^4 + |x|$

In Exercises 33–36, state the (a) domain, (b) range, and (c) x-interval(s), where the function is increasing, decreasing, or constant. Find the values of (d) $f(0)$, (e) $f(-3)$, and (f) $f(3)$.

33.



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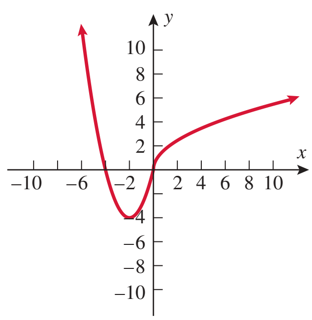

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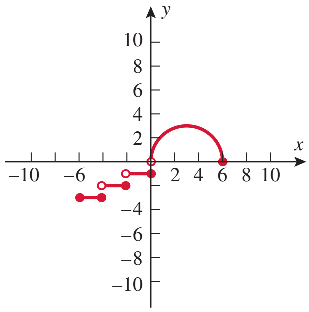

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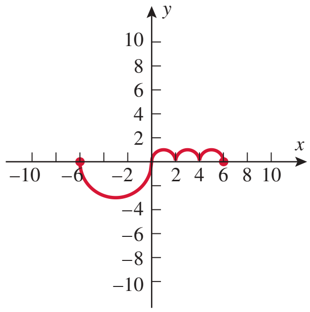


In Exercises 37–40, find the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for each function.

37. $f(x) = x^3 - 1$

38. $f(x) = \dfrac{x-1}{x+2}$

39. $f(x) = x + \dfrac{1}{x}$

40. $f(x) = \sqrt{\dfrac{x}{x+1}}$

41. Find the average rate of change of $f(x) = 4 - x^2$ from $x = 0$ to $x = 2$.

42. Find the average rate of change of $f(x) = |2x - 1|$ from $x = 1$ to $x = 5$.

Graph the piecewise-defined function. State the domain and range in interval notation.

43. $F(x) = \begin{cases} x^2 & x < 0 \\ 2 & x \geq 0 \end{cases}$

44. $f(x) = \begin{cases} -2x - 3 & x \leq 0 \\ 4 & 0 < x \leq 1 \\ x^2 + 4 & x > 1 \end{cases}$

45. $f(x) = \begin{cases} x^2 & x \leq 0 \\ -\sqrt{x} & 0 < x \leq 1 \\ |x + 2| & x > 1 \end{cases}$

46. $F(x) = \begin{cases} x^2 & x < 0 \\ x^3 & 0 < x < 1 \\ -|x| - 1 & x \geq 1 \end{cases}$

Applications

47. Housing Cost. In 2001 the market value of a house was \$135,000; in 2006 the market price of the same house was \$280,000. What is the average rate of the market price as a function of the time, where $t = 0$ corresponds to 2001.

48. Digital TV Conversion. A newspaper reported that by February 2009, only 38% of the urban population was ready for the conversion to digital TV. Ten weeks later, the newspaper reported that 64% of the population was prepared for the broadcasting change. Find the average rate of change of the population percent as a function of the time (in weeks).

1.3 Graphing Techniques: Transformations

Graph the following functions using graphing aids:

49. $y = -(x-2)^2 + 4$

50. $y = |-x + 5| - 7$

51. $y = \sqrt[3]{x-3} + 2$

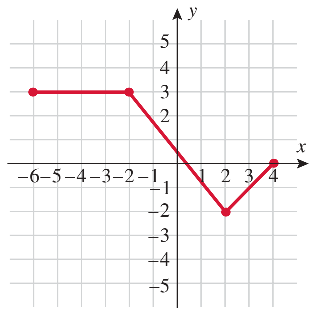
52. $y = \dfrac{1}{x-2} - 4$

53. $y = -\frac{1}{2}x^3$

54. $y = 2x^2 + 3$

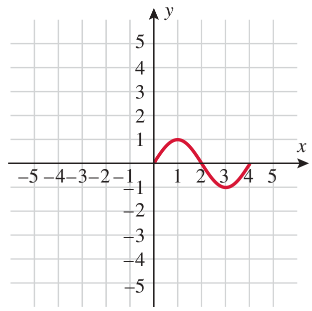
Use the given graph to graph the following:

55.



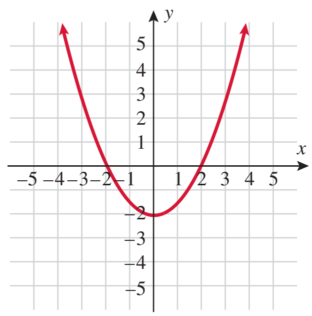
$y = f(x-2)$

56.



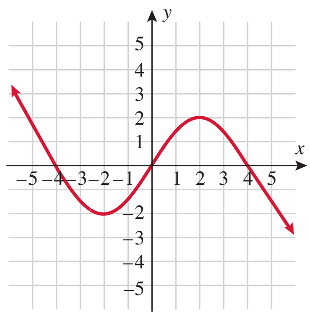
$y = 3f(x)$

57.



$y = -2f(x)$

58.



$y = f(x) + 3$

Write the function whose graph is the graph of $y = \sqrt{x}$, but is transformed accordingly, and state the domain of the resulting function.

59. Shifted to the left three units

60. Shifted down four units

61. Shifted to the right two units and up three units

62. Reflected about the y-axis

63. Stretched by a factor of 5 and shifted vertically down six units

64. Compressed by a factor of 2 and shifted vertically up three units

Transform the function into the form $f(x) = c(x-h)^2 + k$ by completing the square and graph the resulting function using transformations.

65. $y = x^2 + 4x - 8$

66. $y = 2x^2 + 6x - 5$

1.4 Combining Functions

Given the functions g and h, find $g + h$, $g - h$, $g \cdot h$, and $\frac{g}{h}$, and state the domain.

67. $g(x) = -3x - 4$
$h(x) = x - 3$

68. $g(x) = 2x + 3$
$h(x) = x^2 + 6$

69. $g(x) = \dfrac{1}{x^2}$
$h(x) = \sqrt{x}$

70. $g(x) = \dfrac{x + 3}{2x - 4}$
$h(x) = \dfrac{3x - 1}{x - 2}$

71. $g(x) = \sqrt{x - 4}$
$h(x) = \sqrt{2x + 1}$

72. $g(x) = x^2 - 4$
$h(x) = x + 2$

For the given functions f and g, find the composite functions $f \circ g$ and $g \circ f$, and state the domains.

73. $f(x) = 3x - 4$
$g(x) = 2x + 1$

74. $f(x) = x^3 + 2x - 1$
$g(x) = x + 3$

75. $f(x) = \dfrac{2}{x + 3}$
$g(x) = \dfrac{1}{4 - x}$

76. $f(x) = \sqrt{2x^2 - 5}$
$g(x) = \sqrt{x + 6}$

77. $f(x) = \sqrt{x - 5}$
$g(x) = x^2 - 4$

78. $f(x) = \dfrac{1}{\sqrt{x}}$
$g(x) = \dfrac{1}{x^2 - 4}$

Evaluate $f(g(3))$ and $g(f(-1))$, if possible.

79. $f(x) = 4x^2 - 3x + 2$
$g(x) = 6x - 3$

80. $f(x) = \sqrt{4 - x}$
$g(x) = x^2 + 5$

81. $f(x) = \dfrac{x}{|2x - 3|}$
$g(x) = |5x + 2|$

82. $f(x) = \dfrac{1}{x - 1}$
$g(x) = x^2 - 1$

83. $f(x) = x^2 - x + 10$
$g(x) = \sqrt[3]{x - 4}$

84. $f(x) = \dfrac{4}{x^2 - 2}$
$g(x) = \dfrac{1}{x^2 - 9}$

Write the function as a composite $f(g(x))$ of two functions f and g.

85. $h(x) = 3(x - 2)^2 + 4(x - 2) + 7$

86. $h(x) = \dfrac{\sqrt[3]{x}}{1 - \sqrt[3]{x}}$

87. $h(x) = \dfrac{1}{\sqrt{x^2 + 7}}$

88. $h(x) = \sqrt{|3x + 4|}$

Applications

89. Rain. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time, in minutes, $r(t) = 25\sqrt{t + 2}$, find the area of the ripple as a function of time.

90. Geometry. Let the area of a rectangle be given by $42 = l \cdot w$, and let the perimeter be $36 = 2 \cdot l + 2 \cdot w$. Express the perimeter in terms of w.

1.5 One-to-One Functions and Inverse Functions

Determine whether the given function is a one-to-one function.

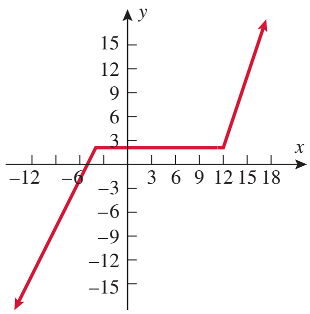
91. $\{(-2, 0), (4, 5), (3, 7)\}$

92. $\{(-8, -6), (-4, 2), (0, 3), (2, -8), (7, 4)\}$

93. $y = \sqrt{x}$ **94.** $y = x^2$ **95.** $f(x) = x^3$ **96.** $f(x) = \dfrac{1}{x^2}$

In Exercises 97–100, determine whether the function is one-to-one.

97.



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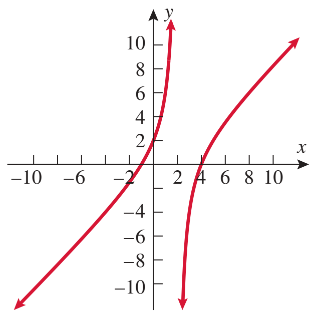

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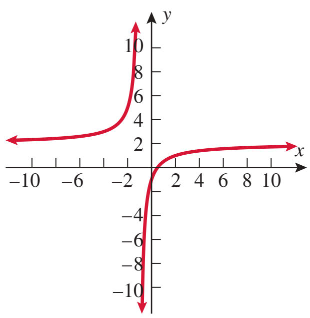
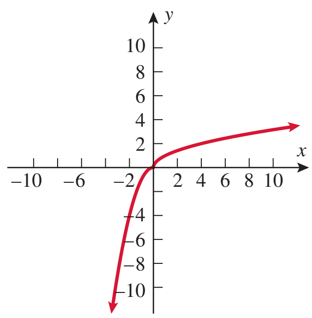


100.



Verify that the function $f^{-1}(x)$ is the inverse of $f(x)$ by showing that $f(f^{-1}(x)) = x$. Graph $f(x)$ and $f^{-1}(x)$ on the same graph and show the symmetry about the line $y = x$.

101. $f(x) = 3x + 4$; $f^{-1}(x) = \dfrac{x - 4}{3}$

102. $f(x) = \dfrac{1}{4x - 7}$; $f^{-1}(x) = \dfrac{1 + 7x}{4x}$

103. $f(x) = \sqrt{x+4}; f^{-1}(x) = x^2 - 4 \quad x \geq 0$

104. $f(x) = \dfrac{x+2}{x-7}; f^{-1}(x) = \dfrac{7x+2}{x-1}$

The function *f* is one-to-one. Find its inverse and check your answer. State the domain and range of both *f* and f^{-1}.

105. $f(x) = 2x + 1$

106. $f(x) = x^5 + 2$

107. $f(x) = \sqrt{x+4}$

108. $f(x) = (x+4)^2 + 3 \quad x \geq -4$

109. $f(x) = \dfrac{x+6}{x+3}$

110. $f(x) = 2\sqrt[3]{x-5} - 8$

Applications

111. Salary. A pharmaceutical salesperson makes \$22,000 base salary a year plus 8% of the total products sold. Write a function $S(x)$ that represents her yearly salary as a function of the total dollars x worth of products sold. Find $S^{-1}(x)$. What does this inverse function tell you?

112. Volume. Express the volume V of a rectangular box that has a square base of length s and is 3 feet high as a function of the square length. Find V^{-1}. If a certain volume is desired, what does the inverse tell you?

Technology Exercises

Section 1.1

113. Use a graphing utility to graph the function and find the domain. Express the domain in interval notation.

$$f(x) = \frac{1}{\sqrt{x^2 - 2x - 3}}$$

114. Use a graphing utility to graph the function and find the domain. Express the domain in interval notation.

$$f(x) = \frac{x^2 - 4x - 5}{x^2 - 9}$$

Section 1.2

115. Use a graphing utility to graph the function. State the (a) domain, (b) range, and (c) x intervals where the function is increasing, decreasing, and constant.

$$f(x) = \begin{cases} 1 - x & x < -1 \\ [[x]] & -1 \leq x < 2 \\ x + 1 & x > 2 \end{cases}$$

116. Use a graphing utility to graph the function. State the (a) domain, (b) range, and (c) x intervals where the function is increasing, decreasing, and constant.

$$f(x) = \begin{cases} |x^2 - 1| & -2 < x < 2 \\ \sqrt{x-2} + 4 & x > 2 \end{cases}$$

Section 1.3

117. Use a graphing utility to graph $f(x) = x^2 - x - 6$ and $g(x) = x^2 - 5x$. Use transforms to describe the relationship between $f(x)$ and $g(x)$?

118. Use a graphing utility to graph $f(x) = 2x^2 - 3x - 5$ and $g(x) = -2x^2 - x + 6$. Use transforms to describe the relationship between $f(x)$ and $g(x)$?

Section 1.4

119. Using a graphing utility, plot $y_1 = \sqrt{2x+3}$, $y_2 = \sqrt{4-x}$, and $y_3 = \dfrac{y_1}{y_2}$. What is the domain of y_3?

120. Using a graphing utility, plot $y_1 = \sqrt{x^2 - 4}$, $y_2 = x^2 - 5$, and $y_3 = y_1^2 - 5$. If y_1 represents a function f and y_2 represents a function g, then y_3 represents the composite function $g \circ f$. The graph of y_3 is only defined for the domain of $g \circ f$. State the domain of $g \circ f$.

Section 1.5

121. Use a graphing utility to graph the function and determine whether it is one-to-one.

$$f(x) = \frac{6}{\sqrt[5]{x^3 - 1}}$$

122. Use a graphing utility to graph the functions f and g and the line $y = x$ in the same screen. Are the two functions inverses of each other?

$$f(x) = \sqrt[4]{x-3} + 1, \; g(x) = x^4 - 4x^3 + 6x^2 - 4x + 3$$

CHAPTER 1 PRACTICE TEST

Assuming that x represents the independent variable and y represents the dependent variable, classify the relationships as:

a. **not a function**
b. **a function, but not one-to-one**
c. **a one-to-one function**

1. $f(x) = |2x + 3|$ **2.** $x = y^2 + 2$ **3.** $y = \sqrt[3]{x + 1}$

Use $f(x) = \sqrt{x - 2}$ and $g(x) = x^2 + 11$, and determine the desired quantity or expression. In the case of an expression, state the domain.

4. $f(11) - 2g(-1)$ **5.** $\left(\dfrac{f}{g}\right)(x)$

6. $\left(\dfrac{g}{f}\right)(x)$ **7.** $g(f(x))$

8. $(f + g)(6)$ **9.** $f(g(\sqrt{7}))$

Determine whether the function is odd, even, or neither.

10. $f(x) = |x| - x^2$

11. $f(x) = 9x^3 + 5x - 3$

12. $f(x) = \dfrac{2}{x}$

Graph the functions. State the domain and range of each function.

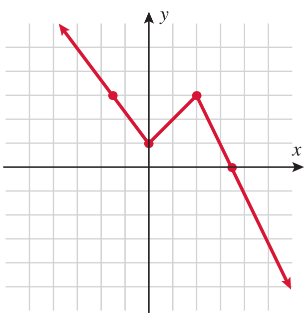
13. $f(x) = -\sqrt{x - 3} + 2$

14. $f(x) = -2(x - 1)^2$

15. $f(x) = \begin{cases} -x & x < -1 \\ 1 & -1 < x < 2 \\ x^2 & x \geq 2 \end{cases}$

Use the graphs of the function to find:

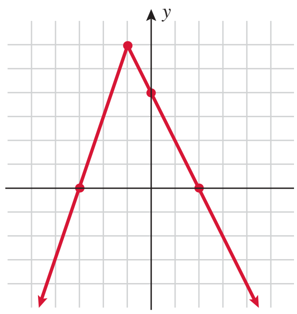
16.



$y = f(x)$

a. $f(3)$ **b.** $f(0)$ **c.** $f(-4)$
d. x, where $f(x) = 3$ **e.** x, where $f(x) = 0$

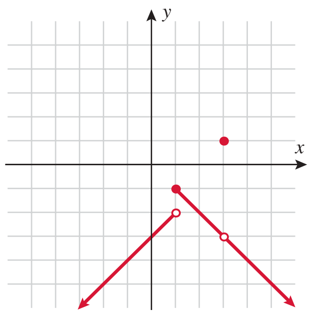
17.



$y = g(x)$

a. $g(3)$ **b.** $g(0)$ **c.** $g(-4)$
d. x, where $g(x) = 0$

18.



$y = f(x)$

a. $p(0)$ **b.** x, where $p(x) = 0$
c. $p(1)$ **d.** $p(3)$

Find the difference equation, $\dfrac{f(x + h) - f(x)}{h}$, for:

19. $f(x) = 3x^2 - 4x + 1$ **20.** $f(x) = x^3 - \dfrac{1}{\sqrt{x}}$

Find the average rate of change of the given functions.

21. $f(x) = 64 - 16x^2$ for $x = 0$ to $x = 2$

22. $f(x) = \sqrt{x - 1}$ for $x = 2$ to $x = 10$

In Exercises 23–26, given the function f, find the inverse if it exists. State the domain and range of both f and f^{-1}.

23. $f(x) = \sqrt{x - 5}$ **24.** $f(x) = x^2 + 5$

25. $f(x) = \dfrac{2x + 1}{5 - x}$ **26.** $f(x) = \begin{cases} -x & x \leq 0 \\ -x^2 & x > 0 \end{cases}$

27. What domain restriction can be made so that $f(x) = x^2$ has an inverse?

28. If the point $(-2, 5)$ lies on the graph of a function, what point lies on the graph of its inverse function?

29. **Pressure.** A mini-submarine descends at a rate of 5 feet per second. The pressure on the submarine structure is a linear function of the depth; when the submarine is on the surface, the pressure is 10 pounds per square inch, and when 100 feet underwater, the pressure is 28 pounds per square inch. Write a function that describes the pressure P as a function of the time t in seconds.

30. **Geometry.** Both the volume V and surface area S of a sphere are functions of the radius R. Write the volume as a function of the surface area.

31. **Circles.** If a quarter circle is drawn by tracing the unit circle in quadrant III, what does the inverse of that function look like? Where is it located?

32. **Sprinkler.** A sprinkler head malfunctions at midfield in an NFL football field. The puddle of water forms a circular pattern around the sprinkler head with a radius in yards that grows as a function of time, in hours: $r(t) = 10\sqrt{t}$. When will the puddle reach the sidelines? (A football field is 30 yards from sideline to sideline.)

33. **Internet.** The cost of airport Internet access is \$15 for the first 30 minutes and \$1 per minute for each minute after that. Write a function describing the cost of the service as a function of minutes used.

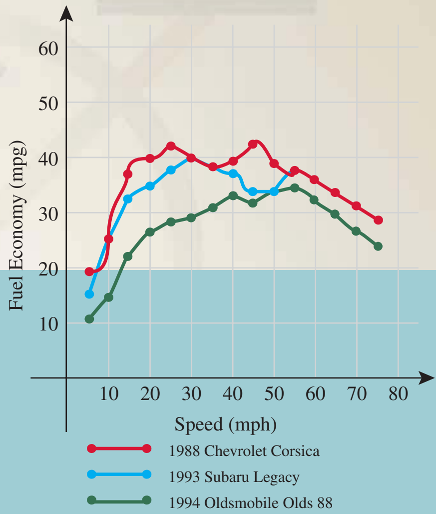
34. **Temperature and CO_2 Emissions.** The following table shows average yearly temperature in degrees Fahrenheit °F and carbon dioxide emissions in parts per million (ppm) for Mauna Loa, Hawaii. Scientists discovered that both temperature and CO_2 emissions are linear functions of the time. Write a function that describes the temperature T as a function of the CO_2 emissions x. Use this function to determine the temperature when the CO_2 reaches the level of 375 ppm.

Year	2000	2005
Temperature (°F)	45.86	46.23
CO_2 emissions (ppm)	369.4	379.7

PRACTICE TEST

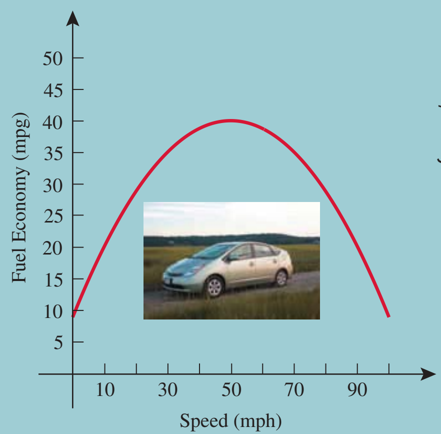
2

Polynomial and Rational Functions



The gas mileage you achieve (in whatever vehicle you drive) is a function of speed, which can be modeled by a *polynomial function*. The number of turning points in the graph of a polynomial function is related to the degree of that polynomial.

We can approximate the above trends with a simple second-degree polynomial function, called a *quadratic function*, whose graph is a parabola. We see from the graph why hypermilers do not drive above posted speed limits (and often drive below them).



IN THIS CHAPTER we will start by discussing quadratic functions (polynomial functions of degree 2), whose graphs are parabolas. We will find the vertex, which is the maximum or minimum point on the graph. Then we will expand our discussion to higher degree polynomial functions. We will discuss techniques to find zeros of polynomial functions and strategies for graphing polynomial functions. Lastly, we will discuss rational functions, which are ratios of polynomial functions.

POLYNOMIAL AND RATIONAL FUNCTIONS

2.1 Quadratic Functions

- Graphs of Quadratic Functions: Parabolas
- Finding the Equation of a Parabola

2.2 Polynomial Functions of Higher Degree

- Identifying Polynomial Functions
- Graphing Polynomial Functions Using Transformations of Power Functions
- Real Zeros of a Polynomial Function
- Graphing General Polynomial Functions

2.3 Dividing Polynomials

- Long Division of Polynomials
- Synthetic Division of Polynomials

2.4 The Real Zeros of a Polynomial Function

- The Remainder Theorem and the Factor Theorem
- The Rational Zero Theorem and Descartes' Rule of Signs
- Factoring Polynomials
- The Intermediate Value Theorem
- Graphing Polynomial Functions

2.5 Complex Zeros: The Fundamental Theorem of Algebra

- Complex Zeros
- Factoring Polynomials

2.6 Rational Functions

- Domain of Rational Functions
- Vertical, Horizontal, and Slant Asymptotes
- Graphing Rational Functions

LEARNING OBJECTIVES

- Given a quadratic function in either standard or general form, find its vertex and sketch its graph.
- Use multiplicity of zeros and end behavior as guides in sketching the graph of a polynomial function.
- Divide polynomials using long division and understand when synthetic division can be used.
- Find all real zeros of a polynomial function (x-intercepts) and use these as guides in sketching the graph of a polynomial function.
- Factor a polynomial function completely over the set of complex numbers.
- Use asymptotes and intercepts as guides in graphing rational functions.

SECTION 2.1 QUADRATIC FUNCTIONS

SKILLS OBJECTIVES

- Graph a quadratic function in standard form.
- Graph a quadratic function in general form.
- Find the equation of a parabola.
- Solve application problems that involve quadratic functions.

CONCEPTUAL OBJECTIVES

- Recognize characteristics of graphs of quadratic functions (parabolas):
  - whether the parabola opens up or down
  - whether the vertex is a maximum or minimum
  - the axis of symmetry

Graphs of Quadratic Functions: Parabolas

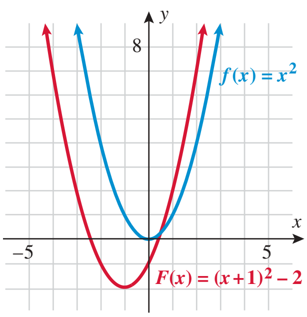
In Chapter 1, we studied functions in general. In this chapter, we will learn about a special group of functions called *polynomial functions*. Polynomial functions are simple functions; often, more complicated functions are approximated by polynomial functions. Polynomial functions model many real-world applications such as the stock market, football punts, business costs, revenues and profits, and the flight path of NASA's "vomit comet." Let's start by defining a polynomial function.

DEFINITION **Polynomial Function**

Let n be a nonnegative integer, and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

is called a **polynomial function of x with degree n**. The coefficient a_n is called the **leading coefficient**, and a_0 is the constant.



Polynomials of particular degrees have special names. In Chapter 1, the library of functions included the constant function $f(x) = b$, which is a horizontal line; the linear function $f(x) = mx + b$, which is a line with slope m and y-intercept $(0, b)$; the square function $f(x) = x^2$; and the cube function $f(x) = x^3$. These are all special cases of a polynomial function.

In Section 1.3, we graphed functions using transformation techniques such as $F(x) = (x + 1)^2 - 2$, which can be graphed by starting with the square function $y = x^2$ and shifting one unit to the left and down two units. See the graph on the left.

Note that if we eliminate the parentheses in $F(x) = (x + 1)^2 - 2$ to get

$$\begin{aligned} F(x) &= x^2 + 2x + 1 - 2 \\ &= x^2 + 2x - 1 \end{aligned}$$

the result is a function defined by a second-degree polynomial (a polynomial with x^2 as the highest degree term), which is also called a *quadratic function*.

DEFINITION **Quadratic Function**

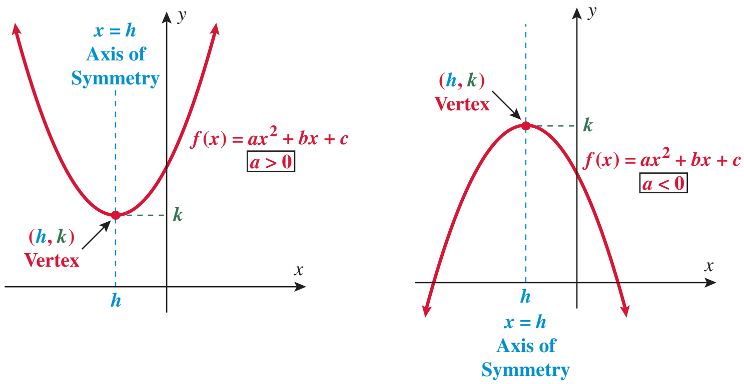
Let a, b, and c be real numbers with $a \neq 0$. The function

$$f(x) = ax^2 + bx + c$$

is called a **quadratic function**.

The graph of any quadratic function is a **parabola**. If the leading coefficient a is *positive*, then the parabola opens *upward.* If the leading coefficient a is *negative*, then the parabola opens *downward.* The **vertex** (or turning point) is the *minimum* point, or low point, on the graph if the parabola opens upward, whereas it is the *maximum* point, or high point, on the graph if the parabola opens downward. The vertical line that intersects the parabola at the vertex is called the **axis of symmetry**.

The axis of symmetry is the line $x = h$, and the vertex is located at the point (h, k), as shown in the following two figures:



Graphing Quadratic Functions in Standard Form

In general, writing a quadratic function in the form

$$f(x) = a(x - h)^2 + k$$

allows the vertex (h, k) and the axis of symmetry $x = h$ to be determined by inspection. This form is a convenient way to express a quadratic function in order to quickly determine its corresponding graph. Hence, this form is called *standard form.*

QUADRATIC FUNCTION: STANDARD FORM

The quadratic function

$$f(x) = a(x - h)^2 + k$$

is in **standard form**. The graph of f is a parabola whose vertex is the point (h, k). The parabola is symmetric with respect to the line $x = h$. If $a > 0$, the parabola opens up. If $a < 0$, the parabola opens down.

Recall that graphing linear functions requires finding two points on the line, or a point and the slope of the line. However, for a quadratic function, simply knowing two points that lie on its graph is no longer sufficient. Below is a general step-by-step procedure for graphing quadratic functions given in standard form.

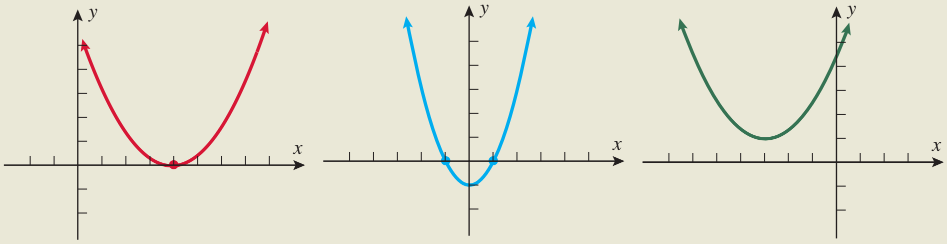
GRAPHING QUADRATIC FUNCTIONS

To graph $f(x) = a(x - h)^2 + k$

Step 1: Determine whether the parabola opens up or down.

$$a > 0 \quad \text{up}$$
$$a < 0 \quad \text{down}$$

Step 2: Determine the vertex (h, k).
Step 3: Find the y-intercept (by setting $x = 0$).
Step 4: Find any x-intercepts [by setting $f(x) = 0$ and solving for x].
Step 5: Plot the vertex and intercepts and connect them with a smooth curve.



Note that Step 4 says to "find any x-intercepts." Parabolas opening up or down will always have a y-intercept. However, they can have **one**, **two**, or **no** x-intercepts. The figures above illustrate this for parabolas opening up, and the same can be said about parabolas opening down.

EXAMPLE 1 Graphing a Quadratic Function Given in Standard Form

Graph the quadratic function $f(x) = (x - 3)^2 - 1$.

Solution:

STEP 1 The parabola opens up. $\quad a = 1$, so $a > 0$

STEP 2 Determine the vertex. $\quad (h, k) = (3, -1)$

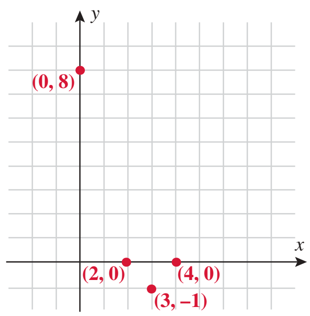
STEP 3 Find the y-intercept. $\quad f(0) = (-3)^2 - 1 = 8$
(0, 8) corresponds to the y-intercept

STEP 4 Find any x-intercepts. $\quad f(x) = (x - 3)^2 - 1 = 0$
$$(x - 3)^2 = 1$$

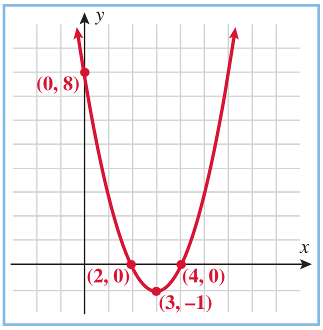
Use the square root method. $\quad x - 3 = \pm 1$

Solve. $\quad x = 2 \quad \text{or} \quad x = 4$
(2, 0) and (4, 0) correspond to the x-intercepts

STEP 5 Plot the vertex and intercepts $(3, -1), (0, 8), (2, 0), (4, 0)$.



Connect the points with a smooth curve opening up.



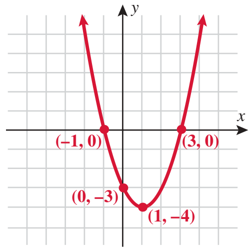
The graph in Example 1 could also have been found by shifting the square function to the right three units and down one unit.

Study Tip

A quadratic function given in standard form can be graphed using the transformation techniques shown in Section 1.3 for the square function.

■ **YOUR TURN** Graph the quadratic function $f(x) = (x - 1)^2 - 4$.

■ **Answer:**



EXAMPLE 2 Graphing a Quadratic Function Given in Standard Form with a Negative Leading Coefficient

Graph the quadratic function $f(x) = -2(x - 1)^2 - 3$.

Solution:

STEP 1 The parabola opens down. $a = -2$, so $a < 0$

STEP 2 Determine the vertex. $(h, k) = (1, -3)$

STEP 3 Find the y-intercept. $f(0) = -2(-1)^2 - 3 = -2 - 3 = -5$

$(0, -5)$ corresponds to the y-intercept

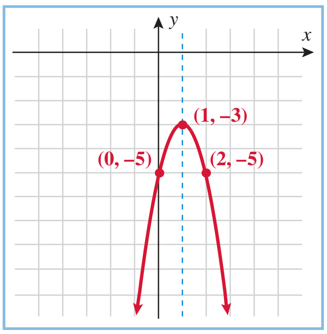
STEP 4 Find any x-intercepts.

$$f(x) = -2(x - 1)^2 - 3 = 0$$
$$-2(x - 1)^2 = 3$$
$$(x - 1)^2 = -\frac{3}{2}$$

A real quantity squared cannot be negative so there are no real solutions. There are no x-intercepts.

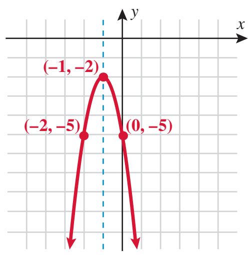
STEP 5 Plot the vertex $(1, -3)$ and y-intercept $(0, -5)$. Connect the points with a smooth curve.

Note that the axis of symmetry is $x = 1$. Because the point $(0, -5)$ lies on the parabola, then by symmetry with respect to $x = 1$, the point $(2, -5)$ also lies on the graph.



■ **YOUR TURN** Graph the quadratic function $f(x) = -3(x + 1)^2 - 2$.

■ **Answer:**



When graphing quadratic functions (parabolas), have *at least three points* labeled on the graph.

- When there are x-intercepts (Example 1), label the vertex, y-intercept, and x-intercepts.
- When there are no x-intercepts (Example 2), label the vertex, y-intercept, and another point.

Graphing Quadratic Functions in General Form

A quadratic function is often written in one of two forms:

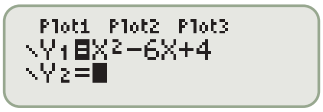
$$\text{Standard form: } f(x) = a(x - h)^2 + k$$
$$\text{General form: } f(x) = ax^2 + bx + c$$

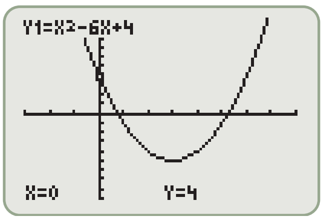
When the quadratic function is expressed in standard form, the graph is easily obtained by identifying the vertex (h, k) and the intercepts and drawing a smooth curve that opens either up or down, depending on the sign of a.

Typically, quadratic functions are expressed in general form and a graph is the ultimate goal, so we must first express the quadratic function in standard form. One technique for transforming a quadratic function from general form to standard form was reviewed in Section 0.2 and is called *completing the square.*

Technology Tip

Use a graphing utility to graph the function $f(x) = x^2 - 6x + 4$ as y_1.





EXAMPLE 3 Graphing a Quadratic Function Given in General Form

Write the quadratic function $f(x) = x^2 - 6x + 4$ in standard form and graph f.

Solution:

Express the quadratic function in standard form by completing the square.

Write the original function. $\quad f(x) = x^2 - 6x + 4$

Group the variable terms together. $\quad = (x^2 - 6x) + 4$

Complete the square.

Half of -6 is -3; -3 squared is 9.

Add and subtract 9 within the parentheses. $\quad = (x^2 - 6x \mathbf{+ 9 - 9}) + 4$

Write the -9 outside the parentheses. $\quad = (x^2 - 6x \mathbf{+ 9)} \mathbf{- 9} + 4$

Write the expression inside the parentheses as a perfect square and simplify. $\quad = (x - 3)^2 - 5$

Now that the quadratic function is written in standard form, $f(x) = (x - 3)^2 - 5$*, we follow our step-by-step procedure for graphing a quadratic function in standard form.*

STEP 1 The parabola opens up. $\quad a = 1$, so $a > 0$

STEP 2 Determine the vertex. $\quad (h, k) = (3, -5)$

STEP 3 Find the y-intercept. $\quad f(x) = x^2 - 6x + 4$

$f(0) = (0)^2 - 6(0) + 4 = 4$

$(0, 4)$ corresponds to the y-intercept

STEP 4 Find any x-intercepts. $\quad f(x) = 0$

Using the standard form:

$$f(x) = (x - 3)^2 - 5 = 0$$
$$(x - 3)^2 = 5$$
$$x - 3 = \pm\sqrt{5}$$
$$x = 3 \pm \sqrt{5}$$

Using the general form:

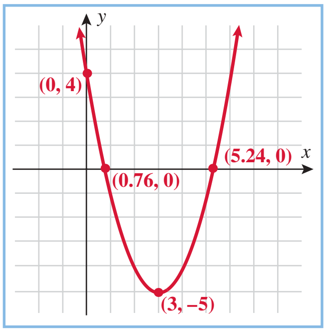
$$f(x) = x - 6x + 4 = 0$$
$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)}$$
$$x = 3 \pm \sqrt{5}$$

$(3 + \sqrt{5}, 0)$ and $(3 - \sqrt{5}, 0)$ correspond to the x-intercepts.

STEP 5 Plot the vertex and intercepts $(3, -5)$, $(0, 4)$, $(3 + \sqrt{5}, 0)$, and $(3 - \sqrt{5}, 0)$.

Connect the points with a smooth parabolic curve.

Note: $3 + \sqrt{5} \approx 5.24$ and $3 - \sqrt{5} \approx 0.76$.

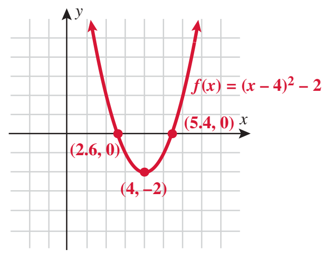


■ **YOUR TURN** Write the quadratic function $f(x) = x^2 - 8x + 14$ in standard form and graph f.

Study Tip

Although either form (standard or general) can be used to find the intercepts, it is often more convenient to use the general form when finding the y-intercept and the standard form when finding the x-intercept.

■ **Answer:**



When the leading coefficient of a quadratic function is not equal to 1, the leading coefficient must be factored out before completing the square.

EXAMPLE 4 Graphing a Quadratic Function Given in General Form with a Negative Leading Coefficient

Graph the quadratic function $f(x) = -3x^2 + 6x + 2$.

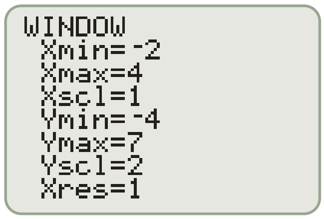
Solution:

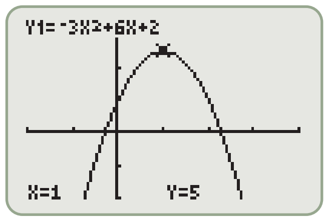
Express the function in standard form by completing the square.

Write the original function.	$f(x) = -3x^2 + 6x + 2$
Group the variable terms together.	$= (-3x^2 + 6x) + 2$
Factor out –3 in order to make the coefficient of x^2 equal to 1 inside the parentheses.	$= -3(x^2 - 2x) + 2$
Add and subtract 1 inside the parentheses to create a perfect square.	$= -3(x^2 - 2x + 1 - 1) + 2$
Regroup the terms.	$= -3(x^2 - 2x + 1) - 3(-1) + 2$
Write the expression inside the parentheses as a perfect square and simplify.	$= -3(x - 1)^2 + 5$

Technology Tip

Use a graphing utility to graph the function $f(x) = -3x^2 + 6x + 2$ as y_1.





Now that the quadratic function is written in standard form, $f(x) = -3(x-1)^2 + 5$, *we follow our step-by-step procedure for graphing a quadratic function in standard form.*

STEP 1 The parabola opens down. $a = -3$; therefore, $a < 0$

STEP 2 Determine the vertex. $(h, k) = (1, 5)$

STEP 3 Find the y-intercept using the general form. $f(0) = -3(0)^2 + 6(0) + 2 = 2$
$(0, 2)$ corresponds to the y-intercept

STEP 4 Find any x-intercepts using the standard form.

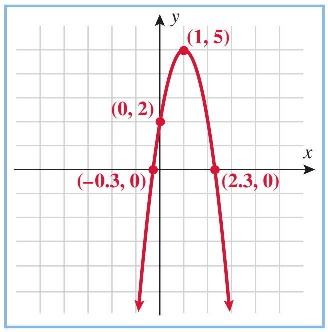
$$f(x) = -3(x-1)^2 + 5 = 0$$
$$-3(x-1)^2 = -5$$
$$(x-1)^2 = \frac{5}{3}$$
$$x - 1 = \pm\sqrt{\frac{5}{3}}$$
$$x = 1 \pm \sqrt{\frac{5}{3}}$$
$$= 1 \pm \frac{\sqrt{15}}{3}$$

The x-intercepts are $\left(1 + \frac{\sqrt{15}}{3}, 0\right)$ and $\left(1 - \frac{\sqrt{15}}{3}, 0\right)$.

STEP 5 Plot the vertex and intercepts $(1, 5)$, $(0, 2)$, $\left(1 + \frac{\sqrt{15}}{3}, 0\right)$, and $\left(1 - \frac{\sqrt{15}}{3}, 0\right)$.

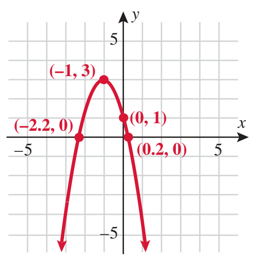
Connect the points with a smooth curve.

Note: $1 + \frac{\sqrt{15}}{3} \approx 2.3$ and $1 - \frac{\sqrt{15}}{3} \approx -0.3$.



YOUR TURN Graph the quadratic function $f(x) = -2x^2 - 4x + 1$.

Answer:



In Examples 3 and 4, the quadratic functions were given in general form and they were transformed into standard form by completing the square. It can be shown (by completing the square) that the vertex of a quadratic function in general form, $f(x) = ax^2 + bx + c$, is located at $x = -\frac{b}{2a}$.

Another approach to sketching the graphs of quadratic functions is to first find the vertex and then find additional points through point-plotting.

VERTEX OF A PARABOLA

The graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola with the **vertex** located at the point

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

GRAPHING A QUADRATIC FUNCTION IN GENERAL FORM

Step 1: Find the vertex.

Step 2: Determine whether the parabola opens up or down.

- If $a > 0$, the parabola opens up.
- If $a < 0$, the parabola opens down.

Step 3: Find additional points near the vertex.

Step 4: Sketch the graph with a parabolic curve.

EXAMPLE 5 Graphing a Quadratic Function Given in General Form

Sketch the graph of $f(x) = -2x^2 + 4x + 5$.

Solution: Let $a = -2$, $b = 4$, and $c = 5$.

STEP 1 Find the vertex.

$$x = -\frac{b}{2a} = -\frac{4}{2(-2)} = 1$$

$$f(1) = -2(1)^2 + 4(1) + 5 = 7$$

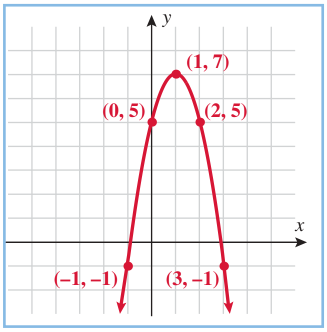
Vertex: $(1, 7)$

STEP 2 The parabola opens down. $a = -2$

STEP 3 Find additional points near the vertex.

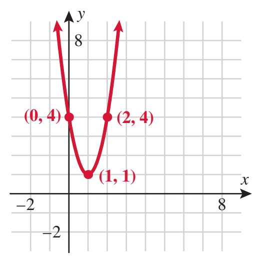
x	-1	0	1	2	3
$f(x)$	$f(-1) = -1$	$f(0) = 5$	$f(1) = 7$	$f(2) = 5$	$f(3) = -1$

STEP 4 Label the vertex and additional points then sketch the graph.



■ **YOUR TURN** Sketch the graph of $f(x) = 3x^2 - 6x + 4$.

■ **Answer:**



Finding the Equation of a Parabola

It is important to understand that the equation $y = x^2$ is equivalent to the quadratic function $f(x) = x^2$. Both have the same parabolic graph. Thus far, we have been given the function and then asked to find characteristics (vertex and intercepts) in order to graph. We now turn our attention to the problem of determining the function, given certain characteristics.

EXAMPLE 6 Finding the Quadratic Function Given the Vertex and a Point That Lies on Its Graph

Find the quadratic function whose graph has a vertex at $(3, 4)$ and which passes through the point $(2, 3)$. Express the quadratic function in both standard and general forms.

Solution:

Write the standard form of a quadratic function. $f(x) = a(x - h)^2 + k$

Substitute the coordinates of the vertex $(h, k) = (3, 4)$. $f(x) = a(x - 3)^2 + 4$

Use the point (2, 3) to find a.

The point (2, 3) implies $f(2) = 3$. $\quad f(2) = a(2 - 3)^2 + 4 = 3$

Solve for a.

$$a(2 - 3)^2 + 4 = 3$$
$$a(-1)^2 + 4 = 3$$
$$a + 4 = 3$$
$$a = -1$$

Write both forms of the quadratic function.

Standard form: $\boxed{f(x) = -(x - 3)^2 + 4}$ $\qquad$ General form: $\boxed{f(x) = -x^2 + 6x - 5}$

YOUR TURN Find the standard form of the equation of a parabola whose graph has a vertex at $(-3, -5)$ and which passes through the point $(-2, -4)$.

Answer: Standard form: $f(x) = (x + 3)^2 - 5$

As we have seen in Example 6, once the vertex is known, the leading coefficient a can be found from any point that lies on the parabola.

Application Problems That Involve Quadratic Functions

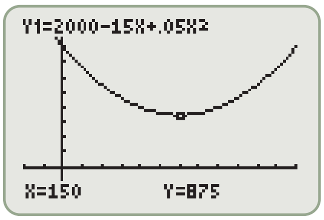
Because the vertex of a parabola represents either the minimum or maximum value of the quadratic function, in application problems it often suffices simply to find the vertex.

Technology Tip

Use a graphing utility to graph the cost function $C(x) = 2000 - 15x + 0.05x^2$ as y_1.

```
WINDOW
 Xmin=-50
 Xmax=300
 Xscl=25
 Ymin=-500
 Ymax=2500
 Yscl=250
 Xres=1
```

```
 Plot1 Plot2 Plot3
\Y1=2000-15X+.05
X²
\Y2=
```



EXAMPLE 7 Finding the Minimum Cost of Manufacturing a Motorcycle

A company that produces motorcycles has a per unit production cost of

$$C(x) = 2000 - 15x + 0.05x^2$$

where C is the cost in dollars to manufacture a motorcycle and x is the number of motorcycles produced. How many motorcycles should be produced in order to minimize the cost of each motorcycle? What is the corresponding minimum cost?

Solution:

The graph of the quadratic function is a parabola.

Rewrite the quadratic function in general form. $\quad C(x) = 0.05x^2 - 15x + 2000$

The parabola opens up, because a is positive. $\quad a = 0.05 > 0$

Because the parabola opens up, the vertex of the parabola is a *minimum*.

Find the x-coordinate of the vertex. $\quad x = -\dfrac{b}{2a} = -\dfrac{(-15)}{2(0.05)} = 150$

The company keeps per unit cost to a minimum when 150 motorcycles are produced.

The minimum cost is \$875 per motorcycle. $\quad C(150) = 875$

YOUR TURN The revenue associated with selling vitamins is

$$R(x) = 500x - 0.001x^2$$

where R is the revenue in dollars and x is the number of bottles of vitamins sold. Determine how many bottles of vitamins should be sold to maximize the revenue.

Answer: 250,000 bottles

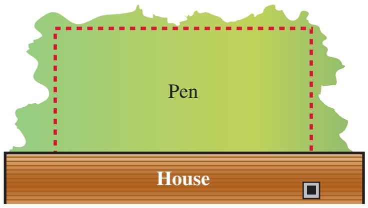
EXAMPLE 8 Finding the Dimensions That Yield a Maximum Area

You have just bought a puppy and want to fence in an area in the backyard for her. You buy 100 linear feet of fence from Home Depot and have decided to make a rectangular fenced-in area using the back of your house as one side. Determine the dimensions of the rectangular pen that will maximize the area in which your puppy may roam. What is the maximum area of the rectangular pen?

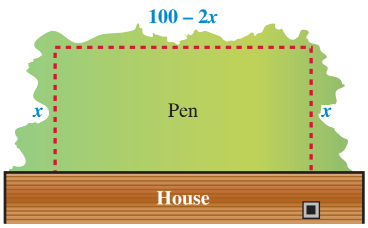
Solution:

STEP 1 Identify the question.
Find the dimensions of the rectangular pen.

STEP 2 Draw a picture.



STEP 3 Set up a function.
If we let x represent the length of one side of the rectangle, then the opposite side is also of length x. Because there are 100 feet of fence, the remaining fence left for the side opposite the house is $100 - 2x$.



The area of a rectangle is equal to length times width:

$$A(x) = x(100 - 2x)$$

STEP 4 Find the maximum value of the function.

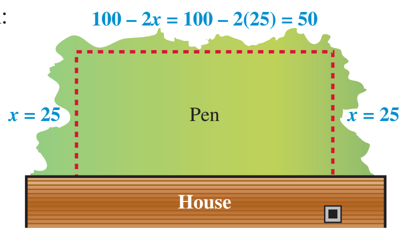
$$A(x) = x(100 - 2x) = -2x^2 + 100x$$

Find the maximum of the parabola that corresponds to the quadratic function for area $A(x) = -2x^2 + 100x$.

$a = -2$ and $b = 100$; therefore, the maximum occurs when

$$x = -\frac{b}{2a} = -\frac{100}{2(-2)} = 25$$

Replacing x with 25 in our original diagram:



The dimensions of the rectangle are $\boxed{\text{25 feet by 50 feet}}$.

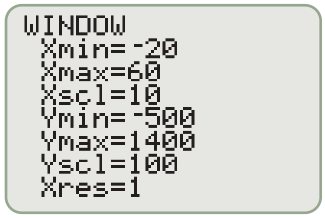
The maximum area $A(25) = 1250$ is $\boxed{\text{1250 square feet}}$.

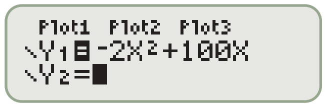
STEP 5 Check the solution.
Two sides are 25 feet and one side is 50 feet, and together they account for all 100 feet of fence.

■ **YOUR TURN** Suppose you have 200 linear feet of fence to enclose a rectangular garden. Determine the dimensions of the rectangle that will yield the greatest area.

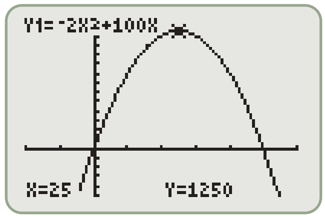
Technology Tip

Use a graphing utility to graph the area function $A(x) = -2x^2 + 100x$.

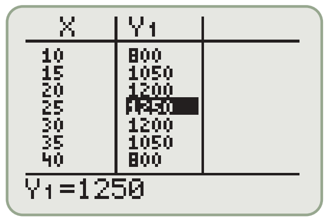




The maximum occurs when $x = 25$. The maximum area is $y = 1250$ sq ft.



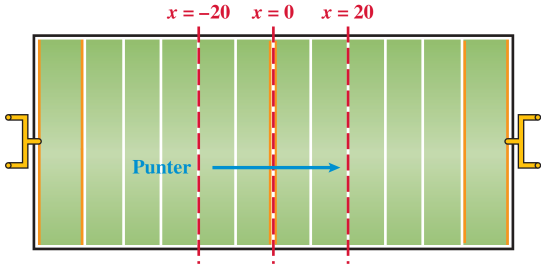
A table of values supports the solution.



■ **Answer:** 50 ft by 50 ft

EXAMPLE 9 **Path of a Punted Football**

The path of a particular punt follows the quadratic function: $h(x) = -\frac{1}{8}(x - 5)^2 + 50$, where $h(x)$ is the height of the ball in yards and x corresponds to the horizontal distance in yards. Assume $x = 0$ corresponds to midfield (the 50 yard line). For example, $x = -20$ corresponds to the punter's own 30 yard line, whereas $x = 20$ corresponds to the other team's 30 yard line.



a. Find the maximum height the ball achieves.
b. Find the horizontal distance the ball covers. Assume the height is zero when the ball is kicked and when the ball is caught.

Solution (a):

Identify the vertex since it is given in standard form. $(h, k) = (5, 50)$

The maximum height of the punt occurs at the other team's 45 yard line, and the height the ball achieves is $\boxed{\text{50 yards (150 feet)}}$.

Solution (b):

The height when the ball is kicked or caught is zero. $h(x) = -\dfrac{1}{8}(x - 5)^2 + 50 = 0$

Solve for x.
$$\frac{1}{8}(x - 5)^2 = 50$$
$$(x - 5)^2 = 400$$
$$(x - 5) = \pm\sqrt{400}$$
$$x = 5 \pm 20$$
$$x = -15 \quad \text{and} \quad x = 25$$

The horizontal distance is the distance between these two points: $|25 - (-15)| = \boxed{40 \text{ yd}}$.

SECTION 2.1 SUMMARY

All quadratic functions $f(x) = ax^2 + bx + c$ or $f(x) = a(x - h)^2 + k$ have graphs that are parabolas:

- If $a > 0$, the parabola opens up.
- If $a < 0$, the parabola opens down.
- The vertex is at the point

$$(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) = \left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

- When the quadratic function is given in general form, completing the square can be used to rewrite the function in standard form.
- At least three points are needed to graph a quadratic function:
  - vertex
  - y-intercept
  - x-intercept(s) or other point(s)

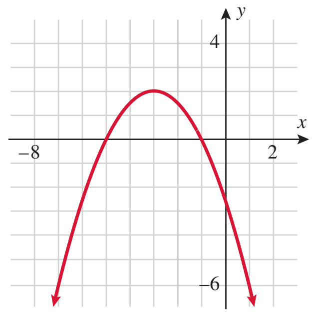
SECTION 2.1 EXERCISES

SKILLS

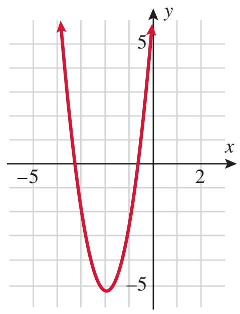
In Exercises 1–4, match the quadratic function with its graph.

1. $f(x) = 3(x + 2)^2 - 5$ **2.** $f(x) = 2(x - 1)^2 + 3$ **3.** $f(x) = -\frac{1}{2}(x + 3)^2 + 2$ **4.** $f(x) = -\frac{1}{3}(x - 2)^2 + 3$

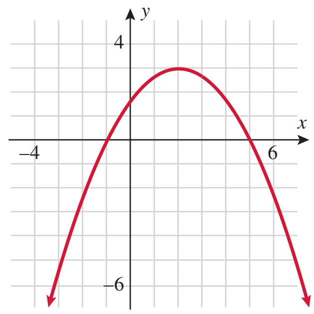
a.



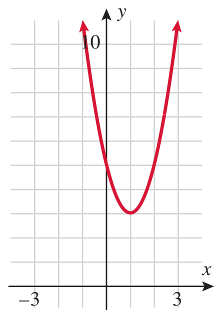
b.



c.



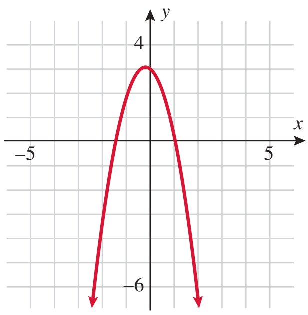
d.



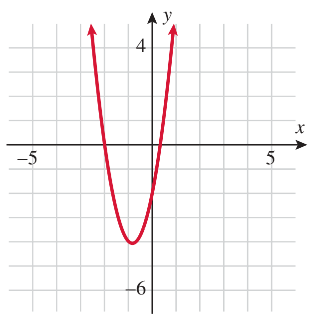
In Exercises 5–8, match the quadratic function with its graph.

5. $f(x) = 3x^2 + 5x - 2$ **6.** $f(x) = 3x^2 - x - 2$ **7.** $f(x) = -x^2 + 2x - 1$ **8.** $f(x) = -2x^2 - x + 3$

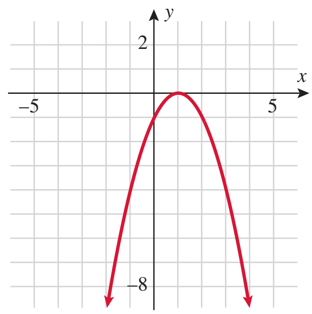
a.



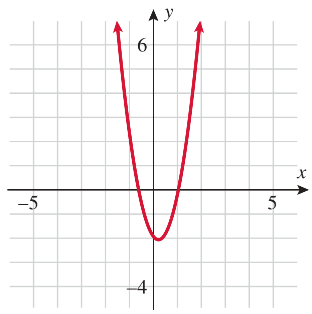
b.



c.



d.



In Exercises 9–22, graph the quadratic function, which is given in standard form.

9. $f(x) = (x + 1)^2 - 2$
10. $f(x) = (x + 2)^2 - 1$
11. $f(x) = (x - 2)^2 - 3$
12. $f(x) = (x - 4)^2 + 2$
13. $f(x) = -(x - 3)^2 + 9$
14. $f(x) = -(x - 5)^2 - 4$
15. $f(x) = -(x + 1)^2 - 3$
16. $f(x) = -(x - 2)^2 + 6$
17. $f(x) = 2(x - 2)^2 + 2$
18. $f(x) = -3(x + 2)^2 - 15$
19. $f(x) = \left(x - \frac{1}{3}\right)^2 + \frac{1}{9}$
20. $f(x) = \left(x + \frac{1}{4}\right)^2 - \frac{1}{2}$
21. $f(x) = -0.5(x - 0.25)^2 + 0.75$
22. $f(x) = -0.2(x + 0.6)^2 + 0.8$

In Exercises 23–34, rewrite the quadratic function in standard form by completing the square.

23. $f(x) = x^2 + 6x - 3$
24. $f(x) = x^2 + 8x + 2$
25. $f(x) = -x^2 - 10x + 3$
26. $f(x) = -x^2 - 12x + 6$
27. $f(x) = 2x^2 + 8x - 2$
28. $f(x) = 3x^2 - 9x + 11$
29. $f(x) = -4x^2 + 16x - 7$
30. $f(x) = -5x^2 + 100x - 36$
31. $f(x) = x^2 + 10x$
32. $f(x) = -4x^2 + 12x - 2$
33. $f(x) = \frac{1}{2}x^2 - 4x + 3$
34. $f(x) = -\frac{1}{3}x^2 + 6x + 4$

In Exercises 35–44, graph the quadratic function.

35. $f(x) = x^2 + 6x - 7$

36. $f(x) = x^2 - 3x + 10$

37. $f(x) = -x^2 - 5x + 6$

38. $f(x) = -x^2 + 3x + 4$

39. $f(x) = 4x^2 - 5x + 10$

40. $f(x) = 3x^2 + 9x - 1$

41. $f(x) = -2x^2 - 12x - 16$

42. $f(x) = -3x^2 + 12x - 12$

43. $f(x) = \frac{1}{2}x^2 - \frac{1}{2}$

44. $f(x) = -\frac{1}{3}x^2 + \frac{4}{3}$

In Exercises 45–54, find the vertex of the parabola associated with each quadratic function.

45. $f(x) = 33x^2 - 2x + 15$

46. $f(x) = 17x^2 + 4x - 3$

47. $f(x) = \frac{1}{2}x^2 - 7x + 5$

48. $f(x) = -\frac{1}{3}x^2 + \frac{2}{5}x + 4$

49. $f(x) = -\frac{2}{5}x^2 + \frac{3}{7}x + 2$

50. $f(x) = -\frac{1}{7}x^2 - \frac{2}{3}x + \frac{1}{9}$

51. $f(x) = -0.002x^2 - 0.3x + 1.7$

52. $f(x) = 0.05x^2 + 2.5x - 1.5$

53. $f(x) = 0.06x^2 - 2.6x + 3.52$

54. $f(x) = -3.2x^2 + 0.8x - 0.14$

In Exercises 55–66, find the quadratic function that has the given vertex and goes through the given point.

55. vertex: $(-1, 4)$ point: $(0, 2)$

56. vertex: $(2, -3)$ point: $(0, 1)$

57. vertex: $(2, 5)$ point: $(3, 0)$

58. vertex: $(1, 3)$ point: $(-2, 0)$

59. vertex: $(-1, -3)$ point: $(-4, 2)$

60. vertex: $(0, -2)$ point: $(3, 10)$

61. vertex: $(-2, -4)$ point: $(-1, 6)$

62. vertex: $(5, 4)$ point: $(2, -5)$

63. vertex: $\left(\frac{1}{2}, -\frac{3}{4}\right)$ point: $\left(\frac{3}{4}, 0\right)$

64. vertex: $\left(-\frac{5}{6}, \frac{2}{3}\right)$ point: $(0, 0)$

65. vertex: $(2.5, -3.5)$ point: $(4.5, 1.5)$

66. vertex: $(1.8, 2.7)$ point: $(-2.2, -2.1)$

APPLICATIONS

67. Business. The annual profit for a company that manufactures cell phone accessories can be modeled by the function

$$P(x) = -0.0001x^2 + 70x + 12{,}500$$

where x is the number of units sold and P is the total profit in dollars.

a. What sales level maximizes the company's annual profit?

b. Find the maximum annual profit for the company.

68. Business. A manufacturer of office supplies has daily production costs of

$$C(x) = 0.5x^2 - 20x + 1600$$

where x is the number of units produced measured in thousands and C is cost in hundreds of dollars.

a. What production level will minimize the manufacturer's daily production costs?

b. Find the minimum daily production costs for the manufacturer.

For Exercises 69 and 70, refer to the following:

An adult male's weight, in kilograms, can be modeled by the function

$$W(t) = -\frac{2}{3}t^2 + \frac{13}{5}t + \frac{433}{5}; \quad 1 \le t \le 18$$

where t measures months ($t = 1$ is January 2010, $t = 2$ is February 2010, etc.) and W is the male's weight.

69. Health/Medicine. During which months was the male losing weight and gaining weight?

70. Health/Medicine. Find the maximum weight to the nearest kilogram of the adult male during the 18 months.

Exercises 71 and 72 concern the path of a punted football. Refer to the diagram in Example 9.

71. Sports. The path of a particular punt follows the quadratic function

$$h(x) = -\frac{8}{125}(x + 5)^2 + 40$$

where $h(x)$ is the height of the ball in yards and x corresponds to the horizontal distance in yards. Assume $x = 0$ corresponds to midfield (the 50 yard line). For example, $x = -20$ corresponds to the punter's own 30 yard line, whereas $x = 20$ corresponds to the other team's 30 yard line.

a. Find the maximum height the ball achieves.

b. Find the horizontal distance the ball covers. Assume the height is zero when the ball is kicked and when the ball is caught.

72. Sports. The path of a particular punt follows the quadratic function

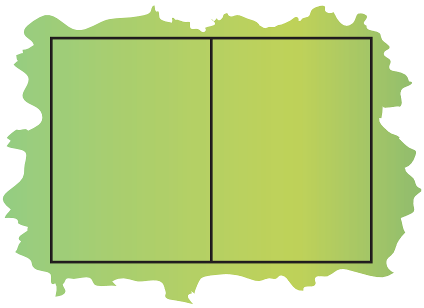
$$h(x) = -\frac{5}{40}(x - 30)^2 + 50$$

where $h(x)$ is the height of the ball in yards and x corresponds to the horizontal distance in yards. Assume $x = 0$ corresponds to midfield (the 50 yard line). For example, $x = -20$ corresponds to the punter's own 30 yard line, whereas $x = 20$ corresponds to the other team's 30 yard line.

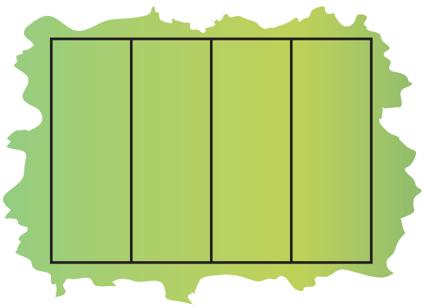
a. Find the maximum height the ball achieves.

b. Find the horizontal distance the ball covers. Assume the height is zero when the ball is kicked and when the ball is caught.

73. Ranching. A rancher has 10,000 linear feet of fencing and wants to enclose a rectangular field and then divide it into two equal pastures with an internal fence parallel to one of the rectangular sides. What is the maximum area of each pasture? Round to the nearest square foot.



74. Ranching. A rancher has 30,000 linear feet of fencing and wants to enclose a rectangular field and then divide it into four equal pastures with three internal fences parallel to one of the rectangular sides. What is the maximum area of each pasture?



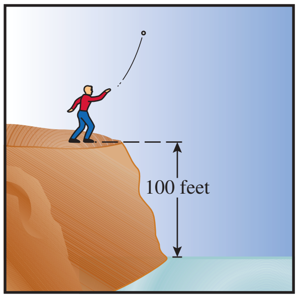
75. Gravity. A person standing near the edge of a cliff 100 feet above a lake throws a rock upward with an initial speed of 32 feet per second. The height of the rock above the lake at the bottom of the cliff is a function of time and is described by

$$h(t) = -16t^2 + 32t + 100$$

a. How many seconds will it take until the rock reaches its maximum height? What is that height?

b. At what time will the rock hit the water?

c. Over what time interval is the rock higher than the cliff?



76. Gravity. A person holds a pistol straight upward and fires. The initial velocity of most bullets is around 1200 feet per second. The height of the bullet is a function of time and is described by

$$h(t) = -16t^2 + 1200t$$

How long, after the gun is fired, does the person have to get out of the way of the bullet falling from the sky?

77. Zero Gravity. As part of their training, astronauts ride the "vomit comet," NASA's reduced gravity KC 135A aircraft that performs parabolic flights to simulate weightlessness. The plane starts at an altitude of 20,000 feet and makes a steep climb at 52° with the horizon for 20–25 seconds and then dives at that same angle back down, repeatedly. The equation governing the altitude of the flight is

$$A(x) = -0.0003x^2 + 9.3x - 46{,}075$$

where $A(x)$ is altitude and x is horizontal distance in feet.

a. What is the maximum altitude the plane attains?

b. Over what horizontal distance is the entire maneuver performed? (Assume the starting and ending altitude is 20,000 feet.)

Courtesy NASA

NASA's "Vomit Comet"

78. Sports. A soccer ball is kicked from the ground at a 45° angle with an initial velocity of 40 feet per second. The height of the soccer ball above the ground is given by $H(x) = -0.0128x^2 + x$, where x is the horizontal distance the ball travels.

a. What is the maximum height the ball reaches?

b. What is the horizontal distance the ball travels?

79. Profit. A small company in Virginia Beach manufactures handcrafted surfboards. The profit of selling x boards is given by

$$P(x) = 20{,}000 + 80x - 0.4x^2$$

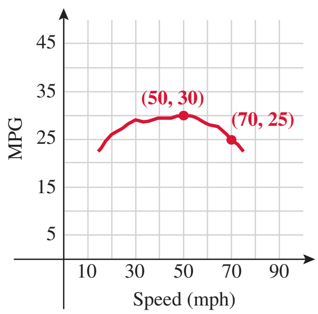
a. How many boards should be made to maximize the profit?

b. What is the maximum profit?

80. Environment: Fuel Economy. Gas mileage (miles per gallon, mpg) can be approximated by a quadratic function of speed. For a particular automobile, assume the vertex occurs when the speed is 50 miles per hour (the mpg will be 30).

a. Write a quadratic function that models this relationship, assuming 70 miles per hour corresponds to 25 mpg.

b. What gas mileage would you expect for this car driving 90 miles per hour?



For Exercises 81 and 82, use the following information:

One function of particular interest in economics is the **profit function**. We denote this function by $P(x)$. It is defined to be the difference between revenue $R(x)$ and cost $C(x)$ so that

$$P(x) = R(x) - C(x)$$

The total revenue received from the sale of x goods at price p is given by

$$R(x) = px$$

The total cost function relates the cost of production to the level of output x. This includes both fixed costs C_f and variable costs C_v (costs per unit produced). The total cost in producing x goods is given by

$$C(x) = C_f + C_v x$$

Thus, the profit function is

$$P(x) = px - C_f - C_v x$$

Assume fixed costs are \$1000, variable costs per unit are \$20, and the demand function is

$$p = 100 - x$$

81. Profit. How many units should the company produce to break even?

82. Profit. What is the maximum profit?

83. Cell Phones. The number of cell phones in the United States can be approximated by a quadratic function. In 1996 there were approximately 16 million cell phones, and in 2005 there were approximately 100 million. Let t be the number of years since 1996. The number of cell phones in 1996 is represented by (0, 16), and the number in 2005 is (9, 100). Let (0, 16) be the vertex.

a. Find a quadratic function that represents the number of cell phones.

b. Based on this model, how many cell phones will be in use in 2010?

84. Underage Smoking. The number of underage cigarette smokers (ages 10–17) has declined in the United States. The peak percent was in 1998 at 49%. In 2006 this had dropped to 36%. Let t be time in years after 1998 ($t = 0$ corresponds to 1998).

a. Find a quadratic function that models the percent of underage smokers as a function of time. Let (0, 49) be the vertex.

b. Now that you have the model, predict the percent of underage smokers in 2010.

85. Drug Concentration. The concentration of a drug in the bloodstream, measured in parts per million, can be modeled with a quadratic function. In 50 minutes the concentration is 93.75 parts per million. The maximum concentration of the drug in the bloodstream occurs in 225 minutes and is 400 parts per million.

a. Find a quadratic function that models the concentration of the drug as a function of time in minutes.

b. After the concentration peaks, eventually the drug will be eliminated from the body. How many minutes will it take until the concentration finally reaches 0?

86. Revenue. Jeff operates a mobile car washing business. When he charged \$20 a car, he washed 70 cars a month. He raised the price to \$25 a car and his business dropped to 50 cars a month.

a. Find a linear function that represents the demand equation (the price per car as a function of the number of cars washed).

b. Find the revenue function $R(x) = xp$.

c. How many cars should he wash to maximize the revenue?

d. What price should he charge to maximize revenue?

CATCH THE MISTAKE

In Exercises 87–90, explain the mistake that is made. There may be a single mistake or there may be more than one mistake.

87. Plot the quadratic function $f(x) = (x + 3)^2 - 1$.

Solution:

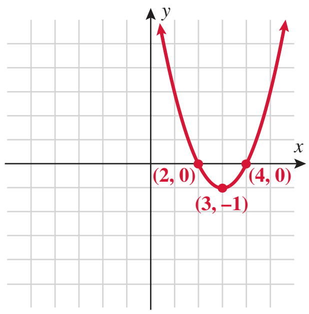
Step 1: The parabola opens up because $a = 1 > 0$.

Step 2: The vertex is $(3, -1)$.

Step 3: The y-intercept is $(0, 8)$.

Step 4: The x-intercepts are $(2, 0)$ and $(4, 0)$.

Step 5: Plot the vertex and intercepts, and connect the points with a smooth curve.



This is incorrect. What mistake(s) was made?

88. Determine the vertex of the quadratic function $f(x) = 2x^2 - 6x - 18$.

Solution:

Step 1: The vertex is given by $(h, k) = \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

In this case, $a = 2$ and $b = 6$.

Step 2: The x-coordinate of the vertex is

$$x = -\frac{6}{2(2)} = -\frac{6}{4} = -\frac{3}{2}$$

Step 3: The y-coordinate of the vertex is

$$f\left(-\frac{3}{2}\right) = -2\left(-\frac{3}{2}\right)^2 + 6\left(-\frac{3}{2}\right) - 18$$
$$= -2\left(\frac{9}{4}\right) - \frac{18}{2} - 18$$
$$= -\frac{9}{2} - 9 - 18$$
$$= -\frac{63}{2}$$

This is incorrect. What mistake(s) was made?

89. Rewrite the following quadratic function in standard form:

$$f(x) = -x^2 + 2x + 3$$

Solution:

Step 1: Group the variables together. $(-x^2 + 2x) + 3$

Step 2: Factor out a negative. $-(x^2 + 2x) + 3$

Step 3: Add and subtract 1 inside the parentheses. $-(x^2 + 2x + 1 - 1) + 3$

Step 4: Factor out the -1. $-(x^2 + 2x + 1) + 1 + 3$

Step 5: Simplify. $-(x + 1)^2 + 4$

This is incorrect. What mistake(s) was made?

90. Find the quadratic function whose vertex is $(2, -3)$ and whose graph passes through the point $(9, 0)$.

Solution:

Step 1: Write the quadratic function in standard form. $f(x) = a(x - h)^2 + k$

Step 2: Substitute $(h, k) = (2, -3)$. $f(x) = a(x - 2)^2 - 3$

Step 3: Substitute the point $(9, 0)$ and solve for a.

$$f(0) = a(0 - 2)^2 - 3 = 9$$
$$4a - 3 = 9$$
$$4a = 12$$
$$a = 3$$

The quadratic function sought is $f(x) = 3(x - 2)^2 - 3$.

This is incorrect. What mistake(s) was made?

CONCEPTUAL

In Exercises 91–94, determine whether each statement is true or false.

91. A quadratic function must have a y-intercept.

92. A quadratic function must have an x-intercept.

93. A quadratic function may have more than one y-intercept.

94. A quadratic function may have more than one x-intercept.

95. For the general quadratic equation, $f(x) = ax^2 + bx + c$, show that the vertex is $(h, k) = \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

96. Given the quadratic function $f(x) = a(x - h)^2 + k$, determine the x- and y-intercepts in terms of a, h, and k.

CHALLENGE

97. A rancher has 1000 feet of fence to enclose a pasture.

a. Determine the maximum area if a rectangular fence is used.

b. Determine the maximum area if a circular fence is used.

98. A 600-room hotel in Orlando is filled to capacity every night when the rate is \$90 per night. For every \$5 increase in the rate, 10 fewer rooms are filled. How much should the hotel charge to produce the maximum income? What is its maximum income?

99. The speed of the river current is $\dfrac{1}{x+4}$ mph. In quiet waters, the speed of a swimmer is $\dfrac{1}{x+11}$ mph. When the swimmer swims down the river, her speed is $\frac{25}{144}$ mph. What is the value of x?

100. When a rectangle is reduced 25% (length and width each reduced by 25%), the new length equals the original width. Find the dimensions of the original rectangle given that the area of the reduced rectangle is 36 sq ft.

TECHNOLOGY

101. On a graphing calculator, plot the quadratic function $f(x) = -0.002x^2 + 5.7x - 23$.

a. Identify the vertex of this parabola.

b. Identify the y-intercept.

c. Identify the x-intercepts (if any).

d. What is the axis of symmetry?

102. Determine the quadratic function whose vertex is $(-0.5, 1.7)$ and whose graph passes through the point $(0, 4)$.

a. Write the quadratic function in general form.

b. Plot this quadratic function with a graphing calculator.

c. Zoom in on the vertex and y-intercept. Do they agree with the given values?

In Exercises 103 and 104, (a) use the calculator commands STAT QuadReg to model the data using a quadratic function; (b) write the quadratic function in standard form and identify the vertex; (c) plot this quadratic function with a graphing calculator and use the TRACE key to highlight the given points. Do they agree with the given values?

103.

x	-2	2	5
y	-29.28	21.92	18.32

104.

x	-9	-2	4
y	-2.72	-16.18	6.62

For Exercises 105 and 106, refer to the following discussion of quadratic regression:

The "least-squares" criterion used to create a *quadratic regression* curve $y = ax^2 + bx + c$ that fits a set of n data points (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n) is that the sum of the squares of the vertical distances from the points to the curve be minimum. This means that we need to determine values of a, b, and c for which

$\sum_{i=1}^{n}(y_i - (ax_i^2 + bx_i + c))^2$ is as small as possible. Calculus can be used to determine formulas for a, b, and c that do the job, but computing them by hand is tedious and unnecessary because the TI-83+ has a built-in program called *QuadReg* that does this. In fact, this was introduced in Section 2.5, Problems 15–48. The following are application problems that involve experimental data for which the best fit curve is a parabola.

Projectile Motion

It is well known that the trajectory of an object thrown with initial velocity v_0 from an initial height s_0 is described by the quadratic function $s(t) = -16t^2 + v_0t + s_0$. If we have data points obtained in such a context, we can apply the procedure outlined in Section 2.5 with *QuadReg* in place of *LinReg*$(ax+b)$ to find a best fit *parabola* of the form $y = ax^2 + bx + c$.

Each year during Halloween season, it is tradition to hold the Pumpkin Launching Contest where students literally hurl their pumpkins in the hope of throwing them the farthest horizontal distance. Ben claims that it is better to use a steeper trajectory since it will have more air time, while Rick believes in throwing the pumpkin with all of his might, but with less inclination. The following data points that describe the pumpkin's horizontal x and vertical y distances (measured in feet) are collected during the flights of their pumpkins:

Ben's Data		Rick's Data	
x	y	x	y
0	4	0	4
1	14.2	1	8.5
2	28.4	2	10.6
3	30.1	3	13.3
4	35.9	4	16.2
5	37.8	5	17.3
6	41.1	6	19.3
7	38.2	7	19.5

105. **a.** Form a scatterplot for Ben's data.

b. Determine the equation of the best fit parabola and report the value of the associated correlation coefficient.

c. Use the best fit curve from (b) to answer the following:

i. What is the initial height of the pumpkin's trajectory and with what initial velocity was it thrown?

ii. What is the maximum height of Ben's pumpkin's trajectory?

iii. How much horizontal distance has the pumpkin traveled by the time it lands?

106. Repeat Exercise 105 for Rick's data.

PREVIEW TO CALCULUS

Parabolas, ellipses, and hyperbolas form a family of curves called conic sections. These curves are studied later in Chapter 9 and in calculus. The general equation of each curve is given below:

Parabola: $(x - h)^2 = 4p(y - k)$

Ellipse: $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Hyperbola: $$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

In Exercises 107–110, write the general equation of each conic section and identify the curve.

107. $4x^2 + 9y^2 = 36y$ **108.** $x^2 + 16y = 4y^2 + 2x + 19$ **109.** $x^2 + 6x - 20y + 5 = 0$ **110.** $x^2 + 105 = 6x - 40y + 4y^2$

SECTION 2.2 POLYNOMIAL FUNCTIONS OF HIGHER DEGREE

SKILLS OBJECTIVES

- Graph polynomial functions using transformations.
- Identify real zeros of a polynomial function and their multiplicities.
- Determine the end behavior of a polynomial function.
- Sketch graphs of polynomial functions using intercepts and end behavior.

CONCEPTUAL OBJECTIVES

- Understand that real zeros of polynomial functions correspond to x-intercepts.
- Understand the intermediate value theorem and how it assists in graphing polynomial functions.
- Realize that end behavior is a result of the leading term dominating.

Identifying Polynomial Functions

DEFINITION **Polynomial Function**

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of x with degree n**. The coefficient a_n is called the leading coefficient.

EXAMPLE 1 **Identifying Polynomials and Their Degree**

For each of the functions given, determine whether the function is a polynomial function. If it is a polynomial function, then state the degree of the polynomial. If it is not a polynomial function, justify your answer.

a. $f(x) = 3 - 2x^5$ **b.** $F(x) = \sqrt{x} + 1$ **c.** $g(x) = 2$

d. $h(x) = 3x^2 - 2x + 5$ **e.** $H(x) = 4x^5(2x - 3)^2$ **f.** $G(x) = 2x^4 - 5x^3 - 4x^{-2}$

Solution:

a. $f(x)$ is a polynomial function of degree 5.

b. $F(x)$ is not a polynomial function. The variable x is raised to the power of $\frac{1}{2}$, which is not an integer.

c. $g(x)$ is a polynomial function of degree zero, also known as a constant function. Note that $g(x) = 2$ can also be written as $g(x) = 2x^0$ (assuming $x \neq 0$).

d. $h(x)$ is a polynomial function of degree 2. A polynomial function of degree 2 is called a quadratic function.

e. $H(x)$ is a polynomial function of degree 7. *Note:* $4x^5\left(4x^2 - 12x + 9\right) = 16x^7 - 48x^6 + 36x^5$.

f. $G(x)$ is not a polynomial function. $-4x^{-2}$ has an exponent that is negative.

YOUR TURN For each of the functions given, determine whether the function is a polynomial function. If it is a polynomial function, then state the degree of the polynomial. If it is not a polynomial function, justify your answer.

a. $f(x) = \dfrac{1}{x} + 2$ **b.** $g(x) = 3x^8(x - 2)^2(x + 1)^3$

Answer:
a. $f(x)$ is not a polynomial because x is raised to the power of -1, which is a negative integer.
b. $g(x)$ is a polynomial of degree 13.

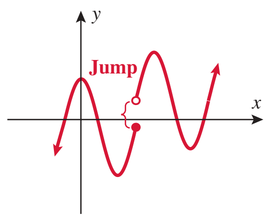
Whenever we have discussed a particular polynomial function of degree 0, 1, or 2, we have graphed it too. These functions are summarized in the table below.

POLYNOMIAL	DEGREE	SPECIAL NAME	GRAPH
$f(x) = c$	0	Constant function	Horizontal line
$f(x) = mx + b$	1	Linear function	Line • Slope $= m$ • y-intercept: $(0, b)$
$f(x) = ax^2 + bx + c$	2	Quadratic function	Parabola • Opens up if $a > 0$. • Opens down if $a < 0$.

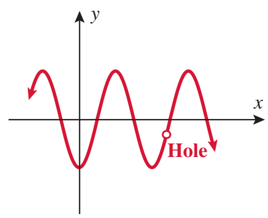
How do we graph polynomial functions that are of degree 3 or higher, and why do we care? Polynomial functions model real-world applications. One example is the percentage of fat in our bodies as we age. We can model the weight of a baby after it comes home from the hospital as a function of time. When a baby comes home from the hospital, it usually experiences weight loss. Then typically there is an increase in the percent of body fat when the baby is nursing. When infants start to walk, the increase in exercise is associated with a drop in the percentage of fat. Growth spurts in children are examples of the percent of body fat increasing and decreasing. Later in life, our metabolism slows down, and typically, the percent of body fat increases. We will model this with a polynomial function. Other examples are stock prices, the federal funds rate, and yo-yo dieting as functions of time.

Graphs of all polynomial functions are both *continuous* and *smooth.* A **continuous** graph is one you can draw completely without picking up your pencil (the graph has no jumps or holes). A **smooth** graph has no sharp corners. The following graphs illustrate what it means to be smooth (no sharp corners or cusps) and continuous (no holes or jumps).

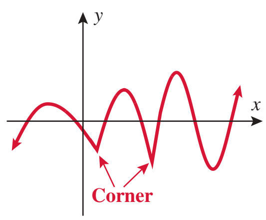
The graph is *not continuous.*



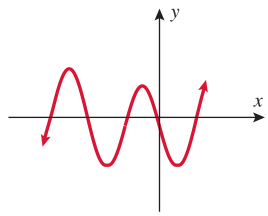
The graph is *not continuous.*



The graph is *continuous* but *not smooth.*



The graph is *continuous* and *smooth.*



Study Tip

All polynomial functions have graphs that are both continuous and smooth.

Graphing Polynomial Functions Using Transformations of Power Functions

Recall from Chapter 1 that graphs of functions can be drawn by hand using graphing aids such as intercepts and symmetry. The graphs of polynomial functions can be graphed using these same aids. Let's start with the simplest types of polynomial functions, called

power functions. Power functions are monomial functions (Appendix) of the form $f(x) = x^n$, where n is a positive integer.

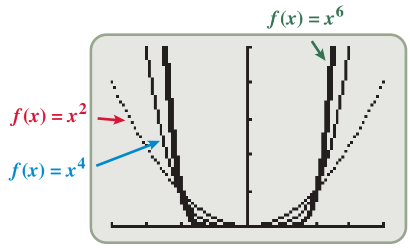
DEFINITION Power Function

Let n be a positive integer and the coefficient $a \neq 0$ be a real number. The function

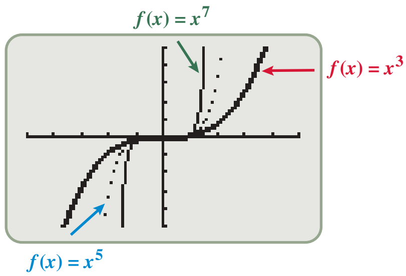
$$f(x) = ax^n$$

is called a **power function of degree n**.

Power functions with *even* powers look similar to the square function.



Power functions with *odd* powers (other than $n = 1$) look similar to the cube function.



All even power functions have similar characteristics to a quadratic function (parabola), and all odd ($n > 1$) power functions have similar characteristics to a cubic function. For example, all even functions are symmetric with respect to the y-axis, whereas all odd functions are symmetric with respect to the origin. This table summarizes their characteristics.

CHARACTERISTICS OF POWER FUNCTIONS: $f(x) = x^n$

	n Even	n Odd
Symmetry	y-axis	Origin
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[0, \infty)$	$(-\infty, \infty)$
Some key points that lie on the graph	$(-1, 1)$, $(0, 0)$, and $(1, 1)$	$(-1, -1)$, $(0, 0)$, and $(1, 1)$
Increasing	$(0, \infty)$	$(-\infty, \infty)$
Decreasing	$(-\infty, 0)$	Nowhere

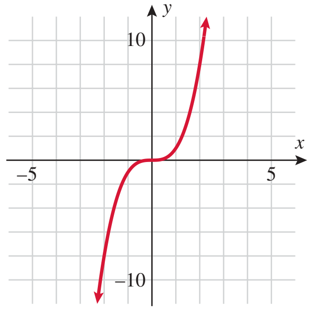
We now have the tools to graph polynomial functions that are transformations of power functions. We will use the power functions combined with our graphing techniques such as horizontal and vertical shifting and reflection (Section 1.3).

EXAMPLE 2 Graphing Transformations of Power Functions

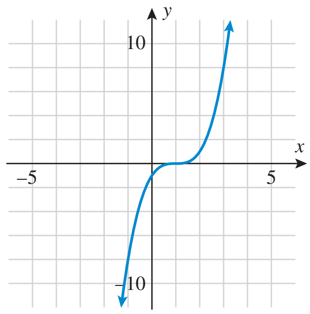
Graph the function $f(x) = (x - 1)^3$.

Solution:

STEP 1 Start with the graph of $y = x^3$.

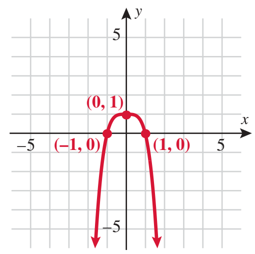


STEP 2 Shift $y = x^3$ to the right one unit to yield the graph of $f(x) = (x - 1)^3$.



■ **YOUR TURN** Graph the function $f(x) = 1 - x^4$.

■ **Answer:** $f(x) = 1 - x^4$



Real Zeros of a Polynomial Function

How do we graph general polynomial functions of degree greater than or equal to 3 if they cannot be written as transformations of power functions? We start by identifying the x-intercepts of the polynomial function. Recall that we determine the x-intercepts by setting the function equal to *zero* and solving for x. Therefore, an alternative name for an x-intercept of a function is a *zero* of the function. In our experience, to set a quadratic function equal to zero, the first step is to factor the quadratic expression into linear factors and then set each factor equal to zero. Therefore, there are four equivalent relationships that are summarized in the following box.

REAL ZEROS OF POLYNOMIAL FUNCTIONS

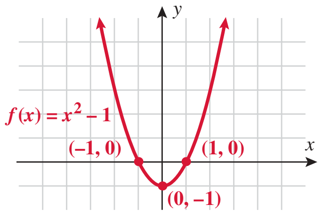
If $f(x)$ is a polynomial function and a is a *real* number, then the following statements are equivalent.

- $x = a$ is a **solution**, or **root**, of the equation $f(x) = 0$.
- $(a, 0)$ is an **x-intercept** of the graph of $f(x)$.
- $x = a$ is a **zero** of the function $f(x)$.
- $(x - a)$ is a **factor** of $f(x)$.

Study Tip

Real zeros correspond to x-intercepts.

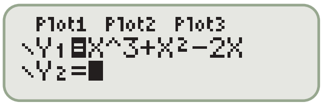
Let's use a simple polynomial function to illustrate these four relationships. We'll focus on the quadratic function $f(x) = x^2 - 1$. The graph of this function is a parabola that opens up and has as its vertex the point $(0, -1)$.

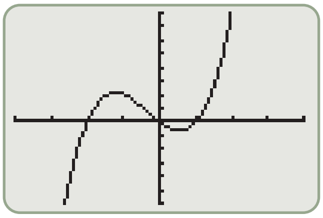
SOLUTION	X-INTERCEPT		ZERO	FACTOR
$x = -1$ and $x = 1$ are solutions, or roots, of the equation $x^2 - 1 = 0$.	The x-intercepts correspond to the points $(-1, 0)$ and $(1, 0)$.		$f(-1) = 0$ $f(1) = 0$	$f(x) = (x - 1)(x + 1)$

We have a good reason for wanting to know the x-intercepts, or zeros. When the value of a continuous function transitions from negative to positive and vice versa, it must pass through zero.

Technology Tip

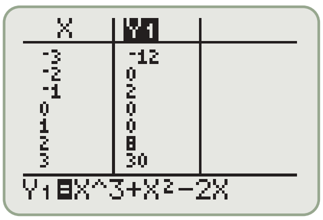
Graph $y_1 = x^3 + x^2 - 2x$.





The zeros of the function -2, 0, and 1 correspond to the x-intercepts $(-2, 0)$, $(0, 0)$, and $(1, 0)$.

The table supports the real zeros shown by the graph.



■ **Answer:** The zeros are 0, 3, and 4.

DEFINITION **Intermediate Value Theorem**

Let a and b be real numbers such that $a < b$ and let f be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero between a and b.

The **intermediate value theorem** will be used later in this chapter to assist us in finding the real zeros of a polynomial function. For now, it tells us that in order to change signs, the graph of a polynomial function must pass through the x-axis. In other words, once we know the zeros, then we know that between two consecutive zeros the graph of a polynomial function is either entirely above the x-axis or entirely below the x-axis. This enables us to break down the x-axis into intervals that we can test, which will assist us in graphing polynomial functions. Keep in mind, though, that the existence of a zero does not imply that the function will change signs—as you will see in the subsection on graphing general polynomial functions.

EXAMPLE 3 Identifying the Real Zeros of a Polynomial Function

Find the zeros of the polynomial function $f(x) = x^3 + x^2 - 2x$.

Solution:

Set the function equal to zero. $\quad x^3 + x^2 - 2x = 0$

Factor out an x common to all three terms. $\quad x(x^2 + x - 2) = 0$

Factor the quadratic expression inside the parentheses. $\quad x(x + 2)(x - 1) = 0$

Apply the zero product property. $\quad x = 0 \quad \text{or} \quad (x + 2) = 0 \quad \text{or} \quad (x - 1) = 0$

Solve. $\quad x = -2,\ x = 0, \text{ and } x = 1$

The zeros are $\boxed{-2, 0, \text{ and } 1}$.

■ **YOUR TURN** Find the zeros of the polynomial function $f(x) = x^3 - 7x^2 + 12x$.

When factoring a quadratic equation, if the factor is raised to a power greater than 1, the corresponding root, or zero, is repeated. For example, the quadratic equation $x^2 - 2x + 1 = 0$ when factored is written as $(x - 1)^2 = 0$. The solution, or root, in this case is $x = 1$, and we say that it is a **repeated** root. Similarly, when determining zeros of higher order polynomial functions, if a factor is repeated, we say that the zero is a repeated, or **multiple**, zero of the function. The number of times that a zero repeats is called its *multiplicity*.

DEFINITION **Multiplicity of a Zero**

If $(x - a)^n$ is a factor of a polynomial f, then a is called a **zero of multiplicity n** of f.

EXAMPLE 4 Finding the Multiplicities of Zeros of a Polynomial Function

Find the zeros, and state their multiplicities, of the polynomial function $g(x) = (x - 1)^2\left(x + \frac{3}{5}\right)^7(x + 5)$.

Solution:

$\boxed{\begin{array}{l} 1 \text{ is a zero of multiplicity } 2. \\ -\frac{3}{5} \text{ is a zero of multiplicity } 7. \\ -5 \text{ is a zero of multiplicity } 1. \end{array}}$

Note: Adding the multiplicities yields the degree of the polynomial. The polynomial $g(x)$ is of degree 10, since $2 + 7 + 1 = 10$.

■ **YOUR TURN** For the polynomial $h(x)$, determine the zeros and state their multiplicities.

$$h(x) = x^2(x - 2)^3\left(x + \frac{1}{2}\right)^5$$

■ **Answer:**
0 is a zero of multiplicity 2.
2 is a zero of multiplicity 3.
$-\frac{1}{2}$ is a zero of multiplicity 5.

EXAMPLE 5 Finding a Polynomial from Its Zeros

Find a polynomial of degree 7 whose zeros are

-2 (multiplicity 2) $\quad$ 0 (multiplicity 4) $\quad$ 1 (multiplicity 1)

Solution:

If $x = a$ is a zero, then $(x - a)$ is a factor.	$f(x) = (x + 2)^2(x - 0)^4(x - 1)^1$
Simplify.	$= x^4(x + 2)^2(x - 1)$
Square the binomial.	$= x^4(x^2 + 4x + 4)(x - 1)$
Multiply the two polynomials.	$= x^4(x^3 + 3x^2 - 4)$
Distribute x^4.	$= \boxed{x^7 + 3x^6 - 4x^4}$

Graphing General Polynomial Functions

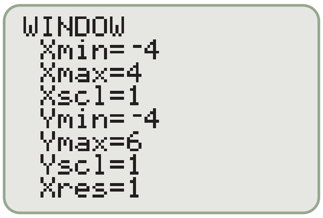
Let's develop a strategy for sketching an approximate graph of any polynomial function. First, we determine the x- and y-intercepts. Then we use the x-intercepts, or zeros, to divide the domain into intervals where the value of the polynomial is positive or negative so that we can find points in those intervals to assist in sketching a smooth and continuous graph. *Note:* It is not always possible to find x-intercepts. Some even degree polynomial functions have no x-intercepts on their graph.

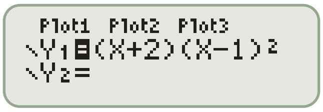
Study Tip

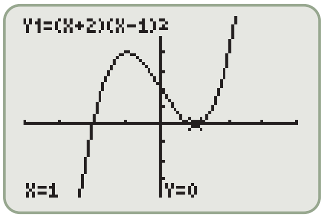
It is not always possible to find x-intercepts. Sometimes there are no x-intercepts (for some even degree polynomial functions).

Technology Tip

The graph of $f(x) = (x + 2)(x - 1)^2$ is shown.

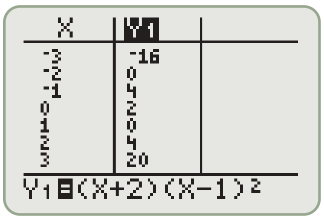






Note: The graph crosses the x-axis at the point $x = -2$ and touches the x-axis at the point $x = 1$.

A table of values supports the graph.



Study Tip

Although there may be up to n x-intercepts for the graph of a polynomial function of degree n, there will always be exactly one y-intercept.

EXAMPLE 6 Using a Strategy for Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = (x + 2)(x - 1)^2$.

Solution:

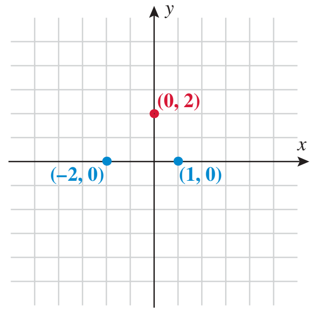
STEP 1 Find the y-intercept. (Let $x = 0$.)

$f(0) = (2)(-1)^2 = 2$
(0, 2) is the y-intercept

STEP 2 Find any x-intercepts. (Set $f(x) = 0$.)

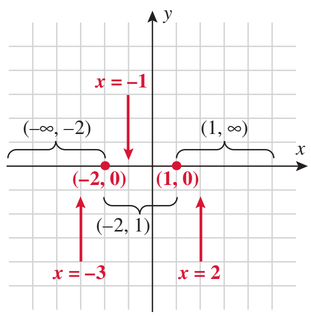
$f(x) = (x + 2)(x - 1)^2 = 0$
$x = -2$ or $x = 1$
(−2, 0) and **(1, 0)** are the x-intercepts

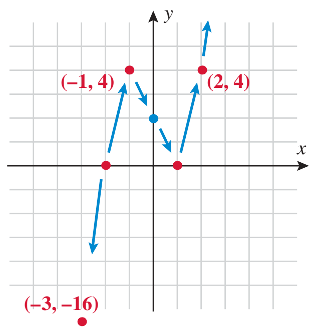
STEP 3 Plot the intercepts.



STEP 4 Divide the x-axis into intervals: $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$

STEP 5 Select a number in each interval and test each interval. The function $f(x)$ either *crosses* the x-axis at an x-intercept or *touches* the x-axis at an x-intercept. Therefore, we need to check each of these intervals to determine whether the function is positive (above the x-axis) or negative (below the x-axis). We do so by selecting numbers in the intervals and determining the value of the function at the corresponding points.

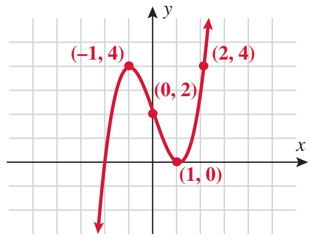




Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, \infty)$
Number Selected in Interval	-3	-1	2
Value of Function	$f(-3) = -16$	$f(-1) = 4$	$f(2) = 4$
Point on Graph	$(-3, -16)$	$(-1, 4)$	$(2, 4)$
Interval Relation to *x*-Axis	Below x-axis	Above x-axis	Above x-axis

From the table, we find three additional points on the graph: $(-3, -16)$, $(-1, 4)$, and $(2, 4)$. The point $(-2, 0)$ is an intercept where the function *crosses* the x-axis, because it is below the x-axis to the left of -2 and above the x-axis to the right of -2. The point $(1, 0)$ is an intercept where the function *touches* the x-axis, because it is above the x-axis on both sides of $x = 1$. Connecting these points with a smooth curve yields the graph.

STEP 6 Sketch a plot of the function.

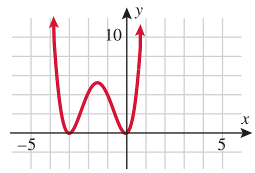


■ **YOUR TURN** Sketch the graph of $f(x) = x^2(x + 3)^2$.

Study Tip

We do not know for sure that the points $(-1, 4)$ and $(1, 0)$ are turning points. We will see later that $(1, 0)$ is a turning point because the graph touches the x-axis at $(1, 0)$, but a graphing utility suggests that $(-1, 4)$ is a turning point, and later in calculus you will learn how to find relative maximum points and relative minimum points.

■ **Answer:**



In Example 6, we found that the function crosses the x-axis at the point $(-2, 0)$. Note that -2 is a zero of multiplicity 1. We also found that the function touches the x-axis at the point $(1, 0)$. Note that 1 is a zero of multiplicity 2. In general, zeros with even multiplicity correspond to intercepts where the function touches the x-axis, and zeros with odd multiplicity correspond to intercepts where the function crosses the x-axis.

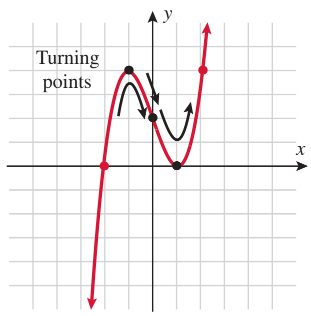
MULTIPLICITY OF A ZERO AND RELATION TO THE GRAPH OF A POLYNOMIAL FUNCTION

If a is a zero of $f(x)$, then

MULTIPLICITY OF a	$f(x)$ ON EITHER SIDE OF $x = a$	GRAPH OF FUNCTION AT THE INTERCEPT
Even	Does not change sign	Touches the x-axis (turns around) at point $(a, 0)$
Odd	Changes sign	Crosses the x-axis at point $(a, 0)$

Study Tip

In general, zeros with *even* multiplicity correspond to intercepts where the function *touches* the x-axis and zeros with *odd* multiplicity correspond to intercepts where the function *crosses* the x-axis.



Also in Example 6, we know that somewhere in the interval $(-2, 1)$ the function must reach a relative or local maximum and then turn back toward the x-axis, because both points $(-2, 0)$ and $(1, 0)$ correspond to x-intercepts. When we sketch the graph, it "appears" that the point $(-1, 4)$ is a *turning point*. The point $(1, 0)$ also corresponds to a turning point. In general, if f is a polynomial of degree n, then the graph of f has at most $n - 1$ turning points.

The point $(-1, 4)$, which we call a turning point, is also a relative or local "high point" on the graph in the vicinity of the point $(-1, 4)$. Also note that the point $(1, 0)$, which we call a turning point, is a relative or local "low point" on the graph in the vicinity of the point $(1, 0)$. We call a "high point" on a graph a **local (relative) maximum** and a "low point" on a graph a **local (relative) minimum**. For quadratic functions we can find the maximum or minimum point by finding the vertex. However, for higher degree polynomial functions, we rely on

Study Tip

If f is a polynomial of degree n, then the graph of f has at most $n - 1$ turning points.

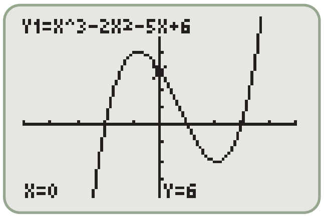
graphing utilities to assist us in locating such points. Later in calculus, techniques will be developed for finding such points exactly. For now, we use the $\boxed{\text{zoom}}$ and $\boxed{\text{trace}}$ features to locate such points on a graph, and we can use the $\boxed{\text{table}}$ feature of a graphing utility to approximate relative minima or maxima.

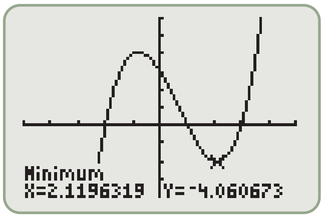
Let us take the polynomial $f(x) = x^3 - 2x^2 - 5x + 6$. Using methods discussed thus far we can find that the x-intercepts of its graph are $(-2, 0)$, $(1, 0)$, and $(3, 0)$ and the y-intercept is the point $(0, 6)$. We can also find additional points that lie on the graph such as $(-1, 8)$ and $(2, -4)$. Plotting these points, we might "think" that the points $(-1, 8)$ and $(2, -4)$ might be turning points, but a graphing utility reveals an approximate relative maximum at the point $(-0.7863, 8.2088207)$ and an approximate relative minimum at the point $(2.1196331, -4.060673)$.

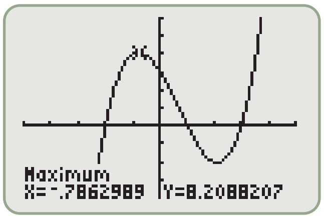
Intercepts and turning points assist us in sketching graphs of polynomial functions. Another piece of information that will assist us in graphing polynomial functions is knowledge of the *end behavior*. All polynomials eventually rise or fall without bound as x gets large in both the positive ($x \to \infty$) and negative ($x \to -\infty$) directions. The highest degree monomial within the polynomial dominates the *end behavior*. In other words, the highest power term is eventually going to overwhelm the other terms as x grows without bound.

Technology Tip

Use TI to graph the function $f(x) = x^3 - 2x^2 - 5x + 6$ as Y_1.







END BEHAVIOR

As x gets large in the positive ($x \to \infty$) and negative ($x \to -\infty$) directions, the graph of the polynomial

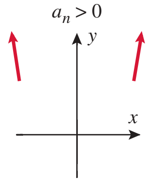
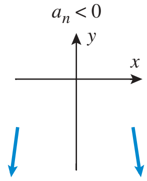
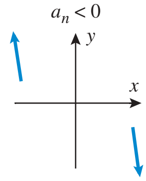
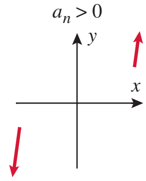
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

has the same behavior as the power function

$$y = a_n x^n$$

Power functions behave much like a quadratic function (parabola) for even-degree polynomial functions and much like a cubic function for odd-degree polynomial functions. There are four possibilities because the leading coefficient can be positive or negative with either an odd or even power.

Let $y = a_n x^n$; then

n	Even	Even	Odd	Odd
a_n	Positive	Negative	Negative	Positive
$x \to -\infty$ **(Left)**	The graph of the function *rises.*	The graph of the function *falls.*	The graph of the function *rises.*	The graph of the function *falls.*
$x \to \infty$ **(Right)**	The graph of the function *rises.*	The graph of the function *falls.*	The graph of the function *falls.*	The graph of the function *rises.*
Graph	$a_n > 0$ 	$a_n < 0$ 	$a_n < 0$ 	$a_n > 0$ 

EXAMPLE 7 Graphing a Polynomial Function

Sketch a graph of the polynomial function $f(x) = 2x^4 - 8x^2$.

Solution:

STEP 1 Determine the y-intercept: $(x = 0)$. $f(0) = 0$

The y-intercept corresponds to the point $(0, 0)$.

STEP 2 Find the zeros of the polynomial. $f(x) = 2x^4 - 8x^2$

Factor out the common $2x^2$. $= 2x^2(x^2 - 4)$

Factor the quadratic binomial. $= 2x^2(x - 2)(x + 2)$

Set $f(x) = 0$. $= 2x^2(x - 2)(x + 2) = 0$

0 is a zero of multiplicity 2. The graph will *touch* the x-axis.

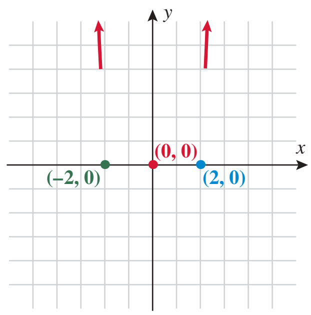
2 is a zero of multiplicity 1. The graph will *cross* the x-axis.

−2 is a zero of multiplicity 1. The graph will *cross* the x-axis.

STEP 3 Determine the end behavior. $f(x) = 2x^4 - 8x^2$ behaves like $y = 2x^4$.

$y = 2x^4$ is of even degree, and the leading coefficient is positive, so the graph rises without bound as x gets large in both the positive and negative directions.

STEP 4 Sketch the intercepts and end behavior.



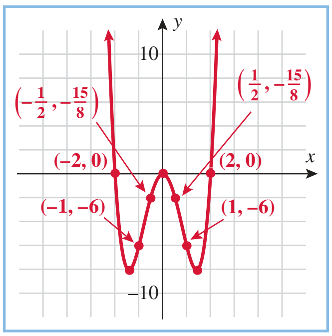
STEP 5 Find additional points.

x	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1
$f(x)$	-6	$-\frac{15}{8}$	$-\frac{15}{8}$	-6

STEP 6 Sketch the graph.

- estimate additional points
- connect with a smooth curve

Note the symmetry about the y-axis. This function is an even function: $f(-x) = f(x)$.

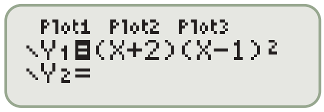


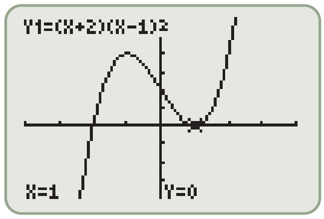
It is important to note that the absolute minimum occurs when $x = \pm\sqrt{2} \approx \pm 1.14$, but at this time can only be illustrated using a graphing utility.

YOUR TURN Sketch a graph of the polynomial function $f(x) = x^5 - 4x^3$.

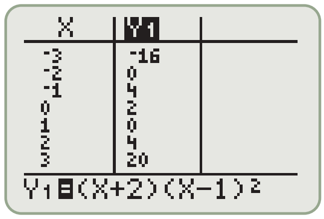
Technology Tip

The graph of $f(x) = (x + 2)(x - 1)^2$ is shown.

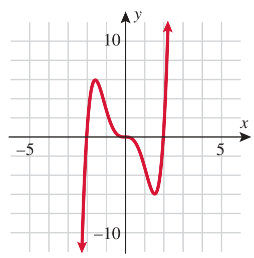




Note: The graph crosses the x-axis at the point $x = -2$ and touches the x-axis at the point $x = 1$. A table of values supports the graph.



Answer:



SECTION 2.2 SUMMARY

In general, polynomials can be graphed in one of two ways:

- Use graph-shifting techniques with power functions.
- General polynomial function.
  1. Identify intercepts.
  2. Determine each real zero and its multiplicity, and ascertain whether the graph crosses or touches the x-axis there.
  3. x-intercepts (real zeros) divide the x-axis into intervals. Test points in the intervals to determine whether the graph is above or below the x-axis.
  4. Determine the end behavior by investigating the end behavior of the highest degree monomial.
  5. Sketch the graph with a smooth curve.

SECTION 2.2 EXERCISES

SKILLS

In Exercises 1–10, determine which functions are polynomials, and for those that are, state their degree.

1. $g(x) = (x + 2)^3\left(x - \frac{3}{5}\right)^2$ **2.** $g(x) = \left(x - \frac{1}{4}\right)^4\left(x + \sqrt{7}\right)^2$ **3.** $g(x) = x^5(x + 2)(x - 6.4)$ **4.** $g(x) = x^4(x - 1)^2(x + 2.5)^3$

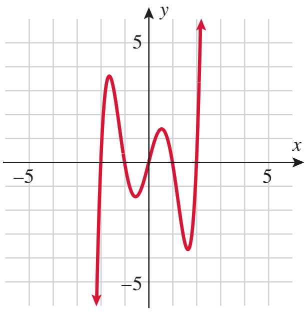
5. $h(x) = \sqrt{x} + 1$ **6.** $h(x) = (x - 1)^{1/2} + 5x$ **7.** $F(x) = x^{1/3} + 7x^2 - 2$ **8.** $F(x) = 3x^2 + 7x - \dfrac{2}{3x}$

9. $G(x) = \dfrac{x + 1}{x^2}$ **10.** $H(x) = \dfrac{x^2 + 1}{2}$

In Exercises 11–18, match the polynomial function with its graph.

11. $f(x) = -3x + 1$ **12.** $f(x) = -3x^2 - x$ **13.** $f(x) = x^2 + x$ **14.** $f(x) = -2x^3 + 4x^2 - 6x$

15. $f(x) = x^3 - x^2$ **16.** $f(x) = 2x^4 - 18x^2$ **17.** $f(x) = -x^4 + 5x^3$ **18.** $f(x) = x^5 - 5x^3 + 4x$

a.


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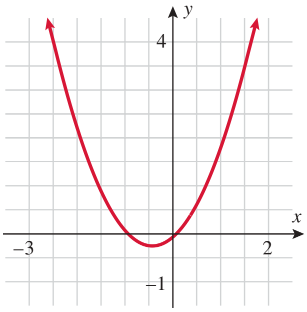

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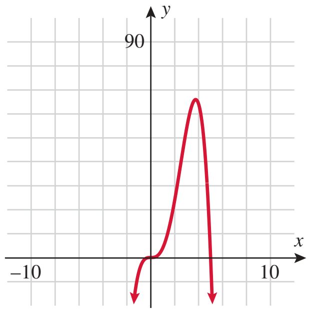

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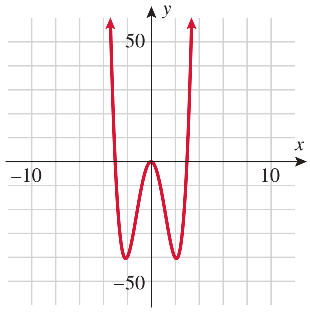

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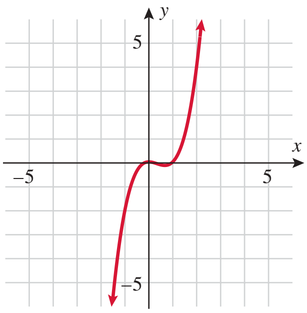

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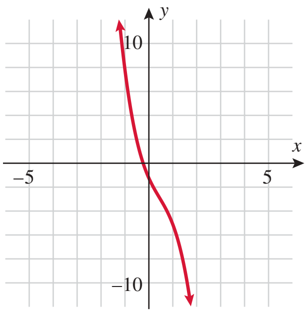

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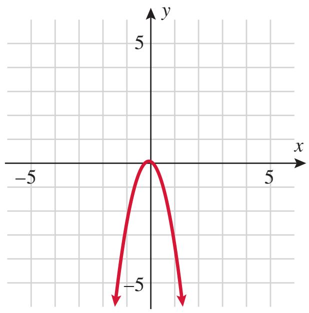

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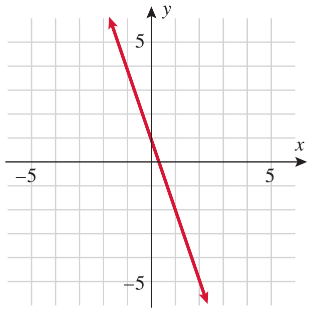


In Exercises 19–24, graph each function by transforming a power function $y = x^n$.

19. $f(x) = (x - 2)^4$

20. $f(x) = (x + 2)^5$

21. $f(x) = x^5 + 3$

22. $f(x) = -x^4 - 3$

23. $f(x) = 3 - (x + 1)^4$

24. $f(x) = (x - 3)^5 - 2$

In Exercises 25–36, find all the real zeros (and state their multiplicities) of each polynomial function.

25. $f(x) = 2(x - 3)(x + 4)^3$

26. $f(x) = -3(x + 2)^3(x - 1)^2$

27. $f(x) = 4x^2(x - 7)^2(x + 4)$

28. $f(x) = 5x^3(x + 1)^4(x - 6)$

29. $f(x) = 4x^2(x - 1)^2(x^2 + 4)$

30. $f(x) = 4x^2(x^2 - 1)(x^2 + 9)$

31. $f(x) = 8x^3 + 6x^2 - 27x$

32. $f(x) = 2x^4 + 5x^3 - 3x^2$

33. $f(x) = -2.7x^3 - 8.1x^2$

34. $f(x) = 1.2x^6 - 4.6x^4$

35. $f(x) = \frac{1}{3}x^6 + \frac{2}{5}x^4$

36. $f(x) = \frac{2}{7}x^5 - \frac{3}{4}x^4 + \frac{1}{2}x^3$

In Exercises 37–50, find a polynomial (there are many) of minimum degree that has the given zeros.

37. $-3, 0, 1, 2$

38. $-2, 0, 2$

39. $-5, -3, 0, 2, 6$

40. $0, 1, 3, 5, 10$

41. $-\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$

42. $-\frac{3}{4}, -\frac{1}{3}, 0, \frac{1}{2}$

43. $1 - \sqrt{2}, 1 + \sqrt{2}$

44. $1 - \sqrt{3}, 1 + \sqrt{3}$

45. -2 (multiplicity 3), 0 (multiplicity 2)

46. -4 (multiplicity 2), 5 (multiplicity 3)

47. -3 (multiplicity 2), 7 (multiplicity 5)

48. 0 (multiplicity 1), 10 (multiplicity 3)

49. $-\sqrt{3}$ (multiplicity 2), -1 (multiplicity 1), 0 (multiplicity 2), $\sqrt{3}$ (multiplicity 2)

50. $-\sqrt{5}$ (multiplicity 2), 0 (multiplicity 1), 1 (multiplicity 2), $\sqrt{5}$ (multiplicity 2)

In Exercises 51–68, for each polynomial function given: (a) list each real zero and its multiplicity; (b) determine whether the graph touches or crosses at each x-intercept; (c) find the y-intercept and a few points on the graph; (d) determine the end behavior; and (e) sketch the graph.

51. $f(x) = (x - 2)^3$

52. $f(x) = -(x + 3)^3$

53. $f(x) = x^3 - 9x$

54. $f(x) = -x^3 + 4x^2$

55. $f(x) = -x^3 + x^2 + 2x$

56. $f(x) = x^3 - 6x^2 + 9x$

57. $f(x) = -x^4 - 3x^3$

58. $f(x) = x^5 - x^3$

59. $f(x) = 12x^6 - 36x^5 - 48x^4$

60. $f(x) = 7x^5 - 14x^4 - 21x^3$

61. $f(x) = 2x^5 - 6x^4 - 8x^3$

62. $f(x) = -5x^4 + 10x^3 - 5x^2$

63. $f(x) = x^3 - x^2 - 4x + 4$

64. $f(x) = x^3 - x^2 - x + 1$

65. $f(x) = -(x + 2)^2(x - 1)^2$

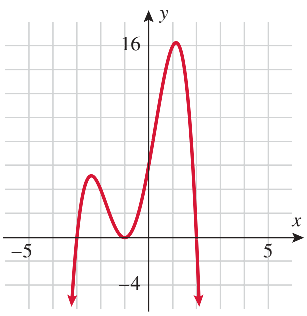
66. $f(x) = (x - 2)^3(x + 1)^3$

67. $f(x) = x^2(x - 2)^3(x + 3)^2$

68. $f(x) = -x^3(x - 4)^2(x + 2)^2$

In Exercises 69–72, for each graph given: (a) list each real zero and its smallest possible multiplicity; (b) determine whether the degree of the polynomial is even or odd; (c) determine whether the leading coefficient of the polynomial is positive or negative; (d) find the y-intercept; and (e) write an equation for the polynomial function (assume the least degree possible).

69.



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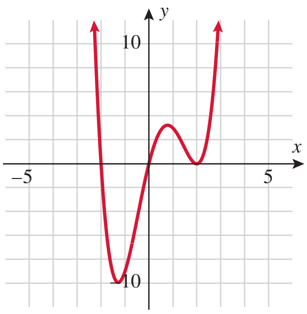

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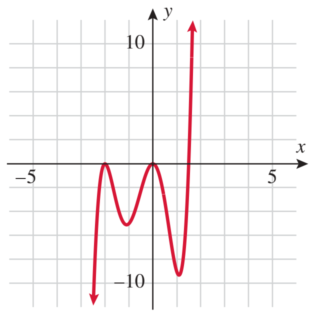

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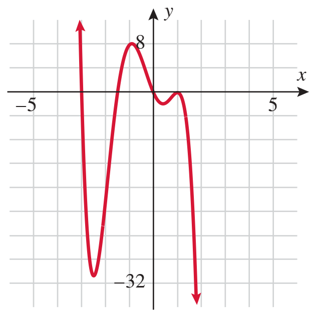
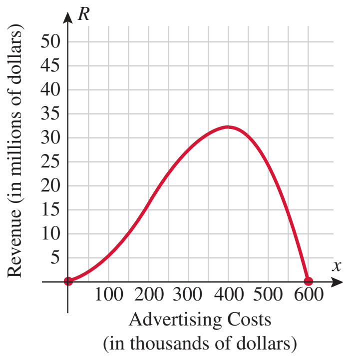


APPLICATIONS

For Exercises 73 and 74, refer to the following:

The relationship between a company's total revenue R (in millions of dollars) is related to its advertising costs x (in thousands of dollars). The relationship between revenue R and advertising costs x is illustrated in the graph.



73. Business. Analyze the graph of the revenue function.

a. Determine the intervals on which revenue is increasing and decreasing.

b. Identify the zeros of the function. Interpret the meaning of zeros for this function.

74. Business. Use the graph to identify the maximum revenue for the company and the corresponding advertising costs that produce maximum revenue.

For Exercises 75 and 76, refer to the following:

During a cough, the velocity v (in meters per second) of air in the trachea may be modeled by the function

$$v(r) = -120r^3 + 80r^2$$

where r is the radius of the trachea (in centimeters) during the cough.

75. Health/Medicine. Graph the velocity function and estimate the intervals on which the velocity of air in the trachea is increasing and decreasing.

76. Health/Medicine. Estimate the radius of the trachea that produces the maximum velocity of air in the trachea. Use this radius to estimate the maximum velocity of air in the trachea.

77. Weight. Jennifer has joined a gym to lose weight and feel better. She still likes to cheat a little and will enjoy the occasional bad meal with an ice cream dream dessert and then miss the gym for a couple of days. Given in the table is Jennifer's weight for a period of 8 months. Her weight can be modeled as a polynomial. Plot these data. How many turning points are there? Assuming these are the minimum number of turning points, what is the lowest degree polynomial that can represent Jennifer's weight?

Month	Weight
1	169
2	158
3	150
4	161
5	154
6	159
7	148
8	153

78. Stock Value. A day trader checks the stock price of Coca-Cola during a 4-hour period (given below). The price of Coca-Cola stock during this 4-hour period can be modeled as a polynomial function. Plot these data. How many turning points are there? Assuming these are the minimum number of turning points, what is the lowest degree polynomial that can represent the Coca-Cola stock price?

Period Watching Stock Market	Price
1	\$53.00
2	\$56.00
3	\$52.70
4	\$51.50

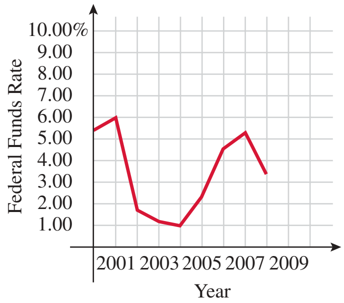
79. Stock Value. The price of Tommy Hilfiger stock during a 4-hour period is given below. If a third-degree polynomial models this stock, do you expect the stock to go up or down in the fifth period?

Period Watching Stock Market	Price
1	\$15.10
2	\$14.76
3	\$15.50
4	\$14.85

80. Stock Value. The stock prices for Coca-Cola during a 4-hour period on another day yield the following results. If a third-degree polynomial models this stock, do you expect the stock to go up or down in the fifth period?

Period Watching Stock Market	Price
1	\$52.80
2	\$53.00
3	\$56.00
4	\$52.70

For Exercises 81 and 82, the following table graph illustrates the average federal funds rate in the month of January (2000 to 2008):



81. Finance. If a polynomial function is used to model the federal funds rate data shown in the graph, determine the degree of the lowest degree polynomial that can be used to model those data.

82. Finance. Should the leading coefficient in the polynomial found in Exercise 81 be positive or negative? Explain.

83. Air Travel. An airline has a daily flight Chicago–Miami. The number of passengers per flight is given in the table below. Which would be the minimum degree of a polynomial that models the number of passengers of the airline?

Day	Passengers
Monday	180
Tuesday	150
Wednesday	175
Thursday	160
Friday	100
Saturday	98
Sunday	120

84. Air Travel. The airline in Exercise 83 discovered that the information about the number of passengers corresponding to Monday and Sunday was mixed. On Sunday, they have 180 passengers, while on Monday, they have 120 passengers. Determine the degree of the lowest degree polynomial that can be used to model those data.

85. Temperature. The weather report indicates that the daily highest temperatures for next week can be described as a cubic polynomial function. The forecasting for Tuesday, Wednesday, and Thursday is 39°, 42°, and 35°, respectively. Thursday will be the coolest day of the week. What can you say about Monday's temperature T?

86. Sports. A basketball player scored more than 20 points on each of the past 9 games.

Game	1	2	3	4	5	6	7	8	9
Points	25	27	30	28	25	24	26	27	25

a. If he scores 24 points in the next game, what is the degree of the polynomial function describing this data?

b. If he scores 26 points in the next game, what is the degree of the polynomial function describing this data?

CATCH THE MISTAKE

In Exercises 87–90, explain the mistake that is made.

87. Find a fourth-degree polynomial function with zeros -2, -1, 3, 4.

Solution: $f(x) = (x - 2)(x - 1)(x + 3)(x + 4)$

This is incorrect. What mistake was made?

88. Determine the end behavior of the polynomial function $f(x) = x(x - 2)^3$.

Solution:

This polynomial has similar end behavior to the graph of $y = x^3$.

End behavior falls to the left and rises to the right.

This is incorrect. What mistake was made?

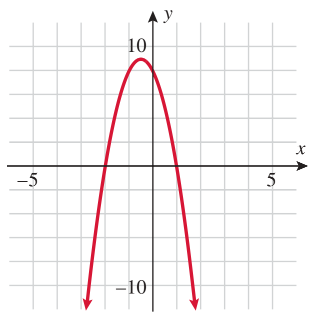
89. Graph the polynomial function $f(x) = (x - 1)^2(x + 2)^3$.

Solution:

The zeros are -2 and 1, and therefore, the x-intercepts are $(-2, 0)$ and $(1, 0)$.

The y-intercept is $(0, 8)$.

Plotting these points and connecting with a smooth curve yield the graph on the right.



This graph is incorrect. What did we forget to do?

90. Graph the polynomial function $f(x) = (x + 1)^2(x - 1)^2$.

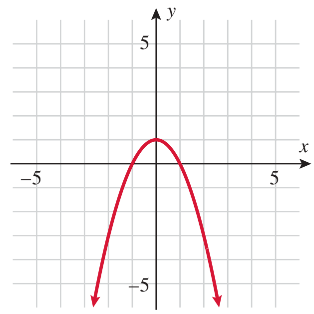
Solution:

The zeros are -1 and 1, so the x-intercepts are $(-1, 0)$ and $(1, 0)$.

The y-intercept is $(0, 1)$.

Plotting these points and connecting with a smooth curve yield the graph on the right.

This graph is incorrect. What did we forget to do?



CONCEPTUAL

In Exercises 91–94, determine whether each statement is true or false.

91. The graph of a polynomial function might not have any y-intercepts.

92. The graph of a polynomial function might not have any x-intercepts.

93. The domain of all polynomial functions is $(-\infty, \infty)$.

94. The range of all polynomial functions is $(-\infty, \infty)$.

95. What is the maximum number of zeros that a polynomial of degree n can have?

96. What is the maximum number of turning points a graph of an nth-degree polynomial can have?

CHALLENGE

97. Find a seventh-degree polynomial that has the following graph characteristics: The graph touches the x-axis at $x = -1$, and the graph crosses the x-axis at $x = 3$. Plot this polynomial function.

98. Find a fifth-degree polynomial that has the following graph characteristics: The graph touches the x-axis at $x = 0$ and crosses the x-axis at $x = 4$. Plot the polynomial function.

99. Determine the zeros of the polynomial $f(x) = x^3 + (b - a)x^2 - abx$ for the positive real numbers a and b.

100. Graph the function $f(x) = x^2(x - a)^2(x - b)^2$ for the positive real numbers a, b, where $b > a$.

TECHNOLOGY

In Exercises 101 and 102, use a graphing calculator or computer to graph each polynomial. From that graph, estimate the x-intercepts (if any). Set the function equal to zero, and solve for the zeros of the polynomial. Compare the zeros with the x-intercepts.

101. $f(x) = x^4 + 2x^2 + 1$

102. $f(x) = 1.1x^3 - 2.4x^2 + 5.2x$

For each polynomial in Exercises 103 and 104, determine the power function that has similar end behavior. Plot this power function and the polynomial. Do they have similar end behavior?

103. $f(x) = -2x^5 - 5x^4 - 3x^3$

104. $f(x) = x^4 - 6x^2 + 9$

In Exercises 105 and 106, use a graphing calculator or a computer to graph each polynomial. From the graph, estimate the x-intercepts and state the zeros of the function and their multiplicities.

105. $f(x) = x^4 - 15.9x^3 + 1.31x^2 + 292.905x + 445.7025$

106. $f(x) = -x^5 + 2.2x^4 + 18.49x^3 - 29.878x^2 - 76.5x + 100.8$

In Exercises 107 and 108, use a graphing calculator or a computer to graph each polynomial. From the graph, estimate the coordinates of the relative maximum and minimum points. Round your answers to two decimal places.

107. $f(x) = 2x^4 + 5x^3 - 10x^2 - 15x + 8$

108. $f(x) = 2x^5 - 4x^4 - 12x^3 + 18x^2 + 16x - 7$

PREVIEW TO CALCULUS

In calculus we study the extreme values of functions; in order to find these values we need to solve different types of equations.

In Exercises 109–112, use the Intermediate Value Theorem to find all the zeros of the polynomial functions in the given interval. Round all your answers to three decimal places.

109. $x^3 + 3x - 5 = 0$, $[0, 2]$

110. $x^5 - x + 0.5 = 0$, $[0, 1]$

111. $x^4 - 3x^3 + 6x^2 - 7 = 0$, $[-2, 2]$

112. $x^3 + x^2 - 2x - 2 = 0$, $[1, 2]$

SECTION 2.3 DIVIDING POLYNOMIALS

SKILLS OBJECTIVES

- Divide polynomials with long division.
- Divide polynomials with synthetic division.

CONCEPTUAL OBJECTIVES

- Extend long division of real numbers to polynomials.
- Understand *when* synthetic division can be used.

Long Division of Polynomials

Let's start with an example whose answer we already know. We know that a quadratic expression can be factored into the product of two linear factors: $x^2 + 4x - 5 = (x + 5)(x - 1)$. Therefore, if we divide both sides of the equation by $(x - 1)$, we get

$$\frac{x^2 + 4x - 5}{x - 1} = x + 5$$

We can state this by saying $x^2 + 4x - 5$ divided by $x - 1$ is equal to $x + 5$. Confirm this statement by long division:

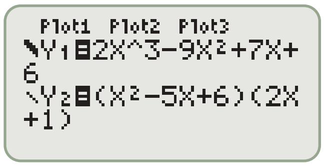
$$x - 1\overline{)x^2 + 4x - 5}$$

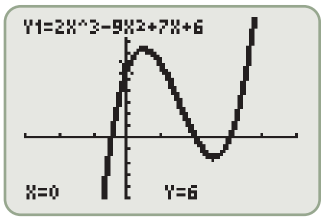
Note that although this is standard division notation, the **dividend**, $x^2 + 4x - 5$, and the **divisor**, $x - 1$, are both polynomials that consist of multiple terms. The *leading* terms of each algebraic expression will guide us.

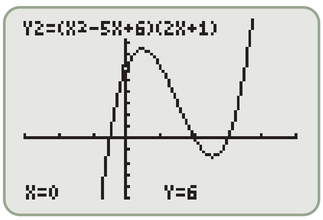
WORDS	MATH
Q: x times what quantity gives x^2? A: x	$\begin{array}{r} \boxed{x} \\ \boxed{x}-1\overline{)\boxed{x^2}+4x-5} \end{array}$
Multiply $x(x - 1) = x^2 - x$.	$\begin{array}{r} x \\ x-1\overline{)x^2+4x-5} \\ x^2-x \end{array}$
Subtract $(x^2 - x)$ from $x^2 + 4x - 5$. *Note:* $-(x^2 - x) = -x^2 + x$. Bring down the -5.	$\begin{array}{r} x \\ x-1\overline{)x^2+4x-5} \\ \underline{-x^2+x} \\ 5x-5 \end{array}$
Q: x times what quantity is $5x$? A: 5 Multiply $5(x - 1) = 5x - 5$.	$\begin{array}{r} x+\boxed{5} \\ \boxed{x}-1\overline{)x^2+4x-5} \\ \underline{-x^2+x} \\ \boxed{5x}-5 \end{array}$
Subtract $(5x - 5)$. *Note:* $-(5x - 5) = -5x + 5$.	$\begin{array}{r} x+5 \\ x-1\overline{)x^2+4x-5} \\ \underline{-x^2+x} \\ 5x-5 \\ \underline{-5x+5} \\ 0 \end{array}$

Technology Tip

Using the graphs of the two functions, a graphing utility can be used to confirm that $(x^2 - 5x + 6)(2x + 1) = 2x^3 - 9x^2 + 7x + 6$.







Notice that the graphs are the same.

The **quotient** is $x + 5$, and as expected, the **remainder** is 0. By long division we have shown that

$$\boxed{\frac{x^2 + 4x - 5}{x - 1} = x + 5}$$

Check: Multiplying the equation by $x - 1$ yields $x^2 + 4x - 5 = (x + 5)(x - 1)$, which we knew to be true.

EXAMPLE 1 Dividing Polynomials Using Long Division; Zero Remainder

Divide $2x^3 - 9x^2 + 7x + 6$ by $2x + 1$.

Solution:

$$\begin{array}{r}
x^2 - 5x + 6 \\
2x + 1\overline{)\,2x^3 - 9x^2 + 7x + 6} \\
-\left(2x^3 + x^2\right) \\
\hline
-10x^2 + 7x \\
-\left(-10x^2 - 5x\right) \\
\hline
12x + 6 \\
-(12x + 6) \\
\hline
0
\end{array}$$

Multiply: $x^2(2x + 1)$.

Subtract: Bring down the $7x$.

Multiply: $-5x(2x + 1)$.

Subtract: Bring down the 6.

Multiply: $6(2x + 1)$.

Subtract.

Quotient: $\boxed{x^2 - 5x + 6}$

Check: $(2x + 1)\left(x^2 - 5x + 6\right) = 2x^3 - 9x^2 + 7x + 6$.

Note: The divisor cannot be equal to zero, $2x + 1 \neq 0$, so we say $x \neq -\frac{1}{2}$.

■ **YOUR TURN** Divide $4x^3 + 13x^2 - 2x - 15$ by $4x + 5$.

■ **Answer:** $x^2 + 2x - 3$, remainder 0.

Why are we interested in dividing polynomials? Because it helps us find zeros of polynomials. In Example 1, using long division, we found that

$$2x^3 - 9x^2 + 7x + 6 = (2x + 1)\left(x^2 - 5x + 6\right)$$

Factoring the quadratic expression enables us to write the cubic polynomial as a product of three linear factors:

$$2x^3 - 9x^2 + 7x + 6 = (2x + 1)\left(x^2 - 5x + 6\right) = (2x + 1)(x - 3)(x - 2)$$

Set the value of the polynomial equal to zero, $(2x + 1)(x - 3)(x - 2) = 0$, and solve for x. The zeros of the polynomial are $-\frac{1}{2}$, 2, and 3. In Example 1 and in the Your Turn, the remainder was 0. Sometimes there is a nonzero remainder (Example 2).

EXAMPLE 2 **Dividing Polynomials Using Long Division; Nonzero Remainder**

Divide $6x^2 - x - 2$ by $x + 1$.

Solution:

$$\begin{array}{rl}
 & 6x - 7 \\
x + 1 & \overline{)6x^2 - x - 2} \\
\text{Multiply } 6x(x+1). & -(6x^2 + 6x) \\
\text{Subtract and bring down } -2. & \quad -7x - 2 \\
\text{Multiply } -7(x+1). & \quad -(-7x - 7) \\
\text{Subtract and identify the remainder.} & \qquad +5
\end{array}$$

$$\underbrace{\overbrace{\frac{6x^2 - x - 2}{x + 1}}^{\text{Dividend}}}_{\text{Divisor}} = \overbrace{6x - 7}^{\text{Quotient}} + \underbrace{\overbrace{\frac{5}{x + 1}}^{\text{Remainder}}}_{\text{Divisor}} \qquad \boxed{x \neq -1}$$

Check: Multiply equation by $x + 1$.

$$\begin{aligned} 6x^2 - x - 2 &= (6x - 7)(x + 1) + \frac{5}{(x + 1)} \cdot (x + 1) \\ &= 6x^2 - x - 7 + 5 \\ &= 6x^2 - x - 2 \checkmark \end{aligned}$$

■ **YOUR TURN** Divide $2x^3 + x^2 - 4x - 3$ by $x - 1$.

■ **Answer:** $2x^2 + 3x - 1$ R: -4 or $2x^2 + 3x - 1 - \dfrac{4}{x - 1}$

In general, when a polynomial is divided by another polynomial, we express the result in the following form:

$$\frac{P(x)}{d(x)} = Q(x) + \frac{r(x)}{d(x)}$$

where $P(x)$ is the **dividend**, $d(x) \neq 0$ is the **divisor**, $Q(x)$ is the **quotient**, and $r(x)$ is the **remainder**. Multiplying this equation by the divisor $d(x)$ leads us to the division algorithm.

THE DIVISION ALGORITHM

If $P(x)$ and $d(x)$ are polynomials with $d(x) \neq 0$, and if the degree of $P(x)$ is greater than or equal to the degree of $d(x)$, then unique polynomials $Q(x)$ and $r(x)$ exist such that

$$P(x) = d(x) \cdot Q(x) + r(x)$$

If the remainder $r(x) = 0$, then we say that $d(x)$ divides $P(x)$ and that $d(x)$ and $Q(x)$ are factors of $P(x)$.

EXAMPLE 3 **Long Division of Polynomials with "Missing" Terms**

Divide $3x^4 + 2x^3 + x^2 + 4$ by $x^2 + 1$.

Solution:

Insert $\mathbf{0x}$ as a placeholder in both the divisor and the dividend.

Multiply $3x^2(x^2 + 0x + 1)$.

Subtract and bring down $0x$.

Multiply $2x(x^2 + 0x + 1)$.

Subtract and bring down 4.

Multiply $-2(x^2 - 2x + 1)$.

Subtract and get remainder $-2x + 6$.

$$
\begin{array}{r}
3x^2 + 2x - 2 \\
x^2 + 0x + 1 \overline{)\, 3x^4 + 2x^3 + x^2 + 0x + 4} \\
-(3x^4 + 0x^3 + 3x^2) \\
\hline
2x^3 - 2x^2 + 0x \\
-(2x^3 + 0x^2 + 2x) \\
\hline
-2x^2 - 2x + 4 \\
-(-2x^2 + 0x - 2) \\
\hline
-2x + 6
\end{array}
$$

$$\boxed{\frac{3x^4 + 2x^3 + x^2 + 4}{x^2 + 1} = 3x^2 + 2x - 2 + \frac{-2x + 6}{x^2 + 1}}$$

■ **Answer:** $2x^2 + 6 + \dfrac{11x^2 + 18x + 36}{x^3 - 3x - 4}$

■ **YOUR TURN** Divide $2x^5 + 3x^2 + 12$ by $x^3 - 3x - 4$.

EXAMPLE 4 **Long Division of Polynomials Resulting in Quotients with Rational Coefficients**

Divide $8x^4 - 5x^3 + 7x - 2$ by $2x^2 + 1$.

Solution:

Insert $0x^2$ as a placeholder in the dividend and $0x$ as a placeholder in the divisor.

Multiply $4x^2(2x^2 + 0x + 1)$.

Subtract and bring down remaining terms.

Multiply $-\frac{5}{2}x(2x^2 + 0x + 1)$.

Subtract and bring down remaining terms.

Multiply $-2(2x^2 + 0x + 1)$.

Subtract and bring down the remainder $\frac{19}{2}x$.

$$
\begin{array}{r}
4x^2 - \frac{5}{2}x - 2 \\
2x^2 + 0x + 1 \overline{)\, 8x^4 - 5x^3 + 0x^2 + 7x - 2} \\
-(8x^4 + 0x^3 + 4x^2) \\
\hline
-5x^3 - 4x^2 + 7x \\
-(-5x^3 + 0x^2 - \frac{5}{2}x) \\
\hline
-4x^2 + \frac{19}{2}x - 2 \\
-(-4x^2 + 0x - 2) \\
\hline
\frac{19}{2}x
\end{array}
$$

$$\frac{8x^4 - 5x^3 + 7x - 2}{2x^2 + 1} = 4x^2 - \frac{5}{2}x - 2 + \frac{\frac{19}{2}x}{2x^2 + 1}$$

■ **Answer:** $5x^2 - \frac{3}{2}x + \frac{5}{2} + \dfrac{\frac{7}{2}x - \frac{3}{2}}{2x^2 - 1}$

■ **YOUR TURN** Divide $10x^4 - 3x^3 + 5x - 4$ by $2x^2 - 1$.

Synthetic Division of Polynomials

In the special case when the *divisor is a linear factor* of the form $x - a$ or $x + a$, there is another, more efficient way to divide polynomials. This method is called **synthetic division**. It is called synthetic because it is a contrived shorthand way of dividing a polynomial by a linear factor. A detailed step-by-step procedure is given below for synthetic division. Let's divide $x^4 - x^3 - 2x + 2$ by $x + 1$ using synthetic division.

Step 1: Write the division in synthetic form.
- List the coefficients of the dividend. **Remember to use 0 for a placeholder.**
- The divisor is $x + 1$, so $x = -1$ is used.

Coefficients of Dividend

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 0 & -2 & 2 \\ & & & & & \\ \hline \end{array}$$

Step 2: *Bring down* the first term (**1**) in the dividend.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 0 & -2 & 2 \\ & \downarrow & \text{Bring down the } 1 & & & \\ \hline & 1 & & & & \end{array}$$

Step 3: *Multiply* the **−1** by this leading coefficient (**1**), and place the product up and to the right in the second column.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 0 & -2 & 2 \\ & & -1 & & & \\ \hline & 1 & & & & \end{array}$$

Step 4: *Add* the values in the second column.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 0 & -2 & 2 \\ & & \downarrow -1 & \text{ADD} & & \\ \hline & 1 & -2 & & & \end{array}$$

Step 5: Repeat Steps 3 and 4 until all columns are filled.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 0 & -2 & 2 \\ & & -1 & 2\downarrow & -2\downarrow & 4\downarrow \\ \hline & 1 & -2 & 2 & -4 & 6 \end{array}$$

Step 6: Identify the **quotient** by assigning powers of x in descending order, beginning with $x^{n-1} = x^{4-1} = x^3$. The last term is the **remainder**.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 0 & -2 & 2 \\ & & -1 & 2 & -2 & 4 \\ \hline & \underbrace{1 \quad -2 \quad 2 \quad -4}_{\text{Quotient Coefficients}} & & & & 6 \end{array}$$

Remainder

Quotient Coefficients: $x^3 - 2x^2 + 2x - 4$

We know that the degree of the first term of the quotient is 3, because a fourth-degree polynomial was divided by a first-degree polynomial. Let's compare dividing $x^4 - x^3 - 2x + 2$ by $x + 1$ using both long division and synthetic division.

Study Tip

If $(x - a)$ is a divisor, then a is the number used in synthetic division.

Study Tip

Synthetic division can only be used when the divisor is of the form $x - a$. Realize that a may be negative, as in the divisor $x + 2$.

Long Division

$$
\begin{array}{r}
x^3 - 2x^2 + 2x - 4 \\
x + 1 \overline{\smash{\big)}\, x^4 - x^3 + 0x^2 - 2x + 2} \\
\underline{x^4 + x^3} \qquad\qquad\qquad\qquad \\
-2x^3 + 0x^2 \qquad\qquad\quad \\
\underline{-(-2x^3 - 2x^2)} \qquad\qquad\quad \\
2x^2 - 2x \qquad\quad \\
\underline{-(2x^2 + 2x)} \qquad\quad \\
-4x + 2 \\
\underline{-(-4x - 4)} \\
+\, 6
\end{array}
$$

Synthetic Division

$$
\begin{array}{r|rrrrr}
-1 & 1 & -1 & 0 & -2 & 2 \\
 & & -1 & 2 & -2 & 4 \\
\hline
 & \underbrace{1 \quad -2 \quad 2 \quad -4}_{x^3 - 2x^2 + 2x - 4} & & & & 6
\end{array}
$$

Both long division and synthetic division yield the same answer.

$$\boxed{\frac{x^4 - x^3 - 2x + 2}{x + 1} = x^3 - 2x^2 + 2x - 4 + \frac{6}{x + 1}}$$

EXAMPLE 5 Synthetic Division

Use synthetic division to divide $3x^5 - 2x^3 + x^2 - 7$ by $x + 2$.

Solution:

STEP 1 Write the division in synthetic form.

- List the coefficients of the dividend. Remember to use **0** for a placeholder.
- The divisor of the original problem is $x + 2$. If we set $x + 2 = 0$, we find that $x = -2$, so -2 is the divisor for synthetic division.

$$
\begin{array}{r|rrrrrr}
-2 & 3 & 0 & -2 & 1 & 0 & -7 \\
 & & & & & & \\
\hline
\end{array}
$$

STEP 2 Perform the synthetic division steps.

$$
\begin{array}{r|rrrrrr}
-2 & 3 & 0 & -2 & 1 & 0 & -7 \\
 & & -6 & 12 & -20 & 38 & -76 \\
\hline
 & 3 & -6 & 10 & -19 & 38 & -83
\end{array}
$$

STEP 3 Identify the quotient and remainder.

$$
\begin{array}{r|rrrrrr}
-2 & 3 & 0 & -2 & 1 & 0 & -7 \\
 & & -6 & 12 & -20 & 38 & -76 \\
\hline
 & \underbrace{3 \quad -6 \quad 10 \quad -19 \quad 38}_{3x^4 - 6x^3 + 10x^2 - 19x + 38} & & & & & \boxed{-83}
\end{array}
$$

$$\boxed{\frac{3x^5 - 2x^3 + x^2 - 7}{x + 2} = 3x^4 - 6x^3 + 10x^2 - 19x + 38 - \frac{83}{x + 2}}$$

■ **YOUR TURN** Use synthetic division to divide $2x^3 - x + 3$ by $x - 1$.

■ **Answer:** $2x^2 + 2x + 1 + \dfrac{4}{x - 1}$

SECTION 2.3 SUMMARY

Division of Polynomials

- Long division can always be used.
- Synthetic division is restricted to when the divisor is of the form $x - a$ or $x + a$.

Expressing Results

- $\dfrac{\text{Dividend}}{\text{Divisor}} = \text{quotient} + \dfrac{\text{remainder}}{\text{divisor}}$
- Dividend = (quotient)(divisor) + remainder

When Remainder Is Zero

- Dividend = (quotient)(divisor)
- Quotient and divisor are factors of the dividend.

SECTION 2.3 EXERCISES

SKILLS

In Exercises 1–26, divide the polynomials using long division. Use exact values and express the answer in the form $Q(x) = ?$, $r(x) = ?$.

1. $(3x^2 - 9x - 5) \div (x - 2)$
2. $(x^2 + 4x - 3) \div (x - 1)$
3. $(3x^2 - 13x - 10) \div (x + 5)$
4. $(3x^2 - 13x - 10) \div (x + 5)$
5. $(x^2 - 4) \div (x + 4)$
6. $(x^2 - 9) \div (x - 2)$
7. $(9x^2 - 25) \div (3x - 5)$
8. $(5x^2 - 3) \div (x + 1)$
9. $(4x^2 - 9) \div (2x + 3)$
10. $(8x^3 + 27) \div (2x + 3)$
11. $(11x + 20x^2 + 12x^3 + 2) \div (3x + 2)$
12. $(12x^3 + 2 + 11x + 20x^2) \div (2x + 1)$
13. $(4x^3 - 2x + 7) \div (2x + 1)$
14. $(6x^4 - 2x^2 + 5) \div (-3x + 2)$
15. $(4x^3 - 12x^2 - x + 3) \div \left(x - \frac{1}{2}\right)$
16. $(12x^3 + 1 + 7x + 16x^2) \div \left(x + \frac{1}{3}\right)$
17. $(-2x^5 + 3x^4 - 2x^2) \div (x^3 - 3x^2 + 1)$
18. $(-9x^6 + 7x^4 - 2x^3 + 5) \div (3x^4 - 2x + 1)$
19. $\dfrac{x^4 - 1}{x^2 - 1}$
20. $\dfrac{x^4 - 9}{x^2 + 3}$
21. $\dfrac{40 - 22x + 7x^3 + 6x^4}{6x^2 + x - 2}$
22. $\dfrac{-13x^2 + 4x^4 + 9}{4x^2 - 9}$
23. $\dfrac{-3x^4 + 7x^3 - 2x + 1}{x - 0.6}$
24. $\dfrac{2x^5 - 4x^3 + 3x^2 + 5}{x - 0.9}$
25. $(x^4 + 0.8x^3 - 0.26x^2 - 0.168x + 0.0441) \div (x^2 + 1.4x + 0.49)$
26. $(x^5 + 2.8x^4 + 1.34x^3 - 0.688x^2 - 0.2919x + 0.0882) \div (x^2 - 0.6x + 0.09)$

In Exercises 27–46, divide the polynomial by the linear factor with synthetic division. Indicate the quotient $Q(x)$ and the remainder $r(x)$.

27. $(3x^2 + 7x + 2) \div (x + 2)$
28. $(2x^2 + 7x - 15) \div (x + 5)$
29. $(7x^2 - 3x + 5) \div (x + 1)$
30. $(4x^2 + x + 1) \div (x - 2)$
31. $(3x^2 + 4x - x^4 - 2x^3 - 4) \div (x + 2)$
32. $(3x^2 - 4 + x^3) \div (x - 1)$
33. $(x^4 + 1) \div (x + 1)$
34. $(x^4 + 9) \div (x + 3)$
35. $(x^4 - 16) \div (x + 2)$
36. $(x^4 - 81) \div (x - 3)$
37. $(2x^3 - 5x^2 - x + 1) \div \left(x + \frac{1}{2}\right)$
38. $(3x^3 - 8x^2 + 1) \div \left(x + \frac{1}{3}\right)$
39. $(2x^4 - 3x^3 + 7x^2 - 4) \div \left(x - \frac{2}{3}\right)$
40. $(3x^4 + x^3 + 2x - 3) \div \left(x - \frac{3}{4}\right)$
41. $(2x^4 + 9x^3 - 9x^2 - 81x - 81) \div (x + 1.5)$
42. $(5x^3 - x^2 + 6x + 8) \div (x + 0.8)$
43. $\dfrac{x^7 - 8x^4 + 3x^2 + 1}{x - 1}$
44. $\dfrac{x^6 + 4x^5 - 2x^3 + 7}{x + 1}$
45. $(x^6 - 49x^4 - 25x^2 + 1225) \div (x - \sqrt{5})$
46. $(x^6 - 4x^4 - 9x^2 + 36) \div (x - \sqrt{3})$

In Exercises 47–60, divide the polynomials by either long division or synthetic division.

47. $(6x^2 - 23x + 7) \div (3x - 1)$

48. $(6x^2 + x - 2) \div (2x - 1)$

49. $(x^3 - x^2 - 9x + 9) \div (x - 1)$

50. $(x^3 + 2x^2 - 6x - 12) \div (x + 2)$

51. $(x^3 + 6x^2 - 2x - 5) \div (x^2 - 1)$

52. $(3x^5 - x^3 + 2x^2 - 1) \div (x^3 + x^2 - x + 1)$

53. $(x^6 - 2x^5 + x^4 - 6x^3 + 7x^2 - 4x + 7) \div (x^2 + 1)$

54. $(x^6 - 1) \div (x^2 + x + 1)$

55. $(x^5 + 4x^3 + 2x^2 - 1) \div (x - 2)$

56. $(x^4 - x^2 + 3x - 10) \div (x + 5)$

57. $(x^4 - 25) \div (x^2 - 1)$

58. $(x^3 - 8) \div (x^2 - 2)$

59. $(x^7 - 1) \div (x - 1)$

60. $(x^6 - 27) \div (x - 3)$

APPLICATIONS

61. Geometry. The area of a rectangle is $6x^4 + 4x^3 - x^2 - 2x - 1$ square feet. If the length of the rectangle is $2x^2 - 1$ feet, what is the width of the rectangle?

62. Geometry. If the rectangle in Exercise 61 is the base of a rectangular box with volume $18x^5 + 18x^4 + x^3 - 7x^2 - 5x - 1$ cubic feet, what is the height of the box?

63. Travel. If a car travels a distance of $x^3 + 60x^2 + x + 60$ miles at an average speed of $x + 60$ miles per hour, how long does the trip take?

64. Sports. If a quarterback throws a ball $-x^2 - 5x + 50$ yards in $5 - x$ seconds, how fast is the football traveling?

CATCH THE MISTAKE

In Exercises 65–68, explain the mistake that is made.

65. Divide $x^3 - 4x^2 + x + 6$ by $x^2 + x + 1$.

Solution:

$$
\begin{array}{r}
x - 3 \\
x^2 + x + 1 \,\overline{)\, x^3 - 4x^2 + x + 6} \\
\underline{x^3 + x^2 + x } \\
-3x^2 + 2x + 6 \\
\underline{-3x^2 - 3x - 3} \\
-x + 3
\end{array}
$$

This is incorrect. What mistake was made?

66. Divide $x^4 - 3x^2 + 5x + 2$ by $x - 2$.

Solution:

$$
\begin{array}{r|rrrr}
-2 & 1 & -3 & 5 & 2 \\
 & & -2 & 10 & -30 \\
\hline
 & 1 & -5 & 15 & \boxed{-28}
\end{array}
\qquad \underbrace{1 \quad -5 \quad 15}_{x^2 - 5x + 15}
$$

This is incorrect. What mistake was made?

67. Divide $x^3 + 4x - 12$ by $x - 3$.

Solution:

$$
\begin{array}{r|rrr}
3 & 1 & 4 & -12 \\
 & & 3 & 21 \\
\hline
 & 1 & 7 & \boxed{9}
\end{array}
\qquad \underbrace{1 \quad 7}_{x + 7}
$$

This is incorrect. What mistake was made?

68. Divide $x^3 + 3x^2 - 2x + 1$ by $x^2 + 1$.

Solution:

$$
\begin{array}{r|rrrr}
-1 & 1 & 3 & -2 & 1 \\
 & & -1 & -2 & 4 \\
\hline
 & 1 & 2 & -4 & \boxed{5}
\end{array}
\qquad \underbrace{1 \quad 2 \quad -4}_{x^2 - 2x - 4}
$$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 69–74, determine whether each statement is true or false.

69. A fifth-degree polynomial divided by a third-degree polynomial will yield a quadratic quotient.

70. A third-degree polynomial divided by a linear polynomial will yield a linear quotient.

71. Synthetic division can be used whenever the degree of the dividend is exactly one more than the degree of the divisor.

72. When the remainder is zero, the divisor is a factor of the dividend.

73. When both the dividend and the divisor have the same degree, the quotient equals one.

74. Long division must be used whenever the degree of the divisor is greater than one.

CHALLENGE

75. Is $x + b$ a factor of $x^3 + (2b - a)x^2 + (b^2 - 2ab)x - ab^2$?

76. Is $x + b$ a factor of $x^4 + (b^2 - a^2)x^2 - a^2b^2$?

77. Divide $x^{3n} + x^{2n} - x^n - 1$ by $x^n - 1$.

78. Divide $x^{3n} + 5x^{2n} + 8x^n + 4$ by $x^n + 1$.

TECHNOLOGY

79. Plot $\dfrac{2x^3 - x^2 + 10x - 5}{x^2 + 5}$. What type of function is it? Perform this division using long division, and confirm that the graph corresponds to the quotient.

80. Plot $\dfrac{x^3 - 3x^2 + 4x - 12}{x - 3}$. What type of function is it? Perform this division using synthetic division, and confirm that the graph corresponds to the quotient.

81. Plot $\dfrac{x^4 + 2x^3 - x - 2}{x + 2}$. What type of function is it? Perform this division using synthetic division, and confirm that the graph corresponds to the quotient.

82. Plot $\dfrac{x^5 - 9x^4 + 18x^3 + 2x^2 - 5x - 3}{x^4 - 6x^3 + 2x + 1}$. What type of function is it? Perform this division using long division, and confirm that the graph corresponds to the quotient.

83. Plot $\dfrac{-6x^3 + 7x^2 + 14x - 15}{2x + 3}$. What type of function is it? Perform this division using long division, and confirm that the graph corresponds to the quotient.

84. Plot $\dfrac{-3x^5 - 4x^4 + 29x^3 + 36x^2 - 18x}{3x^2 + 4x - 2}$. What type of function is it? Perform this division using long division, and confirm that the graph corresponds to the quotient.

PREVIEW TO CALCULUS

For some of the operations in calculus it is convenient to write rational fractions $\dfrac{P(x)}{d(x)}$ in the form $Q(x) + \dfrac{r(x)}{d(x)}$, where $\dfrac{P(x)}{d(x)} = Q(x) + \dfrac{r(x)}{d(x)}$.

In Exercises 85–88, write each rational function $\dfrac{P(x)}{d(x)}$ in the form $Q(x) + \dfrac{r(x)}{d(x)}$.

85. $\dfrac{2x^2 - x}{x + 2}$

86. $\dfrac{5x^3 + 2x^2 - 3x}{x - 3}$

87. $\dfrac{2x^4 + 3x^2 + 6}{x^2 + x + 1}$

88. $\dfrac{3x^5 - 2x^3 + x^2 + x - 6}{x^2 + x + 5}$

SECTION 2.4 THE REAL ZEROS OF A POLYNOMIAL FUNCTION

SKILLS OBJECTIVES

- Apply the remainder theorem to evaluate a polynomial function.
- Apply the factor theorem.
- Use the rational zero (root) theorem to list possible rational zeros.
- Apply Descartes' rule of signs to determine the possible combination of positive and negative real zeros.
- Utilize the upper and lower bound theorems to narrow the search for real zeros.
- Find the real zeros of a polynomial function.
- Factor a polynomial function.
- Employ the intermediate value theorem to approximate a real zero.

CONCEPTUAL OBJECTIVES

- Understand that a polynomial of degree n has at most n real zeros.
- Understand that a real zero can be either rational or irrational and that irrational zeros will not be listed as possible zeros through the rational zero test.
- Realize that rational zeros can be found exactly, whereas irrational zeros must be approximated.

The Remainder Theorem and the Factor Theorem

The zeros of a polynomial function assist us in finding the x-intercepts of the graph of a polynomial function. How do we find the zeros of a polynomial function if we cannot factor them easily? For polynomial functions of degree 2, we have the quadratic formula, which allows us to find the two zeros. For polynomial functions whose degree is greater than 2, much more work is required.* In this section, we focus our attention on finding the *real* zeros of a polynomial function. Later, in Section 2.5, we expand our discussion to *complex* zeros of polynomial functions.

In this section, we start by listing possible rational zeros. As you will see, there are sometimes many possibilities. We can then narrow the search using Descartes' rule of signs, which tells us possible combinations of positive and negative real zeros. We can narrow the search even further with the upper and lower bound rules. Once we have tested possible values and determined a zero, we will employ synthetic division to divide the polynomial by the linear factor associated with the zero. We will continue the process until we have factored the polynomial function into a product of either linear factors or irreducible quadratic factors. Last, we will discuss how to find irrational real zeros using the intermediate value theorem.

If we divide the polynomial function $f(x) = x^3 - 2x^2 + x - 3$ by $x - \mathbf{2}$ using synthetic division, we find the remainder is $\mathbf{-1}$.

$$\begin{array}{r|rrrr} \mathbf{2} & 1 & -2 & 1 & -3 \\ & & 2 & 0 & 2 \\ \hline & 1 & 0 & 1 & \mathbf{-1} \end{array}$$

Notice that if we evaluate the function at $x = \mathbf{2}$, the result is $\mathbf{-1}$. $\qquad f(\mathbf{2}) = \mathbf{-1}$

*There are complicated formulas for finding the zeros of polynomial functions of degree 3 and 4, but there are no such formulas for degree 5 and higher polynomials (according to the Abel–Ruffini theorem).

Words	Math
Recall the Division Algorithm.	$P(x) = d(x) \cdot Q(x) + r(x)$
Let $d(x) = x - a$ for any real number a.	$P(x) = (x - a) \cdot Q(x) + r(x)$
The degree of the remainder is always less than the degree of the divisor: therefore the remainder must be a constant (Call it r, $r(x) = r$).	$P(x) = (x - a) \cdot Q(x) + r$
Let $x = a$.	$P(a) = \underbrace{(a - a)}_{0} \cdot Q(x) + r$
Simplify.	$\boxed{P(a) = r}$

This leads us to the *remainder theorem.*

Remainder Theorem

If a polynomial $P(x)$ is divided by $x - a$, then the remainder is $r = P(a)$.

The remainder theorem tells you that polynomial division can be used to evaluate a polynomial function at a particular point.

Example 1 Two Methods for Evaluating Polynomials

Let $P(x) = 4x^5 - 3x^4 + 2x^3 - 7x^2 + 9x - 5$ and evaluate $P(2)$ by

a. evaluating $P(2)$ directly.
b. the remainder theorem and synthetic division.

Solution:

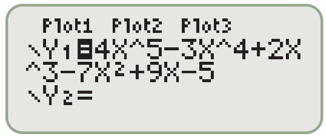
a.
$$\begin{aligned} P(2) &= 4(2)^5 - 3(2)^4 + 2(2)^3 - 7(2)^2 + 9(2) - 5 \\ &= 4(32) - 3(16) + 2(8) - 7(4) + 9(2) - 5 \\ &= 128 - 48 + 16 - 28 + 18 - 5 \\ &= \boxed{81} \end{aligned}$$

b.
$$\begin{array}{r|rrrrrr} 2 & 4 & -3 & 2 & -7 & 9 & -5 \\ & & 8 & 10 & 24 & 34 & 86 \\ \hline & 4 & 5 & 12 & 17 & 43 & \boxed{81} \end{array}$$

YOUR TURN Let $P(x) = -x^3 + 2x^2 - 5x + 2$ and evaluate $P(-2)$ using the remainder theorem and synthetic division.

Technology Tip

A graphing utility can be used to evaluate $P(2)$. Enter $P(x) = 4x^5 - 3x^4 + 2x^3 - 7x^2 + 9x - 5$ as Y_1.



To evaluate $P(2)$, press [VARS] [Y-VARS] [1:Function..] [ENTER] [1:Y_1] [ENTER] [(] 2 [)] [ENTER]

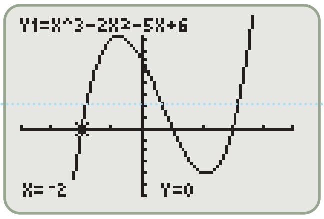
Answer: $P(-2) = 28$

Recall that when a polynomial is divided by $x - a$, if the remainder is zero, we say that $x - a$ is a factor of the polynomial. Through the remainder theorem, we now know that the remainder is related to evaluation of the polynomial at the point $x = a$. We are then led to the *factor theorem*.

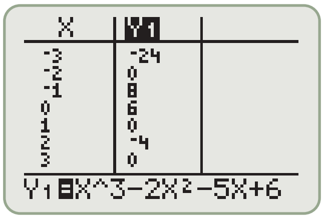
FACTOR THEOREM

If $P(a) = 0$, then $x - a$ is a factor of $P(x)$. Conversely, if $x - a$ is a factor of $P(x)$, then $P(a) = 0$.

Technology Tip



The three zeros of the function give the three factors $x + 2$, $x - 1$, and $x - 3$. A table of values supports the zeros of the graph.



Answer: $(x - 1)$ is a factor; $P(x) = (x - 5)(x - 1)(x + 2)$

EXAMPLE 2 Using the Factor Theorem to Factor a Polynomial

Determine whether $x + 2$ is a factor of $P(x) = x^3 - 2x^2 - 5x + 6$. If so, factor $P(x)$ completely.

Solution:

STEP 1 Divide $P(x) = x^3 - 2x^2 - 5x + 6$ by $x + 2$ using synthetic division.

$$
\begin{array}{r|rrrr}
-2 & 1 & -2 & -5 & 6 \\
 & & -2 & 8 & -6 \\
\hline
 & \underbrace{1 \quad -4 \quad 3}_{x^2 - 4x + 3} & & & \boxed{0}
\end{array}
$$

Since the remainder is zero, $P(-2) = 0$, $\boxed{x + 2 \text{ is a factor}}$ of $P(x) = x^3 - 2x^2 - 5x + 6$.

STEP 2 Write $P(x)$ as a product. $\quad P(x) = (x + 2)(x^2 - 4x + 3)$

STEP 3 Factor the quadratic polynomial. $\quad \boxed{P(x) = (x + 2)(x - 3)(x - 1)}$

YOUR TURN Determine whether $x - 1$ is a factor of $P(x) = x^3 - 4x^2 - 7x + 10$. If so, factor $P(x)$ completely.

EXAMPLE 3 Using the Factor Theorem to Factor a Polynomial

Determine whether $x - 3$ and $x + 2$ are factors of $P(x) = x^4 - 13x^2 + 36$. If so, factor $P(x)$ completely.

Solution:

STEP 1 With synthetic division, divide $P(x) = x^4 - 13x^2 + 36$ by $x - 3$.

$$\begin{array}{r|rrrrr} 3 & 1 & 0 & -13 & 0 & 36 \\ & & 3 & 9 & -12 & -36 \\ \hline & 1 & 3 & -4 & -12 & \boxed{0} \end{array}$$

$$\underbrace{1 \quad 3 \quad -4 \quad -12}_{x^3 + 3x^2 - 4x - 12}$$

Because the remainder is 0, $\boxed{x - 3 \text{ is a factor}}$, and we can write the polynomial as

$$P(x) = (x - 3)\left(x^3 + 3x^2 - 4x - 12\right)$$

STEP 2 With synthetic division, divide the remaining cubic polynomial $\left(x^3 + 3x^2 - 4x - 12\right)$ by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 1 & 3 & -4 & -12 \\ & & -2 & -2 & 12 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array}$$

$$\underbrace{1 \quad 1 \quad -6}_{x^2 + x - 6}$$

Because the remainder is 0, $\boxed{x + 2 \text{ is a factor}}$, and we can now write the polynomial as

$$P(x) = (x - 3)(x + 2)\left(x^2 + x - 6\right)$$

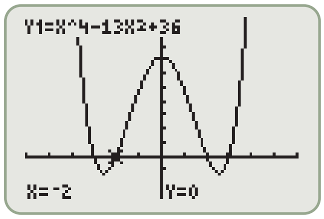
STEP 3 Factor the quadratic polynomial: $x^2 + x - 6 = (x + 3)(x - 2)$.

STEP 4 Write $P(x)$ as a product of linear factors:

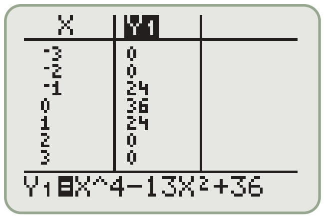
$$\boxed{P(x) = (x - 3)(x - 2)(x + 2)(x + 3)}$$

■ **YOUR TURN** Determine whether $x - 3$ and $x + 2$ are factors of $P(x) = x^4 - x^3 - 7x^2 + x + 6$. If so, factor $P(x)$ completely.

Technology Tip



The zeros of the function at $x = -3$, $x = -2$, $x = 2$, and $x = 3$ are shown in the graph. A table of values supports the zeros of the graph.



■ **Answer:** $(x - 3)$ and $(x + 2)$ are factors; $P(x) = (x - 3)(x + 2)(x - 1)(x + 1)$

The Search for Real Zeros

In all of the examples thus far, the polynomial function and one or more real zeros (or linear factors) were given. Now, we will not be given any real zeros to start with. Instead, we will develop methods to search for them.

Each real zero corresponds to a linear factor and each linear factor is of degree 1. Therefore, the largest number of real zeros a polynomial function can have is equal to the degree of the polynomial.

THE NUMBER OF REAL ZEROS

A polynomial function cannot have more real zeros than its degree.

Study Tip

The largest number of zeros a polynomial can have is equal to the degree of the polynomial.

The following functions illustrate that a polynomial function of degree n can have at most n real zeros:

POLYNOMIAL FUNCTION	DEGREE	REAL ZEROS	COMMENTS
$f(x) = x^2 - 9$	**2**	$x = \pm 3$	**Two** real zeros
$f(x) = x^2 + 4$	**2**	None	**No** real zeros
$f(x) = x^3 - 1$	**3**	$x = 1$	**One** real zero
$f(x) = x^3 - x^2 - 6x$	**3**	$x = -2, 0, 3$	**Three** real zeros

Now that we know the *maximum* number of real zeros a polynomial function can have, let us discuss how to find these zeros.

The Rational Zero Theorem and Descartes' Rule of Signs

When the coefficients of a polynomial are integers, then the *rational zero theorem* (*rational root test*) gives us a list of possible rational zeros. We can then test these possible values to determine whether they really do correspond to actual zeros. *Descartes' rule of signs* tells us the possible combinations of *positive* real zeros and *negative* real zeros. Using Descartes' rule of signs will assist us in narrowing down the large list of possible zeros generated through the rational zero theorem to a (hopefully) shorter list of possible zeros. First, let's look at the rational zero theorem; then we'll turn to Descartes' rule of signs.

THE RATIONAL ZERO THEOREM (RATIONAL ROOT TEST)

If the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ has *integer* coefficients, then every rational zero of $P(x)$ has the form

$$\text{Rational zero} = \frac{\text{integer factors of } a_0}{\text{integer factors of } a_n} = \frac{\text{integer factors of constant term}}{\text{integer factors of leading coefficient}}$$

$$= \pm\frac{\text{positive integer factors of constant term}}{\text{positive integer factors of leading coefficient}}$$

To use this theorem, simply list all combinations of integer factors of both the constant term a_0 and the leading coefficient term a_n and take all appropriate combinations of ratios. This procedure is illustrated in Example 4. Notice that when the leading coefficient is 1, then the possible rational zeros will simply be the possible integer factors of the constant term.

EXAMPLE 4 Using the Rational Zero Theorem

Determine possible rational zeros for the polynomial $P(x) = x^4 - x^3 - 5x^2 - x - 6$ by the rational zero theorem. Test each one to find all rational zeros.

Solution:

STEP 1 List factors of the constant and leading coefficient terms. $\quad a_0 = -6 \quad \pm1, \pm2, \pm3, \pm6$; $\quad a_n = 1 \quad \pm1$

STEP 2 List possible rational zeros $\dfrac{a_0}{a_n}$. $\quad \dfrac{\pm1}{\pm1}, \dfrac{\pm2}{\pm1}, \dfrac{\pm3}{\pm1}, \dfrac{\pm6}{\pm1} = \pm1, \pm2, \pm3, \pm6$

There are three ways to test whether any of these are zeros: Substitute these values into the polynomial to see which ones yield zero, or use either polynomial division or synthetic division to divide the polynomial by these possible zeros, and look for a zero remainder.

STEP 3 Test possible zeros by looking for zero remainders.

$$1 \text{ is not a zero: } \quad P(1) = (1)^4 - (1)^3 - 5(1)^2 - (1) - 6 = -12$$

$$-1 \text{ is not a zero: } P(-1) = (-1)^4 - (-1)^3 - 5(-1)^2 - (-1) - 6 = -8$$

We could continue testing with direct substitution, but let us now use synthetic division as an alternative.

2 is not a zero:

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & -5 & -1 & -6 \\ & & 2 & 2 & -6 & -14 \\ \hline & 1 & 1 & -3 & -7 & \boxed{-20} \end{array}$$

-2 is a zero:

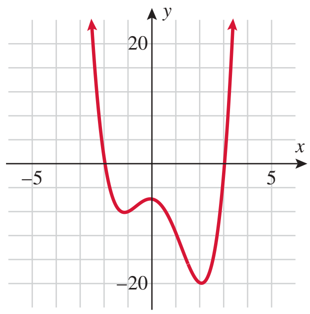
$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -5 & -1 & -6 \\ & & -2 & 6 & -2 & 6 \\ \hline & 1 & -3 & 1 & -3 & \boxed{0} \end{array}$$

Since -2 is a zero, then $x + 2$ is a factor of $P(x)$, and the remaining quotient is $x^3 - 3x^2 + x - 3$. Therefore, if there are any other real roots remaining, we can now use the simpler $x^3 - 3x^2 + x - 3$ for the dividend. Also note that the rational zero theorem can be applied to the new dividend and possibly shorten the list of possible rational zeros. In this case, the possible rational zeros of $F(x) = x^3 - 3x^2 + x - 3$ are ± 1 and ± 3.

3 is a zero:

$$\begin{array}{r|rrrr} 3 & 1 & -3 & 1 & -3 \\ & & 3 & 0 & 3 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}$$

We now know that $\boxed{-2}$ and $\boxed{3}$ are confirmed zeros. If we continue testing, we will find that the other possible zeros fail. This is a fourth-degree polynomial, and we have found two rational real zeros. We see in the graph on the right that these two real zeros correspond to the x-intercepts.



Study Tip

The remainder can be found by evaluating the function or synthetic division. For simple values like $x = \pm 1$, it is easier to evaluate the polynomial function. For other values, it is often easier to use synthetic division.

Study Tip

Notice in Step 3 that the polynomial $F(x) = x^3 - 3x^2 + x - 3$ can be factored by grouping: $F(x) = (x - 3)(x^2 + 1)$.

YOUR TURN List the possible rational zeros of the polynomial $P(x) = x^4 + 2x^3 - 2x^2 + 2x - 3$, and determine rational real zeros.

Answer: Possible rational zeros: ± 1 and ± 3. Rational real zeros: 1 and -3.

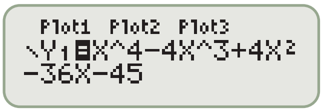
Notice in Example 4 that the polynomial function $P(x) = x^4 - x^3 - 5x^2 - x - 6$ had two rational real zeros, -2 and 3. This implies that $x + 2$ and $x - 3$ are factors of $P(x)$. Also note in the last step when we divided by the zero 3, the quotient was $x^2 + 1$. Therefore, we can write the polynomial in factored form as

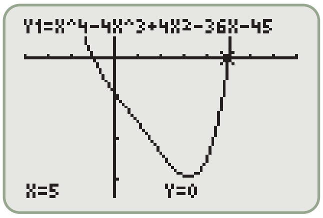
$$P(x) = \underbrace{(x + 2)}_{\substack{\text{linear}\\\text{factor}}}\underbrace{(x - 3)}_{\substack{\text{linear}\\\text{factor}}}\underbrace{(x^2 + 1)}_{\substack{\text{irreducible}\\\text{quadratic}\\\text{factor}}}$$

Notice that the first two factors are of degree 1, so we call them **linear factors**. The third expression, $x^2 + 1$, is of degree 2 and cannot be factored in terms of real numbers.

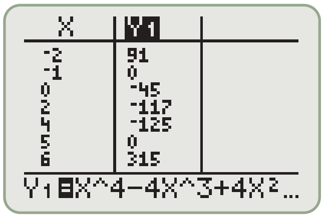
We will discuss complex zeros in the next section. For now, we say that a quadratic expression, $ax^2 + bx + c$, is called **irreducible** if it cannot be factored over the real numbers.

Technology Tip





A table of values supports the real zeros of the graph.



EXAMPLE 5 Factoring a Polynomial Function

Write the following polynomial function as a product of linear and/or irreducible quadratic factors:

$$P(x) = x^4 - 4x^3 + 4x^2 - 36x - 45$$

Solution:

Use the rational zero theorem to list possible rational roots. $x = \pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45$

Test possible zeros by evaluating the function or by utilizing synthetic division.

$x = 1$ is not a zero. $P(1) = -80$

$x = -1$ is a zero. $P(-1) = 0$

Divide $P(x)$ by $x + 1$.

$$\begin{array}{r|rrrrr} -1 & 1 & -4 & 4 & -36 & -45 \\ & & -1 & 5 & -9 & 45 \\ \hline & 1 & -5 & 9 & -45 & \boxed{0} \end{array}$$

$x = 5$ is a zero.

$$\begin{array}{r|rrrr} 5 & 1 & -5 & 9 & -45 \\ & & 5 & 0 & 45 \\ \hline & \underbrace{1 \quad 0 \quad 9}_{x^2 + 9} & & & \boxed{0} \end{array}$$

The factor $x^2 + 9$ is irreducible.

Write the polynomial as a product of linear and/or irreducible quadratic factors. $\boxed{P(x) = (x - 5)(x + 1)(x^2 + 9)}$

Notice that the graph of this polynomial function has x-intercepts at $x = -1$ and $x = 5$.

■ **YOUR TURN** Write the following polynomial function as a product of linear and/or irreducible quadratic factors:

$$P(x) = x^4 - 2x^3 - x^2 - 4x - 6$$

■ **Answer:** $P(x) = (x + 1)(x - 3)(x^2 + 2)$

The rational zero theorem lists possible zeros. It would be helpful if we could narrow that list. Descartes' rule of signs determines the possible combinations of positive real zeros and negative real zeros through variations of sign. A *variation in sign* is a sign difference seen between consecutive coefficients.

Sign Change
− to +

$$P(x) = 2x^6 - 5x^5 - 3x^4 + 2x^3 - x^2 - x - 1$$

Sign Change **+ to −** **Sign Change** **+ to −**

This polynomial experiences three sign changes or variations in sign.

DESCARTES' RULE OF SIGNS

If the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ has real coefficients and $a_0 \neq 0$, then:

- The number of **positive** real zeros of the polynomial is either equal to the number of variations of sign of $P(x)$ or less than that number by an even integer.
- The number of **negative** real zeros of the polynomial is either equal to the number of variations of sign of $P(-x)$ or less than that number by an even integer.

Descartes' rule of signs narrows our search for real zeros, because we don't have to test all of the possible rational zeros. For example, if we know there is one positive real zero, then if we find a positive rational zero, we no longer need to continue to test possible positive zeros.

EXAMPLE 6 Using Descartes' Rule of Signs to Find Possible Combinations of Real Zeros

Determine the possible combinations of real zeros for

$$P(x) = x^4 - 2x^3 + x^2 + 2x - 2$$

Solution:

$P(x)$ has three variations in sign.

Sign Change

$$P(x) = x^4 - 2x^3 + x^2 + 2x - 2$$

Sign Change **Sign Change**

Apply Descartes' rule of signs. $P(x)$ has *either* three or one **positive** real zero.

Find $P(-x)$. $P(-x) = (-x)^4 - 2(-x)^3 + (-x)^2 + 2(-x) - 2$

$= x^4 + 2x^3 + x^2 - 2x - 2$

$P(-x)$ has one variation in sign. $P(-x) = x^4 + 2x^3 + x^2 - 2x - 2$

Sign Change

Apply Descartes' rule of signs. $P(x)$ has one **negative** real zero.

Since $P(x) = x^4 - 2x^3 + x^2 + 2x - 2$ is a *fourth*-degree polynomial, there are at most four real zeros. One zero is a negative real number.

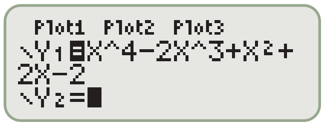
$\boxed{P(x) \text{ has one negative real zero and could have three positive real zeros or one positive real zero.}}$

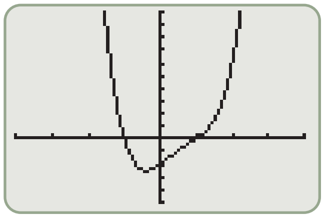
Look at the graph in the Technology Tip to confirm one negative real zero and one positive real zero.

YOUR TURN Determine the possible combinations of zeros for

$$P(x) = x^4 + 2x^3 + x^2 + 8x - 12$$

Technology Tip

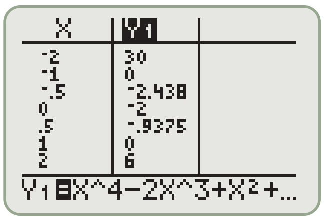




There are one negative real zero and one positive real zero.

The number of negative real zero is 1 and the number of positive real zero is 1. So, it has two complex conjugate zeros.

A table of values supports the zeros of the function.



X	Y1
-2	30
-1	0
-.5	-2.438
0	-2
.5	-.9375
1	0
2	6

Y1=X^4-2X^3+X²+...

■ **Answer:** Positive real zeros: 1
Negative real zeros: 3 or 1

Factoring Polynomials

Now let's draw on the tests discussed in this section thus far to help us in finding all real zeros of a polynomial function. Doing so will enable us to factor polynomials.

EXAMPLE 7 Factoring a Polynomial

Write the polynomial $P(x) = x^5 + 2x^4 - x - 2$ as a product of linear and/or irreducible quadratic factors.

Solution:

STEP 1 Determine variations in sign.

$P(x)$ has one sign change. $\quad P(x) = x^5 + 2x^4 - x - 2$

$P(-x)$ has two sign changes. $\quad P(-x) = -x^5 + 2x^4 + x - 2$

STEP 2 Apply Descartes' rule of signs. $\quad$ Positive Real Zeros: 1

Negative Real Zeros: 2 or 0

STEP 3 Use the rational zero theorem to determine the possible rational zeros. $\quad \pm1, \pm2$

We know (Step 2) that there is one positive real zero, so test the possible positive rational zeros first.

STEP 4 Test possible rational zeros.

$$\begin{array}{r|rrrrrr} 1 & 1 & 2 & 0 & 0 & -1 & -2 \\ & & 1 & 3 & 3 & 3 & 2 \\ \hline & 1 & 3 & 3 & 3 & 2 & \boxed{0} \end{array}$$

1 is a zero:

Now that we have found *the* positive zero, we can test the other two possible negative zeros—because either they both are zeros or neither is a zero (or one is a double root).

$$\begin{array}{r|rrrrr} -1 & 1 & 3 & 3 & 3 & 2 \\ & & -1 & -2 & -1 & -2 \\ \hline & 1 & 2 & 1 & 2 & \boxed{0} \end{array}$$

-1 is a zero:

Let's now try the other possible negative zero, -2.

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 1 & 2 \\ & & -2 & 0 & -2 \\ \hline & \multicolumn{3}{c}{\underbrace{1 \quad 0 \quad 1}_{x^2+1}} & \boxed{0} \end{array}$$

-2 is a zero:

STEP 5 Three of the five have been found to be zeros: -1, -2, and 1.

STEP 6 Write the fifth-degree polynomial as a product of three linear factors and an irreducible quadratic factor.

$$\boxed{P(x) = (x - 1)(x + 1)(x + 2)(x^2 + 1)}$$

■ **Answer:** $P(x) = (x - 2)(x + 1)(x - 1)(x^2 + 2)$

■ **YOUR TURN** Write the polynomial $P(x) = x^5 - 2x^4 + x^3 - 2x^2 - 2x + 4$ as a product of linear and/or irreducible quadratic factors.

EXAMPLE 8 Factoring a Polynomial

Write the polynomial $P(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$ as a product of linear and/or irreducible quadratic factors.

Solution:

STEP 1 Determine variations in sign.

$P(x)$ has two sign changes. $\quad P(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$

$P(-x)$ has two sign changes. $\quad P(-x) = 2x^4 - 3x^3 - 12x^2 + 7x + 6$

STEP 2 Apply Descartes' rule of signs.

Positive Real Zeros: 2 or 0

Negative Real Zeros: 2 or 0

STEP 3 Use the rational zero theorem to determine the possible rational zeros.

$$\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{2}, \pm\frac{3}{2}$$

STEP 4 Test possible rational zeros.

$$\begin{array}{r|rrrrr} -1 & 2 & 3 & -12 & -7 & 6 \\ & & -2 & -1 & 13 & -6 \\ \hline & 2 & 1 & -13 & 6 & \boxed{0} \end{array}$$

-1 is a zero:

Since there are either two or no negative real zeros and we have found one of the negative zeros, try the other negative real zeros.

$$\begin{array}{r|rrrr} -3 & 2 & 1 & -13 & 6 \\ & & -6 & 15 & -6 \\ \hline & \underbrace{2 \quad -5 \quad +2}_{2x^2 - 5x + 2} & & & \boxed{0} \end{array}$$

-3 is a zero:

Factor $2x^2 - 5x + 2$ to find the remaining two zeros. $\quad (2x - 1)(x - 2)$

STEP 5 Write the fourth-degree polynomial as a product of four linear factors.

$$\boxed{P(x) = (2x - 1)(x - 2)(x + 3)(x + 1)}$$

■ **YOUR TURN** Write the polynomial $P(x) = 3x^4 - 5x^3 - 17x^2 + 13x + 6$ as a product of linear and/or irreducible quadratic factors.

■ **Answer:** $P(x) = (3x + 1)(x - 1)(x + 2)(x - 3)$

The rational zero theorem gives us possible rational zeros of a polynomial, and Descartes' rule of signs gives us possible combinations of positive and negative real zeros. Additional aids that help eliminate possible zeros are the *upper* and *lower bound rules*. These rules can give you an upper and lower bound on the real zeros of a polynomial function. If $f(x)$ has a common monomial factor, you should factor it out first, and then follow the upper and lower bound rules.

Study Tip

If $f(x)$ has a common monomial factor, it should be canceled first before applying the bound rules.

UPPER AND LOWER BOUND RULES

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$ using synthetic division.

1. If $c > 0$ and each number in the bottom row is either positive or zero, c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the bottom row are alternately positive and negative (zero entries count as either positive or negative), c is a **lower bound** for the real zeros of f.

EXAMPLE 9 Using Upper and Lower Bounds to Eliminate Possible Zeros

Find the real zeros of $f(x) = 4x^3 - x^2 + 36x - 9$.

Solution:

STEP 1 The rational zero theorem gives possible rational zeros.

$$\frac{\text{Factors of } 9}{\text{Factors of } 4} = \frac{\pm 1, \pm 3, \pm 9}{\pm 1, \pm 2, \pm 4}$$

$$= \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{3}{2}, \pm\frac{9}{4}, \pm 3, \pm\frac{9}{2}, \pm 9$$

STEP 2 Apply Descartes' rule of signs:

$f(x)$ has three sign variations.	three or one positive real zeros
$f(-x)$ has no sign variations.	no negative real zeros

STEP 3 Try $x = 1$.

$$\begin{array}{r|rrrr} 1 & 4 & -1 & 36 & -9 \\ & & 4 & 3 & 39 \\ \hline & 4 & 3 & 39 & 30 \end{array}$$

$x = 1$ is not a zero, but because the last row contains all positive entries, $x = 1$ is an *upper* bound. Since we know there are no negative real zeros, we restrict our search to between 0 and 1.

STEP 4 Try $x = \frac{1}{4}$.

$$\begin{array}{r|rrrr} \frac{1}{4} & 4 & -1 & 36 & -9 \\ & & 1 & 0 & 9 \\ \hline & 4 & 0 & 36 & 0 \end{array}$$

$\frac{1}{4}$ is a zero and the quotient $4x^2 + 36$ has all positive coefficients; therefore, $\frac{1}{4}$ is an upper bound, so $\boxed{\frac{1}{4} \text{ is the only real zero}}$.

Note: If $f(x)$ has a common monomial factor, it should be factored out first before applying the bound rules.

Study Tip

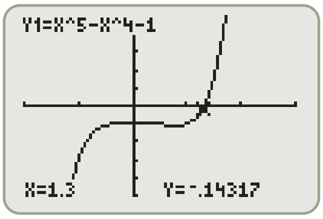
In Example 9, Steps 3 and 4 long division can be used as well as evaluating the function at $x = 1$ and $x = \frac{1}{4}$ to determine if these are zeros.

The Intermediate Value Theorem

In our search for zeros, we sometimes encounter irrational zeros, as in, for example, the polynomial

$$f(x) = x^5 - x^4 - 1$$

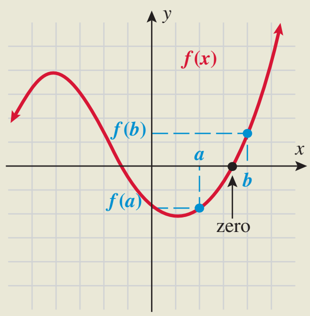
Descartes' rule of signs tells us there is exactly one real positive zero. However, the rational zero test yields only $x = \pm 1$, neither of which is a zero. So if we know there is a real positive zero and we know it's not rational, it must be irrational. Notice that $f(1) = -1$ and $f(2) = 15$. Since polynomial functions are continuous and the function goes from negative to positive between $x = 1$ and $x = 2$, we expect a zero somewhere in that interval. Generating a graph with a graphing utility, we find that there is a zero around $x = 1.3$.



The *intermediate value theorem* is based on the fact that polynomial functions are continuous.

INTERMEDIATE VALUE THEOREM

Let a and b be real numbers such that $a < b$ and $f(x)$ is a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one real zero between a and b.



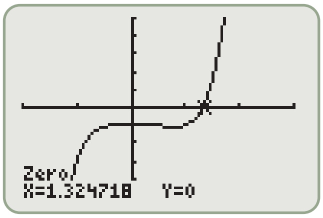
If the intermediate value theorem tells us that there is a real zero in the interval (a, b), how do we approximate that zero? The **bisection method*** is a root-finding algorithm that approximates the solution to the equation $f(x) = 0$. In the bisection method the interval is divided in half and then the subinterval that contains the zero is selected. This is repeated until the bisection method converges to an approximate root of f.

*In calculus you will learn Newton's method, which is a more efficient approximation technique for finding zeros.

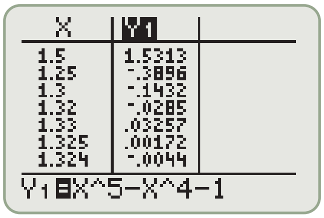
Technology Tip

The graph of $f(x) = x^5 - x^4 - 1$ is shown. To find the zero of the function, press

2nd | TRACE | 2 | Zero | 1 | ENTER | 2 | ENTER | ENTER



A table of values supports this zero of the function.



EXAMPLE 10 Approximating Real Zeros of a Polynomial Function

Approximate the real zero of $f(x) = x^5 - x^4 - 1$.

Note: Descartes' rule of signs tells us that there are no real negative zeros and there is exactly one real positive zero.

Solution:

Find two consecutive integer values for x that have corresponding function values opposite in sign.

x	$f(x)$
1	-1
2	15

Note that a graphing utility would have shown an x-intercept between $x = 1$ and $x = 2$.

Apply the bisection method, with $a = 1$ and $b = 2$. $\quad c = \dfrac{a+b}{2} = \dfrac{1+2}{2} = \dfrac{3}{2}$

Evaluate the function at $x = c$. $\quad f(1.5) \approx 1.53$

Compare the values of f at the endpoints and midpoint. $\quad f(1) = -1, f(1.5) \approx 1.53, f(2) = 15$

Select the subinterval corresponding to the *opposite* signs of f. $\quad (1, 1.5)$

Apply the bisection method again (repeat the algorithm). $\quad \dfrac{1+1.5}{2} = 1.25$

Evaluate the function at $x = 1.25$. $\quad f(1.25) \approx -0.38965$

Compare the values of f at the endpoints and midpoint. $\quad f(1) = -1, f(1.25) \approx -0.38965, f(1.5) \approx 1.53$

Select the subinterval corresponding to the *opposite* signs of f. $\quad (1.25, 1.5)$

Apply the bisection method again (repeat the algorithm). $\quad \dfrac{1.25+1.5}{2} = 1.375$

Evaluate the function at $x = 1.375$. $\quad f(1.375) \approx 0.3404$

Compare the values of f at the endpoints and midpoint. $\quad f(1.25) \approx -0.38965, f(1.375) \approx 0.3404, f(1.5) \approx 1.53$

Select the subinterval corresponding to the *opposite* signs of f. $\quad (1.25, 1.375)$

We can continue this procedure (*applying the bisection method*) to find that the zero is somewhere between $x = 1.32$ and $x = 1.33$, since $f(1.32) \approx -0.285$ and $f(1.33) \approx 0.0326$.

We find that, to three significant digits, $\boxed{1.32}$ is an approximation to the real zero.

Graphing Polynomial Functions

In Section 2.2, we graphed simple polynomial functions that were easily factored. Now that we have procedures for finding real zeros of polynomial functions (rational zero theorem, Descartes' rule of signs, and upper and lower bound rules for rational zeros, and the intermediate value theorem and the bisection method for irrational zeros), let us return to the topic of graphing polynomial functions. Since a real zero of a polynomial function corresponds to an x-intercept of its graph, we now have methods for finding (or estimating) any x-intercepts of the graph of any polynomial function.

EXAMPLE 11 Graphing a Polynomial Function

Graph the function $f(x) = 2x^4 - 2x^3 + 5x^2 + 17x - 22$.

Solution:

STEP 1 Find the y-intercept. $\quad f(0) = -22$

STEP 2 Find any x-intercepts (real zeros).

Apply Descartes' rule of signs.

Three sign changes correspond to three or one positive real zeros. $\quad f(x) = 2x^4 - 2x^3 + 5x^2 + 17x - 22$

One sign change corresponds to one negative real zero. $\quad f(-x) = 2x^4 + 2x^3 + 5x^2 - 17x - 22$

Apply the rational zero theorem.

Let $a_0 = -22$ and $a_n = 2$. $\quad \dfrac{\text{Factors of } a_0}{\text{Factors of } a_n} = \pm\dfrac{1}{2}, \pm1, \pm2, \pm\dfrac{11}{2}, \pm11, \pm22$

Test the possible zeros.

$x = 1$ is a zero. $\quad f(1) = 0$

There are no other rational zeros.

Apply the upper bound rule.

$$\begin{array}{r|rrrrr} 1 & 2 & -2 & 5 & 17 & -22 \\ & & 2 & 0 & 5 & 22 \\ \hline & 2 & 0 & 5 & 22 & \boxed{0} \end{array}$$

Since $x = 1$ is positive and all of the numbers in the bottom row are positive (or zero), $x = 1$ is an upper bound for the real zeros. We know there is exactly one negative real zero, but none of the possible zeros from the rational zero theorem is a zero. Therefore, the negative real zero is irrational.

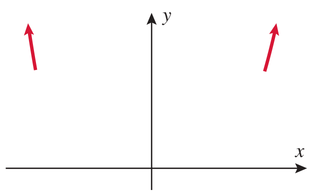
Apply the intermediate value theorem and the bisection method.

f is positive at $x = -2$. $\quad f(-2) = 12$

f is negative at $x = -1$. $\quad f(-1) = -30$

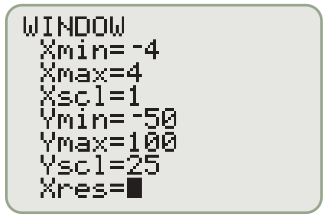
Use the bisection method to find the negative real zero between -2 and -1. $\quad x \approx -1.85$

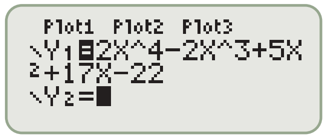
STEP 3 Determine the end behavior. $\quad y = 2x^4$

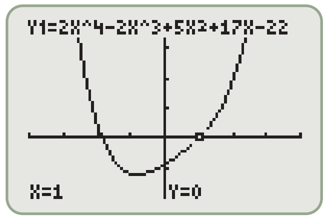


Technology Tip

The graph of $f(x) = 2x^4 - 2x^3 + 5x^2 + 17x - 22$ is shown.

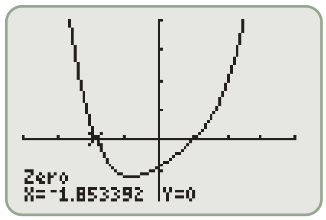




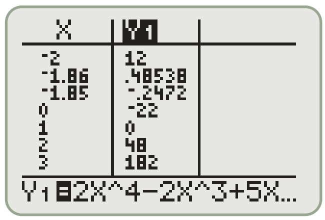


To find the zero of the function, press:

2nd TRACE 2 Zero (−) 3 ENTER (−) 1 ENTER ENTER



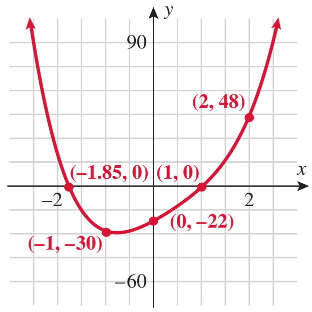
A table of values supports this zero of the function and the graph.



STEP 4 Find additional points.

x	-2	-1.85	-1	0	1	2
$f(x)$	12	0	-30	-22	0	48
Point	$(-2, 12)$	$(-1.85, 0)$	$(-1, -30)$	$(0, -22)$	$(1, 0)$	$(2, 48)$

STEP 5 Sketch the graph.



SECTION 2.4 SUMMARY

In this section, we discussed how to find the real zeros of a polynomial function. Once real zeros are known, it is possible to write the polynomial function as a product of linear and/or irreducible quadratic factors.

The Number of Zeros

- A polynomial of degree n has *at most n* real zeros.
- *Descartes' rule of signs* determines the possible combinations of positive and negative real zeros.
- *Upper and lower bounds* help narrow the search for zeros.

How to Find Zeros

- *Rational zero theorem:* List possible rational zeros:

$$\frac{\text{Factors of constant, } a_0}{\text{Factors of leading coefficient, } a_n}$$

- *Irrational zeros:* Approximate zeros by determining when the polynomial function changes sign (intermediate value theorem).

Procedure for Factoring a Polynomial Function

- List possible rational zeros (rational zero theorem).
- List possible combinations of positive and negative real zeros (Descartes' rule of signs).
- Test possible values until a zero is found.*
- Once a real zero is found, repeat testing on the quotient until linear and/or irreducible quadratic factors remain.
- If there is a real zero but all possible rational roots have failed, then approximate the zero using the *intermediate value theorem* and the *bisection method*.

*Depending on the form of the quotient, upper and lower bounds may eliminate possible zeros.

SECTION 2.4 EXERCISES

SKILLS

In Exercises 1–10, given a real zero of the polynomial, determine all other real zeros, and write the polynomial in terms of a product of linear and/or irreducible quadratic factors.

	Polynomial	Zero		Polynomial	Zero
1.	$P(x) = x^3 - 13x + 12$	1	**2.**	$P(x) = x^3 + 3x^2 - 10x - 24$	3
3.	$P(x) = 2x^3 + x^2 - 13x + 6$	$\frac{1}{2}$	**4.**	$P(x) = 3x^3 - 14x^2 + 7x + 4$	$-\frac{1}{3}$

	Polynomial	Zero		Polynomial	Zero
5.	$P(x) = x^4 - 2x^3 - 11x^2 - 8x - 60$	$-3, 5$	**6.**	$P(x) = x^4 - x^3 + 7x^2 - 9x - 18$	$-1, 2$
7.	$P(x) = x^4 - 5x^2 + 10x - 6$	$1, -3$	**8.**	$P(x) = x^4 - 4x^3 + x^2 + 6x - 40$	$4, -2$
9.	$P(x) = x^4 + 6x^3 + 13x^2 + 12x + 4$	-2 (multiplicity 2)	**10.**	$P(x) = x^4 + 4x^3 - 2x^2 - 12x + 9$	1 (multiplicity 2)

In Exercises 11–18, use the rational zero theorem to list the *possible* rational zeros.

11. $P(x) = x^4 + 3x^2 - 8x + 4$
12. $P(x) = -x^4 + 2x^3 - 5x + 4$
13. $P(x) = x^5 - 14x^3 + x^2 - 15x + 12$
14. $P(x) = x^5 - x^3 - x^2 + 4x + 9$
15. $P(x) = 2x^6 - 7x^4 + x^3 - 2x + 8$
16. $P(x) = 3x^5 + 2x^4 - 5x^3 + x - 10$
17. $P(x) = 5x^5 + 3x^4 + x^3 - x - 20$
18. $P(x) = 4x^6 - 7x^4 + 4x^3 + x - 21$

In Exercises 19–22, list the possible rational zeros, and test to determine all rational zeros.

19. $P(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$
20. $P(x) = x^4 + 2x^3 - 4x^2 - 2x + 3$
21. $P(x) = 2x^3 - 9x^2 + 10x - 3$
22. $P(x) = 3x^3 - 5x^2 - 26x - 8$

In Exercises 23–34, use Descartes' rule of signs to determine the possible number of positive real zeros and negative real zeros.

23. $P(x) = x^4 - 32$
24. $P(x) = x^4 + 32$
25. $P(x) = x^5 - 1$
26. $P(x) = x^5 + 1$
27. $P(x) = x^5 - 3x^3 - x + 2$
28. $P(x) = x^4 + 2x^2 - 9$
29. $P(x) = 9x^7 + 2x^5 - x^3 - x$
30. $P(x) = 16x^7 - 3x^4 + 2x - 1$
31. $P(x) = x^6 - 16x^4 + 2x^2 + 7$
32. $P(x) = -7x^6 - 5x^4 - x^2 + 2x + 1$
33. $P(x) = -3x^4 + 2x^3 - 4x^2 + x - 11$
34. $P(x) = 2x^4 - 3x^3 + 7x^2 + 3x + 2$

For each polynomial in Exercises 35–52: (a) use Descartes' rule of signs to determine the possible combinations of positive real zeros and negative real zeros; (b) use the rational zero test to determine possible rational zeros; (c) test for rational zeros; and (d) factor as a product of linear and/or irreducible quadratic factors.

35. $P(x) = x^3 + 6x^2 + 11x + 6$
36. $P(x) = x^3 - 6x^2 + 11x - 6$
37. $P(x) = x^3 - 7x^2 - x + 7$
38. $P(x) = x^3 - 5x^2 - 4x + 20$
39. $P(x) = x^4 + 6x^3 + 3x^2 - 10x$
40. $P(x) = x^4 - x^3 - 14x^2 + 24x$
41. $P(x) = x^4 - 7x^3 + 27x^2 - 47x + 26$
42. $P(x) = x^4 - 5x^3 + 5x^2 + 25x - 26$
43. $P(x) = 10x^3 - 7x^2 - 4x + 1$
44. $P(x) = 12x^3 - 13x^2 + 2x - 1$
45. $P(x) = 6x^3 + 17x^2 + x - 10$
46. $P(x) = 6x^3 + x^2 - 5x - 2$
47. $P(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$
48. $P(x) = x^4 + 2x^3 + 10x^2 + 18x + 9$
49. $P(x) = x^6 + 12x^4 + 23x^2 - 36$
50. $P(x) = x^4 - x^2 - 16x^2 + 16$
51. $P(x) = 4x^4 - 20x^3 + 37x^2 - 24x + 5$
52. $P(x) = 4x^4 - 8x^3 + 7x^2 + 30x + 50$

In Exercises 53–56, use the information found in Exercises 37, 41, 45, and 51 to assist in sketching a graph of each polynomial function.

53. Exercise 37
54. Exercise 41
55. Exercise 45
56. Exercise 51

In Exercises 57–64, use the intermediate value theorem and the bisection method to approximate the real zero in the indicated interval. Approximate to two decimal places.

57. $f(x) = x^4 - 3x^3 + 4 \quad [1, 2]$
58. $f(x) = x^5 - 3x^3 + 1 \quad [0, 1]$
59. $f(x) = 7x^5 - 2x^2 + 5x - 1 \quad [0, 1]$
60. $f(x) = -2x^3 + 3x^2 + 6x - 7 \quad [-2, -1]$
61. $f(x) = x^3 - 2x^2 - 8x - 3 \quad [-1, 0]$
62. $f(x) = x^4 + 4x^2 - 7x - 13 \quad [-2, -1]$
63. $f(x) = x^5 + 2x^4 - 6x^3 - 25x^2 + 8x - 10 \quad [2, 3]$
64. $f(x) = \frac{1}{2}x^6 + x^4 - 2x^2 - x + 1 \quad [1, 2]$

APPLICATIONS

65. **Geometry.** The distances (in inches) from one vertex of a rectangle to the other three vertices are x, $x + 2$, and $x + 4$. Find the dimensions of the rectangle.

66. **Geometry.** A box is constructed to contain a volume of 97.5 cubic inches. The length of the base is 3.5 inches larger than the width, and the height is 0.5 inch larger than the length. Find the dimensions of the box.

67. **Agriculture.** The weekly volume (in liters) of milk produced in a farm is given by $v(x) = x^3 + 21x^2 - 1480x$, where x is the number of cows. Find the number of cows that corresponds to a total production of 1500 liters of milk in a week.

68. **Profit.** A bakery uses the formula $f(x) = 2x^4 - 7x^3 + 3x^2 + 8x$ to determine the profit of selling x loaves of bread. How many loaves of bread must be sold to have a profit of \$4? Assume $x \geq 1$.

For Exercises 69 and 70, refer to the following:

The demand function for a product is

$$p(x) = 28 - 0.0002x$$

where p is the unit price (in dollars) of the product and x is the number of units produced and sold. The cost function for the product is

$$C(x) = 20x + 1500$$

where C is the total cost (in dollars) and x is the number of units produced. The total profit obtained by producing and selling x units is

$$P(x) = xp(x) - C(x)$$

69. **Business.** Find the total profit function when x units are produced and sold. Use Descartes' rule of signs to determine possible combinations of positive zeros for the profit function.

70. **Business.** Find the break-even point(s) for the product to the nearest unit. Discuss the significance of the break-even point(s) for the product.

71. **Health/Medicine.** During the course of treatment of an illness the concentration of a dose of a drug (in mcg/mL) in the bloodstream fluctuates according to the model

$$C(t) = 15.4 - 0.05t^2$$

where $t = 0$ is when the drug was administered. Assuming a single dose of the drug is administered, in how many hours (to the nearest hour) after being administered will the drug be eliminated from the bloodstream?

72. **Health/Medicine.** During the course of treatment of an illness, the concentration of a dose of a drug (in mcg/mL) in the bloodstream fluctuates according to the model

$$C(t) = 60 - 0.75t^2$$

where $t = 0$ is when the drug was administered. Assuming a single dose of the drug is administered, in how many hours (to the nearest hour) after being administered will the drug be eliminated from the bloodstream?

CATCH THE MISTAKE

In Exercises 73 and 74, explain the mistake that is made.

73. Use Descartes' rule of signs to determine the possible combinations of zeros of

$$P(x) = 2x^5 + 7x^4 + 9x^3 + 9x^2 + 7x + 2$$

Solution:

No sign changes, so no positive real zeros.

$$P(x) = 2x^5 + 7x^4 + 9x^3 + 9x^2 + 7x + 2$$

Five sign changes, so five negative real zeros.

$$P(-x) = -2x^5 + 7x^4 - 9x^3 + 9x^2 - 7x + 2$$

This is incorrect. What mistake was made?

74. Determine whether $x - 2$ is a factor of

$$P(x) = x^3 - 2x^2 - 5x + 6$$

Solution:

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

Yes, $x - 2$ is a factor of $P(x)$.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 75–80, determine whether each statement is true or false.

75. All real zeros of a polynomial correspond to x-intercepts.

76. A polynomial of degree n, $n > 0$, must have at least one zero.

77. A polynomial of degree n, $n > 0$, can be written as a product of n linear factors over real numbers.

78. The number of sign changes in a polynomial is equal to the number of positive real zeros of that polynomial.

79. A polynomial of degree n, $n > 0$, must have exactly n x-intercepts.

80. A polynomial with an odd number of zeros must have odd degree.

CHALLENGE

81. Given that $x = a$ is a zero of $P(x) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$, find the other two zeros, given that a, b, and c are real numbers and $a > b > c$.

82. Given that $x = a$ is a zero of $p(x) = x^3 + (-a + b - c)x^2 - (ab + bc - ac)x + abc$, find the other two real zeros, given that a, b, and c are real positive numbers.

83. Given that b is a zero of $P(x) = x^4 - (a + b)x^3 + (ab - c^2)x^2 + (a + b)c^2x - abc^2$, find the other three real zeros, given that a, b, and c are real positive numbers.

84. Given that a is a zero of $P(x) = x^4 + 2(b - a)x^3 + (a^2 - 4ab + b^2)x^2 + 2ab(a - b)x + a^2b^2$, find the other three real zeros, given that a and b are real positive numbers.

TECHNOLOGY

In Exercises 85 and 86, determine all possible rational zeros of the polynomial. There are many possibilities. Instead of trying them all, use a graphing calculator or software to graph $P(x)$ to help find a zero to test.

85. $P(x) = x^3 - 2x^2 + 16x - 32$

86. $P(x) = x^3 - 3x^2 + 16x - 48$

In Exercises 87 and 88: (a) determine all possible rational zeros of the polynomial, using a graphing calculator or software to graph $P(x)$ to help find the zeros; and (b) factor as a product of linear and/or irreducible quadratic factors.

87. $P(x) = 12x^4 + 25x^3 + 56x^2 - 7x - 30$

88. $P(x) = -3x^3 - x^2 - 7x - 49$

PREVIEW TO CALCULUS

In calculus we use the zeros of the derivative f' of a function f to determine whether the function f is increasing or decreasing around the zeros.

In Exercises 89–92, find the zeros of each polynomial function and determine the intervals over which $f(x) > 0$.

89. $f(x) = x^3 - 4x^2 - 7x + 10$

90. $f(x) = 6x^3 - 13x^2 - 11x + 8$

91. $f(x) = -2x^4 + 5x^3 + 7x^2 - 10x - 6$

92. $f(x) = -3x^4 + 14x^3 - 11x^2 + 14x - 8$

SECTION 2.5 COMPLEX ZEROS: THE FUNDAMENTAL THEOREM OF ALGEBRA

SKILLS OBJECTIVES

- Find the complex zeros of a polynomial function.
- Use the complex conjugate zeros theorem.
- Factor polynomial functions over the complex numbers.

CONCEPTUAL OBJECTIVES

- Extend the domain of polynomial functions to complex numbers.
- Understand how the fundamental theorem of algebra guarantees at least one zero.
- Understand why complex zeros occur in conjugate pairs.

Complex Zeros

Study Tip

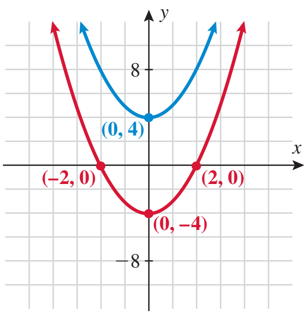
The zeros of a polynomial can be complex numbers. Only when the zeros are real numbers do we interpret zeros as x-intercepts.

In Section 2.4, we found the *real* zeros of a polynomial function. In this section, we find the *complex* zeros of a polynomial function. The domain of polynomial functions thus far has been the set of all real numbers. Now, we consider a more general case, where the domain of a polynomial function is the set of *complex numbers*. Note that the set of real numbers is a subset of the complex numbers. (Choose the imaginary part to be zero.)

It is important to note, however, that when we are discussing *graphs* of polynomial functions, we restrict the domain to the set of real numbers.

A *zero* of a polynomial $P(x)$ is the *solution* or *root* of the equation $P(x) = 0$. The *zeros of a polynomial can be complex numbers*. However, since the axes of the xy-plane represent real numbers, we interpret zeros as x-intercepts only when the zeros are real numbers.

We can illustrate the relationship between real and complex zeros of polynomial functions and their graphs with two similar examples. Let's take the two quadratic functions $f(x) = x^2 - 4$ and $g(x) = x^2 + 4$. The graphs of these two functions are parabolas that open upward with $f(x)$ shifted down four units and $g(x)$ shifted up four units as shown on the left. Setting each function equal to zero and solving for x, we find that the zeros for $f(x)$ are -2 and 2 and the zeros for $g(x)$ are $-2i$ and $2i$. Notice that the x-intercepts for $f(x)$ are $(-2, 0)$ and $(2, 0)$ and $g(x)$ has **no x-intercepts**.



Study Tip

The largest number of zeros a polynomial can have is equal to the degree of the polynomial.

The Fundamental Theorem of Algebra

In Section 2.4, we were able to write a polynomial function as a product of linear and/or irreducible quadratic factors. Now, we consider factors over complex numbers. Therefore, what were irreducible quadratic factors over real numbers will now be a product of two linear factors over the complex numbers.

What are the minimum and maximum number of zeros a polynomial can have? Every polynomial has *at least one zero* (provided the degree is greater than zero). The largest number of zeros a polynomial can have is equal to the degree of the polynomial.

THE FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial $P(x)$ of degree $n > 0$ has *at least one zero* in the complex number system.

The fundamental theorem of algebra and the factor theorem are used to prove the following n zeros theorem.

n ZEROS THEOREM

Every polynomial $P(x)$ of degree $n > 0$ can be expressed as the product of n linear factors in the complex number system. Hence, $P(x)$ has exactly n zeros, not necessarily distinct.

Study Tip

A polynomial of degree n has exactly n zeros (provided we count multiplicities).

These two theorems are illustrated with five polynomials below:

a. The **first-**degree polynomial $f(x) = x + 3$ has exactly **one** zero: $x = -3$.
b. The **second-**degree polynomial $f(x) = x^2 + 10x + 25 = (x + 5)(x + 5)$ has exactly **two** zeros: $x = -5$ and $x = -5$. It is customary to write this as a single zero of multiplicity 2 or refer to it as a repeated root.
c. The **third-**degree polynomial $f(x) = x^3 + 16x = x\left(x^2 + 16\right) = x(x + 4i)(x - 4i)$ has exactly **three** zeros: $x = 0$, $x = -4i$, and $x = 4i$.
d. The **fourth-**degree polynomial $f(x) = x^4 - 1 = \left(x^2 - 1\right)\left(x^2 + 1\right)$ $= (x - 1)(x + 1)(x - i)(x + i)$ has exactly **four** zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.
e. The **fifth-**degree polynomial $f(x) = x^5 = x \cdot x \cdot x \cdot x \cdot x$ has exactly **five** zeros: $x = 0$, which has multiplicity 5.

The fundamental theorem of algebra and the n zeros theorem only tell you that the zeros *exist*—not how to find them. We must rely on techniques discussed in Section 2.4 and additional strategies discussed in this section to determine the zeros.

Complex Conjugate Pairs

Often, at a grocery store or a drugstore, we see signs for special offers—"buy one, get one free." A similar phenomenon occurs for complex zeros of a polynomial function with real coefficients. If we restrict the coefficients of a polynomial to real numbers, complex zeros always come in conjugate pairs. In other words, if a zero of a polynomial function is a complex number, then another zero will always be its complex conjugate. Look at the third-degree polynomial in the above illustration, part (c), where two of the zeros were $-4i$ and $4i$, and in part (d), where two of the zeros were i and $-i$. In general, if we restrict the coefficients of a polynomial to real numbers, complex zeros always come in conjugate pairs.

Study Tip

If we restrict the coefficients of a polynomial to real numbers, complex zeros always come in conjugate pairs.

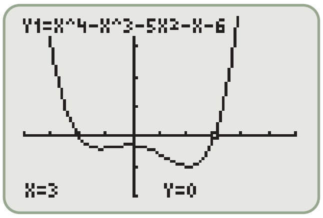
COMPLEX CONJUGATE ZEROS THEOREM

If a polynomial $P(x)$ has real coefficients, and if $a + bi$ is a zero of $P(x)$, then its complex conjugate $a - bi$ is also a zero of $P(x)$.

We use the complex zeros theorem to assist us in factoring a higher degree polynomial.

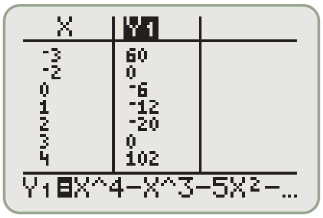
Technology Tip

The graph of $P(x) = x^4 - x^3 - 5x^2 - x - 6$ is shown.



The real zeros of the function at $x = -2$ and $x = 3$ give the factors of $x + 2$ and $x - 3$. Use the synthetic division to find the other factors.

A table of values supports the real zeros of the function and its factors.



■ **Answer:** $P(x) = (x - 2i)(x + 2i)(x - 1)(x - 2)$ *Note:* The zeros of $P(x)$ are 1, 2, $2i$, and $-2i$.

EXAMPLE 1 Factoring a Polynomial with Complex Zeros

Factor the polynomial $P(x) = x^4 - x^3 - 5x^2 - x - 6$ given that i is a zero of $P(x)$.

Since $P(x)$ is a *fourth*-degree polynomial, we expect *four* zeros. The goal in this problem is to write $P(x)$ as a product of four linear factors: $P(x) = (x - a)(x - b)(x - c)(x - d)$, where a, b, c, and d are complex numbers and represent the zeros of the polynomial.

Solution:

Write known zeros and linear factors.

Since i is a zero, we know that $-i$ is a zero. $\quad x = i \quad \text{and} \quad x = -i$

We now know two linear factors of $P(x)$. $\quad (x - i) \quad \text{and} \quad (x + i)$

Write $P(x)$ as a product of four factors. $\quad P(x) = (x - i)(x + i)(x - c)(x - d)$

Multiply the two known factors.

$$\begin{aligned}(x + i)(x - i) &= x^2 - i^2 \\ &= x^2 - (-1) \\ &= x^2 + 1\end{aligned}$$

Rewrite the polynomial. $\quad P(x) = \left(x^2 + 1\right)(x - c)(x - d)$

Divide both sides of the equation by $x^2 + 1$. $\quad \dfrac{P(x)}{x^2 + 1} = (x - c)(x - d)$

Divide $P(x)$ by $x^2 + 1$ using long division.

$$\begin{array}{r}
x^2 - x \quad - 6 \\
x^2 + 0x + 1 \overline{\smash{)}\, x^4 - \; x^3 - 5x^2 - \; x - 6} \\
\underline{-\left(x^4 + 0x^3 + \; x^2\right)} \qquad\quad \\
-x^3 - 6x^2 - \; x \quad\;\; \\
\underline{-\left(-x^3 + 0x^2 - \; x\right)} \quad\;\; \\
-6x^2 + 0x - 6 \\
\underline{-\left(-6x^2 + 0x - 6\right)} \\
0
\end{array}$$

Since the remainder is 0, $x^2 - x - 6$ is a factor. $\quad P(x) = \left(x^2 + 1\right)\left(x^2 - x - 6\right)$

Factor the quotient $x^2 - x - 6$. $\quad x^2 - x - 6 = (x - 3)(x + 2)$

Write $P(x)$ as a product of four linear factors. $\quad \boxed{P(x) = (x - i)(x + i)(x - 3)(x + 2)}$

Check: $P(x)$ is a *fourth*-degree polynomial and we found *four* zeros, two of which are complex conjugates.

■ **YOUR TURN** Factor the polynomial $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$ given that $x - 2i$ is a factor.

EXAMPLE 2 Factoring a Polynomial with Complex Zeros

Factor the polynomial $P(x) = x^4 - 2x^3 + x^2 + 2x - 2$ given that $1 + i$ is a zero of $P(x)$.

Since $P(x)$ is a *fourth*-degree polynomial, we expect *four* zeros. The goal in this problem is to write $P(x)$ as a product of four linear factors: $P(x) = (x - a)(x - b)(x - c)(x - d)$, where a, b, c, and d are complex numbers and represent the zeros of the polynomial.

Solution:

STEP 1 Write known zeros and linear factors.

Since $1 + i$ is a zero, we know that $1 - i$ is a zero. $\quad x = 1 + i$ and $x = 1 - i$

We now know two linear factors of $P(x)$. $\quad [x - (1 + i)]$ and $[x - (1 - i)]$

STEP 2 Write $P(x)$ as a product of four factors. $\quad P(x) = [x - (1 + i)][x - (1 - i)](x - c)(x - d)$

STEP 3 Multiply the first two terms. $\quad [x - (1 + i)][x - (1 - i)]$

First regroup the expressions in each bracket. $\quad [(x - 1) - i][(x - 1) + i]$

Use the special product $(a - b)(a + b) = a^2 - b^2$, where a is $(x - 1)$ and b is i.

$$(x-1)^2 - i^2$$
$$\left(x^2 - 2x + 1\right) - (-1)$$
$$x^2 - 2x + 2$$

STEP 4 Rewrite the polynomial. $\quad P(x) = \left(x^2 - 2x + 2\right)(x - c)(x - d)$

STEP 5 Divide both sides of the equation by $x^2 - 2x + 2$, and substitute in the original polynomial $P(x) = x^4 - 2x^3 + x^2 + 2x - 2$.

$$\frac{x^4 - 2x^3 + x^2 + 2x - 2}{x^2 - 2x + 2} = (x - c)(x - d)$$

STEP 6 Divide the left side of the equation using long division.

$$\frac{x^4 - 2x^3 + x^2 + 2x - 2}{x^2 - 2x + 2} = x^2 - 1$$

STEP 7 Factor $x^2 - 1$. $\quad (x - 1)(x + 1)$

STEP 8 Write $P(x)$ as a product of four linear factors.

$$\boxed{P(x) = [x - (1 + i)][x - (1 - i)][x - 1][x + 1]}$$

or

$$\boxed{P(x) = (x - 1 - i)(x - 1 + i)(x - 1)(x + 1)}$$

■ **YOUR TURN** Factor the polynomial $P(x) = x^4 - 2x^2 + 16x - 15$ given that $1 + 2i$ is a zero.

■ **Answer:** $P(x) = [x - (1 + 2i)] \cdot [x - (1 - 2i)](x - 1)(x + 3)$
Note: The zeros of $P(x)$ are 1, -3, $1 + 2i$, and $1 - 2i$.

Because an n-degree polynomial function has exactly n zeros and since complex zeros always come in conjugate pairs, if the degree of the polynomial is **odd**, there is guaranteed to be **at least one zero that is a real number**. If the degree of the polynomial is even, there is no guarantee that a zero will be real—all the zeros could be complex.

Study Tip

Odd-degree polynomials have at least one real zero.

EXAMPLE 3 **Finding Possible Combinations of Real and Complex Zeros**

List the possible combinations of real and complex zeros for the given polynomials.

a. $17x^5 + 2x^4 - 3x^3 + x^2 - 5$ **b.** $5x^4 + 2x^3 - x + 2$

Solution:

a. Since this is a *fifth*-degree polynomial, there are *five* zeros. Because complex zeros come in conjugate pairs, the table describes the possible five zeros.

Real Zeros	Complex Zeros
1	4
3	2
5	0

Applying Descartes' rule of signs, we find that there are three or one positive real zeros and two or no negative real zeros.

Positive Real Zeros	Negative Real Zeros	Complex Zeros
1	0	4
3	0	2
1	2	2
3	2	0

b. Because this is a *fourth*-degree polynomial, there are *four* zeros. Since complex zeros come in conjugate pairs, the table describes the possible four zeros.

Real Zeros	Complex Zeros
0	4
2	2
4	0

Applying Descartes' rule of signs, we find that there are two or no positive real zeros and two or no negative real zeros.

Positive Real Zeros	Negative Real Zeros	Complex Zeros
0	0	4
2	0	2
0	2	2
2	2	0

YOUR TURN List the possible combinations of real and complex zeros for

$$P(x) = x^6 - 7x^5 + 8x^3 - 2x + 1$$

Answer:

Real Zeros	Complex Zeros
0	6
2	4
4	2
6	0

Factoring Polynomials

Now let's draw on the tests discussed in this chapter to help us find all the zeros of a polynomial. Doing so will enable us to write polynomials as a product of linear factors. Before reading Example 4, reread Section 2.4, Example 7.

EXAMPLE 4 Factoring a Polynomial

Factor the polynomial $P(x) = x^5 + 2x^4 - x - 2$.

Solution:

STEP 1 Determine variations in sign.

$P(x)$ has one sign change. $\quad P(x) = x^5 + 2x^4 - x - 2$

$P(-x)$ has two sign changes. $\quad P(-x) = -x^5 + 2x^4 + x - 2$

STEP 2 Apply Descartes' rule of signs and summarize the results in a table.

Positive Real Zeros	Negative Real Zeros	Complex Zeros
1	2	2
1	0	4

STEP 3 Utilize the rational zero theorem to determine the possible rational zeros. $\quad \pm1, \pm2$

STEP 4 Test possible rational zeros.

1 is a zero:

$$\begin{array}{r|rrrrrr} 1 & 1 & 2 & 0 & 0 & -1 & -2 \\ & & 1 & 3 & 3 & 3 & 2 \\ \hline & 1 & 3 & 3 & 3 & 2 & \boxed{0} \end{array}$$

-1 is a zero:

$$\begin{array}{r|rrrrr} -1 & 1 & 3 & 3 & 3 & 2 \\ & & -1 & -2 & -1 & -2 \\ \hline & 1 & 2 & 1 & 2 & \boxed{0} \end{array}$$

-2 is a zero:

$$\begin{array}{r|rrrr} -2 & 1 & 2 & 1 & 2 \\ & & -2 & 0 & -2 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}$$

$$\underbrace{1 \quad 0 \quad 1}_{x^2 + 1} = (x - i)(x + i)$$

STEP 5 Write $P(x)$ as a product of linear factors.

$$\boxed{P(x) = (x - 1)(x + 1)(x + 2)(x - i)(x + i)}$$

Study Tip

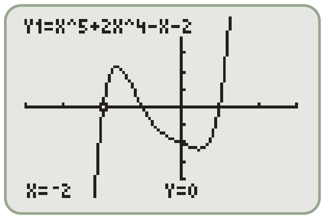
From Step 2 we know there is one positive real zero, so test the positive possible rational zeros first in Step 4.

Study Tip

In Step 4 we could have used synthetic division to show that 1, -1, and -2 are all zeros of $P(x)$.

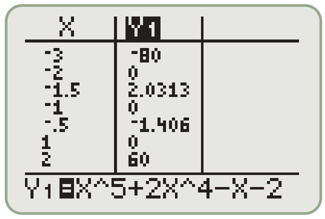
Technology Tip

The graph of $P(x) = x^5 + 2x^4 - x - 2$ is shown.



The real zeros of the function at $x = -2$, $x = -1$, and $x = 1$ give the factors of $x + 2$, $x + 1$, and $x - 1$. Use synthetic division to find the other factors.

A table of values supports the real zeros of the function and its factors.



Study Tip

$P(x)$ is a *fifth*-degree polynomial, so we expect *five* zeros.

SECTION 2.5 SUMMARY

In this section, we discussed **complex zeros** of polynomial functions. A polynomial function $P(x)$ of degree n with real coefficients has the following properties:

- $P(x)$ has at least one zero (if $n > 0$) and no more than n zeros.
- If $a + bi$ is a zero, then $a - bi$ is also a zero.
- The polynomial can be written as a product of linear factors, not necessarily distinct.

SECTION 2.5 EXERCISES

SKILLS

In Exercises 1–8, find all zeros (real and complex). Factor the polynomial as a product of linear factors.

1. $P(x) = x^2 + 4$
2. $P(x) = x^2 + 9$
3. $P(x) = x^2 - 2x + 2$
4. $P(x) = x^2 - 4x + 5$
5. $P(x) = x^4 - 16$
6. $P(x) = x^4 - 81$
7. $P(x) = x^4 - 25$
8. $P(x) = x^4 - 9$

In Exercises 9–16, a polynomial function is described. Find all remaining zeros.

9. Degree: 3 Zeros: $-1, i$
10. Degree: 3 Zeros: $1, -i$
11. Degree: 4 Zeros: $2i, 3 - i$
12. Degree: 4 Zeros: $3i, 2 + i$
13. Degree: 6 Zeros: 2 (multiplicity 2), $1 - 3i, 2 + 5i$
14. Degree: 6 Zeros: -2 (multiplicity 2), $1 - 5i, 2 + 3i$
15. Degree: 6 Zeros: $-i, 1 - i$ (multiplicity 2)
16. Degree: 6 Zeros: $2i, 1 + i$ (multiplicity 2)

In Exercises 17–22, find a polynomial of minimum degree that has the given zeros.

17. $0, 1 - 2i, 1 + 2i$
18. $0, 2 - i, 2 + i$
19. $1, 1 - 5i, 1 + 5i$
20. $2, 4 - i, 4 + i$
21. $1 - i, 1 + i, -3i, 3i$
22. $-i, i, 1 - 2i, 1 + 2i$

In Exercises 23–34, given a zero of the polynomial, determine all other zeros (real and complex) and write the polynomial in terms of a product of linear factors.

	Polynomial	Zero		Polynomial	Zero
23.	$P(x) = x^4 - 2x^3 - 11x^2 - 8x - 60$	$-2i$	24.	$P(x) = x^4 - x^3 + 7x^2 - 9x - 18$	$3i$
25.	$P(x) = x^4 - 4x^3 + 4x^2 - 4x + 3$	i	26.	$P(x) = x^4 - x^3 + 2x^2 - 4x - 8$	$-2i$
27.	$P(x) = x^4 - 2x^3 + 10x^2 - 18x + 9$	$-3i$	28.	$P(x) = x^4 - 3x^3 + 21x^2 - 75x - 100$	$5i$
29.	$P(x) = x^4 - 9x^2 + 18x - 14$	$1 + i$	30.	$P(x) = x^4 - 4x^3 + x^2 + 6x - 40$	$1 - 2i$
31.	$P(x) = x^4 - 6x^3 + 6x^2 + 24x - 40$	$3 - i$	32.	$P(x) = x^4 - 4x^3 + 4x^2 + 4x - 5$	$2 + i$
33.	$P(x) = x^4 - 9x^3 + 29x^2 - 41x + 20$	$2 - i$	34.	$P(x) = x^4 - 7x^3 + 14x^2 + 2x - 20$	$3 + i$

In Exercises 35–58, factor each polynomial as a product of linear factors.

35. $P(x) = x^3 - x^2 + 9x - 9$
36. $P(x) = x^3 - 2x^2 + 4x - 8$
37. $P(x) = x^3 - 5x^2 + x - 5$
38. $P(x) = x^3 - 7x^2 + x - 7$
39. $P(x) = x^3 + x^2 + 4x + 4$
40. $P(x) = x^3 + x^2 - 2$
41. $P(x) = x^3 - x^2 - 18$
42. $P(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$
43. $P(x) = x^4 - 2x^3 - 11x^2 - 8x - 60$
44. $P(x) = x^4 - x^3 + 7x^2 - 9x - 18$
45. $P(x) = x^4 - 4x^3 - x^2 - 16x - 20$
46. $P(x) = x^4 - 3x^3 + 11x^2 - 27x + 18$
47. $P(x) = x^4 - 7x^3 + 27x^2 - 47x + 26$
48. $P(x) = x^4 - 5x^3 + 5x^2 + 25x - 26$
49. $P(x) = -x^4 - 3x^3 + x^2 + 13x + 10$
50. $P(x) = -x^4 - x^3 + 12x^2 + 26x + 24$
51. $P(x) = x^4 - 2x^3 + 5x^2 - 8x + 4$
52. $P(x) = x^4 + 2x^3 + 10x^2 + 18x + 9$
53. $P(x) = x^6 + 12x^4 + 23x^2 - 36$
54. $P(x) = x^6 - 2x^5 + 9x^4 - 16x^3 + 24x^2 - 32x + 16$
55. $P(x) = 4x^4 - 20x^3 + 37x^2 - 24x + 5$
56. $P(x) = 4x^4 - 44x^3 + 145x^2 - 114x + 26$
57. $P(x) = 3x^5 - 2x^4 + 9x^3 - 6x^2 - 12x + 8$
58. $P(x) = 2x^5 - 5x^4 + 4x^3 - 26x^2 + 50x - 25$

APPLICATIONS

In Exercises 59–62, assume the profit model is given by a polynomial function $P(x)$, where x is the number of units sold by the company per year.

59. **Profit.** If the profit function of a given company has all imaginary zeros and the leading coefficient is positive, would you invest in this company? Explain.

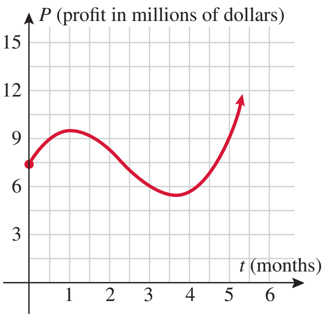
60. **Profit.** If the profit function of a given company has all imaginary zeros and the leading coefficient is negative, would you invest in this company? Explain.

61. Profit. If the profit function of a company is modeled by a third-degree polynomial with a negative leading coefficient and this polynomial has two complex conjugates as zeros and one positive real zero, would you invest in this company? Explain.

62. Profit. If the profit function of a company is modeled by a third-degree polynomial with a positive leading coefficient and this polynomial has two complex conjugates as zeros and one positive real zero, would you invest in this company? Explain.

For Exercises 63 and 64, refer to the following:

The following graph models the profit P of a company where t is months and $t \geq 0$.

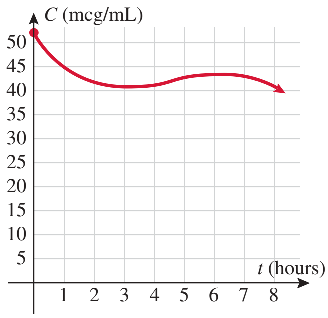


63. Business. If the profit function pictured is a third-degree polynomial, how many real and how many complex zeros does the function have? Discuss the implications of these zeros.

64. Business. If the profit function pictured is a fourth-degree polynomial with a negative leading coefficient, how many real and how many complex zeros does the function have? Discuss the implications of these zeros.

For Exercises 65 and 66, refer to the following:

The following graph models the concentration, C (in μg/mL) of a drug in the bloodstream; and t is time in hours after the drug is administered where $t \geq 0$.



65. Health/Medicine. If the concentration function pictured is a third-degree polynomial, how many real and how many complex zeros does the function have? Discuss the implications of these zeros.

66. Health/Medicine. If the concentration function pictured is a fourth-degree polynomial with a negative leading coefficient, how many real and how many complex zeros does the function have? Discuss the implications of these zeros.

CATCH THE MISTAKE

In Exercises 67 and 68, explain the mistake that is made.

67. Given that 1 is a zero of $P(x) = x^3 - 2x^2 + 7x - 6$, find all other zeros.

Solution:

Step 1: $P(x)$ is a third-degree polynomial, so we expect three zeros.

Step 2: Because 1 is a zero, -1 is a zero, so two linear factors are $(x - 1)$ and $(x + 1)$.

Step 3: Write the polynomial as a product of three linear factors.

$$P(x) = (x - 1)(x + 1)(x - c)$$
$$P(x) = (x^2 - 1)(x - c)$$

Step 4: To find the remaining linear factor, we divide $P(x)$ by $x^2 - 1$.

$$\frac{x^3 - 2x^2 + 7x - 6}{x^2 - 1} = x - 2 + \frac{6x - 8}{x^2 - 1}$$

Which has a nonzero remainder? What went wrong?

68. Factor the polynomial $P(x) = 2x^3 + x^2 + 2x + 1$.

Solution:

Step 1: Since $P(x)$ is an odd-degree polynomial, we are guaranteed one real zero (since complex zeros come in conjugate pairs).

Step 2: Apply the rational zero test to develop a list of potential rational zeros.

Possible zeros: ± 1

Step 3: Test possible zeros.

1 is not a zero: $P(x) = 2(1)^3 + (1)^2 + 2(1) + 1$
$= 6$

-1 is not a zero: $P(x) = 2(-1)^3 + (-1)^2 + 2(-1) + 1$
$= -2$

Note: $-\frac{1}{2}$ is the real zero. Why did we not find it?

CONCEPTUAL

In Exercises 69–72, determine whether each statement is true or false.

69. If $x = 1$ is a zero of a polynomial function, then $x = -1$ is also a zero of the polynomial function.

70. All zeros of a polynomial function correspond to x-intercepts.

71. A polynomial function of degree n, $n > 0$ must have at least one zero.

72. A polynomial function of degree n, $n > 0$ can be written as a product of n linear factors.

73. Is it possible for an odd-degree polynomial to have all imaginary complex zeros? Explain.

74. Is it possible for an even-degree polynomial to have all imaginary zeros? Explain.

CHALLENGE

In Exercises 75 and 76, assume *a* and *b* are nonzero real numbers.

75. Find a polynomial function that has degree 6 and for which bi is a zero of multiplicity 3.

76. Find a polynomial function that has degree 4 and for which $a + bi$ is a zero of multiplicity 2.

77. Find a polynomial function that has degree 6 and for which ai and bi are zeros, where ai has multiplicity 2. Assume $|a| \neq |b|$.

78. Assuming $|a| \neq |b|$, find a polynomial function of lowest degree for which ai and bi are zeros of equal multiplicity.

TECHNOLOGY

For Exercises 79 and 80, determine possible combinations of real and complex zeros. Plot $P(x)$ and identify any real zeros with a graphing calculator or software. Does this agree with your list?

79. $P(x) = x^4 + 13x^2 + 36$

80. $P(x) = x^6 + 2x^4 + 7x^2 - 130x - 288$

For Exercises 81 and 82, find all zeros (real and complex). Factor the polynomial as a product of linear factors.

81. $P(x) = -5x^5 + 3x^4 - 25x^3 + 15x^2 - 20x + 12$

82. $P(x) = x^5 + 2.1x^4 - 5x^3 - 5.592x^2 + 9.792x - 3.456$

PREVIEW TO CALCULUS

In Exercises 83–86, refer to the following:

In calculus we study the integration of rational functions by partial fractions.

a. Factor each polynomial into linear factors. Use complex numbers when necessary.

b. Factor each polynomial using only real numbers.

83. $f(x) = x^3 + x^2 + x + 1$

84. $f(x) = x^3 - 6x^2 + 21x - 26$

85. $f(x) = x^4 + 5x^2 + 4$

86. $f(x) = x^4 - 2x^3 - 7x^2 + 18x - 18$

SECTION 2.6 RATIONAL FUNCTIONS

SKILLS OBJECTIVES

- Find the domain of a rational function.
- Determine vertical, horizontal, and slant asymptotes of rational functions.
- Graph rational functions.

CONCEPTUAL OBJECTIVE

- Interpret the behavior of the graph of a rational function approaching an asymptote.

Domain of Rational Functions

So far in this chapter, we have discussed polynomial functions. We now turn our attention to *rational functions*, which are *ratios* of polynomial functions. Ratios of integers are called *rational numbers*. Similarly, ratios of polynomial functions are called *rational functions*.

DEFINITION **Rational Function**

A function $f(x)$ is a **rational function** if

$$f(x) = \frac{n(x)}{d(x)} \qquad d(x) \neq 0$$

where the numerator $n(x)$ and the denominator $d(x)$ are polynomial functions. The domain of $f(x)$ is the set of all real numbers x such that $d(x) \neq 0$.

Note: If $d(x)$ is a constant, then $f(x)$ is a polynomial function.

The domain of any polynomial function is the set of all real numbers. When we divide two polynomial functions, the result is a *rational function*, and we must exclude any values of x that make the denominator equal to zero.

EXAMPLE 1 **Finding the Domain of a Rational Function**

Find the domain of the rational function $f(x) = \frac{x + 1}{x^2 - x - 6}$. Express the domain in interval notation.

Solution:

Set the denominator equal to zero.	$x^2 - x - 6 = 0$
Factor.	$(x + 2)(x - 3) = 0$
Solve for x.	$x = -2$ or $x = 3$
Eliminate these values from the domain.	$x \neq -2$ or $x \neq 3$
State the domain in interval notation.	$\boxed{(-\infty, -2) \cup (-2, 3) \cup (3, \infty)}$

YOUR TURN Find the domain of the rational function $f(x) = \frac{x - 2}{x^2 - 3x - 4}$. Express the domain in interval notation.

Answer: The domain is the set of all real numbers such that $x \neq -1$ or $x \neq 4$. Interval notation: $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$

It is important to note that there are not always restrictions on the domain. For example, if the denominator is never equal to zero, the domain is the set of all real numbers.

EXAMPLE 2 **When the Domain of a Rational Function Is the Set of All Real Numbers**

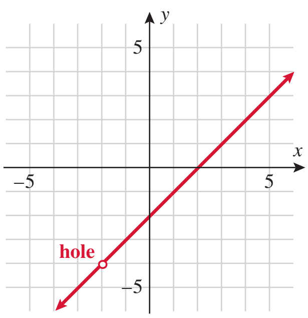
Find the domain of the rational function $g(x) = \frac{3x}{x^2 + 9}$. Express the domain in interval notation.

Solution:

Set the denominator equal to zero.	$x^2 + 9 = 0$
Subtract 9 from both sides.	$x^2 = -9$
Solve for x.	$x = -3i$ or $x = 3i$
There are no *real* solutions; therefore, the domain has no restrictions.	$\mathbb{R}$, the set of all real numbers
State the domain in interval notation.	$\boxed{(-\infty, \infty)}$

■ **YOUR TURN** Find the domain of the rational function $g(x) = \frac{5x}{x^2 + 4}$. Express the domain in interval notation.

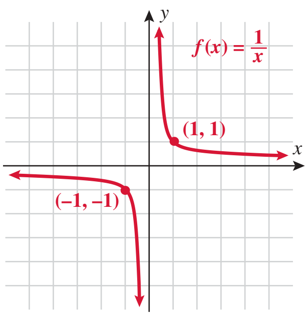
■ **Answer:** The domain is the set of all real numbers. Interval notation: $(-\infty, \infty)$



It is important to note that $f(x) = \frac{x^2 - 4}{x + 2}$, where $x \neq -2$, and $g(x) = x - 2$ are *not* the same function. Although $f(x)$ can be written in the factored form $f(x) = \frac{(x - 2)(x + 2)}{x + 2} = x - 2$, its domain is different. The domain of $g(x)$ is the set of all real numbers, whereas the domain of $f(x)$ is the set of all real numbers such that $x \neq -2$. If we were to plot $f(x)$ and $g(x)$, they would both look like the line $y = x - 2$. However, $f(x)$ would have a hole, or discontinuity, at the point $x = -2$.

Vertical, Horizontal, and Slant Asymptotes

If a function is not defined at a point, then it is still useful to know how the function behaves near that point. Let's start with a simple rational function, the reciprocal function $f(x) = \frac{1}{x}$. This function is defined everywhere except at $x = 0$.



x	$-\frac{1}{10}$	$-\frac{1}{100}$	$-\frac{1}{1000}$	0	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$
$f(x) = \frac{1}{x}$	−10	−100	−1000	undefined	1000	100	10

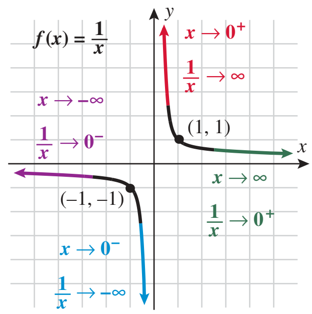
x approaching 0 from the left

x approaching 0 from the right

x	$f(x) = \frac{1}{x}$
−10	$-\frac{1}{10}$
−1	−1
1	1
10	$\frac{1}{10}$

We cannot let $x = 0$, because that point is not in the domain of the function. We should, however, ask the question, "How does $f(x)$ behave as x *approaches* zero?" Let us take values that get closer and closer to $x = 0$, such as $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$ (See the table above.) We use an *arrow* to represent the word *approach*, a *positive* superscript to represent from the *right*, and a *negative* superscript to represent from the *left*. A plot of this function can be generated using point-plotting techniques. The following are observations of the graph $f(x) = \frac{1}{x}$.

Words	Math
As x approaches zero from the *right*, the function $f(x)$ increases without bound.	$x \to 0^+$ $\frac{1}{x} \to \infty$
As x approaches zero from the *left*, the function $f(x)$ decreases without bound.	$x \to 0^-$ $\frac{1}{x} \to -\infty$
As x approaches infinity (increases without bound), the function $f(x)$ approaches zero from *above*.	$x \to \infty$ $\frac{1}{x} \to 0^+$
As x approaches negative infinity (decreases without bound), the function $f(x)$ approaches zero from *below*.	$x \to -\infty$ $\frac{1}{x} \to 0^-$



The symbol ∞ does not represent an actual real number. This symbol represents growing without bound.

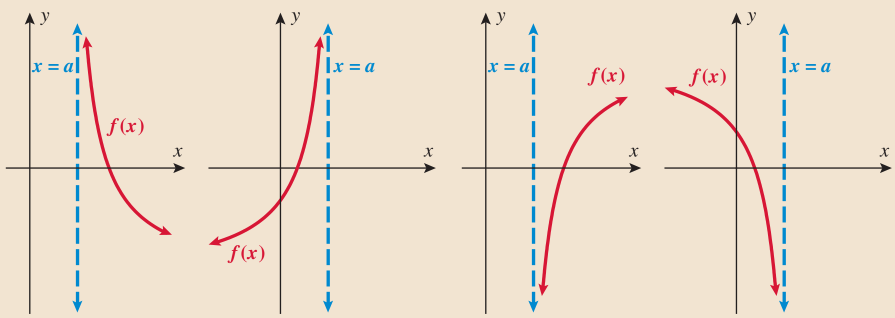
1. Notice that the function is not defined at $x = 0$. The y-axis, or the vertical line $x = 0$, represents the *vertical asymptote*.
2. Notice that the value of the function is never equal to zero. The x-axis is approached but not actually reached by the function. The x-axis, or $y = 0$, is a *horizontal asymptote*.

Asymptotes are lines that the graph of a function approaches. Suppose a football team's defense is its own 8 yard line and the team gets an "offsides" penalty that results in loss of "half the distance to the goal." Then the offense would get the ball on the 4 yard line. Suppose the defense gets another penalty on the next play that results in "half the distance to the goal." The offense would then get the ball on the 2 yard line. If the defense received 10 more penalties all resulting in "half the distance to the goal," would the referees *give* the offense a touchdown? No, because although the offense may appear to be snapping the ball from the goal line, technically it has not actually reached the goal line. Asymptotes utilize the same concept.

We will start with *vertical asymptotes*. Although the function $f(x) = \frac{1}{x}$ had one vertical asymptote, in general, rational functions can have *none*, *one*, or *several* vertical asymptotes. We will first formally define what a vertical asymptote is and then discuss how to find it.

Definition **Vertical Asymptotes**

The line $x = a$ is a **vertical asymptote** for the graph of a function if $f(x)$ either increases or decreases without bound as x approaches a from either the left or the right.



As $x \to a^+$ $f(x) \to \infty$ | As $x \to a^-$ $f(x) \to \infty$ | As $x \to a^+$ $f(x) \to -\infty$ | As $x \to a^-$ $f(x) \to -\infty$

Vertical asymptotes assist us in graphing rational functions since they essentially "steer" the function in the vertical direction. How do we locate the vertical asymptotes of a rational function? Set the denominator equal to zero. If the numerator and denominator have no common factors, then any numbers that are excluded from the domain of a rational function locate vertical asymptotes.

A rational function $f(x) = \dfrac{n(x)}{d(x)}$ is said to be in **lowest terms** if the numerator $n(x)$ and denominator $d(x)$ have no common factors. Let $f(x) = \dfrac{n(x)}{d(x)}$ be a rational function in lowest terms; then any zeros of the numerator $n(x)$ correspond to x-intercepts of the graph of f, and any zeros of the denominator $d(x)$ correspond to vertical asymptotes of the graph of f. If a rational function does have a common factor (is not in lowest terms), then the common factor(s) should be canceled, resulting in an equivalent rational function $R(x)$ in lowest terms. If $(x - a)^p$ is a factor of the numerator and $(x - a)^q$ is a factor of the denominator, then there is a *hole* in the graph at $x = a$ provided $p \geq q$ and $x = a$ is a vertical asymptote if $p < q$.

Study Tip

The vertical asymptotes of a rational function in *lowest terms* occur at x-values that make the denominator equal to zero.

LOCATING VERTICAL ASYMPTOTES

Let $f(x) = \dfrac{n(x)}{d(x)}$ be a rational function in lowest terms [that is, assume $n(x)$ and $d(x)$ are polynomials with no common factors]; then the graph of f has a vertical asymptote at any real zero of the denominator $d(x)$. That is, if $d(a) = 0$, then $x = a$ corresponds to a vertical asymptote on the graph of f.

Note: If f is a rational function that is not in lowest terms, then divide out the common factors, resulting in a rational function R that is in lowest terms. Any common factor $x - a$ of the function f corresponds to a hole in the graph of f at $x = a$ provided the multiplicity of a in the numerator is greater than or equal to the multiplicity of a in the denominator.

EXAMPLE 3 Determining Vertical Asymptotes

Locate any vertical asymptotes of the rational function $f(x) = \dfrac{5x + 2}{6x^2 - x - 2}$.

Solution:

Factor the denominator. $\quad f(x) = \dfrac{5x + 2}{(2x + 1)(3x - 2)}$

The numerator and denominator have no common factors, which means that all zeros of the denominator correspond to vertical asymptotes.

Set the denominator equal to zero. $\quad 2x + 1 = 0 \quad$ and $\quad 3x - 2 = 0$

Solve for x. $\quad x = -\dfrac{1}{2} \quad$ and $\quad x = \dfrac{2}{3}$

The vertical asymptotes are $\boxed{x = -\frac{1}{2}}$ and $\boxed{x = \frac{2}{3}}$.

YOUR TURN Locate any vertical asymptotes of the following rational function:

$$f(x) = \frac{3x - 1}{2x^2 - x - 15}$$

Answer: $x = -\frac{5}{2}$ and $x = 3$

EXAMPLE 4 **Determining Vertical Asymptotes When the Rational Function Is Not in Lowest Terms**

Locate any vertical asymptotes of the rational function $f(x) = \dfrac{x+2}{x^3 - 3x^2 - 10x}$.

Solution:

Factor the denominator.
$$x^3 - 3x^2 - 10x = x(x^2 - 3x - 10) = x(x-5)(x+2)$$

Write the rational function in factored form.
$$f(x) = \frac{(x+2)}{x(x-5)(x+2)}$$

Cancel (divide out) the common factor $(x + 2)$.
$$R(x) = \frac{1}{x(x-5)} \qquad x \neq -2$$

Find the values when the denominator of R is equal to zero.
$$x = 0 \quad \text{and} \quad x = 5$$

The vertical asymptotes are $\boxed{x = 0}$ and $\boxed{x = 5}$.

Note: $x = -2$ is not in the domain of $f(x)$, even though there is no vertical asymptote there. There is a "hole" in the graph at $x = -2$. Graphing calculators do not always show such "holes."

■ **YOUR TURN** Locate any vertical asymptotes of the following rational function:

$$f(x) = \frac{x^2 - 4x}{x^2 - 7x + 12}$$

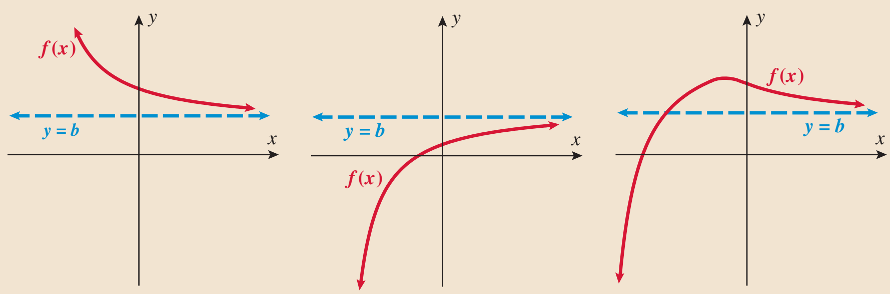
■ **Answer:** $x = 3$

We now turn our attention to *horizontal asymptotes.* As we have seen, rational functions can have several vertical asymptotes. However, rational functions can have *at most* one horizontal asymptote. Horizontal asymptotes imply that a function approaches a constant value as x becomes large in the positive or negative direction. Another difference between vertical and horizontal asymptotes is that the graph of a function never touches a vertical asymptote but, as you will see in the next box, the graph of a function may cross a horizontal asymptote, just not at the "ends" ($x \to \pm\infty$).

DEFINITION **Horizontal Asymptote**

The line $y = b$ is a **horizontal asymptote** of the graph of a function if $f(x)$ approaches b as x increases or decreases without bound. The following are three examples:

As $x \to \infty$, $f(x) \to b$



Note: A horizontal asymptote steers a function as x gets large. Therefore, when x is not large, the function may cross the asymptote.

How do we determine whether a horizontal asymptote exists? And, if it does, how do we locate it? We investigate the value of the rational function as $x \to \infty$ or as $x \to -\infty$. One of two things will happen: Either the rational function will increase or decrease without bound or the rational function will approach a constant value.

We say that a rational function is **proper** if the degree of the numerator is less than the degree of the denominator. Proper rational functions, like $f(x) = \frac{1}{x}$, approach zero as x gets large. Therefore, all proper rational functions have the specific horizontal asymptote, $y = 0$ (see Example 5a).

We say that a rational function is **improper** if the degree of the numerator is greater than or equal to the degree of the denominator. In this case, we can divide the numerator by the denominator and determine how the quotient behaves as x increases without bound.

- If the quotient is a constant (resulting when the degrees of the numerator and denominator are equal), then as $x \to \infty$ or as $x \to -\infty$, the rational function approaches the constant quotient (see Example 5b).
- If the quotient is a polynomial function of degree 1 or higher, then the quotient depends on x and does not approach a constant value as x increases (see Example 5c). In this case, we say that there is no horizontal asymptote.

We find horizontal asymptotes by comparing the degree of the numerator and the degree of the denominator. There are three cases to consider:

1. The degree of the numerator is less than the degree of the denominator.
2. The degree of the numerator is equal to the degree of the denominator.
3. The degree of the numerator is greater than the degree of the denominator.

Locating horizontal asymptotes

Let f be a rational function given by

$$f(x) = \frac{n(x)}{d(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where $n(x)$ and $d(x)$ are polynomials.

1. When $n < m$, the x-axis ($y = 0$) is the horizontal asymptote.
2. When $n = m$, the line $y = \frac{a_n}{b_m}$ (ratio of leading coefficients) is the horizontal asymptote.
3. When $n > m$, there is no horizontal asymptote.

In other words:

1. When the degree of the numerator is less than the degree of the denominator, then $y = 0$ is the horizontal asymptote.
2. When the degree of the numerator is the same as the degree of the denominator, then the horizontal asymptote is the ratio of the leading coefficients.
3. If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.

EXAMPLE 5 Finding Horizontal Asymptotes

Determine whether a horizontal asymptote exists for the graph of each of the given rational functions. If it does, locate the horizontal asymptote.

a. $f(x) = \dfrac{8x + 3}{4x^2 + 1}$ **b.** $g(x) = \dfrac{8x^2 + 3}{4x^2 + 1}$ **c.** $h(x) = \dfrac{8x^3 + 3}{4x^2 + 1}$

Solution (a):

The degree of the numerator $8x + 3$ is 1. $n = 1$

The degree of the denominator $4x^2 + 1$ is 2. $m = 2$

The degree of the numerator is less than the degree of the denominator. $n < m$

The x-axis is the horizontal asymptote for the graph of $f(x)$. $y = 0$

The line $\boxed{y = 0}$ is the horizontal asymptote for the graph of $f(x)$.

Solution (b):

The degree of the numerator $8x^2 + 3$ is 2. $n = 2$

The degree of the denominator $4x^2 + 1$ is 2. $m = 2$

The degree of the numerator is equal to the degree of the denominator. $n = m$

The ratio of the leading coefficients is the horizontal asymptote for the graph of $g(x)$. $y = \dfrac{8}{4} = 2$

The line $\boxed{y = 2}$ is the horizontal asymptote for the graph of $g(x)$.

If we divide the numerator by the denominator, the resulting quotient is the constant 2.

$$g(x) = \frac{8x^2 + 3}{4x^2 + 1} = 2 + \frac{1}{4x^2 + 1}$$

Solution (c):

The degree of the numerator $8x^3 + 3$ is 3. $n = 3$

The degree of the denominator $4x^2 + 1$ is 2. $m = 2$

The degree of the numerator is greater than the degree of the denominator. $n > m$

The graph of the rational function $h(x)$ has $\boxed{\text{no horizontal asymptote}}$.

If we divide the numerator by the denominator, the resulting quotient is a linear function and corresponds to the slant asymptote $y = 2x$.

$$h(x) = \frac{8x^3 + 3}{4x^2 + 1} = 2x + \frac{-2x + 3}{4x^2 + 1}$$

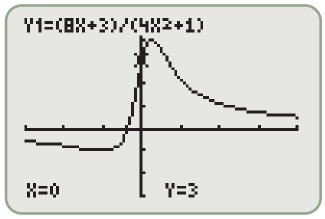
■ **YOUR TURN** Find the horizontal asymptote (if one exists) for the graph of the rational function $f(x) = \dfrac{7x^3 + x - 2}{-4x^3 + 1}$.

There are three types of lines: horizontal (slope is zero), vertical (slope is undefined), and slant (nonzero slope). Similarly, there are three types of linear asymptotes: horizontal, vertical, and *slant*.

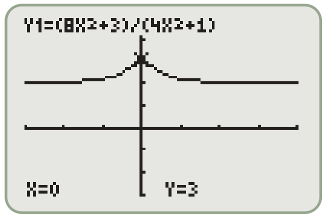
Technology Tip

The following graphs correspond to the rational functions given in Example 5. The horizontal asymptotes are apparent, but are not drawn in the graph.

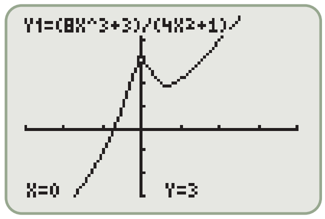
a. Graph $f(x) = \dfrac{8x + 3}{4x^2 + 1}$.



b. Graph $g(x) = \dfrac{8x^2 + 3}{4x^2 + 1}$.



c. Graph $h(x) = \dfrac{8x^3 + 3}{4x^2 + 1}$.



■ **Answer:** $y = -\frac{7}{4}$ is the horizontal asymptote.

Study Tip

There are three types of linear asymptotes: horizontal, vertical, and *slant*.

Recall that in dividing polynomials, the degree of the quotient is always the difference between the degree of the numerator and the degree of the denominator. For example, a cubic (third-degree) polynomial divided by a quadratic (second-degree) polynomial results in a linear (first-degree) polynomial. A fifth-degree polynomial divided by a fourth-degree polynomial results in a first-degree (linear) polynomial. When the degree of the numerator is exactly one more than the degree of the denominator, the quotient is linear and represents a *slant asymptote.*

Slant Asymptotes

Let f be a rational function given by $f(x) = \dfrac{n(x)}{d(x)}$, where $n(x)$ and $d(x)$ are polynomials and the degree of $n(x)$ is *one more than* the degree of $d(x)$. On dividing $n(x)$ by $d(x)$, the rational function can be expressed as

$$f(x) = mx + b + \frac{r(x)}{d(x)}$$

where the degree of the remainder $r(x)$ is less than the degree of $d(x)$ and the line $y = mx + b$ is a **slant asymptote** for the graph of f.

Note that as $x \to -\infty$ or $x \to \infty$, $f(x) \to mx + b$.

Technology Tip

The graph of $f(x) = \dfrac{4x^3 + x^2 + 3}{x^2 - x + 1}$ has a slant asymptote of $y = 4x + 5$.

■ **Answer:** $y = x + 5$

EXAMPLE 6 Finding Slant Asymptotes

Determine the slant asymptote of the rational function $f(x) = \dfrac{4x^3 + x^2 + 3}{x^2 - x + 1}$.

Solution:

Divide the numerator by the denominator with long division.

$$\begin{array}{r}
4x + 5 \phantom{{}+ 0x + 3} \\
x^2 - x + 1 \overline{\smash{)}\, 4x^3 + x^2 + 0x + 3} \\
-(4x^3 - 4x^2 + 4x) \phantom{{}+ 3} \\
\hline
5x^2 - 4x + 3 \\
-(5x^2 - 5x + 5) \\
\hline
x - 2
\end{array}$$

Note that as $x \to \pm\infty$, the rational expression approaches 0.

$$f(x) = \mathbf{4x + 5} + \underbrace{\frac{x - 2}{x^2 - x + 1}}_{\substack{\to 0 \text{ as} \\ x \to \pm\infty}}$$

The quotient is the slant asymptote.

$$\boxed{y = 4x + 5}$$

■ **YOUR TURN** Find the slant asymptote of the rational function $f(x) = \dfrac{x^2 + 3x + 2}{x - 2}$.

Study Tip

Common factors need to be divided out first; then the remaining x-values corresponding to a denominator value of 0 are vertical asymptotes.

Graphing Rational Functions

We can now graph rational functions using asymptotes as graphing aids. The following box summarizes the six-step procedure for graphing rational functions.

GRAPHING RATIONAL FUNCTIONS

Let f be a rational function given by $f(x) = \dfrac{n(x)}{d(x)}$.

Step 1: Find the domain of the rational function f.

Step 2: Find the **intercept(s)**.
- y-intercept: evaluate $f(0)$.
- x-intercept: solve the equation $n(x) = 0$ for x in the domain of f.

Step 3: Find any **holes**.
- Factor the numerator and denominator.
- Divide out common factors.
- A common factor $x - a$ corresponds to a hole in the graph of f at $x = a$ if the multiplicity of a in the numerator is greater than or equal to the multiplicity of a in the denominator.
- The result is an equivalent rational function $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms.

Step 4: Find any **asymptotes**.
- Vertical asymptotes: solve $q(x) = 0$.
- Compare the degree of the numerator and the degree of the denominator to determine whether either a horizontal or slant asymptote exists. If one exists, find it.

Step 5: Find **additional points** on the graph of f—particularly near asymptotes.

Step 6: **Sketch** the graph; draw the asymptotes, label the intercept(s) and additional points, and complete the graph with a smooth curve between and beyond the vertical asymptotes.

Study Tip

Any real number excluded from the domain of a rational function corresponds to either a vertical asymptote or a hole on its graph.

It is important to note that any real number eliminated from the domain of a rational function corresponds to either a vertical asymptote or a hole on its graph.

EXAMPLE 7 Graphing a Rational Function

Graph the rational function $f(x) = \dfrac{x}{x^2 - 4}$.

Solution:

STEP 1 Find the **domain**.

Set the denominator equal to zero.	$x^2 - 4 = 0$
Solve for x.	$x = \pm 2$
State the domain.	$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

STEP 2 Find the **intercepts**.

y-intercept:	$f(0) = \dfrac{0}{-4} = 0$	$y = 0$
x-intercepts:	$f(x) = \dfrac{x}{x^2 - 4} = 0$	$x = 0$

The only intercept is at the point $\boxed{(0, 0)}$.

STEP 3 Find any holes. $f(x) = \dfrac{x}{(x+2)(x-2)}$

There are no common factors, so f is in lowest terms.
Since there are no common factors, there are no holes on the graph of f.

STEP 4 Find any **asymptotes**.

Vertical asymptotes: $d(x) = (x+2)(x-2) = 0$

$\boxed{x = -2}$ and $\boxed{x = 2}$

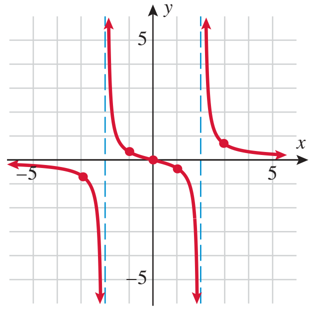
Horizontal asymptote: $\dfrac{\text{Degree of numerator}}{\text{Degree of denominator}} = \dfrac{1}{2}$

Degree of numerator < Degree of denominator $\boxed{y = 0}$

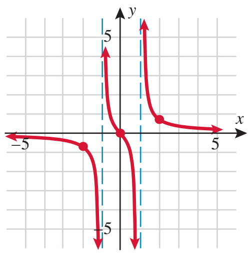
STEP 5 Find **additional points** on the graph.

x	-3	-1	1	3
$f(x)$	$-\frac{3}{5}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{3}{5}$

STEP 6 **Sketch** the graph; label the intercepts, asymptotes, and additional points and complete with a smooth curve approaching the asymptotes.



■ **Answer:**



■ **YOUR TURN** Graph the rational function $f(x) = \dfrac{x}{x^2 - 1}$.

EXAMPLE 8 Graphing a Rational Function with No Horizontal or Slant Asymptotes

State the asymptotes (if there are any) and graph the rational function $f(x) = \dfrac{x^4 - x^3 - 6x^2}{x^2 - 1}$.

Solution:

STEP 1 Find the domain.

Set the denominator equal to zero. $x^2 - 1 = 0$

Solve for x. $x = \pm 1$

State the domain. $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

STEP 2 Find the **intercepts**.

y-intercept: $f(0) = \dfrac{0}{-1} = 0$

x-intercepts: $n(x) = x^4 - x^3 - 6x^2 = 0$

Factor. $x^2(x-3)(x+2) = 0$

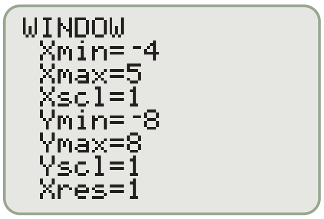
Solve. $x = 0, x = 3$, and $x = -2$

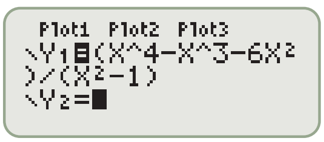
The **intercepts** are the points **(0, 0)**, **(3, 0)**, and **(−2, 0)**.

Technology Tip

The behavior of each function as x approaches ∞ or $-\infty$ can be shown using tables of values.

Graph $f(x) = \dfrac{x^4 - x^3 - 6x^2}{x^2 - 1}$.





STEP 3 Find any **holes**.
$$f(x) = \frac{x^2(x-3)(x+2)}{(x-1)(x+1)}$$

There are no common factors, so f is in lowest terms.
Since there are no common factors, there are no holes on the graph of f.

STEP 4 Find the **asymptotes**.

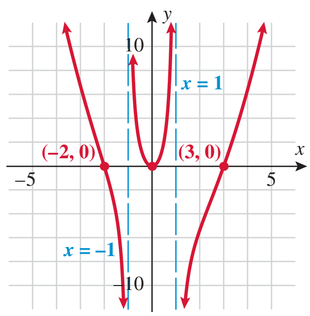
Vertical asymptote:	$d(x) = x^2 - 1 = 0$
Factor.	$(x+1)(x-1) = 0$
Solve.	$x = -1$ and $x = 1$
No horizontal asymptote: degree of $n(x) >$ degree of $d(x)$	$[4 > 2]$
No slant asymptote: degree of $n(x) -$ degree of $d(x) > 1$	$[4 - 2 = 2 > 1]$

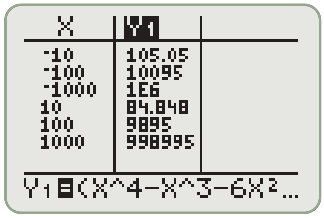
The **asymptotes** are $x = -1$ and $x = 1$.

STEP 5 Find **additional points** on the graph.

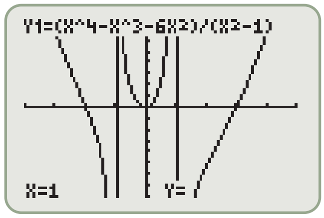
x	-3	-0.5	0.5	2	4
$f(x)$	6.75	1.75	2.08	-5.33	6.4

STEP 6 **Sketch** the graph; label the **intercepts** and **asymptotes**, and complete with a smooth curve between and beyond the vertical asymptote.



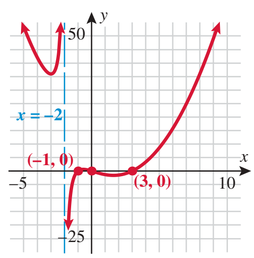


The graph of $f(x)$ shows that the vertical asymptotes are at $x = \pm 1$ and there is no horizontal asymptote or slant asymptote.



★ **YOUR TURN** State the asymptotes (if there are any) and graph the rational function:
$$f(x) = \frac{x^3 - 2x^2 - 3x}{x + 2}$$

■ **Answer:** Vertical asymptote: $x = -2$. No horizontal or slant asymptotes.



EXAMPLE 9 Graphing a Rational Function with a Horizontal Asymptote

State the asymptotes (if there are any) and graph the rational function
$$f(x) = \frac{4x^3 + 10x^2 - 6x}{8 - x^3}$$

Solution:

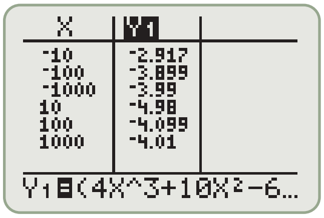
STEP 1 Find the **domain**.

Set the denominator equal to zero.	$8 - x^3 = 0$
Solve for x.	$x = 2$
State the domain.	$(-\infty, 2) \cup (2, \infty)$

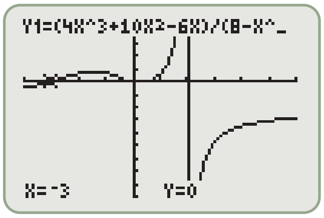
Technology Tip

The behavior of each function as x approaches ∞ or $-\infty$ can be shown using tables of values.

Graph $f(x) = \dfrac{4x^3 - 10x^2 - 6x}{8 - x^3}$.



The graph of $f(x)$ shows that the vertical asymptote is at $x = 1$ and the horizontal asymptote is at $y = -4$.



Step 2 Find the **intercepts**.

y-intercept: $f(0) = \dfrac{0}{8} = 0$

x-intercepts: $n(x) = 4x^3 + 10x^2 - 6x = 0$

Factor. $2x(2x - 1)(x + 3) = 0$

Solve. $x = 0,\ x = \dfrac{1}{2}$, and $x = -3$

The **intercepts** are the points **(0, 0)**, $\left(\frac{1}{2}, 0\right)$, and **(−3, 0)**.

Step 3 Find the **holes**. $f(x) = \dfrac{2x(2x - 1)(x + 3)}{(2 - x)(x^2 + 2x + 4)}$

There are no common factors, so f is in lowest terms (no holes).

Step 4 Find the **asymptotes**.

Vertical asymptote: $d(x) = 8 - x^3 = 0$

Solve. $x = 2$

Horizontal asymptote: degree of $n(x)$ = degree of $d(x)$

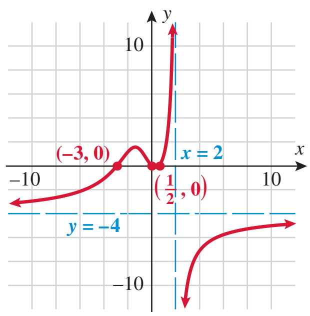
Use leading coefficients. $y = \dfrac{4}{-1} = -4$

The **asymptotes** are $x = 2$ and $y = -4$.

Step 5 Find **additional points** on the graph.

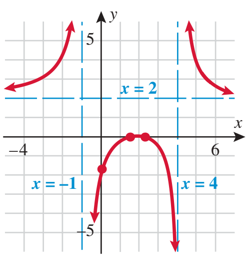
x	-4	-1	$\frac{1}{4}$	1	3
$f(x)$	-1	1.33	-0.10	1.14	-9.47

Step 6 **Sketch** the graph; label the intercepts and asymptotes and complete with a smooth curve.



■ **YOUR TURN** Graph the rational function $f(x) = \dfrac{2x^2 - 7x + 6}{x^2 - 3x - 4}$. Give equations of the vertical and horizontal asymptotes and state the intercepts.

■ **Answer:** Vertical asymptotes: $x = 4,\ x = -1$
Horizontal asymptote: $y = 2$
Intercepts: $\left(0, -\frac{3}{2}\right), \left(\frac{3}{2}, 0\right), (2, 0)$



EXAMPLE 10 Graphing a Rational Function with a Slant Asymptote

Graph the rational function $f(x) = \dfrac{x^2 - 3x - 4}{x + 2}$.

Solution:

STEP 1 Find the **domain**.

Set the denominator equal to zero. $x + 2 = 0$

Solve for x. $x = -2$

State the domain. $(-\infty, -2) \cup (-2, \infty)$

STEP 2 Find the **intercepts**.

y-intercept: $f(0) = -\dfrac{4}{2} = -2$

x-intercepts: $n(x) = x^2 - 3x - 4 = 0$

Factor. $(x + 1)(x - 4) = 0$

Solve. $x = -1$ and $x = 4$

The **intercepts** are the points **(0, −2)**, **(−1, 0)**, and **(4, 0)**.

STEP 3 Find any **holes**. $f(x) = \dfrac{(x - 4)(x + 1)}{(x + 2)}$

There are no common factors, so f is in lowest terms.
Since there are no common factors, there are no holes on the graph of f.

STEP 4 Find the **asymptotes**.

Vertical asymptote: $d(x) = x + 2 = 0$

Solve. $x = -2$

Slant asymptote: degree of $n(x)$ − degree of $d(x) = 1$

Divide $n(x)$ by $d(x)$. $f(x) = \dfrac{x^2 - 3x - 4}{x + 2} = x - 5 + \dfrac{6}{x + 2}$

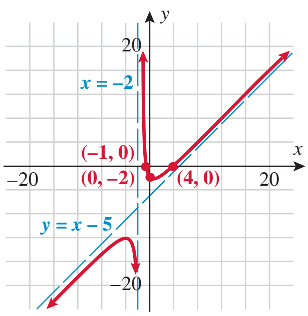
Write the equation of the asymptote. $y = x - 5$

The **asymptotes** are $x = -2$ and $y = x - 5$.

STEP 5 Find **additional points** on the graph.

x	−6	−5	−3	5	6
$f(x)$	−12.5	−12	−14	0.86	1.75

STEP 6 **Sketch** the graph; label the intercepts and asymptotes, and complete with a smooth curve between and beyond the vertical asymptote.



■ **YOUR TURN** For the function $f(x) = \dfrac{x^2 + x - 2}{x - 3}$, state the asymptotes (if any exist) and graph the function.

Technology Tip

The behavior of each function as x approaches ∞ or $-\infty$ can be shown using tables of values.

Graph $f(x) = \dfrac{x^2 - 3x - 4}{x + 2}$.

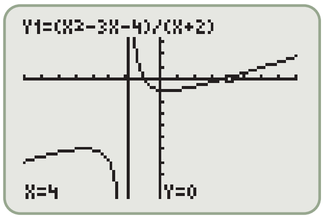
```
WINDOW
 Xmin=-8
 Xmax=8
 Xscl=1
 Ymin=-20
 Ymax=10
 Yscl=2
 Xres=1
```

```
 Plot1 Plot2 Plot3
\Y1=(X²-3X-4)/(X
+2)
```

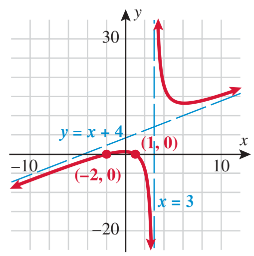
X	Y1
-10	-15.75
-100	-105.1
-1000	-1005
10	5.5
100	95.059
1000	995.01

Y1=(X²-3X-4)/(X...

The graph of $f(x)$ shows that the vertical asymptote is at $x = -2$ and the slant asymptote is at $y = x - 5$.



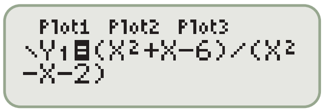
■ **Answer:**
Vertical asymptote: $x = 3$
Slant asymptote: $y = x + 4$

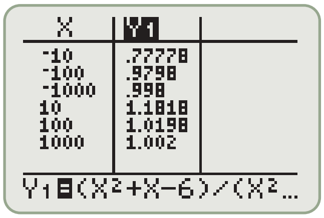


Technology Tip

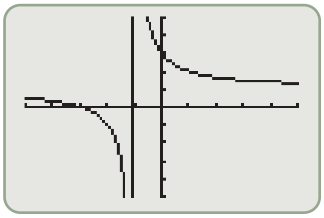
The behavior of each function as x approaches ∞ or $-\infty$ can be shown using tables of values.

Graph $f(x) = \dfrac{x^2 + x - 6}{x^2 - x - 2}$.

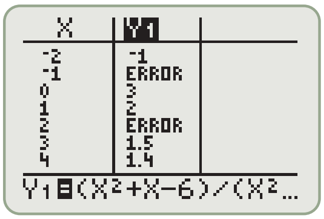




The graph of $f(x)$ shows that the vertical asymptote is at $x = -1$ and the horizontal asymptote is at $y = 1$.



Notice that the hole at $x = 2$ is not apparent in the graph. A table of values supports the graph.



EXAMPLE 11 Graphing a Rational Function with a Hole in the Graph

Graph the rational function $f(x) = \dfrac{x^2 + x - 6}{x^2 - x - 2}$.

Solution:

STEP 1 Find the **domain**.

Set the denominator equal to zero. $\qquad x^2 - x - 2 = 0$

Solve for x. $\qquad (x - 2)(x + 1) = 0$

$x = -1 \quad \text{or} \quad x = 2$

State the domain. $\qquad (-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

STEP 2 Find the **intercepts**.

y-intercept: $\qquad f(0) = \dfrac{-6}{-2} = 3 \qquad y = 3$

x-intercepts: $\qquad n(x) = x^2 + x - 6 = 0$

$(x + 3)(x - 2) = 0$

$x = -3 \quad \text{or} \quad x = 2$

The intercepts correspond to the points $\boxed{(0, 3)}$ and $\boxed{(-3, 0)}$. The point $(2, 0)$ appears to be an x-intercept; however, $x = 2$ is not in the domain of the function.

STEP 3 Find any **holes**. $\qquad f(x) = \dfrac{(x - 2)(x + 3)}{(x - 2)(x + 1)}$

Since $x - 2$ is a common factor, there is a *hole* in the graph of f at $x = 2$.

Dividing out the common factor generates an equivalent rational function in lowest terms. $\qquad R(x) = \dfrac{(x + 3)}{(x + 1)}$

STEP 4 Find the **asymptotes**.

Vertical asymptotes: $\qquad x + 1 = 0$

$\boxed{x = -1}$

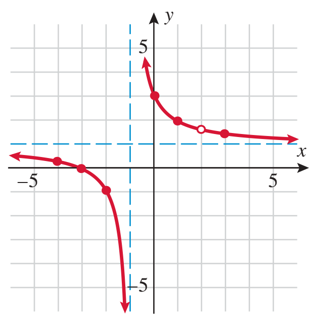
Horizontal asymptote: $\qquad \dfrac{\text{Degree of numerator}}{\text{Degree of denominator}} = \dfrac{\overbrace{2}^{f}}{2} = \dfrac{\overbrace{1}^{R}}{1}$

Since the degree of the numerator equals the degree of the denominator, use the leading coefficients. $\qquad \boxed{y = \dfrac{1}{1} = 1}$

STEP 5 Find **additional points** on the graph.

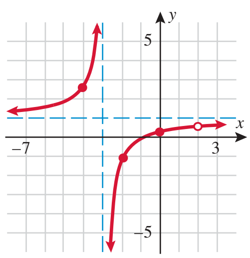
x	-4	-2	$-\frac{1}{2}$	1	3
$f(x)$ or $R(x)$	$\frac{1}{3}$	-1	5	2	$\frac{3}{2}$

STEP 6 **Sketch** the graph; label the intercepts, asymptotes, and additional points and complete with a smooth curve approaching asymptotes.



■ **YOUR TURN** Graph the rational function $f(x) = \dfrac{x^2 - x - 2}{x^2 + x - 6}$.

■ **Answer:**



SECTION 2.6 SUMMARY

In this section, rational functions were discussed.

$$f(x) = \frac{n(x)}{d(x)}$$

- **Domain:** All real numbers except the x-values that make the denominator equal to zero, $d(x) = 0$.
- **Vertical Asymptotes:** Vertical lines, $x = a$, where $d(a) = 0$, after all common factors have been divided out. Vertical asymptotes steer the graph and are never touched.
- **Horizontal Asymptote:** Horizontal line, $y = b$, that steers the graph as $x \longrightarrow \pm\infty$.
  1. If degree of the numerator $<$ degree of the denominator, then $y = 0$ is a horizontal asymptote.
  2. If degree of the numerator $=$ degree of the denominator, then $y = c$ is a horizontal asymptote, where c is the ratio of the leading coefficients of the numerator and denominator, respectively.
  3. If degree of the numerator $>$ degree of the denominator, then there is no horizontal asymptote.
- **Slant Asymptote:** Slant line, $y = mx + b$, that steers the graph as $x \longrightarrow \pm\infty$.
  1. If degree of the numerator $-$ degree of the denominator $= 1$, then there is a slant asymptote.
  2. Divide the numerator by the denominator. The quotient corresponds to the equation of the line (slant asymptote).

Procedure for Graphing Rational Functions

1. Find the domain of the function.
2. Find the intercept(s).
   - y-intercept (does not exist if $x = 0$ is a vertical asymptote)
   - x-intercepts (if any)
3. Find any holes.
   - If $x - a$ is a common factor of the numerator and denominator, then $x = a$ corresponds to a hole in the graph of the rational function if the multiplicity of a in the numerator is greater than or equal to the multiplicity of a in the denominator. The result after the common factor is canceled is an equivalent rational function in lowest terms (no common factor).
4. Find any asymptotes.
   - Vertical asymptotes
   - Horizontal/slant asymptotes
5. Find additional points on the graph.
6. Sketch the graph: Draw the asymptotes and label the intercepts and points and connect with a smooth curve.

SECTION 2.6 EXERCISES

SKILLS

In Exercises 1–8, find the domain of each rational function.

1. $f(x) = \dfrac{x + 4}{x^2 + x - 12}$

2. $f(x) = \dfrac{x - 1}{x^2 + 2x - 3}$

3. $f(x) = \dfrac{x - 2}{x^2 - 4}$

4. $f(x) = \dfrac{x + 7}{2(x^2 - 49)}$

5. $f(x) = \dfrac{7x}{x^2 + 16}$

6. $f(x) = -\dfrac{2x}{x^2 + 9}$

7. $f(x) = -\dfrac{3(x^2 + x - 2)}{2(x^2 - x - 6)}$

8. $f(x) = \dfrac{5(x^2 - 2x - 3)}{(x^2 - x - 6)}$

In Exercises 9–16, find all vertical asymptotes and horizontal asymptotes (if there are any).

9. $f(x) = \dfrac{1}{x + 2}$

10. $f(x) = \dfrac{1}{5 - x}$

11. $f(x) = \dfrac{7x^3 + 1}{x + 5}$

12. $f(x) = \dfrac{2 - x^3}{2x - 7}$

13. $f(x) = \dfrac{6x^5 - 4x^2 + 5}{6x^2 + 5x - 4}$

14. $f(x) = \dfrac{6x^2 + 3x + 1}{3x^2 - 5x - 2}$

15. $f(x) = \dfrac{\frac{1}{3}x^2 + \frac{1}{3}x - \frac{1}{4}}{x^2 + \frac{1}{9}}$

16. $f(x) = \dfrac{\frac{1}{10}\left(x^2 - 2x + \frac{3}{10}\right)}{2x - 1}$

In Exercises 17–22, find the slant asymptote corresponding to the graph of each rational function.

17. $f(x) = \dfrac{x^2 + 10x + 25}{x + 4}$ **18.** $f(x) = \dfrac{x^2 + 9x + 20}{x - 3}$ **19.** $f(x) = \dfrac{2x^2 + 14x + 7}{x - 5}$

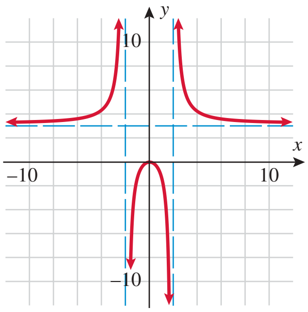
20. $f(x) = \dfrac{3x^3 + 4x^2 - 6x + 1}{x^2 - x - 30}$ **21.** $f(x) = \dfrac{8x^4 + 7x^3 + 2x - 5}{2x^3 - x^2 + 3x - 1}$ **22.** $f(x) = \dfrac{2x^6 + 1}{x^5 - 1}$

In Exercises 23–28, match the function to the graph.

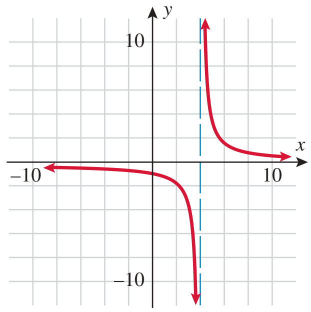
23. $f(x) = \dfrac{3}{x - 4}$ **24.** $f(x) = \dfrac{3x}{x - 4}$ **25.** $f(x) = \dfrac{3x^2}{x^2 - 4}$

26. $f(x) = -\dfrac{3x^2}{x^2 + 4}$ **27.** $f(x) = \dfrac{3x^2}{4 - x^2}$ **28.** $f(x) = \dfrac{3x^2}{x + 4}$

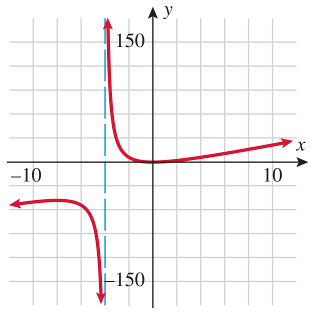
a.



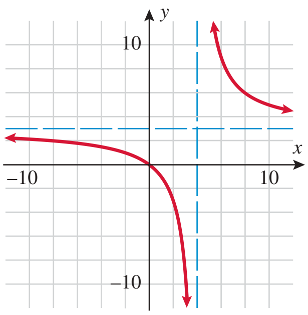
b.



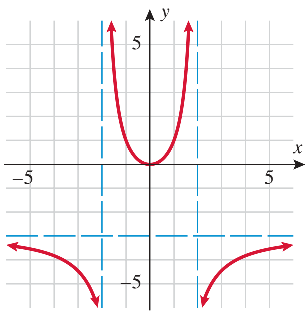
c.



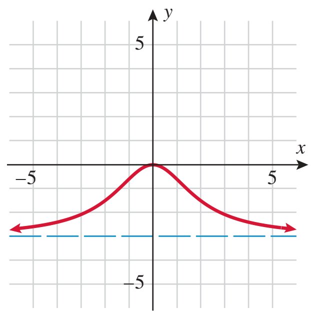
d.



e.



f.



In Exercises 29–56, graph the rational functions. Locate any asymptotes on the graph.

29. $f(x) = \dfrac{2}{x + 1}$ **30.** $f(x) = \dfrac{4}{x - 2}$ **31.** $f(x) = \dfrac{2x}{x - 1}$ **32.** $f(x) = \dfrac{4x}{x + 2}$

33. $f(x) = \dfrac{x - 1}{x}$ **34.** $f(x) = \dfrac{2 + x}{x - 1}$ **35.** $f(x) = \dfrac{2(x^2 - 2x - 3)}{x^2 + 2x}$ **36.** $f(x) = \dfrac{3(x^2 - 1)}{x^2 - 3x}$

37. $f(x) = \dfrac{x^2}{x + 1}$ **38.** $f(x) = \dfrac{x^2 - 9}{x + 2}$ **39.** $f(x) = \dfrac{2x^3 - x^2 - x}{x^2 - 4}$ **40.** $f(x) = \dfrac{3x^3 + 5x^2 - 2x}{x^2 + 4}$

41. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$ **42.** $f(x) = \dfrac{1 - x^2}{x^2 + 1}$ **43.** $f(x) = \dfrac{7x^2}{(2x + 1)^2}$ **44.** $f(x) = \dfrac{12x^4}{(3x + 1)^4}$

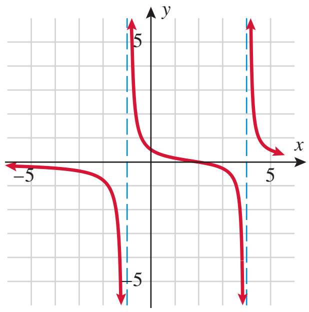
45. $f(x) = \dfrac{1 - 9x^2}{(1 - 4x^2)^3}$ **46.** $f(x) = \dfrac{25x^2 - 1}{(16x^2 - 1)^2}$ **47.** $f(x) = 3x + \dfrac{4}{x}$ **48.** $f(x) = x - \dfrac{4}{x}$

49. $f(x) = \dfrac{(x - 1)^2}{(x^2 - 1)}$ **50.** $f(x) = \dfrac{(x + 1)^2}{(x^2 - 1)}$ **51.** $f(x) = \dfrac{(x - 1)(x^2 - 4)}{(x - 2)(x^2 + 1)}$ **52.** $f(x) = \dfrac{(x - 1)(x^2 - 9)}{(x - 3)(x^2 + 1)}$

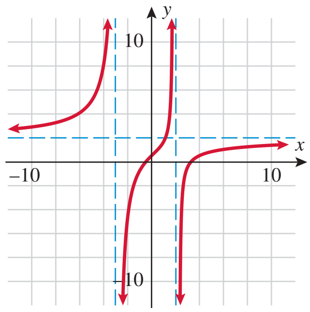
53. $f(x) = \dfrac{3x(x - 1)}{x(x^2 - 4)}$ **54.** $f(x) = \dfrac{-2x(x - 3)}{x(x^2 + 1)}$ **55.** $f(x) = \dfrac{x^2(x + 5)}{2x(x^2 + 3)}$ **56.** $f(x) = \dfrac{4x(x - 1)(x + 2)}{x^2(x^2 - 4)}$

In Exercises 57–60, for each graph of the rational function given determine: (a) all intercepts, (b) all asymptotes, and (c) an equation of the rational function.

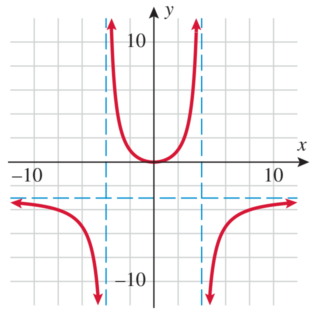
57.



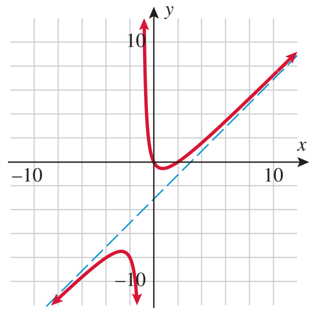
58.



59.



60.



APPLICATIONS

61. Epidemiology. Suppose the number of individuals infected by a virus can be determined by the formula

$$n(t) = \frac{9500t - 2000}{4 + t}$$

where $t > 0$ is the time in months.

a. Find the number of infected people by the end of the fourth month.

b. After how many months are there 5500 infected people?

c. What happens with the number of infected people if the trend continues?

62. Investment. A financial institution offers to its investors a variable annual interest rate using the formula

$$r(x) = \frac{4x^2}{x^2 + 2x + 5}$$

where x is the amount invested in thousands of dollars.

a. What is the annual interest rate for an investment of \$8000?

b. What is the annual interest rate for an investment of \$20,000?

c. What is the maximum annual interest rate offered by them?

63. Medicine. The concentration C of a particular drug in a person's bloodstream t minutes after injection is given by

$$C(t) = \frac{2t}{t^2 + 100}$$

a. What is the concentration in the bloodstream after 1 minute?

b. What is the concentration in the bloodstream after 1 hour?

c. What is the concentration in the bloodstream after 5 hours?

d. Find the horizontal asymptote of $C(t)$. What do you expect the concentration to be after several days?

64. Medicine. The concentration C of aspirin in the bloodstream t hours after consumption is given by $C(t) = \frac{t}{t^2 + 40}$.

a. What is the concentration in the bloodstream after $\frac{1}{2}$ hour?

b. What is the concentration in the bloodstream after 1 hour?

c. What is the concentration in the bloodstream after 4 hours?

d. Find the horizontal asymptote for $C(t)$. What do you expect the concentration to be after several days?

65. Typing. An administrative assistant is hired after graduating from high school and learns to type on the job. The number of words he can type per minute is given by

$$N(t) = \frac{130t + 260}{t + 5} \qquad t \geq 0$$

where t is the number of months he has been on the job.

a. How many words per minute can he type the day he starts?

b. How many words per minute can he type after 12 months?

c. How many words per minute can he type after 3 years?

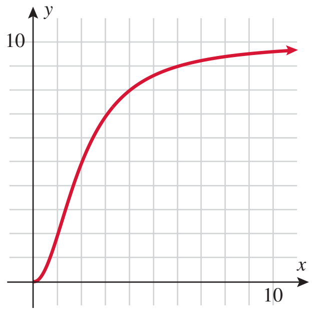
d. How many words per minute would you expect him to type if he worked there until he retired?

66. Memorization. A professor teaching a large lecture course tries to learn students' names. The number of names she can remember $N(t)$ increases with each week in the semester t and is given by the rational function

$$N(t) = \frac{600t}{t + 20}$$

How many students' names does she know by the third week in the semester? How many students' names should she know by the end of the semester (16 weeks)? According to this function, what are the most names she can remember?

67. Food. The amount of food that cats typically eat increases as their weight increases. A rational function that describes this is $F(x) = \dfrac{10x^2}{x^2 + 4}$, where the amount of food $F(x)$ is given in ounces and the weight of the cat x is given in pounds. Calculate the horizontal asymptote. How many ounces of food will most adult cats eat?



68. Memorization. The *Guinness Book of World Records, 2004* states that Dominic O'Brien (England) memorized on a single sighting a random sequence of 54 separate packs of cards all shuffled together (2808 cards in total) at Simpson's-In-The-Strand, London, England, on May 1, 2002. He memorized the cards in 11 hours 42 minutes, and then recited them in exact sequence in a time of 3 hours 30 minutes. With only a 0.5% margin of error allowed (no more than 14 errors), he broke the record with just 8 errors. If we let x represent the time (hours) it takes to memorize the cards and y represent the number of cards memorized, then a rational function that models this event is given by $y = \dfrac{2800x^2 + x}{x^2 + 2}$. According to this model, how many cards could be memorized in an hour? What is the greatest number of cards that can be memorized?

69. Gardening. A 500-square-foot rectangular garden will be enclosed with fencing. Write a rational function that describes how many linear feet of fence will be needed to enclose the garden as a function of the width of the garden w.

70. Geometry. A rectangular picture has an area of 414 square inches. A border (matting) is used when framing. If the top and bottom borders are each 4 inches and the side borders are 3.5 inches, write a function that represents the area $A(l)$ of the entire frame as a function of the length of the picture l.

For Exercises 71 and 72, refer to the following:

The monthly profit function for a product is given by

$$P(x) = -x^3 + 10x^2$$

where x is the number of units sold measured in thousands and P is profit measured in thousands of dollars. The average profit, which represents the profit per thousand units sold, for this product is given by

$$\overline{P}(x) = \frac{-x^3 + 10x^2}{x}$$

where x is units sold measured in thousands and P is profit measured in thousands of dollars.

71. Business. Find the number of units that must be sold to produce an average profit of \$16,000 per thousand units. Convert the answer to average profit in dollars per unit.

72. Business. Find the number of units that must be sold to produce an average profit of \$25,000 per thousand units. Convert the answer to average profit in dollars per unit.

For Exercises 73 and 74, refer to the following:

Some medications, such as Synthroid, are prescribed as a maintenance drug because they are taken regularly for an ongoing condition, such as hypothyroidism. Maintenance drugs function by maintaining a therapeutic drug level in the bloodstream over time. The concentration of a maintenance drug over a 24-hour period is modeled by the function

$$C(t) = \frac{22(t - 1)}{t^2 + 1} + 24$$

where t is time in hours after the dose was administered and C is the concentration of the drug in the bloodstream measured in μg/mL. This medication is designed to maintain a consistent concentration in the bloodstream of approximately 25 μg/mL. *Note:* This drug will become inert; that is, the concentration will drop to 0 μg/mL, during the 25th hour after taking the medication.

73. Health/Medicine. Find the concentration of the drug, to the nearest tenth of μg/mL, in the bloodstream 15 hours after the dose is administered. Is this the only time this concentration of the drug is found in the bloodstream? At what other times is this concentration reached? Round to the nearest hour. Discuss the significance of this answer.

74. Health/Medicine. Find the time, after the first hour and a half, at which the concentration of the drug in the bloodstream has dropped to 25 μg/mL. Find the concentration of the drug 24 hours after taking a dose to the nearest tenth of a μg/mL. Discuss the importance of taking the medication every 24 hours rather than every day.

CATCH THE MISTAKE

In Exercises 75–78, explain the mistake that is made.

75. Determine the vertical asymptotes of the function $f(x) = \dfrac{x-1}{x^2-1}$.

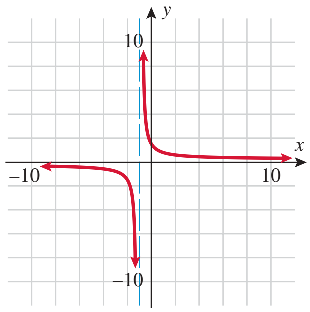
Solution:

Set the denominator equal to zero. $x^2 - 1 = 0$

Solve for x. $x = \pm 1$

The vertical asymptotes are $x = -1$ and $x = 1$.

The following is a correct graph of the function:



Note that only $x = -1$ is an asymptote. What went wrong?

76. Determine the vertical asymptotes of $f(x) = \dfrac{2x}{x^2+1}$.

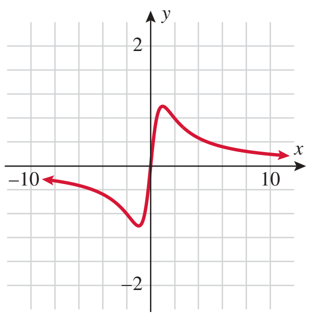
Solution:

Set the denominator equal to zero. $x^2 + 1 = 0$

Solve for x. $x = \pm 1$

The vertical asymptotes are $x = -1$ and $x = 1$.

The following is a correct graph of the function:



Note that there are no vertical asymptotes. What went wrong?

77. Determine whether a horizontal or a slant asymptote exists for the function $f(x) = \dfrac{9-x^2}{x^2-1}$. If one does, find it.

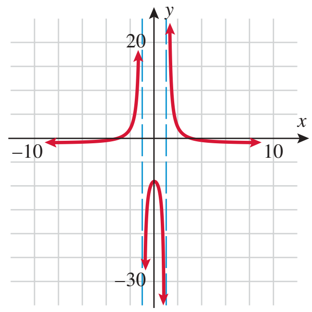
Solution:

Step 1: The degree of the numerator equals the degree of the denominator, so there is a horizontal asymptote.

Step 2: The horizontal asymptote is the ratio of the lead coefficients: $y = \frac{9}{1} = 9$.

The horizontal asymptote is $y = 9$.

The following is a correct graph of the function.



Note that there is no horizontal asymptote at $y = 9$. What went wrong?

78. Determine whether a horizontal or a slant asymptote exists for the function $f(x) = \dfrac{x^2+2x-1}{3x^3-2x^2-1}$. If one does, find it.

Solution:

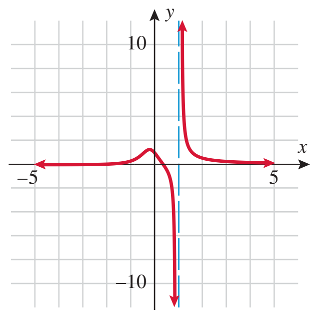
Step 1: The degree of the denominator is exactly one more than the degree of the numerator, so there is a slant asymptote.

Step 2: Divide.

$$
\begin{array}{r}
3x - 8 \\
x^2 + 2x - 1 \overline{\smash{\big)}\, 3x^3 - 2x^2 + 0x - 1} \\
\underline{3x^3 + 6x^2 - 3x } \\
-8x^2 + 3x - 1 \\
\underline{8x^2 - 16x + 8} \\
19x - 9
\end{array}
$$

The slant asymptote is $y = 3x - 8$.

The following is the correct graph of the function.



Note that $y = 3x - 8$ is not an asymptote. What went wrong?

CONCEPTUAL

For Exercises 79–82, determine whether each statement is true or false.

79. A rational function can have either a horizontal asymptote or an oblique asymptote, but not both.

80. A rational function can have at most one vertical asymptote.

81. A rational function can cross a vertical asymptote.

82. A rational function can cross a horizontal or an oblique asymptote.

83. Determine the asymptotes of the rational function $f(x) = \dfrac{(x - a)(x + b)}{(x - c)(x + d)}$.

84. Determine the asymptotes of the rational function $f(x) = \dfrac{3x^2 + b^2}{x^2 + a^2}$.

CHALLENGE

85. Write a rational function that has vertical asymptotes at $x = -3$ and $x = 1$ and a horizontal asymptote at $y = 4$.

86. Write a rational function that has no vertical asymptotes, approaches the x-axis as a horizontal asymptote, and has an x-intercept of $(3, 0)$.

87. Write a rational function that has no vertical asymptotes and oblique asymptote $y = x$, y-intercept $(0, 1)$, and x-intercept $(-1, 0)$. Round your answers to two decimal places.

88. Write a rational function that has vertical asymptotes at $x = -3$ and $x = 1$ and oblique asymptote $y = 3x$, y-intercept $(0, 2)$, and x-intercept $(2, 0)$. Round your answers to two decimal places.

TECHNOLOGY

89. Determine the vertical asymptotes of $f(x) = \dfrac{x - 4}{x^2 - 2x - 8}$. Graph this function utilizing a graphing utility. Does the graph confirm the asymptotes?

90. Determine the vertical asymptotes of $f(x) = \dfrac{2x + 1}{6x^2 + x - 1}$. Graph this function utilizing a graphing utility. Does the graph confirm the asymptotes?

91. Find the asymptotes and intercepts of the rational function $f(x) = \dfrac{1}{3x + 1} - \dfrac{2}{x}$. (*Note:* Combine the two expressions into a single rational expression.) Graph this function utilizing a graphing utility. Does the graph confirm what you found?

92. Find the asymptotes and intercepts of the rational function $f(x) = -\dfrac{1}{x^2 + 1} + \dfrac{1}{x}$. (*Note:* Combine the two expressions into a single rational expression.) Graph this function utilizing a graphing utility. Does the graph confirm what you found?

For Exercises 93 and 94: (a) Identify all asymptotes for each function. (b) Plot $f(x)$ and $g(x)$ in the same window. How does the end behavior of the function f differ from that of g? (c) Plot $g(x)$ and $h(x)$ in the same window. How does the end behavior of g differ from that of h? (d) Combine the two expressions into a single rational expression for the functions g and h. Does the strategy of finding horizontal and slant asymptotes agree with your findings in (b) and (c)?

93. $f(x) = \dfrac{1}{x - 3}$, $g(x) = 2 + \dfrac{1}{x - 3}$, and $h(x) = -3 + \dfrac{1}{x - 3}$

94. $f(x) = \dfrac{2x}{x^2 - 1}$, $g(x) = x + \dfrac{2x}{x^2 - 1}$, and $h(x) = x - 3 + \dfrac{2x}{x^2 - 1}$

PREVIEW TO CALCULUS

In calculus the integral of a rational function f on an interval $[a, b]$ might not exist if f has a vertical asymptote in $[a, b]$.

In Exercises 95–98, find the vertical asymptotes of each rational function.

95. $f(x) = \dfrac{x - 1}{x^3 - 2x^2 - 13x - 10}$ $\quad [0, 3]$

96. $f(x) = \dfrac{x^2 + x + 2}{x^3 + 2x^2 - 25x - 50}$ $\quad [-3, 2]$

97. $f(x) = \dfrac{5x + 2}{6x^2 - x - 2}$ $\quad [-2, 0]$

98. $f(x) = \dfrac{6x - 2x^2}{x^3 + x}$ $\quad [-1, 1]$

CHAPTER 2 INQUIRY-BASED LEARNING PROJECT

Discovering the Connection between the Standard Form of a Quadratic Function and Transformations of the Square Function

In Chapter 1, you saw that if you are familiar with the graphs of a small library of common functions, you can sketch the graphs of many related functions using transformation techniques. These ideas will help you here as you discover the relationship between the standard form of a quadratic function and its graph.

Let G and H be functions with:

$$G(x) = F(x - 1) + 3 \quad \text{and} \quad H(x) = -F(x + 2) - 4$$

where $F(x) = x^2$.

1. For this part, consider the function G.

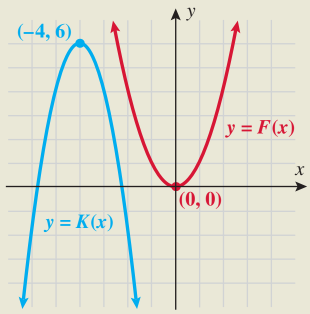
- **a.** List the transformation you'd use to sketch the graph of G from the graph of F.
- **b.** Write an equation for $G(x)$ in the form $G(x) = a(x - h)^2 + k$. This is called the **standard form** of a quadratic function. What are the values of a, h, and k?
- **c.** The **vertex**, or turning point, of the graph of $F(x) = x^2$ is (0, 0). How can you use the transformations you listed in part (a) to determine the coordinates of the vertex of the graph of G?
- **d.** The vertical line that passes through the vertex of a parabola is called its **axis of symmetry**. The axis of symmetry of the graph of $F(x) = x^2$ is the y-axis, or the vertical line with equation $x = 0$. How can you determine the axis of symmetry of the graph of G? Write the equation of this line.
- **e.** Sketch graphs of F and G.

2. Next consider the function H given above.

- **a.** List the transformations that will produce the graph of H from the graph of F.
- **b.** Write an equation for $H(x)$ in standard form. What are the values of a, h, and k?
- **c.** What are the coordinates of the vertex of the graph of H? How do the transformations you listed in part (a) help you determine this?
- **d.** Determine the equation of the axis of symmetry of the graph of H.
- **e.** Sketch graphs of F and H.

3. a. What do you know about the graph of a quadratic function just by looking at its equation in standard form, $f(x) = a(x - h)^2 + k$?

b. Shown below are the graphs of $F(x) = x^2$ and another quadratic function, $y = K(x)$. Write the equation of K in standard form. *Hint:* Think about the transformations.



MODELING OUR WORLD

The following table summarizes average yearly temperature in degrees Fahrenheit (°F) and carbon dioxide emissions in parts per million (ppm) for **Mauna Loa, Hawaii**.

YEAR	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
TEMPERATURE	44.45	43.29	43.61	43.35	46.66	45.71	45.53	47.53	45.86	46.23
CO_2 EMISSIONS (ppm)	316.9	320.0	325.7	331.1	338.7	345.9	354.2	360.6	369.4	379.7

In the Modeling Our World in Chapter 0, the temperature and carbon emissions were modeled with *linear functions*. Now, let us model these same data using *polynomial functions*.

1. Plot the temperature data with time on the horizontal axis and temperature on the vertical axis. Let $t = 0$ correspond to 1960. Adjust the vertical range of the graph to (43, 48). How many turning points (local maxima and minima) do these data exhibit? What is the lowest degree polynomial function whose graph can pass through these data?

2. Find a *polynomial function* that models the temperature in Mauna Loa.
   - **a.** Find a quadratic function: Let the data from 1995 correspond to the vertex of the graph and apply the 2005 data to determine the function.
   - **b.** Find a quadratic function: Let the data from 2000 correspond to the vertex of the graph and apply the 2005 data to determine the function.
   - **c.** Utilize regression and all data given to find a polynomial function whose degree is found in 1.

3. Predict what the temperature will be in Mauna Loa in 2020.
   - **a.** Use the line found in Exercise 2(a).
   - **b.** Use the line found in Exercise 2(b).
   - **c.** Use the line found in Exercise 2(c).

4. Predict what the temperature will be in Mauna Loa in 2100.
   - **a.** Use the line found in Exercise 2(a).
   - **b.** Use the line found in Exercise 2(b).
   - **c.** Use the line found in Exercise 2(c).

5. Do your models support the claim of "global warming"? Explain.

6. Plot the carbon dioxide emissions data with time on the horizontal axis and carbon dioxide emissions on the vertical axis. Let $t = 0$ correspond to 1960. Adjust the vertical range of the graph to (315, 380).

7. Find a *quadratic function* that models the CO_2 emissions (ppm) in Mauna Loa.

a. Let the data from 1960 correspond to the vertex of the graph and apply the 2005 data to determine the function.

b. Let the data from 1980 correspond to the vertex of the graph and apply the 2005 data to determine the function.

c. Utilize regression and all data given.

8. Predict the expected CO_2 levels in Mauna Loa in 2020.

a. Use the line found in Exercise 7(a).

b. Use the line found in Exercise 7(b).

c. Use the line found in Exercise 7(c).

9. Predict the expected CO_2 levels in Mauna Loa in 2100.

a. Use the line found in Exercise 7(a).

b. Use the line found in Exercise 7(b).

c. Use the line found in Exercise 7(c).

10. Do your models support the claim of "global warming"? Explain. Do these models give similar predictions to the linear models found in Chapter 0?

11. Discuss differences in models and predictions found in parts (a), (b), and (c) and also discuss difference in linear and polynomial functions.

SECTION	CONCEPT	KEY IDEAS/FORMULAS
2.1	**Quadratic functions**	
	Graphs of quadratic functions: parabolas	**Graphing quadratic functions in standard form** $f(x) = a(x - h)^2 + k$ ■ Vertex: (h, k) ■ Opens up: $a > 0$ ■ Opens down: $a < 0$ **Graphing quadratic functions in general form** $f(x) = ax^2 + bx + c$, vertex is $(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
	Finding the equation of a parabola	Applications
2.2	**Polynomial functions of higher degree**	
	Identifying polynomial functions	$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ is a polynomial of degree n.
	Graphing polynomial functions using transformations of power functions	$y = x^n$ behave similar to ■ $y = x^2$, when n is even. ■ $y = x^3$, when n is odd.
	Real zeros of a polynomial function	$P(x) = (x - a)(x - b)^n = 0$ ■ a is a zero of multiplicity 1. ■ b is a zero of multiplicity n.
	Graphing general polynomial functions	Intercepts; zeros and multiplicities; end behavior
2.3	**Dividing polynomials**	Use zero placeholders for missing terms.
	Long division of polynomials	Can be used for all polynomial division.
	Synthetic division of polynomials	Can only be used when dividing by $(x \pm a)$.
2.4	**The real zeros of a polynomial function**	$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ If $P(c) = 0$, then c is a zero of $P(x)$.
	The remainder theorem and the factor theorem	If $P(x)$ is divided by $x - a$, then the remainder r is $r = P(a)$.
	The rational zero theorem and Descartes' rule of signs	Possible zeros $= \dfrac{\text{Factors of } a_0}{\text{Factors of } a_n}$ Number of positive or negative real zeros is related to the number of sign variations in $P(x)$ or $P(-x)$.

Section	Concept	Key Ideas/Formulas
	Factoring polynomials	1. List possible rational zeros (rational zero theorem). 2. List possible combinations of positive and negative real zeros (Descartes' rule of signs). 3. Test possible values until a zero is found. 4. Once a real zero is found, use synthetic division. Then repeat testing on the quotient until linear and/or irreducible quadratic factors are reached. 5. If there is a real zero but all possible rational roots have failed, then approximate the real zero using the intermediate value theorem/bisection method.
	The intermediate value theorem	Intermediate value theorem and the bisection method are used to approximate irrational zeros.
	Graphing polynomial functions	1. Find the intercepts. 2. Determine end behavior. 3. Find additional points. 4. Sketch a smooth curve.
2.5	**Complex zeros: The fundamental theorem of algebra**	$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ $P(x) = \underbrace{(x - c_1)(x - c_2) \cdots (x - c_n)}_{n \text{ factors}}$ where the c's represent complex (not necessarily distinct) zeros.
	Complex zeros	**The fundamental theorem of algebra** $P(x)$ of degree n has at least one zero and at most n zeros.
	Complex conjugate pairs	If $a + bi$ is a zero of $P(x)$, then $a - bi$ is also a zero.
	Factoring polynomials	The polynomial can be written as a product of linear factors, not necessarily distinct.
2.6	**Rational functions**	$f(x) = \dfrac{n(x)}{d(x)} \qquad d(x) \neq 0$
	Domain of rational functions	**Domain:** All real numbers except x-values that make the denominator equal to zero; that is, $d(x) = 0$. A rational function $f(x) = \dfrac{n(x)}{d(x)}$ is said to be in *lowest terms* if $n(x)$ and $d(x)$ have no common factors.
	Vertical, horizontal, and slant asymptotes	A rational function that has a common factor $x - a$ in both the numerator and denominator has a hole at $x = a$ in its graph if the multiplicity of a in the numerator is greater than or equal to the multiplicity of a in the denominator. *Vertical Asymptotes* A rational function in lowest terms has a vertical asymptote corresponding to any x-values that make the denominator equal to zero.

CHAPTER REVIEW

Section	Concept	Key Ideas/Formulas
		Horizontal Asymptote ■ $y = 0$ if degree of $n(x) <$ degree of $d(x)$. ■ No horizontal asymptote if degree of $n(x) >$ degree of $d(x)$. ■ $y = \dfrac{\text{Leading coefficient of } n(x)}{\text{Leading coefficient of } d(x)}$ if degree of $n(x) =$ degree of $d(x)$. *Slant Asymptote* If degree of $n(x) -$ degree of $d(x) = 1$. Divide $n(x)$ by $d(x)$ and the quotient determines the slant asymptote; that is, $y =$ quotient.
	Graphing rational functions	1. Find the domain of the function. 2. Find the intercept(s). 3. Find any holes. 4. Find any asymptote. 5. Find additional points on the graph. 6. *Sketch the graph:* Draw the asymptotes and label the intercepts and points and connect with a smooth curve.

CHAPTER 2 REVIEW EXERCISES

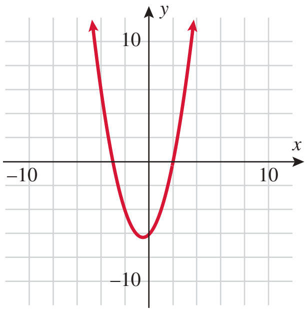
2.1 Quadratic Functions

Match the quadratic function with its graph.

1. $f(x) = -2(x+6)^2 + 3$ **2.** $f(x) = \frac{1}{4}(x-4)^2 + 2$

3. $f(x) = x^2 + x - 6$ **4.** $f(x) = -3x^2 - 10x + 8$

a.



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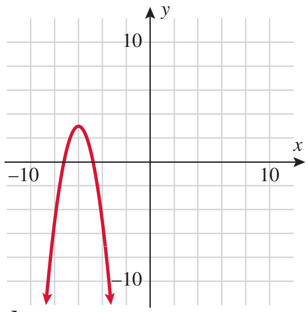

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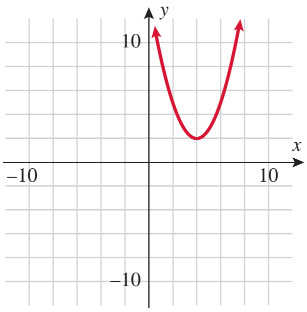

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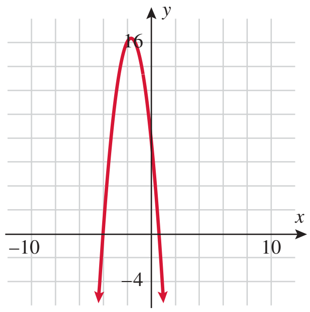


Graph the quadratic function given in standard form.

5. $f(x) = -(x-7)^2 + 4$ **6.** $f(x) = (x+3)^2 - 5$

7. $f(x) = -\frac{1}{2}\left(x - \frac{1}{3}\right)^2 + \frac{2}{5}$ **8.** $f(x) = 0.6(x - 0.75)^2 + 0.5$

Rewrite the quadratic function in standard form by completing the square.

9. $f(x) = x^2 - 3x - 10$ **10.** $f(x) = x^2 - 2x - 24$

11. $f(x) = 4x^2 + 8x - 7$ **12.** $f(x) = -\frac{1}{4}x^2 + 2x - 4$

Graph the quadratic function given in general form.

13. $f(x) = x^2 - 3x + 5$

14. $f(x) = -x^2 + 4x + 2$

15. $f(x) = -4x^2 + 2x + 3$

16. $f(x) = -0.75x^2 + 2.5$

Find the vertex of the parabola associated with each quadratic function.

17. $f(x) = 13x^2 - 5x + 12$

18. $f(x) = \frac{2}{5}x^2 - 4x + 3$

19. $f(x) = -0.45x^2 - 0.12x + 3.6$

20. $f(x) = -\frac{3}{4}x^2 + \frac{2}{5}x + 4$

Find the quadratic function that has the given vertex and goes through the given point.

21. vertex: $(-2, 3)$ point: $(1, 4)$

22. vertex: $(4, 7)$ point: $(-3, 1)$

23. vertex: $(2.7, 3.4)$ point: $(3.2, 4.8)$

24. vertex: $\left(-\frac{5}{2}, \frac{7}{4}\right)$ point: $\left(\frac{1}{2}, \frac{3}{5}\right)$

Applications

25. Profit. The revenue and the cost of a local business are given below as functions of the number of units x in thousands produced and sold. Use the cost and the revenue to answer the questions that follow.

$$C(x) = \frac{1}{3}x + 2 \quad \text{and} \quad R(x) = -2x^2 + 12x - 12$$

a. Determine the profit function.

b. State the break-even points.

c. Graph the profit function.

d. What is the range of units to make and sell that will correspond to a profit?

26. Geometry. Given the length of a rectangle is $2x - 4$ and the width is $x + 7$, find the area of the rectangle. What dimensions correspond to the largest area?

27. Geometry. A triangle has a base of $x + 2$ units and a height of $4 - x$ units. Determine the area of the triangle. What dimensions correspond to the largest area?

28. Geometry. A person standing at a ridge in the Grand Canyon throws a penny upward and toward the pit of the canyon. The height of the penny is given by the function:

$$h(t) = -12t^2 + 80t$$

a. What is the maximum height that the penny will reach?

b. How many seconds will it take the penny to hit the ground below?

2.2 Polynomial Functions of Higher Degree

Determine which functions are polynomials, and for those, state their degree.

29. $f(x) = x^6 - 2x^5 + 3x^2 + 9x - 42$

30. $f(x) = (3x-4)^3(x+6)^2$

31. $f(x) = 3x^4 - x^3 + x^2 + \sqrt[4]{x} + 5$

32. $f(x) = 5x^3 - 2x^2 + \dfrac{4x}{7} - 3$

Match the polynomial function with its graph.

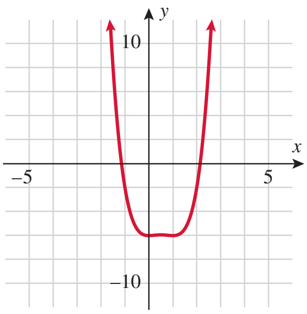
33. $f(x) = 2x - 5$

34. $f(x) = -3x^2 + x - 4$

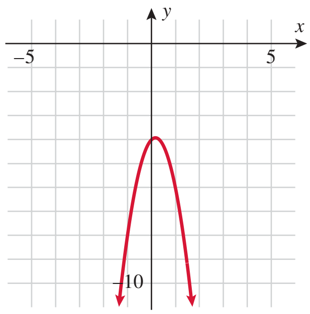
35. $f(x) = x^4 - 2x^3 + x^2 - 6$

36. $f(x) = x^7 - x^5 + 3x^4 + 3x + 7$

a.



b.



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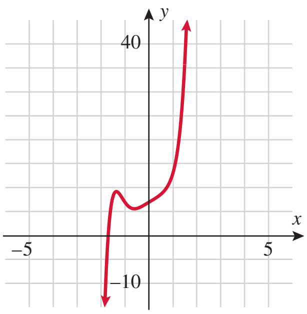

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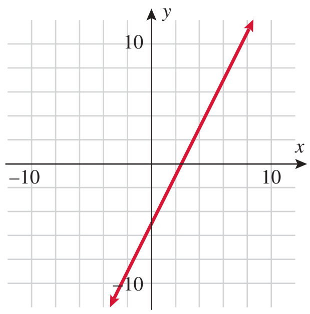

Graph each function by transforming a power function $y = x^n$.

37. $f(x) = -x^7$

38. $f(x) = (x - 3)^3$

39. $f(x) = x^4 - 2$

40. $f(x) = -6 - (x + 7)^5$

Find all the real zeros of each polynomial function, and state their multiplicities.

41. $f(x) = 3(x + 4)^2(x - 6)^5$

42. $f(x) = 7x(2x - 4)^3(x + 5)$

43. $f(x) = x^5 - 13x^3 + 36x$

44. $f(x) = 4.2x^4 - 2.6x^2$

Find a polynomial of minimum degree that has the given zeros.

45. $-3, 0, 4$

46. $2, 4, 6, -8$

47. $-\frac{2}{5}, \frac{3}{4}, 0$

48. $2 - \sqrt{5}, 2 + \sqrt{5}$

49. -2 (multiplicity of 2), 3 (multiplicity of 2)

50. 3 (multiplicity of 2), -1 (multiplicity of 2), 0 (multiplicity of 3)

For each polynomial function given: (a) list each real zero and its multiplicity; (b) determine whether the graph touches or crosses at each x-intercept; (c) find the y-intercept and a few points on the graph; (d) determine the end behavior; and (e) sketch the graph.

51. $f(x) = x^2 - 5x - 14$

52. $f(x) = -(x - 5)^5$

53. $f(x) = 6x^7 + 3x^5 - x^2 + x - 4$

54. $f(x) = -x^4(3x + 6)^3(x - 7)^3$

Applications

55. Salary. Tiffany has started tutoring students x hours per week. The tutoring job corresponds to the following additional income:

$$f(x) = (x - 1)(x - 3)(x - 7)$$

a. Graph the polynomial function.

b. Give any real zeros that occur.

c. How many hours of tutoring are financially beneficial to Tiffany?

56. Profit. The following function is the profit for Walt Disney World, where $P(x)$ represents profit in millions of dollars and x represents the month ($x = 1$ corresponds to January):

$$P(x) = 3(x - 2)^2(x - 5)^2(x - 10)^2 \quad 1 \le x \le 12$$

Graph the polynomial. When are the peak seasons?

2.3 Dividing Polynomials

Divide the polynomials with long division. If you choose to use a calculator, do not round off. Keep the exact values instead. Express the answer in the form $Q(x) = ?, \quad r(x) = ?$.

57. $(x^2 + 2x - 6) \div (x - 2)$

58. $(2x^2 - 5x - 1) \div (2x - 3)$

59. $(4x^4 - 16x^3 + x - 9 + 12x^2) \div (2x - 4)$

60. $(6x^2 + 2x^3 - 4x^4 + 2 - x) \div (2x^2 + x - 4)$

Use synthetic division to divide the polynomial by the linear factor. Indicate the quotient $Q(x)$ and the remainder $r(x)$.

61. $(x^4 + 4x^3 + 5x^2 - 2x - 8) \div (x + 2)$

62. $(x^3 - 10x + 3) \div (2 + x)$

63. $(x^6 - 64) \div (x + 8)$

64. $(2x^5 + 4x^4 - 2x^3 + 7x + 5) \div \left(x - \frac{3}{4}\right)$

Divide the polynomials with either long division or synthetic division.

65. $(5x^3 + 8x^2 - 22x + 1) \div (5x^2 - 7x + 3)$

66. $(x^4 + 2x^3 - 5x^2 + 4x + 2) \div (x - 3)$

67. $(x^3 - 4x^2 + 2x - 8) \div (x + 1)$

68. $(x^3 - 5x^2 + 4x - 20) \div (x^2 + 4)$

Applications

69. Geometry. The area of a rectangle is given by the polynomial $6x^4 - 8x^3 - 10x^2 + 12x - 16$. If the width is $2x - 4$, what is the length of the rectangle?

70. Volume. A 10 inch by 15 inch rectangular piece of cardboard is used to make a box. Square pieces x inches on a side are cut out from the corners of the cardboard and then the sides are folded up. Find the volume of the box.

2.4 The Real Zeros of a Polynomial Function

Find the following values by applying synthetic division. Check by substituting the value into the function.

$f(x) = 6x^5 + x^4 - 7x^2 + x - 1 \qquad g(x) = x^3 + 2x^2 - 3$

71. $f(-2)$ **72.** $f(1)$ **73.** $g(1)$ **74.** $g(-1)$

Determine whether the number given is a zero of the polynomial.

75. -3, $P(x) = x^3 - 5x^2 + 4x + 2$

76. 2 and -2, $P(x) = x^4 - 16$

77. 1, $P(x) = 2x^4 - 2x$

78. 4, $P(x) = x^4 - 2x^3 - 8x$

Given a zero of the polynomial, determine all other real zeros, and write the polynomial in terms of a product of linear or irreducible factors.

	Polynomial	Zero
79.	$P(x) = x^4 - 6x^3 + 32x$	-2
80.	$P(x) = x^3 - 7x^2 + 36$	3
81.	$P(x) = x^5 - x^4 - 8x^3 + 12x^2$	0
82.	$P(x) = x^4 - 32x^2 - 144$	6

Use Descartes' rule of signs to determine the possible number of positive real zeros and negative real zeros.

83. $P(x) = x^4 + 3x^3 - 16$

84. $P(x) = x^5 + 6x^3 - 4x - 2$

85. $P(x) = x^9 - 2x^7 + x^4 - 3x^3 + 2x - 1$

86. $P(x) = 2x^5 - 4x^3 + 2x^2 - 7$

Use the rational zero theorem to list the possible rational zeros.

87. $P(x) = x^3 - 2x^2 + 4x + 6$

88. $P(x) = x^5 - 4x^3 + 2x^2 - 4x - 8$

89. $P(x) = 2x^4 + 2x^3 - 36x^2 - 32x + 64$

90. $P(x) = -4x^5 - 5x^3 + 4x + 2$

List the possible rational zeros, and test to determine all rational zeros.

91. $P(x) = 2x^3 - 5x^2 + 1$

92. $P(x) = 12x^3 + 8x^2 - 13x + 3$

93. $P(x) = x^4 - 5x^3 + 20x - 16$

94. $P(x) = 24x^4 - 4x^3 - 10x^2 + 3x - 2$

For each polynomial: (a) use Descartes' rule of signs to determine the possible combinations of positive real zeros and negative real zeros; (b) use the rational zero test to determine possible rational zeros; (c) determine, if possible, the smallest value in the list of possible rational zeros for $P(x)$ that serves as a lower bound for all real zeros; (d) test for rational zeros; (e) factor as a product of linear and/or irreducible quadratic factors; and (f) graph the polynomial function.

95. $P(x) = x^3 + 3x - 5$

96. $P(x) = x^3 + 3x^2 - 6x - 8$

97. $P(x) = x^3 - 9x^2 + 20x - 12$

98. $P(x) = x^4 - x^3 - 7x^2 + x + 6$

99. $P(x) = x^4 - 5x^3 - 10x^2 + 20x + 24$

100. $P(x) = x^5 - 3x^3 - 6x^2 + 8x$

2.5 Complex Zeros: The Fundamental Theorem of Algebra

Find all zeros. Factor the polynomial as a product of linear factors.

101. $P(x) = x^2 + 25$ **102.** $P(x) = x^2 + 16$

103. $P(x) = x^2 - 2x + 5$ **104.** $P(x) = x^2 + 4x + 5$

A polynomial function is described. Find all remaining zeros.

105. Degree: 4 Zeros: $-2i$, $3 + i$

106. Degree: 4 Zeros: $3i$, $2 - i$

107. Degree: 6 Zeros: i, $2 - i$ (multiplicity 2)

108. Degree: 6 Zeros: $2i$, $1 - i$ (multiplicity 2)

Given a zero of the polynomial, determine all other zeros (real and complex) and write the polynomial in terms of a product of linear factors.

	Polynomial	Zero
109.	$P(x) = x^4 - 3x^3 - 3x^2 - 3x - 4$	i
110.	$P(x) = x^4 - 4x^3 + x^2 + 16x - 20$	$2 - i$
111.	$P(x) = x^4 - 2x^3 + 11x^2 - 18x + 18$	$-3i$
112.	$P(x) = x^4 - 5x^2 + 10x - 6$	$1 + i$

REVIEW EXERCISES

Factor each polynomial as a product of linear factors.

113. $P(x) = x^4 - 81$

114. $P(x) = x^3 - 6x^2 + 12x$

115. $P(x) = x^3 - x^2 + 4x - 4$

116. $P(x) = x^4 - 5x^3 + 12x^2 - 2x - 20$

2.6 Rational Functions

Determine the vertical, horizontal, or slant asymptotes (if they exist) for the following rational functions.

117. $f(x) = \dfrac{7 - x}{x + 2}$

118. $f(x) = \dfrac{2 - x^2}{(x - 1)^3}$

119. $f(x) = \dfrac{4x^2}{x + 1}$

120. $f(x) = \dfrac{3x^2}{x^2 + 9}$

121. $f(x) = \dfrac{2x^2 - 3x + 1}{x^2 + 4}$

122. $f(x) = \dfrac{-2x^2 + 3x + 5}{x + 5}$

Graph the rational functions.

123. $f(x) = -\dfrac{2}{x - 3}$

124. $f(x) = \dfrac{5}{x + 1}$

125. $f(x) = \dfrac{x^2}{x^2 + 4}$

126. $f(x) = \dfrac{x^2 - 36}{x^2 + 25}$

127. $f(x) = \dfrac{x^2 - 49}{x + 7}$

128. $f(x) = \dfrac{2x^2 - 3x - 2}{2x^2 - 5x - 3}$

Technology Exercises

Section 2.1

129. On a graphing calculator, plot the quadratic function:

$$f(x) = 0.005x^2 - 4.8x - 59$$

a. Identify the vertex of this parabola.

b. Identify the y-intercept.

c. Identify the x-intercepts (if any).

d. What is the axis of symmetry?

130. Determine the quadratic function whose vertex is $(2.4, -3.1)$ and passes through the point $(0, 5.54)$.

a. Write the quadratic function in general form.

b. Plot this quadratic function with a graphing calculator.

c. Zoom in on the vertex and y-intercept. Do they agree with the given values?

Section 2.2

Use a graphing calculator or a computer to graph each polynomial. From the graph, estimate the x-intercepts and state the zeros of the function and their multiplicities.

131. $f(x) = 5x^3 - 11x^2 - 10.4x + 5.6$

132. $f(x) = -x^3 - 0.9x^2 + 2.16x - 2.16$

Section 2.3

133. Plot $\dfrac{15x^3 - 47x^2 + 38x - 8}{3x^2 - 7x + 2}$. What type of function is it? Perform this division using long division, and confirm that the graph corresponds to the quotient.

134. Plot $\dfrac{-4x^3 + 14x^2 - x - 15}{x - 3}$. What type of function is it? Perform this division using synthetic division, and confirm that the graph corresponds to the quotient.

Section 2.4

(a) Determine all possible rational zeros of the polynomial. Use a graphing calculator or software to graph $P(x)$ to help find the zeros. (b) Factor as a product of linear and/or irreducible quadratic factors.

135. $P(x) = x^4 - 3x^3 - 12x^2 + 20x + 48$

136. $P(x) = -5x^5 - 18x^4 - 32x^3 - 24x^2 + x + 6$

Section 2.5

Find all zeros (real and complex). Factor the polynomial as a product of linear factors.

137. $P(x) = 2x^3 + x^2 - 2x - 91$

138. $P(x) = -2x^4 + 5x^3 + 37x^2 - 160x + 150$

Section 2.6

(a) Graph the function $f(x)$ utilizing a graphing utility to determine whether it is a one-to-one function. (b) If it is, find its inverse. (c) Graph both functions in the same viewing window.

139. $f(x) = \dfrac{2x - 3}{x + 1}$

140. $f(x) = \dfrac{4x + 7}{x - 2}$

1. Graph the quadratic function $f(x) = -(x - 4)^2 + 1$.
2. Write the quadratic function in standard form $f(x) = -x^2 + 4x - 1$.
3. Find the vertex of the parabola $f(x) = -\frac{1}{2}x^2 + 3x - 4$.
4. Find a quadratic function whose graph has a vertex at $(-3, -1)$ and whose graph passes through the point $(-4, 1)$.
5. Find a sixth-degree polynomial function with the given zeros:

   2 of multiplicity 3   1 of multiplicity 2   0 of multiplicity 1
6. For the polynomial function $f(x) = x^4 + 6x^3 - 7x$:
   a. List each real zero and its multiplicity.
   b. Determine whether the graph touches or crosses at each x-intercept.
   c. Find the y-intercept and a few points on the graph.
   d. Determine the end behavior.
   e. Sketch the graph.
7. Divide $-4x^4 + 2x^3 - 7x^2 + 5x - 2$ by $2x^2 - 3x + 1$.
8. Divide $17x^5 - 4x^3 + 2x - 10$ by $x + 2$.
9. Is $x - 3$ a factor of $x^4 + x^3 - 13x^2 - x + 12$?
10. Determine whether -1 is a zero of $P(x) = x^{21} - 2x^{18} + 5x^{12} + 7x^3 + 3x^2 + 2$.
11. Given that $x - 7$ is a factor of $P(x) = x^3 - 6x^2 - 9x + 14$, factor the polynomial in terms of linear factors.
12. Given that $3i$ is a zero of $P(x) = x^4 - 3x^3 + 19x^2 - 27x + 90$, find all other zeros.
13. Can a polynomial have zeros that are not x-intercepts? Explain.
14. Apply Descartes' rule of signs to determine the possible combinations of positive real zeros, negative real zeros, and complex zeros of $P(x) = 3x^5 + 2x^4 - 3x^3 + 2x^2 - x + 1$.
15. From the rational zero test, list all possible rational zeros of $P(x) = 3x^4 - 7x^2 + 3x + 12$.

In Exercises 16–18, determine all zeros of the polynomial function and graph.

16. $P(x) = -x^3 + 4x$
17. $P(x) = 2x^3 - 3x^2 + 8x - 12$
18. $P(x) = x^4 - 6x^3 + 10x^2 - 6x + 9$
19. **Sports.** A football player shows up in August at 300 pounds. After 2 weeks of practice in the hot sun, he is down to 285 pounds. Ten weeks into the season he is up to 315 pounds because of weight training. In the spring he does not work out, and he is back to 300 pounds by the next August. Plot these points on a graph. What degree polynomial could this be?
20. **Profit.** The profit of a company is governed by the polynomial $P(x) = x^3 - 13x^2 + 47x - 35$, where x is the number of units sold in thousands. How many units does the company have to sell to break even?
21. **Interest Rate.** The interest rate for a 30-year fixed mortgage fluctuates with the economy. In 1970 the mortgage interest rate was 8%, and in 1988 it peaked at 13%. In 2002 it dipped down to 4%, and in 2005 it was up to 6%. What is the lowest degree polynomial that can represent this function?

In Exercises 22–25, determine (if any) the:

a. x- and y-intercepts
b. vertical asymptotes
c. horizontal asymptotes
d. slant asymptotes
e. graph

22. $f(x) = \dfrac{2x - 9}{x + 3}$
23. $g(x) = \dfrac{x}{x^2 - 4}$
24. $h(x) = \dfrac{3x^3 - 3}{x^2 - 4}$
25. $F(x) = \dfrac{x - 3}{x^2 - 2x - 8}$
26. **Food.** After a sugary snack, the glucose level of the average body almost doubles. The percentage increase in glucose level y can be approximated by the rational function $y = \dfrac{25x}{x^2 + 50}$, where x represents the number of minutes after eating the snack. Graph the function.
27. **a.** Use the calculator commands $\boxed{\text{STAT}}$ $\boxed{\text{QuadReg}}$ to model the data using a quadratic function.
    b. Write the quadratic function in standard form and identify the vertex.
    c. Find the x-intercepts.
    d. Plot this quadratic function with a graphing calculator. Do they agree with the given values?

x	-3	2.2	7.5
y	10.01	-9.75	25.76

28. Find the asymptotes and intercepts of the rational function $f(x) = \dfrac{x(2x - 3)}{x^2 - 3x} + 1$. (*Note:* Combine the two expressions into a single rational expression.) Graph this function utilizing a graphing utility. Does the graph confirm what you found?

1. If $f(x) = 4x - \dfrac{1}{\sqrt{x+2}}$, find $f(2)$, $f(-1)$, $f(1+h)$, and $f(-x)$.

2. If $f(x) = (x-1)^4 - \sqrt{2x+3}$, $f(1)$, $f(3)$, and $f(x+h)$.

3. If $f(x) = \dfrac{3x-5}{2-x-x^2}$, find $f(-3)$, $f(0)$, $f(1)$, and $f(4)$.

4. If $f(x) = 4x^3 - 3x^2 + 5$, evaluate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

5. If $f(x) = \sqrt{x} - \dfrac{1}{x^2}$, evaluate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

6. If $f(x) = \begin{cases} 0 & x < 0 \\ 3x + x^2 & 0 \le x \le 4 \\ |2x - x^3| & x > 4 \end{cases}$

find $f(-5)$, $f(0)$, $f(3)$, $f(4)$, and $f(5)$.

In Exercises 7 and 8, (a) Graph the piecewise-defined functions. (b) State the domain and range in interval notation. (c) Determine the intervals where the function is increasing, decreasing, or constant.

7. $f(x) = \begin{cases} |6 - 2x| & x \le 8 \\ 10 & 8 < x < 10 \\ \dfrac{1}{x-10} & x > 10 \end{cases}$

8. $f(x) = \begin{cases} (x+5)^2 - 6 & x < -2 \\ \sqrt{x-1} + 3 & -2 \le x < 10 \\ 26 - 2x & 10 \le x \le 14 \end{cases}$

9. The position of a particle is described by the curve $y = \dfrac{2t}{t^2+3}$, where t is time (in seconds). What is the average rate of change of the position as a function of time from $t = 5$ to $t = 9$?

10. Express the domain of the function $f(x) = \sqrt{6x-7}$ with interval notation.

11. Determine whether the function $g(x) = \sqrt{x+10}$ is even, odd, or neither.

12. For the function $y = -(x+1)^2 + 2$, identify all of the transformations of $y = x^2$.

13. Sketch the graph of $y = \sqrt{x-1} + 3$ and identify all transformations.

14. Find the composite function $f \circ g$ and state the domain for $f(x) = x^2 - 3$ and $g(x) = \sqrt{x+2}$.

15. Evaluate $g(f(-1))$ for $f(x) = 7 - 2x^2$ and $g(x) = 2x - 10$.

16. Find the inverse of the function $f(x) = (x-4)^2 + 2$, where $x \ge 4$.

17. Find a quadratic function whose graph has a vertex at $(-2, 3)$ and passes through the point $(-1, 4)$.

18. Find all of the real zeros and state the multiplicity of the function $f(x) = -3.7x^4 - 14.8x^3$.

19. Use long division to find the quotient $Q(x)$ and the remainder $r(x)$ of $\left(-20x^3 - 8x^2 + 7x - 5\right) \div (-5x + 3)$.

20. Use synthetic division to find the quotient $Q(x)$ and the remainder $r(x)$ of $\left(2x^3 + 3x^2 - 11x + 6\right) \div (x - 3)$.

21. List the possible rational zeros, and test to determine all rational zeros for $P(x) = 12x^3 + 29x^2 + 7x - 6$.

22. Given the real zero $x = 5$ of the polynomial $P(x) = 2x^3 - 3x^2 - 32x - 15$, determine all the other zeros and write the polynomial in terms of a product of linear factors.

23. Factor the polynomial $P(x) = x^3 - 5x^2 + 2x + 8$ completely.

24. Factor the polynomial $P(x) = x^5 + 7x^4 + 15x^3 + 5x^2 - 16x - 12$ completely.

25. Find all vertical and horizontal asymptotes of $f(x) = \dfrac{3x-5}{x^2-4}$.

26. Graph the function $f(x) = \dfrac{2x^3 - x^2 - x}{x^2 - 1}$.

27. Find the asymptotes and intercepts of the rational function $f(x) = \dfrac{5}{2x-3} - \dfrac{1}{x}$. (*Note:* Combine the two expressions into a single rational expression.) Graph this function utilizing a graphing utility. Does the graph confirm what you found?

28. Find the asymptotes and intercepts of the rational function $f(x) = \dfrac{6x}{3x+1} - \dfrac{6x}{4x-1}$. (*Note:* Combine the two expressions into a single rational expression.) Graph this function utilizing a graphing utility. Does the graph confirm what you found?

3

Exponential and Logarithmic Functions

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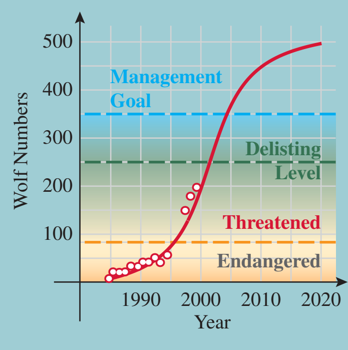
Most populations initially grow *exponentially*, but then as resources become limited, the population reaches a *carrying capacity*. This exponential increase followed by a saturation at some carrying capacity is called *logistic growth*. Often when a particular species is placed on the endangered species list, it is protected from human predators and then its population size increases until naturally leveling off at some carrying capacity.

The U.S. Fish and Wildlife Service removed the gray wolf (*canis lupus*) from the Wisconsin list of endangered and threatened species in 2004, and placed it on the list of protected wild animals. About 537 to 564 wolves existed in Wisconsin in the late winter of 2008.



Wisconsin Wolf Population Growth if Carrying Capacity is 500 Wolves

IN THIS CHAPTER we will discuss exponential functions and their inverses, logarithmic functions. We will graph these functions and use their properties to solve exponential and logarithmic equations. We will then discuss particular exponential and logarithmic models that represent phenomena such as compound interest, world populations, conservation biology models, carbon dating, pH values in chemistry, and the bell curve that is fundamental in statistics for describing how quantities vary in the real world.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

3.1 Exponential Functions and Their Graphs	3.2 Logarithmic Functions and Their Graphs	3.3 Properties of Logarithms	3.4 Exponential and Logarithmic Equations	3.5 Exponential and Logarithmic Models
• Evaluating Exponential Functions • Graphs of Exponential Functions • The Natural Base *e* • Applications of Exponential Functions	• Evaluating Logarithms • Common and Natural Logarithms • Graphs of Logarithmic Functions • Applications of Logarithms	• Properties of Logarithms • Change-of-Base Formula	• Solving Exponential Equations • Solving Logarithmic Equations	• Exponential Growth Models • Exponential Decay Models • Gaussian (Normal) Distribution Models • Logistic Growth Models • Logarithmic Models

LEARNING OBJECTIVES

- Evaluate exponential functions for particular values and understand the characteristics of the graph of an exponential function.
- Evaluate logarithmic functions for particular values and understand the characteristics of the graph of a logarithmic function.
- Understand that logarithmic functions are inverses of exponential functions and derive the properties of logarithms.
- Solve exponential and logarithmic equations.
- Use the exponential growth, exponential decay, logarithmic, logistic growth, and Gaussian distribution models to represent real-world phenomena.

SECTION 3.1 EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

SKILLS OBJECTIVES

- Evaluate exponential functions.
- Graph exponential functions.
- Find the domain and range of exponential functions.
- Define the number *e*.
- Solve real-world problems using exponential functions.

CONCEPTUAL OBJECTIVES

- Understand the difference between algebraic and exponential functions.
- Understand that irrational exponents lead to approximations.

Evaluating Exponential Functions

Most of the functions (polynomial, rational, radical, etc.) we have studied thus far have been **algebraic functions**. Algebraic functions involve basic operations, powers, and roots. In this chapter, we discuss *exponential functions* and *logarithmic functions*. The following table illustrates the difference between algebraic functions and *exponential functions*:

FUNCTION	VARIABLE IS IN THE	CONSTANT IS IN THE	EXAMPLE	EXAMPLE
Algebraic	Base	Exponent	$f(x) = x^2$	$g(x) = x^{1/3}$
Exponential	Exponent	Base	$F(x) = 2^x$	$G(x) = \left(\frac{1}{3}\right)^x$

DEFINITION **Exponential Function**

An **exponential function** with **base *b*** is denoted by

$$f(x) = b^x$$

where b and x are any real numbers such that $b > 0$ and $b \neq 1$.

Note:

- We eliminate $b = 1$ as a value for the base because it merely yields the constant function $f(x) = 1^x = 1$.
- We eliminate negative values for b because they would give nonreal-number values such as $(-9)^{1/2} = \sqrt{-9} = 3i$.
- We eliminate $b = 0$ because 0^x corresponds to an undefined value when x is negative.

Our experience with integer exponents has implied a constant multiplication:

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$$

We can now think of $f(x) = 2^x$ as a continuous function that for positive integers will result in a constant multiplication.

Sometimes the value of an exponential function for a specific argument can be found by inspection as an *exact* number.

x	-3	-1	0	1	3
$F(x) = 2^x$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^3 = 8$

If an exponential function cannot be evaluated exactly, then we find the decimal *approximation* using a calculator.

x	-2.7	$-\frac{4}{5}$	$\frac{5}{7}$	2.7
$F(x) = 2^x$	$2^{-2.7} \approx 0.154$	$2^{-4/5} \approx 0.574$	$2^{5/7} \approx 1.641$	$2^{2.7} \approx 6.498$

The domain of exponential functions, $f(x) = b^x$, is the set of all real numbers. All of the arguments discussed in the first two tables have been rational numbers. What happens if x is irrational? We can approximate the irrational number with a decimal approximation such as $b^\pi \approx b^{3.14}$ or $b^{\sqrt{2}} \approx b^{1.41}$.

Consider $7^{\sqrt{3}}$, and realize that the irrational number $\sqrt{3}$ is a decimal that never terminates or repeats: $\sqrt{3} \approx 1.7320508$. We can show in advanced mathematics that there is a number $7^{\sqrt{3}}$, and although we cannot write it exactly, we can approximate the number. In fact, the closer the exponent is to $\sqrt{3}$, the closer the approximation is to $7^{\sqrt{3}}$.

$$7^{1.7} \approx 27.3317$$
$$7^{1.73} \approx 28.9747$$
$$7^{1.732} \approx 29.0877$$
$$\vdots$$
$$7^{\sqrt{3}} \approx 29.0906$$

It is important to note that the properties of exponents (Appendix) hold when the exponent is any real number (rational or irrational).

EXAMPLE 1 Evaluating Exponential Functions

Let $f(x) = 3^x$, $g(x) = \left(\frac{1}{4}\right)^x$ and $h(x) = 10^{x-2}$. Find the following values:

a. $f(2)$ **b.** $f(\pi)$ **c.** $g\left(-\frac{3}{2}\right)$ **d.** $h(2.3)$ **e.** $f(0)$ **f.** $g(0)$

If an approximation is required, approximate to four decimal places.

Solution:

a. $f(2) = 3^2 = \boxed{9}$

b. $f(\pi) = 3^\pi \approx \boxed{31.5443}$ *

c. $g\left(-\frac{3}{2}\right) = \left(\frac{1}{4}\right)^{-3/2} = 4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = \boxed{8}$

d. $h(2.3) = 10^{2.3-2} = 10^{0.3} \approx \boxed{1.9953}$

e. $f(0) = 3^0 = \boxed{1}$

f. $g(0) = \left(\frac{1}{4}\right)^0 = \boxed{1}$

Notice that parts (a) and (c) were evaluated exactly, whereas parts (b) and (d) required approximation using a calculator.

■ **YOUR TURN** Let $f(x) = 2^x$ and $g(x) = \left(\frac{1}{9}\right)^x$ and $h(x) = 5^{x-2}$. Find the following values:

a. $f(4)$ **b.** $f(\pi)$ **c.** $g\left(-\frac{3}{2}\right)$ **d.** $h(2.9)$

Evaluate exactly when possible, and round to four decimal places when a calculator is needed.

■ **Answer:** **a.** 16 **b.** 8.8250 **c.** 27 **d.** 4.2567

*In part (b), the π button on the calculator is selected. If we instead approximate π by 3.14, we get a slightly different approximation for the function value:

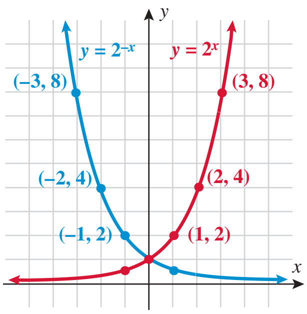
$$f(\pi) = 3^\pi \approx 3^{3.14} \approx 31.4891$$

Graphs of Exponential Functions

Let's graph two exponential functions, $y = 2^x$ and $y = 2^{-x} = \left(\frac{1}{2}\right)^x$, by plotting points.

x	$y = 2^x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	(0, 1)
1	$2^1 = 2$	(1, 2)
2	$2^2 = 4$	(2, 4)
3	$2^3 = 8$	(3, 8)

x	$y = 2^{-x}$	(x, y)
-3	$2^{-(-3)} = 2^3 = 8$	$(-3, 8)$
-2	$2^{-(-2)} = 2^2 = 4$	$(-2, 4)$
-1	$2^{-(-1)} = 2^1 = 2$	$(-1, 2)$
0	$2^0 = 1$	(0, 1)
1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$
2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$

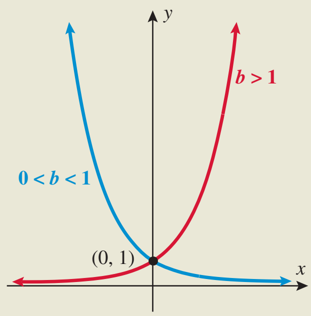


Notice that both graphs' y-intercept is (0, 1) (as shown to the left) and neither graph has an x-intercept. The x-axis is a horizontal asymptote for both graphs. The following box summarizes general characteristics of the graphs of exponential functions.

Characteristics of Graphs of Exponential Functions

$$f(x) = b^x, \qquad b > 0, \qquad b \neq 1$$

- Domain: $(-\infty, \infty)$
- Range: $(0, \infty)$
- x-intercepts: none
- y-intercept: (0, 1)
- Horizontal asymptote: x-axis
- The graph passes through $(1, b)$ and $\left(-1, \frac{1}{b}\right)$.
- As x increases, $f(x)$ increases if $b > 1$ and decreases if $0 < b < 1$.
- The function f is one-to-one.



Since exponential functions, $f(x) = b^x$, all go through the point (0, 1) and have the x-axis as a horizontal asymptote, we can find the graph by finding two additional points as outlined in the following procedure.

Procedure for Graphing $f(x) = b^x$

Step 1: Label the point (0, 1) corresponding to the y-intercept $f(0)$.

Step 2: Find and label two additional points corresponding to $f(-1)$ and $f(1)$.

Step 3: Connect the three points with a *smooth* curve with the x-axis as the horizontal asymptote.

EXAMPLE 2 Graphing Exponential Functions for $b > 1$

Graph the function $f(x) = 5^x$.

Solution:

STEP 1: Label the y-intercept $(0, 1)$. $f(0) = 5^0 = 1$

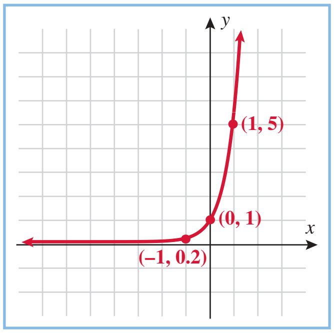
STEP 2: Label the point $(1, 5)$. $f(1) = 5^1 = 5$

Label the point $(-1, 0.2)$. $f(-1) = 5^{-1} = \dfrac{1}{5} = 0.2$

STEP 3: Sketch a smooth curve through the three points with the x-axis as a horizontal asymptote.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

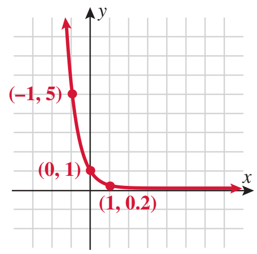


■ **YOUR TURN** Graph the function $f(x) = 5^{-x}$.

Technology Tip

The graph of $f(x) = 5^x$ is shown.

■ **Answer:**



EXAMPLE 3 Graphing Exponential Functions for $b < 1$

Graph the function $f(x) = \left(\frac{2}{5}\right)^x$.

Solution:

STEP 1: Label the y-intercept $(0, 1)$. $f(0) = \left(\dfrac{2}{5}\right)^0 = 1$

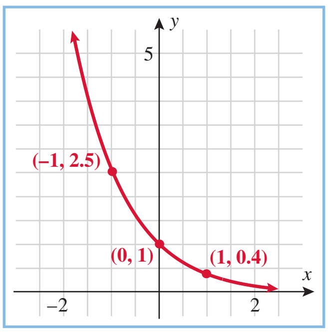
STEP 2: Label the point $(-1, 2.5)$. $f(-1) = \left(\dfrac{2}{5}\right)^{-1} = \dfrac{5}{2} = 2.5$

Label the point $(1, 0.4)$. $f(1) = \left(\dfrac{2}{5}\right)^1 = \dfrac{2}{5} = 0.4$

STEP 3: Sketch a smooth curve through the three points with the x-axis as a horizontal asymptote.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$



Exponential functions, like all functions, can be graphed by point-plotting. We can also use transformations (horizontal and vertical shifting and reflection; Section 1.3) to graph exponential functions.

EXAMPLE 4 Graphing Exponential Functions Using a Horizontal or Vertical Shift

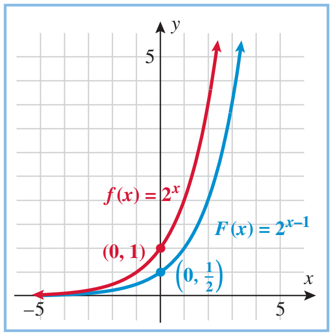
a. Graph the function $F(x) = 2^{x-1}$. State the domain and range of F.
b. Graph the function $G(x) = 2^x + 1$. State the domain and range of G.

Solution (a):

Identify the base function.	$f(x) = 2^x$
Identify the base function y-intercept and horizontal asymptote.	$(0, 1)$ and $y = 0$
The graph of the function F is found by shifting the graph of the function f to the right one unit.	$F(x) = f(x - 1)$
Shift the y-intercept to the right one unit.	$(0, 1)$ shifts to $(1, 1)$
The horizontal asymptote is not altered by a horizontal shift.	$y = 0$
Find additional points on the graph.	$F(0) = 2^{0-1} = 2^{-1} = \frac{1}{2}$
	y-intercept: $\left(0, \frac{1}{2}\right)$
	$F(2) = 2^{2-1} = 2^1 = 2$

Sketch the graph of $F(x) = 2^{x-1}$ with a *smooth* curve.

Domain: $(-\infty, \infty)$
Range: $(0, \infty)$

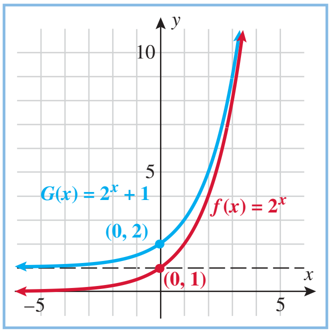


Solution (b):

Identify the base function.	$f(x) = 2^x$
Identify the base function y-intercept and horizontal asymptote.	$(0, 1)$ and $y = 0$
The graph of the function G is found by shifting the graph of the function f up one unit.	$G(x) = f(x) + 1$
Shift the y-intercept up one unit.	$(0, 1)$ shifts to $(0, 2)$
Shift the horizontal asymptote up one unit.	$y = 0$ shifts to $y = 1$
Find additional points on the graph.	$G(1) = 2^1 + 1 = 2 + 1 = 3$
	$G(-1) = 2^{-1} + 1 = \frac{1}{2} + 1 = \frac{3}{2}$

Sketch the graph of $G(x) = 2^x + 1$ with a *smooth* curve.

Domain: $(-\infty, \infty)$
Range: $(1, \infty)$



EXAMPLE 5 **Graphing Exponential Functions Using Both Horizontal and Vertical Shifts**

Graph the function $F(x) = 3^{x+1} - 2$. State the domain and range of F.

Solution:

Identify the base function. $f(x) = 3^x$

Identify the base function y-intercept and horizontal asymptote. $(0, 1)$ and $y = 0$

The graph of the function F is found by shifting the graph of the function f to the left one unit and down two units. $F(x) = f(x + 1) - 2$

Shift the y-intercept to the left one unit and down two units. $(0, 1)$ shifts to $(-1, -1)$

Shift the horizontal asymptote down two units. $y = 0$ shifts to $y = -2$

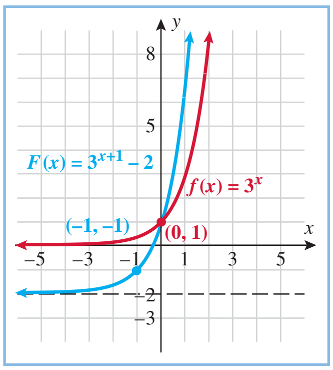
Find additional points on the graph.

$F(0) = 3^{0+1} - 2 = 3 - 2 = 1$

$F(1) = 3^{1+1} - 2 = 9 - 2 = 7$

Sketch the graph of $F(x) = 3^{x+1} - 2$ with a *smooth* curve.

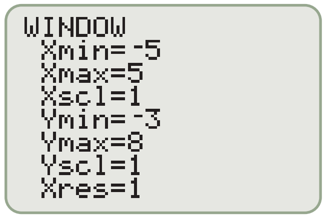
Domain: $(-\infty, \infty)$
Range: $(-2, \infty)$

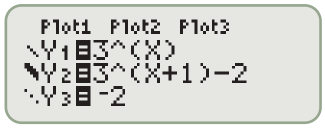


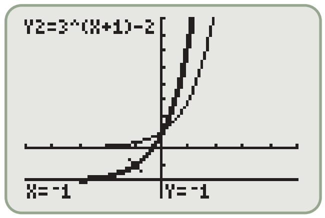
■ **YOUR TURN** Graph $f(x) = 2^{x+3} - 1$. State the domain and range of f.

Technology Tip

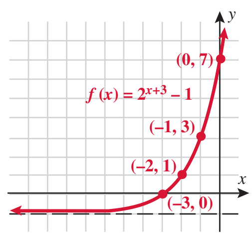
The graph of $f(x) = 3^{x+1} - 2$ is shown.







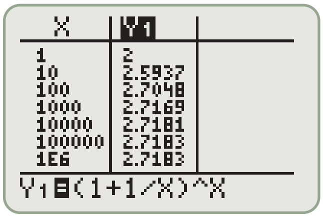
■ **Answer:** Domain: $(-\infty, \infty)$
Range: $(-1, \infty)$



The Natural Base e

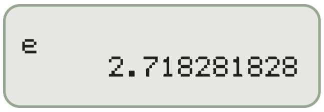
Technology Tip

Use a graphing utility to set up the table for $\left(1+\frac{1}{m}\right)^m$. Enter x for m.



Technology Tip

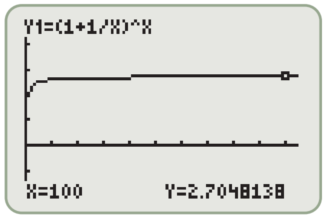
Evaluate e^1. To find the value of e, press the e key. This can be done by pressing $\boxed{\text{2nd}}\boxed{\div}$.



Any positive real number can serve as the base for an exponential function. A particular irrational number, denoted by the letter e, appears as the base in many applications, as you will soon see when we discuss continuous compounded interest. Although you will see 2 and 10 as common bases, the base that appears most often is e, because $\boldsymbol{e}$, as you will come to see in your further studies of mathematics, is the **natural base**. The exponential function with base e, $f(x) = e^x$, is called the **exponential function** or the **natural exponential function**. Mathematicians did not pull this irrational number out of a hat. The number e has many remarkable properties, but most simply, it comes from evaluating the expression $\left(1+\frac{1}{m}\right)^m$ as m gets large (increases without bound).

$\boxed{e \approx 2.71828}$

m	$\left(1+\frac{1}{m}\right)^m$
1	2
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	**2.71828**



Calculators have an $\boxed{e^x}$ button for approximating the natural exponential function.

EXAMPLE 6 Evaluating the Natural Exponential Function

Evaluate $f(x) = e^x$ for the given x-values. Round your answers to four decimal places.

a. $x = 1$ **b.** $x = -1$ **c.** $x = 1.2$ **d.** $x = -0.47$

Solution:

a. $f(1) = e^1 \approx 2.718281828 \approx \boxed{2.7183}$

b. $f(-1) = e^{-1} \approx 0.367879441 \approx \boxed{0.3679}$

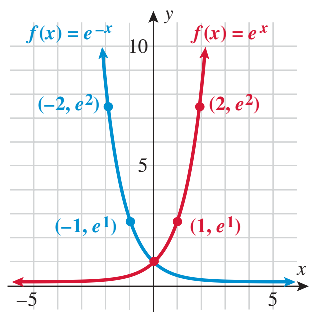
c. $f(1.2) = e^{1.2} \approx 3.320116923 \approx \boxed{3.3201}$

d. $f(-0.47) = e^{-0.47} \approx 0.625002268 \approx \boxed{0.6250}$

Study Tip

A **common base** is 10, that is, $f(x) = 10^x$. The *natural base* is e, that is, $f(x) = e^x$.

Like all exponential functions of the form $f(x) = b^x$, $f(x) = e^x$ and $f(x) = e^{-x}$ have (0, 1) as their y-intercept and the x-axis as a horizontal asymptote as shown in the figure on the right.

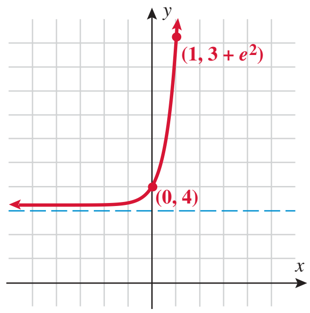


EXAMPLE 7 **Graphing Exponential Functions with Base e**

Graph the function $f(x) = 3 + e^{2x}$.

Solution:

x	$f(x) = 3 + e^{2x}$	(x, y)
-2	3.02	$(-2, 3.02)$
-1	3.14	$(1, 3.14)$
0	4	$(0, 4)$
1	10.39	$(1, 10.39)$
2	57.60	$(2, 57.60)$



Note: The y-intercept is $(0, 4)$, and the line $y = 3$ is the horizontal asymptote.

■ **YOUR TURN** Graph the function $f(x) = e^{x+1} - 2$.

Applications of Exponential Functions

Exponential functions describe either *growth* or *decay*. Populations and investments are often modeled with exponential growth functions, while the declining value of a used car and the radioactive decay of isotopes are often modeled with exponential decay functions. In Section 3.5, various exponential models will be discussed. In this section, we discuss doubling time, half-life, and compound interest.

A successful investment program, growing at about 7.2% per year, will double in size every 10 years. Let's assume that you will retire at the age of 65. There is a saying: *It's not the first time your money doubles, it's the last time that makes such a difference.* As you may already know or as you will soon find, it is important to start investing early.

Suppose Maria invests \$5000 at age 25 and David invests \$5000 at age 35. Let's calculate how much will accrue from the initial \$5000 investment by the time they each retire, assuming their money doubles every 10 years and that they both retire at age 65.

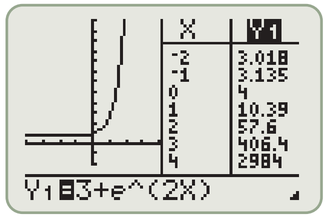
AGE	MARIA	DAVID
25	\$5,000	
35	\$10,000	\$5,000
45	\$20,000	\$10,000
55	\$40,000	\$20,000
65	**\$80,000**	**\$40,000**

They each made a one-time investment of \$5000. By investing 10 years sooner, Maria made twice what David made.

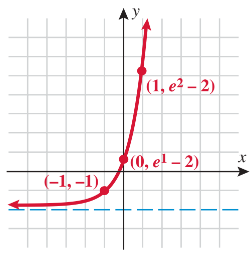
A measure of growth rate is the *doubling time*, the time it takes for something to double. Often doubling time is used to describe populations.

Technology Tip

The graph of $y_1 = 3 + e^{2x}$ is shown. To setup the split screen, press MODE G-T ENTER GRAPH.



■ **Answer:**



DOUBLING TIME GROWTH MODEL

The doubling time growth model is given by

$$P = P_0 2^{t/d}$$

where P = Population at time t

P_0 = Population at time $t = 0$

d = Doubling time

Note that when $t = d$, $P = 2P_0$ (population is equal to twice the original).

The units for P and P_0 are the same and can be any quantity (people, dollars, etc.). The units for t and d must be the same (years, weeks, days, hours, seconds, etc.).

In the investment scenario with Maria and David, $P_0 = \$5000$ and $d = 10$ years, so the model used to predict how much money the original \$5000 investment yielded is $P = 5000(2)^{t/10}$. Maria retired 40 years after the original investment, $t = 40$, and David retired 30 years after the original investment, $t = 30$.

$$\text{Maria: } P = 5000(2)^{40/10} = 5000(2)^4 = 5000(16) = 80{,}000$$
$$\text{David: } P = 5000(2)^{30/10} = 5000(2)^3 = 5000(8) = 40{,}000$$

EXAMPLE 8 Doubling Time of Populations

In 2004 the population in Kazakhstan, a country in Asia, reached 15 million. It is estimated that the population doubles in 30 years. If the population continues to grow at the same rate, what will the population be in 2024? Round to the nearest million.

Solution:

Write the doubling model.	$P = P_0 2^{t/d}$
Substitute $P_0 = 15$ million, $d = 30$ years, and $t = 20$ years.	$P = 15(2)^{20/30}$
Simplify.	$P = 15(2)^{2/3} \approx 23.8110$

In 2024, there will be approximately $\boxed{\text{24 million people}}$ in Kazakhstan.

■ **YOUR TURN** What will the approximate population in Kazakhstan be in 2044? Round to the nearest million.

■ **Answer:** 38 million

We now turn our attention from exponential growth to exponential decay, or negative growth. Suppose you buy a brand-new car from a dealership for \$24,000. The value of a car decreases over time according to an exponential decay function. The **half-life** of this particular car, or the time it takes for the car to depreciate 50%, is approximately 3 years. The exponential decay is described by

$$A = A_0\left(\frac{1}{2}\right)^{t/h}$$

where A_0 is the amount the car is worth (in dollars) when new (that is, when $t = 0$), A is the amount the car is worth (in dollars) after t years, and h is the half-life in years. In our car scenario, $A_0 = 24{,}000$ and $h = 3$:

$$A = 24{,}000\left(\frac{1}{2}\right)^{t/3}$$

How much is the car worth after three years? Six years? Nine years? Twenty-four years?

$$t = 3: \quad A = 24{,}000\left(\frac{1}{2}\right)^{3/3} = 24{,}000\left(\frac{1}{2}\right) = 12{,}000$$

$$t = 6: \quad A = 24{,}000\left(\frac{1}{2}\right)^{6/3} = 24{,}000\left(\frac{1}{2}\right)^{2} = 6000$$

$$t = 9: \quad A = 24{,}000\left(\frac{1}{2}\right)^{9/3} = 24{,}000\left(\frac{1}{2}\right)^{3} = 3000$$

$$t = 24: \quad A = 24{,}000\left(\frac{1}{2}\right)^{24/3} = 24{,}000\left(\frac{1}{2}\right)^{8} = 93.75 \approx 100$$

The car that was worth \$24,000 new is worth \$12,000 in 3 years, \$6000 in 6 years, \$3000 in 9 years, and about \$100 in the junkyard in 24 years.

EXAMPLE 9 Radioactive Decay

The radioactive isotope of potassium ^{42}K, which is used in the diagnosis of brain tumors, has a half-life of 12.36 hours. If 500 milligrams of potassium 42 are taken, how many milligrams will remain after 24 hours? Round to the nearest milligram.

Solution:

Write the half-life formula. $\quad A = A_0\left(\frac{1}{2}\right)^{t/h}$

Substitute $A_0 = 500$ mg, $h = 12.36$ hours, $t = 24$ hours. $\quad A = 500\left(\frac{1}{2}\right)^{24/12.36}$

Simplify. $\quad A \approx 500(0.2603) \approx 130.15$

After 24 hours, there are approximately $\boxed{\text{130 milligrams}}$ of potassium 42 left.

■ **YOUR TURN** How many milligrams of potassium 42 are expected to be left in the body after 1 week?

■ **Answer:** 0.04 mg (less than 1 mg)

In Section 0.1, *simple interest* was defined where the interest I is calculated based on the principal P, the annual interest rate r, and the time t in years, using the formula $I = Prt$.

If the interest earned in a period is then reinvested at the same rate, future interest is earned on both the principal and the reinvested interest during the next period. Interest paid on both the principal and interest is called *compound interest*.

COMPOUND INTEREST

If a **principal** P is invested at an annual **rate** $\boldsymbol{r}$ **compounded** n times a year, then the **amount** A in the account at the end of t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

The annual interest rate r is expressed as a decimal.

The following list shows the typical number of times interest is compounded:

Annually	$n = 1$	Monthly	$n = 12$
Semiannually	$n = 2$	Weekly	$n = 52$
Quarterly	$n = 4$	Daily	$n = 365$

EXAMPLE 10 Compound Interest

If \$3000 is deposited in an account paying 3% compounded quarterly, how much will you have in the account in 7 years?

Solution:

Write the compound interest formula. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Substitute $P = 3000$, $r = 0.03$, $n = 4$, and $t = 7$. $A = 3000\left(1 + \frac{0.03}{4}\right)^{(4)(7)}$

Simplify. $A = 3000(1.0075)^{28} \approx 3698.14$

You will have $\boxed{\$3698.14}$ in the account.

■ **YOUR TURN** If \$5000 is deposited in an account paying 6% compounded annually, how much will you have in the account in 4 years?

Technology Tip

Enter $3000\left(1 + \frac{0.03}{4}\right)^{(4)(7)}$ into the graphing calculator.

```
3000(1+0.03/4)^(
4*7)
        3698.135243
```

■ **Answer:** \$6312.38

Notice in the compound interest formula that as n increases the amount A also increases. In other words, the more times the interest is compounded per year, the more money you make. Ideally, your bank will compound your interest infinitely many times. This is called *compounding continuously*. We will now show the development of the continuous compounding formula, $A = Pe^{rt}$.

WORDS	**MATH**
Write the compound interest formula.	$A = P\left(1 + \frac{r}{n}\right)^{nt}$
Note that $\frac{r}{n} = \frac{1}{n/r}$ and $nt = \left(\frac{n}{r}\right)rt$.	$A = P\left(1 + \frac{1}{n/r}\right)^{(n/r)rt}$
Let $m = \frac{n}{r}$.	$A = P\left(1 + \frac{1}{m}\right)^{mrt}$
Use the exponential property: $x^{mrt} = (x^m)^{rt}$.	$A = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$

Recall that as m increases, $\left(1 + \frac{1}{m}\right)^{m}$ approaches e. Therefore, as the number of times the interest is compounded approaches infinity, or as $n \to \infty$, the amount in an account $A = P\left(1 + \frac{r}{n}\right)^{nt}$ approaches $A = Pe^{rt}$.

CONTINUOUS COMPOUND INTEREST

If a **principal** P is invested at an annual **rate r compounded continuously**, then the **amount** A in the account at the end of t years is given by

$$A = Pe^{rt}$$

The annual interest rate r is expressed as a decimal.

It is important to note that for a given interest rate, the highest return you can earn is by compounding continuously.

EXAMPLE 11 Continuously Compounded Interest

If \$3000 is deposited in a savings account paying 3% a year compounded continuously, how much will you have in the account in 7 years?

Solution:

Write the continuous compound interest formula. $A = Pe^{rt}$

Substitute $P = 3000$, $r = 0.03$, and $t = 7$. $A = 3000e^{(0.03)(7)}$

Simplify. $A \approx 3701.034$

There will be $\boxed{\$3701.03}$ in the account in 7 years.

Note: In Example 10, we worked this same problem compounding *quarterly,* and the result was \$3698.14.

If the number of times per year interest is compounded increases, then the total interest earned that year also increases.

YOUR TURN If \$5000 is deposited in an account paying 6% compounded continuously, how much will be in the account in 4 years?

Technology Tip

Enter $3000e^{(0.03)(7)}$ into the graphing calculator.

```
3000e^(0.03*7)
        3701.03418
```

Study Tip

If the number of times per year interest is compounded increases, then the total interest earned that year also increases.

Answer: \$6356.25

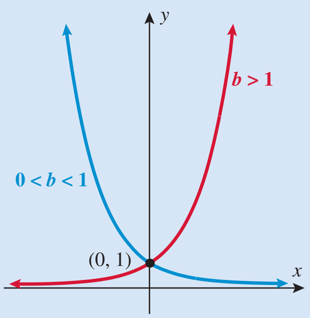
SECTION 3.1 SUMMARY

In this section, we discussed exponential functions (constant base, variable exponent).

General Exponential Functions: $f(x) = b^x$, $b \neq 1$, and $b > 0$

1. Evaluating exponential functions
   - Exact (by inspection): $f(x) = 2^x$ $\quad f(3) = 2^3 = 8$.
   - Approximate (with the aid of a calculator): $f(x) = 2^x$ $f(\sqrt{3}) = 2^{\sqrt{3}} \approx 3.322$
2. Graphs of exponential functions
   - Domain: $(-\infty, \infty)$ and range: $(0, \infty)$.
   - The point $(0, 1)$ corresponds to the y-intercept.
   - The graph passes through the points $(1, b)$ and $\left(-1, \frac{1}{b}\right)$.
   - The x-axis is a horizontal asymptote.
   - The function f is one-to-one.

$$f(x) = b^x, \qquad b > 0, \qquad b \neq 1$$



Procedure for Graphing: $f(x) = b^x$

Step 1: Label the point $(0, 1)$ corresponding to the y-intercept $f(0)$.

Step 2: Find and label two additional points corresponding to $f(-1)$ and $f(1)$.

Step 3: Connect the three points with a smooth curve with the x-axis as the horizontal asymptote.

The Natural Exponential Function: $f(x) = e^x$

- The irrational number e is called the natural base.
- $e = \left(1 + \frac{1}{m}\right)^m$ as $m \to \infty$
- $e \approx 2.71828$

Applications of Exponential Functions (all variables expressed in consistent units)

1. Doubling time: $P = P_0 2^{t/d}$
   - d is doubling time.
   - P is population at time t.
   - P_0 is population at time $t = 0$.
2. Half-life: $A = A_0\left(\frac{1}{2}\right)^{t/h}$
   - h is the half-life.
   - A is amount at time t.
   - A_0 is amount at time $t = 0$.
3. Compound interest (P = principal, A = amount after t years, r = interest rate)
   - Compounded n times a year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
   - Compounded continuously: $A = Pe^{rt}$

SECTION 3.1 EXERCISES

SKILLS

In Exercises 1–6, evaluate *exactly* (without using a calculator). For rational exponents, consider converting to radical form first.

1. 5^{-2} **2.** 4^{-3} **3.** $8^{2/3}$ **4.** $27^{2/3}$ **5.** $\left(\frac{1}{9}\right)^{-3/2}$ **6.** $\left(\frac{1}{16}\right)^{-3/2}$

In Exercises 7–12, approximate with a calculator. Round your answer to four decimal places.

7. $5^{\sqrt{2}}$ **8.** $6^{\sqrt{3}}$ **9.** e^{2} **10.** $e^{1/2}$ **11.** $e^{-\pi}$ **12.** $e^{-\sqrt{2}}$

In Exercises 13–20, for the functions $f(x) = 3^x$, $g(x) = \left(\frac{1}{16}\right)^x$, and $h(x) = 10^{x+1}$, find the function value at the indicated points.

13. $f(3)$ **14.** $h(1)$ **15.** $g(-1)$ **16.** $f(-2)$

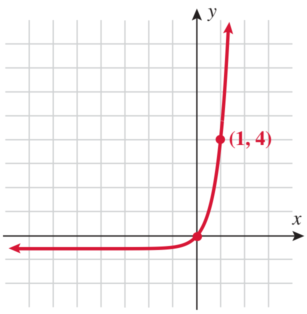
17. $g\left(-\frac{1}{2}\right)$ **18.** $g\left(-\frac{3}{2}\right)$ **19.** $f(e)$ **20.** $g(\pi)$

In Exercises 21–26, match the graph with the function.

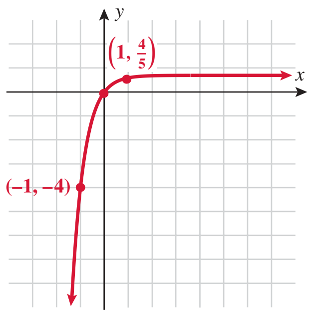
21. $y = 5^{x-1}$ **22.** $y = 5^{1-x}$ **23.** $y = -5^{x}$

24. $y = -5^{-x}$ **25.** $y = 1 - 5^{-x}$ **26.** $y = 5^{x} - 1$

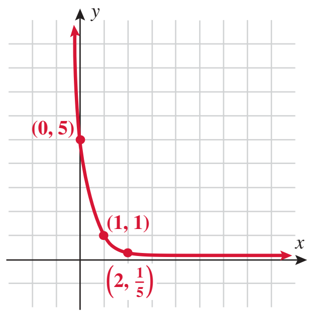
a.



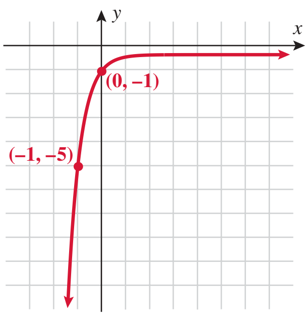
b.



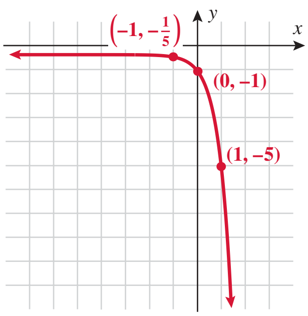
c.



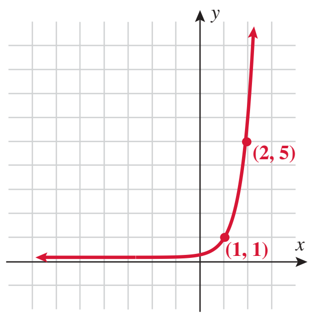
d.



e.



f.



In Exercises 27–46, graph the exponential function using transformations. State the *y*-intercept, two additional points, the domain, the range, and the horizontal asymptote.

27. $f(x) = 6^x$

28. $f(x) = 7^x$

29. $f(x) = 10^{-x}$

30. $f(x) = 4^{-x}$

31. $f(x) = e^x$

32. $f(x) = -e^{-x}$

33. $f(x) = e^{-x}$

34. $f(x) = -e^x$

35. $f(x) = 2^x - 1$

36. $f(x) = 3^x - 1$

37. $f(x) = 2 - e^x$

38. $f(x) = 1 + e^{-x}$

39. $f(x) = 5 + 4^{-x}$

40. $f(x) = 5^x - 2$

41. $f(x) = e^{x+1} - 4$

42. $f(x) = e^{x-1} + 2$

43. $f(x) = 3e^{x/2}$

44. $f(x) = 2e^{-x}$

45. $f(x) = 1 + \left(\frac{1}{2}\right)^{x-2}$

46. $f(x) = 2 - \left(\frac{1}{3}\right)^{x+1}$

APPLICATIONS

47. Population Doubling Time. In 2002 there were 7.1 million people living in London, England. If the population is expected to double by 2090, what is the expected population in London in 2050?

48. Population Doubling Time. In 2004 the population in Morganton, Georgia, was 43,000. The population in Morganton doubled by 2010. If the growth rate remains the same, what is the expected population in Morganton in 2020?

49. Investments. Suppose an investor buys land in a rural area for \$1500 an acre and sells some of it 5 years later at \$3000 an acre and the rest of it 10 years later at \$6000. Write a function that models the value of land in that area, assuming the growth rate stays the same. What would the expected cost per acre be 30 years after the initial investment of \$1500?

50. Salaries. Twin brothers, Collin and Cameron, get jobs immediately after graduating from college at the age of 22. Collin opts for the higher starting salary, \$55,000, and stays with the same company until he retires at 65. His salary doubles every 15 years. Cameron opts for a lower starting salary, \$35,000, but moves to a new job every 5 years; he doubles his salary every 10 years until he retires at 65. What is the annual salary of each brother upon retirement?

51. Radioactive Decay. A radioactive isotope of selenium, ^{75}Se, which is used in medical imaging of the pancreas, has a half-life of 119.77 days. If 200 milligrams are given to a patient, how many milligrams are left after 30 days?

52. Radioactive Decay. The radioactive isotope indium-111 (^{111}In), used as a diagnostic tool for locating tumors associated with prostate cancer, has a half-life of 2.807 days. If 300 milligrams are given to a patient, how many milligrams will be left after a week?

53. Radioactive Decay. A radioactive isotope of beryllium-11 decays to borom-11 with a half-life of 13.81 seconds. Beryllium is given to patients that suffer Chronic Beryllium Disease (CBD). If 800 milligrams are given to a CBD patient, how much beryllium is present after 2 minutes? Round your answer to the nearest milligram.

54. Radioactive Decay. If the CBD patient in Exercise 53 is given 1000 milligrams, how much beryllium is present after 1 minute? Round your answer to the nearest milligram.

55. Depreciation of Furniture. A couple buy a new bedroom set for \$8000 and 10 years later sell it for \$4000. If the depreciation continues at the same rate, how much would the bedroom set be worth in 4 more years?

56. Depreciation of a Computer. A student buys a new laptop for \$1500 when she arrives as a freshman. A year later, the computer is worth approximately \$750. If the depreciation continues at the same rate, how much would she expect to sell her laptop for when she graduates 4 years after she bought it?

57. Compound Interest. If you put \$3200 in a savings account that earns 2.5% interest per year compounded quarterly, how much would you expect to have in that account in 3 years?

58. Compound Interest. If you put \$10,000 in a savings account that earns 3.5% interest per year compounded annually, how much would you expect to have in that account in 5 years?

59. Compound Interest. How much money should you put in a savings account now that earns 5% a year compounded daily if you want to have \$32,000 in 18 years?

60. Compound Interest. How much money should you put in a savings account now that earns 3.0% a year compounded weekly if you want to have \$80,000 in 15 years?

61. Compound Interest. If you put \$3200 in a savings account that pays 2% a year compounded continuously, how much will you have in the account in 15 years?

62. Compound Interest. If you put \$7000 in a money market account that pays 4.3% a year compounded continuously, how much will you have in the account in 10 years?

63. Compound Interest. How much money should you deposit into a money market account that pays 5% a year compounded continuously to have \$38,000 in the account in 20 years?

64. Compound Interest. How much money should you deposit into a certificate of deposit that pays 6% a year compounded continuously to have \$80,000 in the account in 18 years?

For Exercises 65 and 66, refer to the following:

Exponential functions can be used to model the concentration of a drug in a patient's body. Suppose the concentration of Drug X in a patient's bloodstream is modeled by

$$C(t) = C_0 e^{-rt}$$

where $C(t)$ represents the concentration at time t (in hours), C_0 is the concentration of the drug in the blood immediately after injection, and $r > 0$ is a constant indicating the removal of the drug by the body through metabolism and/or excretion. The rate constant r has units of 1/time (1/hr). It is important to note that this model assumes that the blood concentration of the drug C_0 peaks immediately when the drug is injected.

65. Health/Medicine. After an injection of Drug Y, the concentration of the drug in the bloodstream drops at the rate of 0.020 1/hr. Find the concentration, to the nearest tenth, of the drug 20 hours after receiving an injection with initial concentration of 5.0 mg/L.

66. Health/Medicine. After an injection of Drug Y, the concentration of the drug in the bloodstream drops at the rate of 0.009 1/hr. Find the concentration, to the nearest tenth, of the drug 4 hours after receiving an injection with initial concentration of 4.0 mg/L.

For Exercises 67 and 68, refer to the following:

The demand for a product, in thousands of units, can be expressed by the exponential demand function

$$D(p) = 2300(0.85)^p$$

where p is the price per unit.

67. Economics. Find the demand for the product by completing the following table.

p (PRICE PER UNIT)	$D(p)$–DEMAND FOR PRODUCT IN UNITS
1.00	
5.00	
10.00	
20.00	
40.00	
60.00	
80.00	
90.00	

68. Economics. Evaluate $D(91)$ and interpret what this means in terms of demand.

CATCH THE MISTAKE

In Exercises 69–72, explain the mistake that is made.

69. Evaluate the expression $4^{-1/2}$.

Solution: $4^{-1/2} = 4^2 = 16$

The correct value is $\frac{1}{2}$. What mistake was made?

70. Evaluate the function for the given x: $f(x) = 4^x$ for $x = \frac{3}{2}$.

Solution:

$$f\left(\frac{3}{2}\right) = 4^{3/2}$$

$$= \frac{4^3}{4^2} = \frac{64}{16} = 4$$

The correct value is 8. What mistake was made?

71. If \$2000 is invested in a savings account that earns 2.5% interest compounding continuously, how much will be in the account in one year?

Solution:

Write the compound continuous interest formula. $A = Pe^{rt}$

Substitute $P = 2000$, $r = 2.5$, and $t = 1$. $A = 2000e^{(2.5)(1)}$

Simplify. $A = 24{,}364.99$

This is incorrect. What mistake was made?

72. If \$5000 is invested in a savings account that earns 3% interest compounding continuously, how much will be in the account in 6 months?

Solution:

Write the compound continuous interest formula. $A = Pe^{rt}$

Substitute $P = 5000$, $r = 0.03$, and $t = 6$. $A = 5000e^{(0.03)(6)}$

Simplify. $A = 5986.09$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 73–76, determine whether each statement is true or false.

73. The function $f(x) = -e^{-x}$ has the y-intercept (0, 1).

74. The function $f(x) = -e^{-x}$ has a horizontal asymptote along the x-axis.

75. The functions $y = 3^{-x}$ and $y = \left(\frac{1}{3}\right)^x$ have the same graphs.

76. $e = 2.718$.

77. Plot $f(x) = 3^x$ and its inverse on the same graph.

78. Plot $f(x) = e^x$ and its inverse on the same graph.

79. Graph $f(x) = e^{|x|}$.

80. Graph $f(x) = e^{-|x|}$.

CHALLENGE

81. Find the y-intercept and horizontal asymptote of $f(x) = be^{-x+1} - a$.

82. Find the y-intercept and horizontal asymptote of $f(x) = a + be^{x+1}$.

83. Graph $f(x) = b^{|x|}$, $b > 1$, and state the domain.

84. Graph the function $f(x) = \begin{cases} a^x & x < 0 \\ a^{-x} & x \geq 0 \end{cases}$ where $a > 1$.

85. Graph the function

$$f(x) = \begin{cases} -a^x & x < 0 \\ -a^{-x} & x \geq 0 \end{cases} \quad \text{where } 0 < a < 1.$$

86. Find the y-intercept and horizontal asymptote(s) of $f(x) = 2^x + 3^x$.

TECHNOLOGY

87. Plot the function $y = \left(1 + \frac{1}{x}\right)^x$. What is the horizontal asymptote as x increases?

88. Plot the functions $y = 2^x$, $y = e^x$, and $y = 3^x$ in the same viewing screen. Explain why $y = e^x$ lies between the other two graphs.

89. Plot $y_1 = e^x$ and $y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ in the same viewing screen. What do you notice?

90. Plot $y_1 = e^{-x}$ and $y_2 = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$ in the same viewing screen. What do you notice?

91. Plot the functions $f(x) = \left(1 + \frac{1}{x}\right)^x$, $g(x) = \left(1 + \frac{2}{x}\right)^x$, and $h(x) = \left(1 + \frac{2}{x}\right)^{2x}$ in the same viewing screen. Compare their horizontal asymptotes as x increases. What can you say about the function values of f, g, and h in terms of the powers of e as x increases?

92. Plot the functions $f(x) = \left(1 + \frac{1}{x}\right)^x$, $g(x) = \left(1 - \frac{1}{x}\right)^x$, and $h(x) = \left(1 - \frac{2}{x}\right)^x$ in the same viewing screen. Compare their horizontal asymptotes as x increases. What can you say about the function values of f, g, and h in terms of the powers of e as x increases?

For Exercises 93 and 94, refer to the following:

Newton's Law of Heating and Cooling: Have you ever heated soup in a microwave and, upon taking it out, have it seem to cool considerably in the matter of minutes? Or has your ice-cold soda become tepid in just moments while outside on a hot summer's day? This phenomenon is based on the so-called *Newton's Law of Heating and Cooling*. Eventually, the soup will cool so that its temperature is the same as the temperature of the room in which it is being kept, and the soda will warm until its temperature is the same as the outside temperature.

93. Consider the following data:

TIME (IN MINUTES)	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
TEMPERATURE OF SOUP (IN DEGREES FAHRENHEIT)	203	200	195	188	180	171	160	151

a. Form a scatterplot for this data.

b. Use *ExpReg* to find the best fit exponential function for this data set, and superimpose its graph on the scatterplot. How good is the fit?

c. Use the best fit exponential curve from (b) to answer the following:

i. What will the predicted temperature of the soup be at 6 minutes?

ii. What was the temperature of the soup the moment it was taken out of the microwave?

d. Assume the temperature of the house is 72°F. According to Newton's Law of Heating and Cooling, the temperature of the soup should approach 72°. In light of this, comment on the shortcomings of the best fit exponential curve.

94. Consider the following data:

TIME (IN MINUTES)	1	2	3	4	5	6	7	8
TEMPERATURE OF SODA (IN DEGREES FAHRENHEIT)	45	48	49	53	57	61	68	75

a. Form a scatterplot for this data.
b. Use *ExpReg* to find the best fit exponential function for this data set, and superimpose its graph on the scatterplot. How good is the fit?
c. Use the best fit exponential curve from (b) to answer the following:
  i. What will the predicted temperature of the soda be at 10 minutes?
  ii. What was the temperature of the soda the moment it was taken out of the refrigerator?
d. Assume the temperature of the house is 90°F. According to Newton's Law of Heating and Cooling, the temperature of the soda should approach 90°. In light of this, comment on the shortcomings of the best fit exponential curve.

PREVIEW TO CALCULUS

In calculus the following two functions are studied:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

95. Determine whether $f(x) = \sinh x$ is an even function or an odd function.

96. Determine whether $f(x) = \cosh x$ is an even function or an odd function.

97. Show that $\cosh^2 x - \sinh^2 x = 1$.

98. Show that $\cosh x + \sinh x = e^x$.

SECTION 3.2 LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

SKILLS OBJECTIVES

- Convert exponential expressions to logarithmic expressions.
- Convert logarithmic expressions to exponential expressions.
- Evaluate logarithmic expressions exactly by inspection.
- Approximate common and natural logarithms using a calculator.
- Graph logarithmic functions.
- Determine domain restrictions on logarithmic functions.

CONCEPTUAL OBJECTIVES

- Interpret logarithmic functions as inverses of exponential functions.
- Understand that logarithmic functions allow very large ranges of numbers in science and engineering applications to be represented on a smaller scale.

Evaluating Logarithms

In Section 3.1, we found that the graph of an exponential function, $f(x) = b^x$, passes through the point (0, 1), with the x-axis as a horizontal asymptote. The graph passes both the vertical line test (for a function) and the horizontal line test (for a one-to-one function), and therefore an inverse exists. We will now apply the technique outlined in Section 1.5 to find the inverse of $f(x) = b^x$:

WORDS	MATH
Let $y = f(x)$.	$y = b^x$
Interchange x and y.	$x = b^y$
Solve for y.	$y = ?$

DEFINITION **Logarithmic Function**

For $x > 0$, $b > 0$, and $b \neq 1$, the **logarithmic function with base b** is denoted $f(x) = \log_b x$, where

$$y = \log_b x \quad \text{if and only if} \quad x = b^y$$

We read $\log_b x$ as "log base b of x."

Study Tip

- $\log_b x = y$ is equivalent to $b^y = x$.
- The exponent y is called a logarithm (or "log" for short).

This definition says that $x = b^y$ (**exponential form**) and $y = \log_b x$ (**logarithmic form**) are equivalent. One way to remember this relationship is by adding arrows to the logarithmic form:

$$\log_b x = y \Leftrightarrow b^y = x$$

EXAMPLE 1 Changing from Logarithmic Form to Exponential Form

Express each equation in its equivalent exponential form.

a. $\log_2 8 = 3$ **b.** $\log_9 3 = \frac{1}{2}$ **c.** $\log_5\left(\frac{1}{25}\right) = -2$

Solution:

a. $\log_2 8 = 3$ is equivalent to $\boxed{2^3 = 8}$

b. $\log_9 3 = \frac{1}{2}$ is equivalent to $\boxed{9^{1/2} = 3}$

c. $\log_5\left(\frac{1}{25}\right) = -2$ is equivalent to $\boxed{5^{-2} = \frac{1}{25}}$

■ **YOUR TURN** Write each equation in its equivalent exponential form.

a. $\log_3 9 = 2$ **b.** $\log_{16} 4 = \frac{1}{2}$ **c.** $\log_2\left(\frac{1}{8}\right) = -3$

■ **Answer: a.** $9 = 3^2$
b. $4 = 16^{1/2}$
c. $\frac{1}{8} = 2^{-3}$

EXAMPLE 2 Changing from Exponential Form to Logarithmic Form

Write each equation in its equivalent logarithmic form.

a. $16 = 2^4$ **b.** $9 = \sqrt{81}$ **c.** $\frac{1}{9} = 3^{-2}$ **d.** $x^a = z$

Solution:

a. $16 = 2^4$ is equivalent to $\boxed{\log_2 16 = 4}$

b. $9 = \sqrt{81} = 81^{1/2}$ is equivalent to $\boxed{\log_{81} 9 = \frac{1}{2}}$

c. $\frac{1}{9} = 3^{-2}$ is equivalent to $\boxed{\log_3\left(\frac{1}{9}\right) = -2}$

d. $x^a = z$ is equivalent to $\boxed{\log_x z = a \quad \text{for } x > 0}$

■ **YOUR TURN** Write each equation in its equivalent logarithmic form.

a. $81 = 9^2$ **b.** $12 = \sqrt{144}$ **c.** $\frac{1}{49} = 7^{-2}$ **d.** $y^b = w$

■ **Answer: a.** $\log_9 81 = 2$
b. $\log_{144} 12 = \frac{1}{2}$
c. $\log_7\left(\frac{1}{49}\right) = -2$
d. $\log_y w = b \quad \text{for } y > 0$

Some logarithms can be found exactly, while others must be approximated. Example 3 illustrates how to find the exact value of a logarithm. Example 4 illustrates approximating values of logarithms with a calculator.

EXAMPLE 3 Finding the Exact Value of a Logarithm

Find the exact value of

a. $\log_3 81$ **b.** $\log_{169} 13$ **c.** $\log_5\left(\frac{1}{5}\right)$

Solution (a):

The logarithm has some value. Let's call it x. $\log_3 81 = x$

Change from logarithmic to exponential form. $3^x = 81$

3 raised to what power is 81? $3^4 = 81 \quad x = 4$

Change from exponential to logarithmic form. $\boxed{\log_3 81 = 4}$

Solution (b):

The logarithm has some value. Let's call it x. $\log_{169} 13 = x$

Change from logarithmic to exponential form. $169^x = 13$

169 raised to what power is 13? $169^{1/2} = \sqrt{169} = 13 \quad x = \frac{1}{2}$

Change from exponential to logarithmic form. $\boxed{\log_{169} 13 = \frac{1}{2}}$

Solution (c):

The logarithm has some value. Let's call it x. $\log_5\left(\frac{1}{5}\right) = x$

Change from logarithmic to exponential form. $5^x = \frac{1}{5}$

5 raised to what power is $\frac{1}{5}$? $5^{-1} = \frac{1}{5} \quad x = -1$

Change from exponential to logarithmic form. $\boxed{\log_5\left(\frac{1}{5}\right) = -1}$

YOUR TURN Evaluate the given logarithms exactly.

a. $\log_2 \frac{1}{2}$ **b.** $\log_{100} 10$ **c.** $\log_{10} 1000$

Answer: a. $\log_2 \frac{1}{2} = -1$
b. $\log_{100} 10 = \frac{1}{2}$
c. $\log_{10} 1000 = 3$

Common and Natural Logarithms

Two logarithmic bases that arise frequently are base 10 and base e. The logarithmic function of base 10 is called the **common logarithmic function**. Since it is common, $f(x) = \log_{10} x$ is often expressed as $f(x) = \log x$. Thus, if no explicit base is indicated, base 10 is implied. The logarithmic function of base e is called the **natural logarithmic function**. The natural logarithmic function $f(x) = \log_e x$ is often expressed as $f(x) = \ln x$. Both the LOG and LN buttons appear on scientific and graphing calculators. For the ***logarithms*** (not the functions), we say "the log" (for base 10) and "the natural log" (for base e).

Earlier in this section, we evaluated logarithms exactly by converting to exponential form and identifying the exponent. For example, to evaluate $\log_{10} 100$, we ask the question, 10 raised to what power is 100? The answer is 2.

Calculators enable us to approximate logarithms. For example, evaluate $\log_{10} 233$. We are unable to evaluate this exactly by asking the question, 10 raised to what power is 233? Since $10^2 < 10^x < 10^3$, we know the answer x must lie between 2 and 3. Instead, we use a calculator to find an approximate value 2.367.

Study Tip

- $\log_{10} x = \log x$. No explicit base implies base 10.
- $\log_e x = \ln x$

EXAMPLE 4 Using a Calculator to Evaluate Common and Natural Logarithms

Use a calculator to evaluate the common and natural logarithms. Round your answers to four decimal places.

a. $\log 415$ **b.** $\ln 415$ **c.** $\log 1$ **d.** $\ln 1$ **e.** $\log(-2)$ **f.** $\ln(-2)$

Solution:

a. $\log(415) \approx 2.618048097 \approx \boxed{2.6180}$ **b.** $\ln(415) \approx 6.02827852 \approx \boxed{6.0283}$

c. $\log(1) = \boxed{0}$ **d.** $\ln(1) = \boxed{0}$

e. $\log(-2)$ $\boxed{\text{undefined}}$ **f.** $\ln(-2)$ $\boxed{\text{undefined}}$

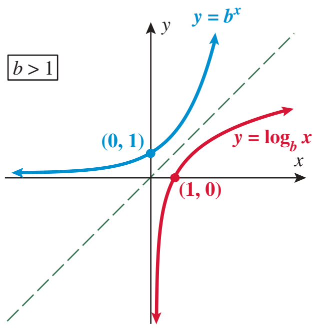
Study Tip

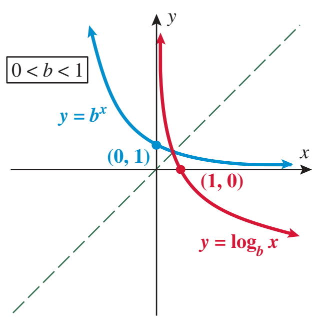
Logarithms can only be evaluated for positive arguments.

Parts (c) and (d) in Example 4 illustrate that all logarithmic functions pass through the point (1, 0). Parts (e) and (f) in Example 4 illustrate that the domains of logarithmic functions are positive real numbers.

Graphs of Logarithmic Functions

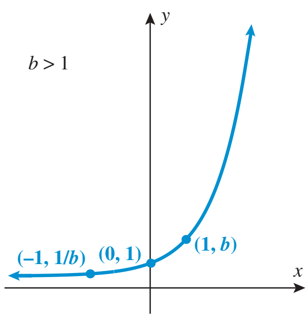
The general logarithmic function $y = \log_b x$ is defined as the inverse of the exponential function $y = b^x$. Therefore, when these two functions are plotted on the same graph, they are symmetric about the line $y = x$. Notice the symmetry about the line $y = x$ when $y = b^x$ and $y = \log_b x$ are plotted on the same graph.

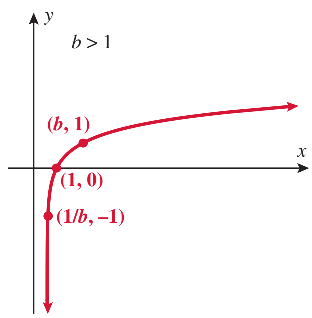




Comparison of Inverse Functions: $f(x) = \log_b x$ and $f^{-1}(x) = b^x$

EXPONENTIAL FUNCTION	LOGARITHMIC FUNCTION
$y = b^x$	$y = \log_b x$
y-intercept (0, 1)	x-intercept (1, 0)
Domain $(-\infty, \infty)$	Domain $(0, \infty)$
Range $(0, \infty)$	Range $(-\infty, \infty)$
Horizontal asymptote: x-axis	Vertical asymptote: y-axis





Additionally, the domain of one function is the range of the other, and vice versa. When dealing with logarithmic functions, special attention must be paid to the domain of the function. The domain of $y = \log_b x$ is $(0, \infty)$. In other words, you can only take the log of a positive real number, $x > 0$.

EXAMPLE 5 Finding the Domain of a Shifted Logarithmic Function

Find the domain of each of the given logarithmic functions.

a. $f(x) = \log_b(x - 4)$ **b.** $g(x) = \log_b(5 - 2x)$

Solution (a):

Set the argument greater than zero. $x - 4 > 0$

Solve the inequality. $x > 4$

Write the domain in interval notation. $\boxed{(4, \infty)}$

Solution (b):

Set the argument greater than zero. $5 - 2x > 0$

Solve the inequality. $-2x > -5$

$$2x < 5$$

$$x < \frac{5}{2}$$

Write the domain in interval notation. $\boxed{\left(-\infty, \frac{5}{2}\right)}$

■ **YOUR TURN** Find the domain of the given logarithmic functions.

a. $f(x) = \log_b(x + 2)$ **b.** $g(x) = \log_b(3 - 5x)$

■ **Answer: a.** $(-2, \infty)$
b. $\left(-\infty, \frac{3}{5}\right)$

It is important to note that when finding the domain of a logarithmic function, we set the argument strictly greater than zero and solve.

Study Tip

Review solving inequalities in Section 0.4.

EXAMPLE 6 Finding the Domain of a Logarithmic Function with a Complicated Argument

Find the domain of each of the given logarithmic functions.

a. $\ln(x^2 - 9)$ **b.** $\log(|x + 1|)$

Solution (a):

Set the argument greater than zero $x^2 - 9 > 0$

Solve the inequality. $\boxed{(-\infty, -3) \cup (3, \infty)}$

Solution (b):

Set the argument greater than zero $|x + 1| > 0$

Solve the inequality. $x \neq -1$

Write the domain in interval notation. $\boxed{(-\infty, -1) \cup (-1, \infty)}$

■ **YOUR TURN** Find the domain of each of the given logarithmic functions.

a. $\ln(x^2 - 4)$ **b.** $\log(|x - 3|)$

■ **Answer: a.** $(-\infty, -2) \cup (2, \infty)$
b. $(-\infty, 3) \cup (3, \infty)$

Recall from Section 1.3 that a technique for graphing general functions is transformations of known functions. For example, to graph $f(x) = (x - 3)^2 + 1$, we start with the known parabola $y = x^2$, whose vertex is at (0, 0), and we shift that graph to the right three units and up one unit. We use the same techniques for graphing logarithmic functions. To graph $y = \log_b(x + 2) - 1$, we start with the graph of $y = \log_b(x)$ and shift the graph to the left two units and down one unit.

EXAMPLE 7 Graphing Logarithmic Functions Using Horizontal and Vertical Shifts

Graph the functions, and state the domain and range of each.

a. $y = \log_2(x - 3)$ **b.** $\log_2 x - 3$

Solution:

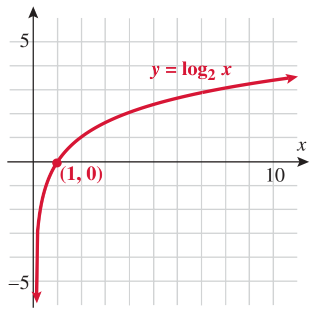
Identify the base function. $y = \log_2 x$

Label key features of $y = \log_2 x$.

x-intercept: **(1, 0)**

Vertical asymptote: $x = 0$

Additional points: **(2, 1), (4, 2)**



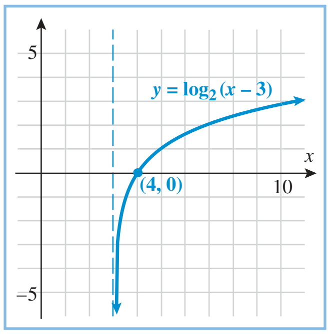
a. Shift base function to the *right* three units.

x-intercept: **(4, 0)**

Vertical asymptote: $x = 3$

Additional points: **(5, 1), (7, 2)**

$\boxed{\text{Domain: } (3, \infty) \qquad \text{Range: } (-\infty, \infty)}$



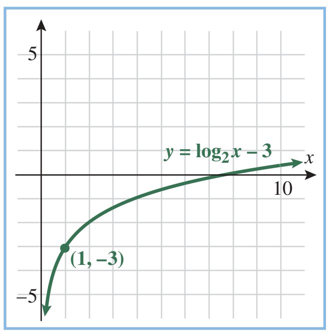
b. Shift base function *down* three units.

x-intercept: **(1, −3)**

Vertical asymptote: $x = 0$

Additional points: **(2, −2), (4, −1)**

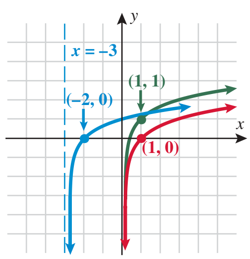
$\boxed{\text{Domain: } (0, \infty) \qquad \text{Range: } (-\infty, \infty)}$



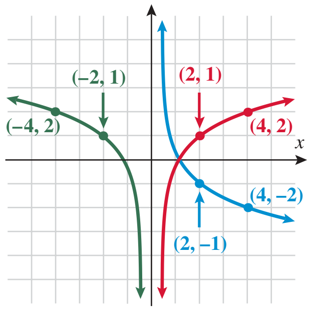
■ **YOUR TURN** Graph the functions and state the domain and range of each.

a. $y = \log_3 x$ **b.** $y = \log_3(x + 3)$ **c.** $\log_3 x + 1$

■ **Answer:**
a. Domain: $(0, \infty)$ Range: $(-\infty, \infty)$
b. Domain: $(-3, \infty)$ Range: $(-\infty, \infty)$
c. Domain: $(0, \infty)$ Range: $(-\infty, \infty)$



All of the transformation techniques (shifting, reflection, and compression) discussed in Chapter 1 also apply to logarithmic functions. For example, the graphs of $-\log_2 x$ and $\log_2(-x)$ are found by reflecting the graph of $y = \log_2 x$ about the x-axis and y-axis, respectively.



EXAMPLE 8 Graphing Logarithmic Functions Using Transformations

Graph the function $f(x) = -\log_2(x - 3)$ and state its domain and range.

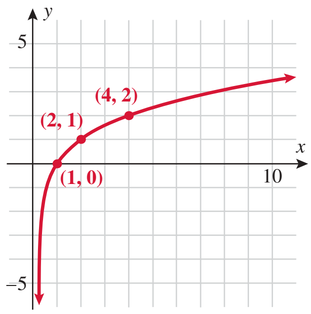
Solution:

Graph $y = \log_2 x$.

x-intercept: $(1, 0)$

Vertical asymptote: $x = 0$

Additional points: $(2, 1)$, $(4, 2)$

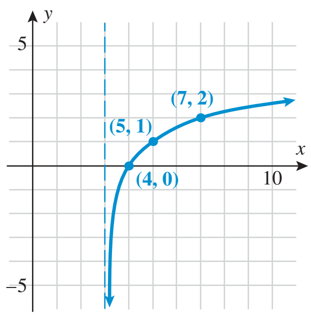


Graph $y = \log_2(x - 3)$ by shifting $y = \log_2 x$ to the *right* three units.

x-intercept: $(4, 0)$

Vertical asymptote: $x = 3$

Additional points: $(5, 1)$, $(7, 2)$

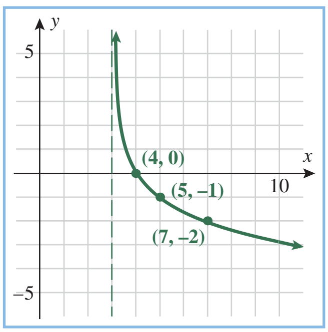


Graph $y = -\log_2(x - 3)$ by reflecting $y = \log_2(x - 3)$ about the x-axis.

x-intercept: $(4, 0)$

Vertical asymptote: $x = 3$

Additional points: $(5, -1)$, $(7, -2)$



Domain: $(3, \infty)$ Range: $(-\infty, \infty)$

Applications of Logarithms

Logarithms are used to make a large range of numbers manageable. For example, to create a scale to measure a human's ability to hear, we must have a way to measure the sound intensity of an explosion, even though that intensity can be more than a trillion (10^{12}) times greater than that of a soft whisper. Decibels in engineering and physics, pH in chemistry, and the Richter scale for earthquakes are all applications of logarithmic functions.

The **decibel** is a logarithmic unit used to measure the magnitude of a physical quantity relative to a specified reference level. The *decibel* (dB) is employed in many engineering and science applications. The most common application is the intensity of sound.

DEFINITION **Decibel (Sound)**

The **decibel** is defined as $$D = 10\log\left(\frac{I}{I_T}\right)$$

where D is the decibel level (dB), I is the intensity of the sound measured in watts per square meter, and I_T is the intensity threshold of the least audible sound a human can hear.

The human average threshold is $I_T = 1 \times 10^{-12}$ W/m^2.

Notice that when $I = I_T$, then $D = 10\overbrace{\log 1}^{0} = 0$ dB. People who work professionally with sound, such as acoustics engineers or medical hearing specialists, refer to this threshold level I_T as "0 dB." The following table illustrates typical sounds we hear and their corresponding decibel levels.

Sound Source	Sound Intensity (W/m^2)	Decibels (dB)
Threshold of hearing	1.0×10^{-12}	0
Vacuum cleaner	1.0×10^{-4}	80
iPod	1.0×10^{-2}	100
Jet engine	1.0×10^{3}	150

For example, a whisper (approximately 0 dB) from someone standing next to a jet engine (150 dB) might go unheard because when these are added, we get approximately 150 dB (the jet engine).

EXAMPLE 9 Calculating Decibels of Sounds

Suppose you have seats to a concert given by your favorite musical artist. Calculate the approximate decibel level associated with the typical sound intensity, given $I = 1 \times 10^{-2}$ W/m^2.

Solution:

Write the decibel-scale formula. $$D = 10\log\left(\frac{I}{I_T}\right)$$

Substitute $I = 1 \times 10^{-2}$ W/m^2 and $I_T = 1 \times 10^{-12}$ W/m^2. $$D = 10\log\left(\frac{1 \times 10^{-2}}{1 \times 10^{-12}}\right)$$

Simplify. $D = 10\log(10^{10})$

Recall that the implied base for log is 10. $D = 10\log_{10}(10^{10})$

Evaluate the right side. $[\log_{10}(10^{10}) = 10]$ $D = 10 \cdot 10$

$D = 100$

The typical sound level in the front row of a rock concert is $\boxed{100 \text{ dB}}$.

■ **YOUR TURN** Calculate the approximate decibels associated with a sound so loud it will cause instant perforation of the eardrums, $I = 1 \times 10^4 \text{ W/m}^2$.

■ **Answer:** 160 dB

The Richter scale (earthquakes) is another application of logarithms.

DEFINITION Richter Scale

The magnitude M of an earthquake is measured using the **Richter scale**

$$M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$$

where

M is the magnitude
E is the seismic energy released by the earthquake (in joules)
E_0 is the energy released by a reference earthquake $E_0 = 10^{4.4}$ joules

EXAMPLE 10 Calculating the Magnitude of an Earthquake

On October 17, 1989, just moments before game 3 of the World Series between the Oakland A's and the San Francisco Giants was about to start—with 60,000 fans in Candlestick Park—a devastating earthquake erupted. Parts of interstates and bridges collapsed, and President George H. W. Bush declared the area a disaster zone. The earthquake released approximately 1.12×10^{15} joules. Calculate the magnitude of the earthquake using the Richter scale.

Solution:

Write the Richter scale formula. $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$

Substitute $E = 1.12 \times 10^{15}$ and $E_0 = 10^{4.4}$. $M = \frac{2}{3}\log\left(\frac{1.12 \times 10^{15}}{10^{4.4}}\right)$

Simplify. $M = \frac{2}{3}\log(1.12 \times 10^{10.6})$

Approximate the logarithm using a calculator. $M \approx \frac{2}{3}(10.65) \approx 7.1$

The 1989 earthquake in California measured $\boxed{7.1}$ on the Richter scale.

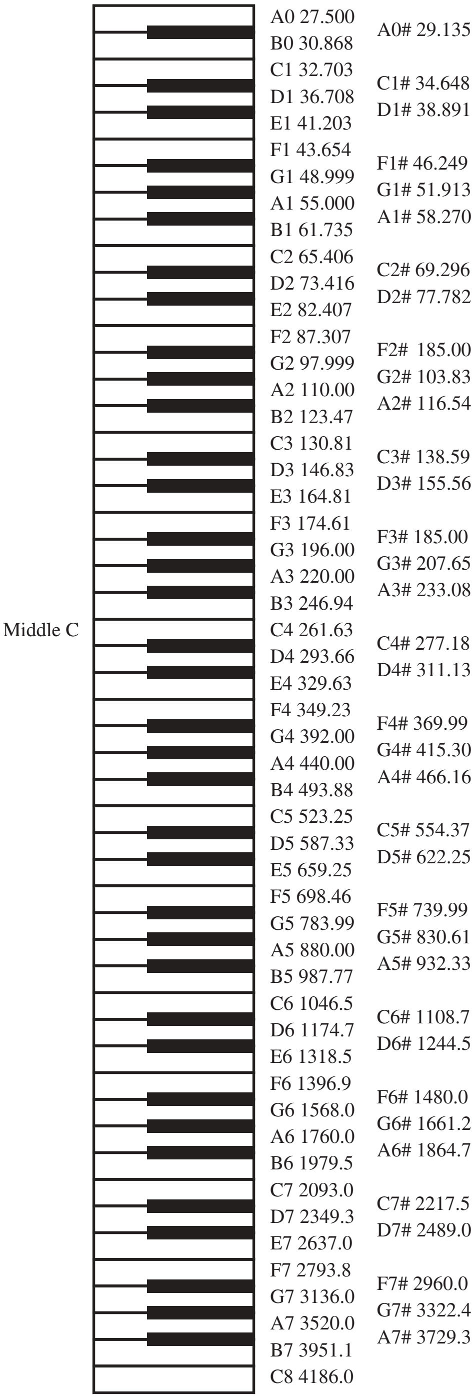
■ **YOUR TURN** On May 3, 1996, Seattle experienced a moderate earthquake. The energy that the earthquake released was approximately 1.12×10^{12} joules. Calculate the magnitude of the 1996 Seattle earthquake using the Richter scale.

■ **Answer:** 5.1

Technology Tip

Enter the number 1.12×10^{15} using the scientific notation key EXP or EE.

```
2/3log(1.12E15/1
0^4.4)
             7.099478682
```

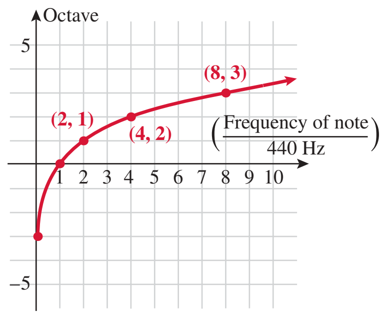


A **logarithmic scale** expresses the logarithm of a physical quantity instead of the quantity itself. In music, the pitch is the perceived fundamental frequency of sound. The note A above middle C on a piano has the pitch associated with a pure tone of 440 hertz. An octave is the interval between one musical pitch and another with either double or half its frequency. For example, if a note has a frequency of 440 hertz, then the note an octave above it has a frequency of 880 hertz, and the note an octave below it has a frequency of 220 hertz. Therefore, the ratio of two notes an octave apart is 2:1.

The following table lists the frequencies associated with A notes.

Note	A_1	A_2	A_3	A_4	A_5	A_6	A_7
Frequency (Hz)	55	110	220	440	880	1760	3520
Octave with respect to A_4	-3	-2	-1	0	$+1$	$+2$	$+3$

We can graph $\dfrac{\text{Frequency of note}}{440 \text{ Hz}}$ on the horizontal axis and the octave (with respect to A_4) on the vertical axis.



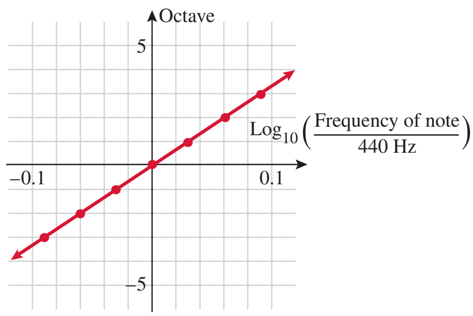
If we instead graph the logarithm of this quantity, $\log\left[\dfrac{\text{Frequency of note}}{440 \text{ Hz}}\right]$, we see that using a logarithmic scale expresses octaves linearly (up or down an octave). In other words, an "octave" is a purely logarithmic concept.

When a logarithmic scale is used we typically classify a graph one of two ways:

- Log-log plot (both the horizontal and vertical axes use logarithmic scales)
- Semilog plot (one of the axes uses a logarithmic scale)

The second graph with octaves on the vertical axis and the log of the ratio of frequencies on the horizontal axis is called a semilog plot.

Semilog Plot



EXAMPLE 11 **Graphing Using a Logarithmic Scale**

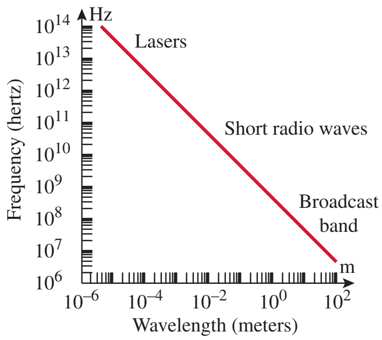
Frequency is inversely proportional to the wavelength: In a vacuum $f = \frac{c}{\lambda}$, where f is the frequency (in hertz), $c = 3.0 \times 10^8$ m/s is the speed of light in a vacuum, and λ is the wavelength in meters. Graph frequency versus wavelength using a log-log plot.

Solution:

Let wavelength range from microns (10^{-6}) to hundreds of meters (10^2) by powers of 10 along the horizontal axis.

λ	$f = \frac{3.0 \times 10^8}{\lambda}$	
10^{-6}	3.0×10^{14}	
10^{-5}	3.0×10^{13}	
10^{-4}	3.0×10^{12}	(TeraHertz: THz)
10^{-3}	3.0×10^{11}	
10^{-2}	3.0×10^{10}	
10^{-1}	3.0×10^{9}	(GigaHertz: GHz)
10^{0}	3.0×10^{8}	
10^{1}	3.0×10^{7}	
10^{2}	3.0×10^{6}	(MegaHertz: MHz)

Log-Log Plot



The logarithmic scales allow us to represent a large range of numbers. In this graph, the x-axis ranges from microns, 10^{-6} meters, to hundreds of meters, and the y-axis ranges from megahertz (MHz), 10^6 hertz, to hundreds of terahertz (THz), 10^{12} hertz.

SECTION 3.2 SUMMARY

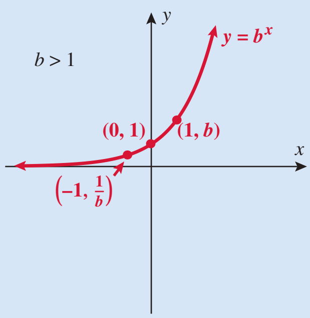
In this section, logarithmic functions were defined as inverses of exponential functions.

$$y = \log_b x \text{ is equivalent to } x = b^y$$

NAME	EXPLICIT BASE	IMPLICIT BASE
Common logarithm	$f(x) = \log_{10} x$	$f(x) = \log x$
Natural logarithm	$f(x) = \log_e x$	$f(x) = \ln x$

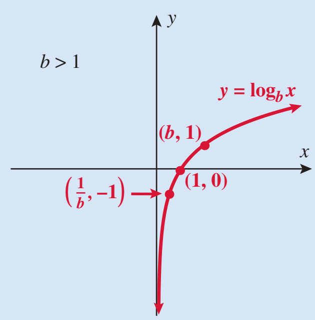
Evaluating Logarithms

- *Exact:* Convert to exponential form first, then evaluate.
- *Approximate:* Natural and common logarithms with calculators.



Graphs of Logarithmic Functions

EXPONENTIAL FUNCTION	LOGARITHMIC FUNCTION
$y = b^x$	$y = \log_b x$
y-intercept: $(0, 1)$	x-intercept: $(1, 0)$
Domain: $(-\infty, \infty)$	Domain: $(0, \infty)$
Range: $(0, \infty)$	Range: $(-\infty, \infty)$
Horizontal asymptote: x-axis	Vertical asymptote: y-axis



SECTION 3.2 EXERCISES

SKILLS

In Exercises 1–20, write each logarithmic equation in its equivalent exponential form.

1. $\log_{81} 3 = \frac{1}{4}$ **2.** $\log_{121} 11 = \frac{1}{2}$ **3.** $\log_2\left(\frac{1}{32}\right) = -5$ **4.** $\log_3\left(\frac{1}{81}\right) = -4$ **5.** $\log 0.01 = -2$

6. $\log 0.0001 = -4$ **7.** $\log 10{,}000 = 4$ **8.** $\log 1000 = 3$ **9.** $\log_{1/4}(64) = -3$ **10.** $\log_{1/6}(36) = -2$

11. $-1 = \ln\left(\frac{1}{e}\right)$ **12.** $1 = \ln e$ **13.** $\ln 1 = 0$ **14.** $\log 1 = 0$ **15.** $\ln 5 = x$

16. $\ln 4 = y$ **17.** $z = \log_x y$ **18.** $y = \log_x z$ **19.** $x = \log_y(x + y)$ **20.** $z = \ln x^y$

In Exercises 21–34, write each exponential equation in its equivalent logarithmic form.

21. $0.00001 = 10^{-5}$ **22.** $3^6 = 729$ **23.** $78{,}125 = 5^7$ **24.** $100{,}000 = 10^5$ **25.** $15 = \sqrt{225}$

26. $7 = \sqrt[3]{343}$ **27.** $\frac{8}{125} = \left(\frac{2}{5}\right)^3$ **28.** $\frac{8}{27} = \left(\frac{2}{3}\right)^3$ **29.** $3 = \left(\frac{1}{27}\right)^{-1/3}$ **30.** $4 = \left(\frac{1}{1024}\right)^{-1/5}$

31. $e^x = 6$ **32.** $e^{-x} = 4$ **33.** $x = y^z$ **34.** $z = y^x$

In Exercises 35–46, evaluate the logarithms exactly (if possible).

35. $\log_2 1$ **36.** $\log_5 1$ **37.** $\log_5 3125$ **38.** $\log_3 729$

39. $\log 10^7$ **40.** $\log 10^{-2}$ **41.** $\log_{1/4} 4096$ **42.** $\log_{1/7} 2401$

43. $\log 0$ **44.** $\ln 0$ **45.** $\log(-100)$ **46.** $\ln(-1)$

In Exercises 47–54, approximate (if possible) the common and natural logarithms using a calculator. Round to two decimal places.

47. $\log 29$ **48.** $\ln 29$ **49.** $\ln 380$ **50.** $\log 380$

51. $\log 0$ **52.** $\ln 0$ **53.** $\ln 0.0003$ **54.** $\log 0.0003$

In Exercises 55–66, state the domain of the logarithmic function in interval notation.

55. $f(x) = \log_2(x + 5)$ **56.** $f(x) = \log_2(4x - 1)$ **57.** $f(x) = \log_3(5 - 2x)$ **58.** $f(x) = \log_3(5 - x)$

59. $f(x) = \ln(7 - 2x)$ **60.** $f(x) = \ln(3 - x)$ **61.** $f(x) = \log|x|$ **62.** $f(x) = \log|x + 1|$

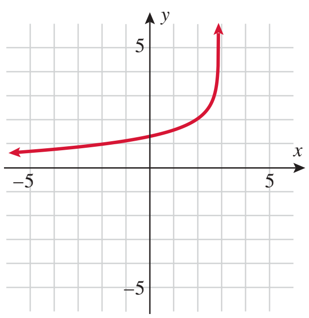
63. $f(x) = \log(x^2 + 1)$ **64.** $f(x) = \log(1 - x^2)$ **65.** $f(x) = \log(10 + 3x - x^2)$ **66.** $f(x) = \log_3(x^3 - 3x^2 + 3x - 1)$

In Exercises 67–72, match the graph with the function.

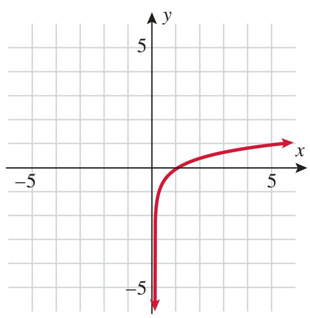
67. $y = \log_5 x$ **68.** $y = \log_5(-x)$ **69.** $y = -\log_5(-x)$

70. $y = \log_5(x + 3) - 1$ **71.** $y = \log_5(1 - x) - 2$ **72.** $y = -\log_5(3 - x) + 2$

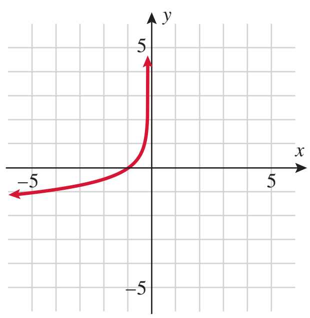
a.



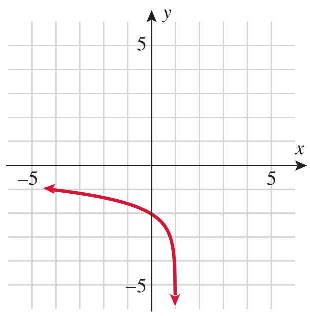
b.



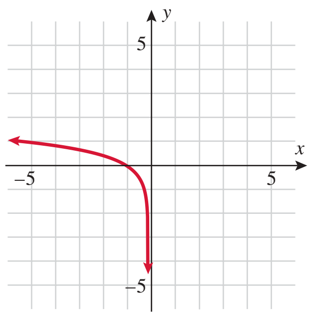
c.



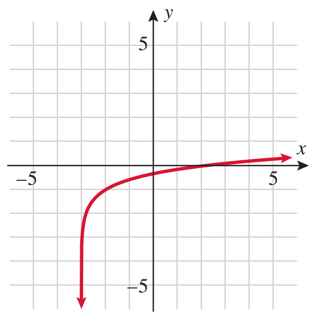
d.



e.



f.



In Exercises 73–84, graph the logarithmic function using transformation techniques. State the domain and range of *f*.

73. $f(x) = \log(x - 1)$ **74.** $f(x) = \log(x + 2)$ **75.** $\ln(x + 2)$ **76.** $\ln(x - 1)$

77. $f(x) = \log_3(x + 2) - 1$ **78.** $f(x) = \log_3(x + 1) - 2$ **79.** $f(x) = -\log(x) + 1$ **80.** $f(x) = \log(-x) + 2$

81. $f(x) = \ln(x + 4)$ **82.** $f(x) = \ln(4 - x)$ **83.** $f(x) = \log(2x)$ **84.** $f(x) = 2\ln(-x)$

APPLICATIONS

For Exercises 85–88, refer to the following:

$$\textbf{Decibel: } D = 10\log\left(\frac{I}{I_T}\right) \qquad I_T = 1 \times 10^{-12}\ \text{W/m}^2$$

85. Sound. Calculate the decibels associated with *normal conversation* if the intensity is $I = 1 \times 10^{-6}$ W/m^2.

86. Sound. Calculate the decibels associated with the *onset of pain* if the intensity is $I = 1 \times 10^{1}$ W/m^2.

87. Sound. Calculate the decibels associated with attending *a football game in a loud college stadium* if the intensity is $I = 1 \times 10^{-0.3}$ W/m^2.

88. Sound. Calculate the decibels associated with a *doorbell* if the intensity is $I = 1 \times 10^{-4.5}$ W/m^2.

For Exercises 89–92, refer to the following:

$$\textbf{Richter scale: } M = \frac{2}{3}\log\left(\frac{E}{E_0}\right) \qquad E_0 = 10^{4.4}\ \text{joules}$$

89. Earthquakes. On Good Friday 1964, one of the most severe North American earthquakes ever recorded struck Alaska. The energy released measured 1.41×10^{17} joules. Calculate the magnitude of the 1964 Alaska earthquake using the Richter scale.

90. Earthquakes. On January 22, 2003, Colima, Mexico, experienced a major earthquake. The energy released measured 6.31×10^{15} joules. Calculate the magnitude of the 2003 Mexican earthquake using the Richter scale.

91. Earthquakes. On December 26, 2003, a major earthquake rocked southeastern Iran. In Bam, 30,000 people were killed, and 85% of buildings were damaged or destroyed. The energy released measured 2×10^{14} joules. Calculate the magnitude of the 2003 Iran earthquake with the Richter scale.

92. Earthquakes. On November 1, 1755, Lisbon was destroyed by an earthquake, which killed 90,000 people and destroyed 85% of the city. It was one of the most destructive earthquakes in history. The energy released measured 8×10^{17} joules. Calculate the magnitude of the 1755 Lisbon earthquake with the Richter scale.

For Exercises 93–98, refer to the following:

The pH of a solution is a measure of the molar concentration of hydrogen ions, H^+, in moles per liter, in the solution, which means that it is a measure of the acidity or basicity of the solution. The letters pH stand for "power of hydrogen," and the numerical value is defined as

$$\text{pH} = -\log_{10}[H^+]$$

Very acid corresponds to pH values near 1, neutral corresponds to a pH near 7 (pure water), and very basic corresponds to values near 14. In the next six exercises you will be asked to calculate the pH value of wine, Pepto-Bismol, normal rainwater, bleach, and fruit. List these six liquids and use your intuition to classify them as neutral, acidic, very acidic, basic, or very basic before you calculate their actual pH values.

93. Chemistry. If wine has an approximate hydrogen ion concentration of 5.01×10^{-4}, calculate its pH value.

94. Chemistry. Pepto-Bismol has a hydrogen ion concentration of about 5.01×10^{-11}. Calculate its pH value.

95. Chemistry. Normal rainwater is slightly acidic and has an approximate hydrogen ion concentration of $10^{-5.6}$. Calculate its pH value. Acid rain and tomato juice have similar approximate hydrogen ion concentrations of 10^{-4}. Calculate the pH value of acid rain and tomato juice.

96. Chemistry. Bleach has an approximate hydrogen ion concentration of 5.0×10^{-13}. Calculate its pH value.

97. Chemistry. An apple has an approximate hydrogen ion concentration of $10^{-3.6}$. Calculate its pH value.

98. Chemistry. An orange has an approximate hydrogen ion concentration of $10^{-4.2}$. Calculate its pH value.

99. Archaeology. Carbon dating is a method used to determine the age of a fossil or other organic remains. The age t in years is related to the mass C (in milligrams) of carbon 14 through a logarithmic equation:

$$t = -\frac{\ln\left(\frac{C}{500}\right)}{0.0001216}$$

How old is a fossil that contains 100 milligrams of carbon 14?

100. Archaeology. Repeat Exercise 99, only now the fossil contains 40 milligrams of carbon 14.

101. Broadcasting. Decibels are used to quantify losses associated with atmospheric interference in a communication system. The ratio of the power (watts) received to the power transmitted (watts) is often compared. Often, *watts* are transmitted, but losses due to the atmosphere typically correspond to *milliwatts* being received:

$$\text{dB} = 10 \log\left(\frac{\text{Power received}}{\text{Power transmitted}}\right)$$

If 1 watt of power is transmitted and 3 megawatts is received, calculate the power loss in decibels.

102. Broadcasting. Repeat Exercise 101, assuming 3 watts of power is transmitted and 0.2 megawatt is received.

For Exercises 103 and 104, refer to the following:

The range of all possible frequencies of electromagnetic radiation is called the electromagnetic spectrum. In a vacuum the frequency of electromagnetic radiation is modeled by

$$f = \frac{c}{\lambda}$$

where c is 3.0×10^8 m/s and λ is wavelength in meters.

103. Physics/Electromagnetic Spectrum. The *radio spectrum* is the portion of the electromagnetic spectrum that corresponds to radio frequencies. The radio spectrum is used for various transmission technologies and is government regulated. Ranges of the radio spectrum are often allocated based on usage; for example, AM radio, cell phones, and television. (Source: http://en.wikipedia.org/wiki/Radio_spectrum)

a. Complete the following table for the various usages of the radio spectrum.

Usage	Wavelength	Frequency
Super Low Frequency—Communication with Submarines	10,000,000 m	30 Hz
Ultra Low Frequency—Communication within Mines	1,000,000 m	
Very Low Frequency—Avalanche Beacons	100,000 m	
Low Frequency—Navigation, AM Long-wave Broadcasting	10,000 m	
Medium Frequency—AM Broadcasts, Amateur Radio	1000 m	
High Frequency—Shortwave broadcasts, Citizens Band Radio	100 m	
Very High Frequency—FM Radio, Television	10 m	
Ultra High Frequency—Television, Mobile Phones	0.050 m	

b. Graph the frequency within the radio spectrum (in hertz) as a function of wavelength (in meters).

104. Physics/Electromagnetic Spectrum. The *visible spectrum* is the portion of the electromagnetic spectrum that is visible to the human eye. Typically, the human eye can see wavelengths between 390 and 750 nm (nanometers or 10^{-9} m).

a. Complete the following table for the following colors of the visible spectrum.

Color	Wavelength	Frequency
Violet	400 nm	750×10^{12} Hz
Cyan	470 nm	
Green	480 nm	
Yellow	580 nm	
Orange	610 nm	
Red	630 nm	

b. Graph the frequency (in hertz) of the colors as a function of wavelength (in meters) on a log-log plot.

CATCH THE MISTAKE

In Exercises 105–108, explain the mistake that is made.

105. Evaluate the logarithm $\log_2 4$.

Solution:

Set the logarithm equal to x. $\quad \log_2 4 = x$

Write the logarithm in exponential form. $\quad x = 2^4$

Simplify. $\quad x = 16$

Answer: $\quad \log_2 4 = 16$

This is incorrect. The correct answer is $\log_2 4 = 2$. What went wrong?

106. Evaluate the logarithm $\log_{100} 10$.

Solution:

Set the logarithm equal to x. $\quad \log_{100} 10 = x$

Express the equation in exponential form. $\quad 10^x = 100$

Solve for x. $\quad x = 2$

Answer: $\quad \log_{100} 10 = 2$

This is incorrect. The correct answer is $\log_{100} 10 = \frac{1}{2}$. What went wrong?

107. State the domain of the logarithmic function $f(x) = \log_2(x + 5)$ in interval notation.

Solution:

The domain of all logarithmic functions is $x > 0$.

Interval notation: $(0, \infty)$

This is incorrect. What went wrong?

108. State the domain of the logarithmic function $f(x) = \ln|x|$ in interval notation.

Solution:

Since the absolute value eliminates all negative numbers, the domain is the set of all real numbers.

Interval notation: $(-\infty, \infty)$

This is incorrect. What went wrong?

CONCEPTUAL

In Exercises 109–112, determine whether each statement is true or false.

109. The domain of the standard logarithmic function, $y = \ln x$, is the set of nonnegative real numbers.

110. The horizontal axis is the horizontal asymptote of the graph of $y = \ln x$.

111. The graphs of $y = \log x$ and $y = \ln x$ have the same x-intercept $(1, 0)$.

112. The graphs of $y = \log x$ and $y = \ln x$ have the same vertical asymptote, $x = 0$.

CHALLENGE

113. State the domain, range, and x-intercept of the function $f(x) = -\ln(x - a) + b$ for a and b real positive numbers.

114. State the domain, range, and x-intercept of the function $f(x) = \log(a - x) - b$ for a and b real positive numbers.

115. Graph the function $f(x) = \begin{cases} \ln(-x) & x < 0 \\ \ln(x) & x > 0 \end{cases}$.

116. Graph the function $f(x) = \begin{cases} -\ln(-x) & x < 0 \\ -\ln(x) & x > 0 \end{cases}$.

TECHNOLOGY

117. Apply a graphing utility to graph $y = e^x$ and $y = \ln x$ in the same viewing screen. What line are these two graphs symmetric about?

118. Apply a graphing utility to graph $y = 10^x$ and $y = \log x$ in the same viewing screen. What line are these two graphs symmetric about?

119. Apply a graphing utility to graph $y = \log x$ and $y = \ln x$ in the same viewing screen. What are the two common characteristics?

120. Using a graphing utility, graph $y = \ln|x|$. Is the function defined everywhere?

121. Apply a graphing utility to graph $f(x) = \ln(3x)$, $g(x) = \ln 3 + \ln x$, and $h(x) = (\ln 3)(\ln x)$ in the same viewing screen. Determine which two functions give the same graph, then state the domain of the function.

122. Apply a graphing utility to graph $f(x) = \ln(x^2 - 4)$, $g(x) = \ln(x + 2) + \ln(x - 2)$, and $h(x) = \ln(x + 2)\ln(x - 2)$ in the same viewing screen. Determine the domain where two functions give the same graph.

For Exercises 123 and 124, refer to the following:

Experimental data is collected all the time in biology and chemistry labs as scientists seek to understand natural phenomena. In biochemistry, the *Michaelis–Menten* kinetics law describes the rates of enzyme reactions using the relationship between the rate of the reaction and the concentration of the substrate involved.

The following data has been collected, where the velocity v is measured in μmol/min of the enzyme reaction and the substrate level [S] is measured in mol/L.

[S] (IN MOL/L)	v (IN μMOL/MIN)
4.0×10^{-4}	130
2.0×10^{-4}	110
1.0×10^{-4}	89
5.0×10^{-5}	62
4.0×10^{-5}	52
2.5×10^{-5}	38
2.0×10^{-5}	32

123. **a.** Create a scatterplot of this data by identifying [S] with the x-axis and v with the y-axis.

b. The graph *seems* to be leveling off. Give an estimate of the maximum value the velocity might achieve. Call this estimate $V_{\max}$.

c. Another constant of importance in describing the relationship between v and [S] is K_m. This is the value of [S] that results in the velocity being half its maximum value. Estimate this value.

Note: K_m measures the affinity level of a particular enzyme to a particular substrate. The lower the value of K_m, the higher the affinity. The higher the value of K_m, the lower the affinity.

d. The actual equation that governs the relationship between v and [S] is $v = \dfrac{V_{\max}}{1 + K_m/[S]}$, which is NOT linear. This is the *simple Michaelis–Menten kinetics equation*.

i. Use *LnReg* to get a best fit logarithmic curve for this data. Although the relationship between v and $[S]$ is not logarithmic (rather, it is logistic), the best fit logarithmic curve does not grow very quickly and so it serves as a reasonably good fit.

ii. At what $[S]$ value, approximately, is the velocity 100 μmol/min?

124. The Michaelis–Menten equation can be arranged into various other forms that give a straight line (rather than a logistic curve) when one variable is plotted against another. One such rearrangement is the *double-reciprocal Lineweaver–Burk equation*. This equation plots the data values of the reciprocal of velocity ($1/v$) versus the reciprocal of the substrate level (1/[S]). The equation is as follows:

$$\frac{1}{v} = \frac{K_m}{V_{\max}}\frac{1}{[S]} + \frac{1}{V_{\max}}$$

Think of y as $\dfrac{1}{v}$ and x as $\dfrac{1}{[S]}$.

a. What is the slope of the "line"? How about its y-intercept?

b. Using the data from Exercise 123, we create two new columns for $1/v$ and 1/[S] to obtain the following data set:

[S] (MOL/L)	1/[S]	v (μMOL/MIN)	$1/v$
4.00E−04	2.50E+03	130	0.00769231
2.00E−04	5.00E+03	110	0.00909091
1.00E−04	1.00E+04	89	0.01123596
5.00E−05	2.00E+04	62	0.01612903
4.00E−05	2.50E+04	53	0.01886792
2.50E−05	4.00E+04	38	0.02631579
2.00E−05	5.00E+04	32	0.03125

Create a scatterplot for the new data, treating x as $1/[S]$ and y as $1/v$.

c. Determine the best fit line and value of r.

d. Use the equation of the best fit line in (c) to calculate $V_{\max}$.

e. Use the above information to determine K_m.

PREVIEW TO CALCULUS

In Exercises 125–126, refer to the following:

Recall that the derivative of f can be found by letting $h \to 0$ in the difference quotient $\frac{f(x+h)-f(x)}{h}$. In calculus we prove that $\frac{e^h - 1}{h} = 1$, when h approaches 0; that is, for really small values of h, $\frac{e^h - 1}{h}$ gets very close to 1.

125. Use this information to find the derivative of $f(x) = e^x$.

126. Use this information to find the derivative of $f(x) = e^{2x}$. *Hint:* $e^{2h} - 1 = (e^h - 1)(e^h + 1)$

We also prove in calculus that the derivative of the inverse function f^{-1} is given by $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$.

127. Given $f(x) = e^x$, find

a. $f^{-1}(x)$ **b.** $(f^{-1})'(x)$

128. Given $f(x) = e^{2x}$, find

a. $f^{-1}(x)$ **b.** $(f^{-1})'(x)$

SECTION 3.3 PROPERTIES OF LOGARITHMS

SKILLS OBJECTIVES

- Write a single logarithm as a sum or difference of logarithms.
- Write a logarithmic expression as a single logarithm.
- Evaluate logarithms of a general base (other than base 10 or e).

CONCEPTUAL OBJECTIVES

- Derive the seven basic logarithmic properties.
- Derive the change-of-base formula.

Properties of Logarithms

Since exponential functions and logarithmic functions are inverses of one another, properties of exponents are related to properties of logarithms. We will start by reviewing properties of exponents (Appendix), and then proceed to properties of logarithms.

PROPERTIES OF EXPONENTS

Let a, b, m, and n be any real numbers and $m > 0$, $n > 0$, and $b \neq 0$; then the following are true:

1. $b^m \cdot b^n = b^{m+n}$

2. $b^{-m} = \frac{1}{b^m} = \left(\frac{1}{b}\right)^m$

3. $\frac{b^m}{b^n} = b^{m-n}$

4. $(b^m)^n = b^{mn}$

5. $(ab)^m = a^m \cdot b^m$

6. $b^0 = 1$

7. $b^1 = b$

From these properties of exponents, we can develop similar properties for logarithms. We list seven basic properties.

Properties of Logarithms

If b, M, and N are positive real numbers, where $b \neq 1$, and p and x are real numbers, then the following are true:

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^x = x$

4. $b^{\log_b x} = x \quad x > 0$

5. $\log_b MN = \log_b M + \log_b N$ — *Product rule:* Log of a product is the sum of the logs.

6. $\log_b \left(\dfrac{M}{N}\right) = \log_b M - \log_b N$ — *Quotient rule:* Log of a quotient is the difference of the logs.

7. $\log_b M^p = p \log_b M$ — *Power rule:* Log of a number raised to an exponent is the exponent times the log of the number.

We will devote this section to proving and illustrating these seven properties.

The first two properties follow directly from the definition of a logarithmic function and properties of exponentials.

$$\text{Property (1):} \quad \log_b 1 = 0 \text{ since } b^0 = 1$$

$$\text{Property (2):} \quad \log_b b = 1 \text{ since } b^1 = b$$

The third and fourth properties follow from the fact that exponential functions and logarithmic functions are inverses of one another. Recall that inverse functions satisfy the relationship that $f^{-1}(f(x)) = x$ for all x in the domain of $f(x)$, and $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}. Let $f(x) = b^x$ and $f^{-1}(x) = \log_b x$.

Property (3):

Write the inverse identity.	$f^{-1}(f(x)) = x$
Substitute $f^{-1}(x) = \log_b x$.	$\log_b(f(x)) = x$
Substitute $f(x) = b^x$.	$\log_b b^x = x$

Property (4):

Write the inverse identity.	$f\left(f^{-1}(x)\right) = x$
Substitute $f(x) = b^x$.	$b^{f^{-1}(x)} = x$
Substitute $f^{-1}(x) = \log_b x \quad x > 0$.	$b^{\log_b x} = x$

The first four properties are summarized below for common and natural logarithms:

Common and Natural Logarithm Properties

Common Logarithm (base 10)	Natural Logarithm (base e)
1. $\log 1 = 0$	**1.** $\ln 1 = 0$
2. $\log 10 = 1$	**2.** $\ln e = 1$
3. $\log 10^x = x$	**3.** $\ln e^x = x$
4. $10^{\log x} = x \quad x > 0$	**4.** $e^{\ln x} = x \quad x > 0$

EXAMPLE 1 **Using Logarithmic Properties**

Use properties (1)–(4) to simplify the expressions.

a. $\log_{10} 10$ **b.** $\ln 1$ **c.** $10^{\log(x+8)}$
d. $e^{\ln(2x+5)}$ **e.** $\log 10^{x^2}$ **f.** $\ln e^{x+3}$

Solution:

a. Use property (2). $\log_{10} 10 = \boxed{1}$

b. Use property (1). $\ln 1 = \boxed{0}$

c. Use property (4). $10^{\log(x+8)} = \boxed{x + 8}$ $x > -8$

d. Use property (4). $e^{\ln(2x+5)} = \boxed{2x + 5}$ $x > -\frac{5}{2}$

e. Use property (3). $\log 10^{x^2} = \boxed{x^2}$

f. Use property (3). $\ln e^{x+3} = \boxed{x + 3}$

The fifth through seventh properties follow from the properties of exponents and the definition of logarithms. We will prove the product rule and leave the proofs of the quotient and power rules for the exercises.

$$\text{Property (5):} \quad \log_b MN = \log_b M + \log_b N$$

Words	**Math**
Assume two logs that have the same base.	Let $u = \log_b M$ and $v = \log_b N$ $M > 0, N > 0$
Change to equivalent exponential forms.	$b^u = M$ and $b^v = N$
Write the log of a product.	$\log_b MN$
Substitute $M = b^u$ and $N = b^v$.	$= \log_b(b^u b^v)$
Use properties of exponents.	$= \log_b(b^{u+v})$
Apply property (3).	$= u + v$
Substitute $u = \log_b M$, $v = \log_b N$.	$= \log_b M + \log_b N$

$$\boxed{\log_b MN = \log_b M + \log_b N}$$

In other words, the log of a product is the sum of the logs. Let us illustrate this property with a simple example.

$$\overbrace{\log_2 8}^{3} + \overbrace{\log_2 4}^{2} = \overbrace{\log_2 32}^{5}$$

Notice that $\log_2 8 + \log_2 4 \neq \log_2 12$.

EXAMPLE 2 **Writing a Logarithmic Expression as a Sum of Logarithms**

Use the logarithmic properties to write the expression $\log_b(u^2\sqrt{v})$ as a sum of simpler logarithms.

Solution:

Convert the radical to exponential form. $\log_b(u^2\sqrt{v}) = \log_b(u^2v^{1/2})$

Use the product property (5). $= \log_b u^2 + \log_b v^{1/2}$

Use the power property (7). $= \boxed{2\log_b u + \frac{1}{2}\log_b v}$

YOUR TURN Use the logarithmic properties to write the expression $\log_b(x^4\sqrt[3]{y})$ as a sum of simpler logarithms.

Answer: $\log_b(x^4\sqrt[3]{y}) = 4\log_b x + \frac{1}{3}\log_b y$

EXAMPLE 3 **Writing a Sum of Logarithms as a Single Logarithmic Expression: The Right Way and the Wrong Way**

Use properties of logarithms to write the expression $2\log_b 3 + 4\log_b u$ as a single logarithmic expression.

CAUTION

$\log_b M + \log_b N = \log_b(MN)$

$\log_b M + \log_b N \neq \log_b(M + N)$

COMMON MISTAKE

A common mistake is to write the sum of the logs as a log of the sum.

$$\log_b M + \log_b N \neq \log_b(M + N)$$

CORRECT

Use the power property (7).

$$2\log_b 3 + 4\log_b u = \log_b 3^2 + \log_b u^4$$

Simplify.

$$\log_b 9 + \log_b u^4$$

Use the product property (5).

$$\boxed{= \log_b(9u^4)}$$

INCORRECT

$\neq \log_b(9 + u^4)$ **ERROR**

YOUR TURN Express $2\ln x + 3\ln y$ as a single logarithm.

Answer: $\ln(x^2y^3)$

EXAMPLE 4 **Writing a Logarithmic Expression as a Difference of Logarithms**

Write the expression $\ln\left(\frac{x^3}{y^2}\right)$ as a difference of logarithms.

Solution:

Apply the quotient property (6). $\ln\left(\frac{x^3}{y^2}\right) = \ln(x^3) - \ln(y^2)$

Apply the power property (7). $= \boxed{3\ln x - 2\ln y}$

YOUR TURN Write the expression $\log\left(\frac{a^4}{b^5}\right)$ as a difference of logarithms.

Answer: $4\log a - 5\log b$

Another common mistake is misinterpreting the quotient rule.

EXAMPLE 5 Writing the Difference of Logarithms as a Logarithm of a Quotient

Write the expression $\frac{2}{3}\ln x - \frac{1}{2}\ln y$ as a logarithm of a quotient.

COMMON MISTAKE

$$\log_b M - \log_b N \neq \frac{\log_b M}{\log_b N}$$

CORRECT

Use the power property (7).

$$\frac{2}{3}\ln x - \frac{1}{2}\ln y = \ln x^{2/3} - \ln y^{1/2}$$

Use the quotient property (6).

$$\boxed{\ln\left(\frac{x^{2/3}}{y^{1/2}}\right)}$$

INCORRECT

$$\frac{\ln x^{2/3}}{\ln y^{1/2}} \quad \textbf{ERROR}$$

CAUTION

$$\log_b M - \log_b N = \log_b\left(\frac{M}{N}\right)$$

$$\log_b M - \log_b N \neq \frac{\log_b M}{\log_b N}$$

YOUR TURN Write the expression $\frac{1}{2}\log a - 3\log b$ as a single logarithm.

Answer: $\log\left(\frac{a^{1/2}}{b^3}\right)$

EXAMPLE 6 Combining Logarithmic Expressions into a Single Logarithm

Write the expression $3\log_b x + \log_b(2x + 1) - 2\log_b 4$ as a single logarithm.

Solution:

Use the power property (7) on the first and third terms. $\quad = \log_b x^3 + \log_b(2x + 1) - \log_b 4^2$

Use the product property (5) on the first two terms. $\quad = \log_b\left[x^3(2x + 1)\right] - \log_b 16$

Use the quotient property (6). $\quad = \boxed{\log_b\left[\frac{x^3(2x + 1)}{16}\right]}$

YOUR TURN Write the expression $2\ln x - \ln(3y) + 3\ln z$ as a single logarithm.

Answer: $\ln\left(\frac{x^2z^3}{3y}\right)$

EXAMPLE 7 Expanding a Logarithmic Expression into a Sum or Difference of Logarithms

Write $\ln\left(\frac{x^2 - x - 6}{x^2 + 7x + 6}\right)$ as a sum or difference of logarithms.

Solution:

Factor the numerator and denominator. $\quad = \ln\left[\frac{(x - 3)(x + 2)}{(x + 6)(x + 1)}\right]$

Use the quotient property (6). $\quad = \ln[(x - 3)(x + 2)] - \ln[(x + 6)(x + 1)]$

Use the product property (5). $\quad = \ln(x - 3) + \ln(x + 2) - [\ln(x + 6) + \ln(x + 1)]$

Eliminate brackets. $\quad = \boxed{\ln(x - 3) + \ln(x + 2) - \ln(x + 6) - \ln(x + 1)}$

Change-of-Base Formula

Recall that in the last section, we were able to evaluate logarithms two ways: (1) exactly by writing the logarithm in exponential form and identifying the exponent and (2) using a calculator if the logarithms were base 10 or e. How do we evaluate a logarithm of general base if we cannot identify the exponent? We use the *change-of-base formula.*

EXAMPLE 8 Using Properties of Logarithms to Change the Base to Evaluate a General Logarithm

Evaluate $\log_3 8$. Round the answer to four decimal places.

Solution:

Let $y = \log_3 8$.	$y = \log_3 8$
Write the logarithm in exponential form.	$3^y = 8$
Take the log of both sides.	$\log 3^y = \log 8$
Use the power property (7).	$y \log 3 = \log 8$
Divide both sides by log 3.	$y = \dfrac{\log 8}{\log 3}$
Let $y = \log_3 8$.	$\boxed{\log_3 8 = \dfrac{\log 8}{\log 3}}$

Example 8 illustrated our ability to use properties of logarithms to change from base 3 to base 10, which our calculators can handle. This leads to the general change-of-base formula.

CHANGE-OF-BASE FORMULA

For any logarithmic bases a and b and any positive number M, the change-of-base formula says that

$$\log_b M = \frac{\log_a M}{\log_a b}$$

In the special case when a is either 10 or e, this relationship becomes

Common Logarithms		**Natural Logarithms**
$\log_b M = \dfrac{\log M}{\log b}$	or	$\log_b M = \dfrac{\ln M}{\ln b}$

It does not matter what base we select (10, e, or any other base); the ratio will be the same.

Proof of Change-of-Base Formula

WORDS	**MATH**
Let y be the logarithm we want to evaluate.	$y = \log_b M$
Write $y = \log_b M$ in exponential form.	$b^y = M$
Let a be any positive real number (where $a \neq 1$).	
Take the log of base a of both sides of the equation.	$\log_a b^y = \log_a M$
Use the power rule on the left side of the equation.	$y \log_a b = \log_a M$
Divide both sides of the equation by $\log_a b$.	$y = \dfrac{\log_a M}{\log_a b}$

EXAMPLE 9 Using the Change-of-Base Formula

Use the change-of-base formula to evaluate $\log_4 17$. Round to four decimal places.

Solution:

We will illustrate this in two ways (choosing common and natural logarithms) using a scientific calculator.

Common Logarithms

Use the change-of-base formula with base 10. $\log_4 17 = \dfrac{\log 17}{\log 4}$

Approximate with a calculator. ≈ 2.043731421

$\approx \boxed{2.0437}$

Natural Logarithms

Use the change-of-base formula with base *e*. $\log_4 17 = \dfrac{\ln 17}{\ln 4}$

Approximate with a calculator. ≈ 2.043731421

$\approx \boxed{2.0437}$

YOUR TURN Use the change-of-base formula to approximate $\log_7 34$. Round to four decimal places.

Answer: $\log_7 34 \approx 1.8122$

SECTION 3.3 SUMMARY

Properties of Logarithms

If b, M, and N are positive real numbers, where $b \neq 1$, and p and x are real numbers, then the following are true:

- Product Property: $\log_b MN = \log_b M + \log_b N$
- Quotient Property: $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$
- Power Property: $\log_b M^p = p \log_b M$

General Logarithm	Common Logarithm	Natural Logarithm
$\log_b 1 = 0$	$\log 1 = 0$	$\ln 1 = 0$
$\log_b b = 1$	$\log 10 = 1$	$\ln e = 1$
$\log_b b^x = x$	$\log 10^x = x$	$\ln e^x = x$
$b^{\log_b x} = x \quad x > 0$	$10^{\log x} = x \quad x > 0$	$e^{\ln x} = x \quad x > 0$
$\log_b M = \dfrac{\log_a M}{\log_a b}$	$\log_b M = \dfrac{\log M}{\log b}$	$\log_b M = \dfrac{\ln M}{\ln b}$

SECTION 3.3 EXERCISES

SKILLS

In Exercises 1–20, apply the properties of logarithms to simplify each expression. Do not use a calculator.

1. $\log_9 1$ **2.** $\log_{69} 1$ **3.** $\log_{1/2}\left(\frac{1}{2}\right)$ **4.** $\log_{3.3} 3.3$ **5.** $\log_{10} 10^8$

6. $\ln e^3$ **7.** $\log_{10} 0.001$ **8.** $\log_3 3^7$ **9.** $\log_2 \sqrt{8}$ **10.** $\log_5 \sqrt[3]{5}$

11. $8^{\log_8 5}$ **12.** $2^{\log_2 5}$ **13.** $e^{\ln(x+5)}$ **14.** $10^{\log(3x^2+2x+1)}$ **15.** $5^{3\log_5 2}$

16. $7^{2\log_7 5}$ **17.** $7^{-2\log_7 3}$ **18.** $e^{-2\ln 10}$ **19.** $7e^{-3\ln x}$ **20.** $-19e^{-2\ln x^2}$

In Exercises 21–36, write each expression as a sum or difference of logarithms.

Example: $\log(m^2 n^5) = 2\log m + 5\log n$

21. $\log_b(x^3y^5)$ **22.** $\log_b(x^{-3}y^{-5})$ **23.** $\log_b(x^{1/2}y^{1/3})$ **24.** $\log_b(\sqrt{r}\,\sqrt[3]{t})$

25. $\log_b\left(\frac{r^{1/3}}{s^{1/2}}\right)$ **26.** $\log_b\left(\frac{r^4}{s^2}\right)$ **27.** $\log_b\left(\frac{x}{yz}\right)$ **28.** $\log_b\left(\frac{xy}{z}\right)$

29. $\log(x^2\sqrt{x+5})$ **30.** $\log[(x-3)(x+2)]$ **31.** $\ln\left[\frac{x^3(x-2)^2}{\sqrt{x^2+5}}\right]$ **32.** $\ln\left[\frac{\sqrt{x+3}\,\sqrt[3]{x-4}}{(x+1)^4}\right]$

33. $\log\left(\frac{x^2-2x+1}{x^2-9}\right)$ **34.** $\log\left(\frac{x^2-x-2}{x^2+3x-4}\right)$ **35.** $\ln\sqrt{\frac{x^2+3x-10}{x^2-3x+2}}$ **36.** $\ln\left[\frac{\sqrt[3]{x-1}(3x-2)^4}{(x+1)\sqrt{x-1}}\right]^2$

In Exercises 37–48, write each expression as a single logarithm.

Example: $2\log m + 5\log n = \log(m^2n^5)$

37. $3\log_b x + 5\log_b y$ **38.** $2\log_b u + 3\log_b v$ **39.** $5\log_b u - 2\log_b v$ **40.** $3\log_b x - \log_b y$

41. $\frac{1}{2}\log_b x + \frac{2}{3}\log_b y$ **42.** $\frac{1}{2}\log_b x - \frac{2}{3}\log_b y$ **43.** $2\log u - 3\log v - 2\log z$ **44.** $3\log u - \log 2v - \log z$

45. $\ln(x+1) + \ln(x-1) - 2\ln(x^2+3)$ **46.** $\ln\sqrt{x-1} + \ln\sqrt{x+1} - 2\ln(x^2-1)$

47. $\frac{1}{2}\ln(x+3) - \frac{1}{3}\ln(x+2) - \ln(x)$ **48.** $\frac{1}{3}\ln(x^2+4) - \frac{1}{2}\ln(x^2-3) - \ln(x-1)$

In Exercises 49–58, evaluate the logarithms using the change-of-base formula. Round to four decimal places.

49. $\log_5 7$ **50.** $\log_4 19$ **51.** $\log_{1/2} 5$ **52.** $\log_5 \frac{1}{2}$ **53.** $\log_{2.7} 5.2$

54. $\log_{7.2} 2.5$ **55.** $\log_\pi 10$ **56.** $\log_\pi 2.7$ **57.** $\log_{\sqrt{3}} 8$ **58.** $\log_{\sqrt{2}} 9$

APPLICATIONS

59. Sound. Sitting in the front row of a rock concert exposes us to a sound pressure (or sound level) of 1×10^{-1} W/m^2 (or 110 decibels), and a normal conversation is typically around 1×10^{-6} W/m^2 (or 60 decibels). How many decibels are you exposed to if a friend is talking in your ear at a rock concert? *Note:* 160 decibels causes perforation of the eardrums. *Hint:* Add the sound pressures and convert to decibels.

60. Sound. A whisper corresponds to 1×10^{-10} W/m^2 (or 20 decibels) and a normal conversation is typically around 1×10^{-6} W/m^2 (or 60 decibels). How many decibels are you exposed to if one friend is whispering in your ear, while the other one is talking at a normal level? *Hint:* Add the sound pressures and convert to decibels.

For Exercises 61 and 62, refer to the following:

There are two types of waves associated with an earthquake: *compression* and *shear*. The compression, or longitudinal, waves displace material behind the earthquake's path. Longitudinal waves travel at great speeds and are often called "primary waves" or simply "P" waves. Shear, or transverse, waves displace material at right angles to the earthquake's path. Transverse waves do not travel as rapidly through the Earth's crust and mantle as do longitudinal waves, and they are called "secondary" or "S" waves.

61. Earthquakes. If a seismologist records the energy of P waves as 4.5×10^{12} joules and the energy of S waves as 7.8×10^{8} joules, what is the total energy (sum the two energies)? What would the combined effect be on the Richter scale?

62. Earthquakes. Repeat Exercise 61, assuming the energy associated with the P waves is 5.2×10^{11} joules and the energy associated with the S waves is 4.1×10^{9} joules.

63. Photography. In photographic quality assurance, logarithms are used to determine, for instance, the density. Density is the common logarithm of the opacity, which is the quotient of the amount of incident light and the amount of transmitted light. What is the density of a photographic material that only transmits 90% of the incident light?

64. pH Scale. The pH scale measures how acidic or basic a substance is. pH is defined as the negative logarithm of the hydrogen ion activity in an aqueous solution, a_{H}. Thus, if $a_{\text{H}} = 0.01$, then $\text{pH} = -\log 0.01 = 2$. Determine the pH of a liquid with $a_{\text{H}} = 0.00407$. Round your answer to the nearest hundredth.

65. pH Scale. How many times more acidic is a substance with $\text{pH} = 3.2$ than a substance with $\text{pH} = 4.4$? Round your answer to the nearest integer.

66. Information Theory. In information theory, logarithms in base 2 are often used. The capacity C of a noisy channel with bandwidth W and signal and noise powers S and N is $C = W \log_2\left(1 + \dfrac{S}{N}\right)$. The signal noise ratio R is given by $R = 10 \log\left(\dfrac{S}{N}\right)$. Assuming a channel with a bandwidth of 3 megahertz and a signal noise ratio $R = 2$ dB, calculate the channel capacity.

CATCH THE MISTAKE

In Exercises 67–70, simplify if possible and explain the mistake that is made.

67. $3 \log 5 - \log 25$

Solution:

Apply the quotient property (6). $\dfrac{3 \log 5}{\log 25}$

Write $25 = 5^2$. $\dfrac{3 \log 5}{\log 5^2}$

Apply the power property (7). $\dfrac{3 \log 5}{2 \log 5}$

Simplify. $\dfrac{3}{2}$

This is incorrect. The correct answer is log 5. What mistake was made?

68. $\ln 3 + 2 \ln 4 - 3 \ln 2$

Solution:

Apply the power property (7). $\ln 3 + \ln 4^2 - \ln 2^3$

Simplify. $\ln 3 + \ln 16 - \ln 8$

Apply property (5). $\ln(3 + 16 - 8)$

Simplify. $\ln 11$

This is incorrect. The correct answer is ln 6. What mistake was made?

69. $\log_2 x + \log_3 y - \log_4 z$

Solution:

Apply the product property (5). $\log_6 xy - \log_4 z$

Apply the quotient property (6). $\log_{24} xyz$

This is incorrect. What mistake was made?

70. $2(\log 3 - \log 5)$

Solution:

Apply the quotient property (6). $2\left(\log \dfrac{3}{5}\right)$

Apply the power property (7). $\left(\log \dfrac{3}{5}\right)^2$

Apply a calculator to approximate. ≈ 0.0492

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 71–74, determine whether each statement is true or false.

71. $\log e = \dfrac{1}{\ln 10}$

72. $\ln e = \dfrac{1}{\log 10}$

73. $\ln(xy)^3 = (\ln x + \ln y)^3$

74. $\dfrac{\ln a}{\ln b} = \dfrac{\log a}{\log b}$

75. $\log 12x^3 = 36 \log x$

76. $e^{\ln x^2} = x^2$

CHALLENGE

77. Prove the quotient rule: $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$.

Hint: Let $u = \log_b M$ and $v = \log_b N$. Write both in exponential form and find the quotient $\log_b\left(\dfrac{M}{N}\right)$.

78. Prove the power rule: $\log_b M^p = p \log_b M$. *Hint:* Let $u = \log_b M$. Write this log in exponential form and find $\log_b M^p$.

79. Write in terms of simpler logarithmic forms.

$$\log_b\left(\sqrt{\frac{x^2}{y^3 z^{-5}}}\right)^6$$

80. Show that $\log_b\left(\dfrac{1}{x}\right) = -\log_b x$.

81. Show that $\log_b\left(\dfrac{a^2}{b^3}\right)^{-3} = 9 - \dfrac{6}{\log_a b}$.

82. Given that $\log_b 2 = 0.4307$ and $\log_b 3 = 0.6826$, find $\log_b \sqrt{48}$. Do not use a calculator.

TECHNOLOGY

83. Use a graphing calculator to plot $y = \ln(2x)$ and $y = \ln 2 + \ln x$. Are they the same graph?

84. Use a graphing calculator to plot $y = \ln(2 + x)$ and $y = \ln 2 + \ln x$. Are they the same graph?

85. Use a graphing calculator to plot $y = \dfrac{\log x}{\log 2}$ and $y = \log x - \log 2$. Are they the same graph?

86. Use a graphing calculator to plot $y = \log\left(\dfrac{x}{2}\right)$ and $y = \log x - \log 2$. Are they the same graph?

87. Use a graphing calculator to plot $y = \ln\left(x^2\right)$ and $y = 2 \ln x$. Are they the same graph?

88. Use a graphing calculator to plot $y = (\ln x)^2$ and $y = 2 \ln x$. Are they the same graph?

89. Use a graphing calculator to plot $y = \ln x$ and $y = \dfrac{\log x}{\log e}$. Are they the same graph?

90. Use a graphing calculator to plot $y = \log x$ and $y = \dfrac{\ln x}{\ln 10}$. Are they the same graph?

PREVIEW TO CALCULUS

In calculus we prove that the derivative of $f + g$ is $f' + g'$ and that the derivative of $f - g$ is $f' - g'$. It is also shown in calculus that if $f(x) = \ln x$ then $f'(x) = \dfrac{1}{x}$.

91. Use these properties to find the derivative of $f(x) = \ln x^2$.

92. Use these properties to find the derivative of $f(x) = \ln \dfrac{1}{x}$.

93. Find the derivative of $f(x) = \ln \dfrac{1}{x^2}$.

94. Find the derivative of $f(x) = \ln x^2 + \ln x^3$.

SECTION 3.4 EXPONENTIAL AND LOGARITHMIC EQUATIONS

SKILLS OBJECTIVES

- Solve exponential equations.
- Solve logarithmic equations.
- Solve application problems involving exponential and logarithmic equations.

CONCEPTUAL OBJECTIVE

- Understand how exponential and logarithmic equations are solved using properties of one-to-one functions and inverses.

Solving Exponential Equations

To solve algebraic equations such as $x^2 - 9 = 0$, the goal is to solve for the variable, x, by finding the values of x that make the statement true. Exponential and logarithmic equations have the variable (x) buried within an exponent or a logarithm, but the goal is the same. Find the value(s) of x that makes the statement true.

Exponential equation: $e^{2x+1} = 5$

Logarithmic equation: $\log(3x - 1) = 7$

There are two methods for solving exponential and logarithmic equations that are based on the properties of one-to-one functions and inverses. To solve simple exponential and logarithmic equations, we will use one-to-one properties. To solve more complicated exponential and logarithmic equations, we will use properties of inverses. The following box summarizes the one-to-one and inverse properties that hold true when $b > 0$ and $b \neq 1$.

ONE-TO-ONE PROPERTIES

$b^x = b^y$ if and only if $x = y$

$\log_b x = \log_b y$ if and only if $x = y$

INVERSE PROPERTIES

$b^{\log_b x} = x \qquad x > 0$

$\log_b b^x = x$

The following strategies are outlined for solving simple and complicated exponential equations using the one-to-one and inverse properties.

STRATEGIES FOR SOLVING EXPONENTIAL EQUATIONS

TYPE OF EQUATION	STRATEGY	EXAMPLE
Simple	1. Rewrite both sides of the equation in terms of the same base.	$2^{x-3} = 32$ $2^{x-3} = 2^5$
	2. Use the one-to-one property to equate the exponents.	$x - 3 = 5$
	3. Solve for the variable.	$\boxed{x = 8}$
Complicated	1. Isolate the exponential expression.	$3e^{2x} - 2 = 7$ $3e^{2x} = 9$ $e^{2x} = 3$
	2. Take the same logarithm* of both sides.	$\ln e^{2x} = \ln 3$
	3. Simplify using the inverse properties.	$2x = \ln 3$
	4. Solve for the variable.	$\boxed{x = \frac{1}{2}\ln 3}$

*Take the logarithm with base that is equal to the base of the exponent and use the property $\log_b b^x = x$ or take the natural logarithm and use the property $\ln M^p = p \ln M$.

EXAMPLE 1 Solving a Simple Exponential Equation

Solve the exponential equations using the one-to-one property.

a. $3^x = 81$ **b.** $5^{7-x} = 125$ **c.** $\left(\frac{1}{2}\right)^{4y} = 16$

Solution (a):

Substitute $81 = 3^4$. $3^x = 3^4$

Use the one-to-one property to identify x. $\boxed{x = 4}$

Solution (b):

Substitute $125 = 5^3$. $5^{7-x} = 5^3$

Use the one-to-one property. $7 - x = 3$

Solve for x. $\boxed{x = 4}$

Solution (c):

Substitute $\left(\dfrac{1}{2}\right)^{4y} = \left(\dfrac{1}{2^{4y}}\right) = 2^{-4y}$. $2^{-4y} = 16$

Substitute $16 = 2^4$. $2^{-4y} = 2^4$

Use the one-to-one property to identify y. $\boxed{y = -1}$

■ **YOUR TURN** Solve the following equations:

a. $2^{x-1} = 8$ **b.** $\left(\frac{1}{3}\right)^y = 27$

■ **Answer: a.** $x = 4$ **b.** $y = -3$

In Example 1, we were able to rewrite the equation in a form with the same bases so that we could use the one-to-one property. In Example 2, we will not be able to write both sides in a form with the same bases. Instead, we will use properties of inverses.

EXAMPLE 2 Solving a More Complicated Exponential Equation with a Base Other Than 10 or *e*

Solve the exponential equations exactly and then approximate the answers to four decimal places.

a. $5^{3x} = 16$ **b.** $4^{3x+2} = 71$

Solution (a):

Take the natural logarithm of both sides of the equation. $\ln 5^{3x} = \ln 16$

Use the power property on the left side of the equation. $3x \ln 5 = \ln 16$

Divide both sides of the equation by 3 ln 5. $\boxed{x = \dfrac{\ln 16}{3 \ln 5}}$

Use a calculator to approximate x to four decimal places. $\boxed{x \approx 0.5742}$

Solution (b):

Rewrite in logarithmic form. $3x + 2 = \log_4 71$

Subtract 2 from both sides. $3x = \log_4 71 - 2$

Divide both sides by 3. $x = \dfrac{\log_4 71 - 2}{3}$

Use the change-of-base formula, $\log_4 71 = \dfrac{\ln 71}{\ln 4}$. $\boxed{x = \dfrac{\dfrac{\ln 71}{\ln 4} - 2}{3}}$

Use a calculator to approximate x to four decimal places. $x \approx \dfrac{3.07487356 - 2}{3} \approx \boxed{0.3583}$

We could have proceeded in an alternative way by taking either the natural log or the common log of both sides and using the power property (instead of using the change-of-base formula) to evaluate the logarithm with base 4.

Take the natural logarithm of both sides. $\ln\left(4^{3x+2}\right) = \ln 71$

Use the power property (7). $(3x + 2)\ln 4 = \ln 71$

Divide by ln 4. $3x + 2 = \dfrac{\ln 71}{\ln 4}$

Subtract 2 and divide by 3. $\boxed{x = \dfrac{\dfrac{\ln 71}{\ln 4} - 2}{3}}$

Use a calculator to approximate x. $x \approx \dfrac{3.07487356 - 2}{3} \approx \boxed{0.3583}$

■ **YOUR TURN** Solve the equation $5^{y^2} = 27$ exactly and then approximate the answer to four decimal places.

Technology Tip

```
((ln(71)/ln(4)-2
)/3
        .3582911866
```

■ **Answer:**
$y = \pm\sqrt{\log_5 27} \approx \pm 1.4310$

EXAMPLE 3 Solving a More Complicated Exponential Equation with Base 10 or *e*

Solve the exponential equation $4e^{x^2} = 64$ exactly and then approximate the answer to four decimal places.

Solution:

Divide both sides by 4. $e^{x^2} = 16$

Take the natural logarithm (ln) of both sides. $\ln\left(e^{x^2}\right) = \ln 16$

Simplify the left side with the property of inverses. $x^2 = \ln 16$

Solve for x using the square-root method. $\boxed{x = \pm\sqrt{\ln 16}}$

Use a calculator to approximate x to four decimal places. $\boxed{x \approx \pm 1.6651}$

■ **YOUR TURN** Solve the equation $10^{2x-3} = 7$ exactly and then approximate the answer to four decimal places.

■ **Answer:** $x = \dfrac{\log_{10} 7 + 3}{2} \approx 1.9225$

EXAMPLE 4 Solving an Exponential Equation Quadratic in Form

Solve the equation $e^{2x} - 4e^x + 3 = 0$ exactly and then approximate the answer to four decimal places.

Solution:

Let $u = e^x$. $\left(\textit{Note: } u^2 = e^x \cdot e^x = e^{2x}.\right)$ $u^2 - 4u + 3 = 0$

Factor. $(u - 3)(u - 1) = 0$

Solve for u. $u = 3$ or $u = 1$

Substitute $u = e^x$. $e^x = 3$ or $e^x = 1$

Take the natural logarithm (ln) of both sides. $\ln(e^x) = \ln 3$ or $\ln(e^x) = \ln 1$

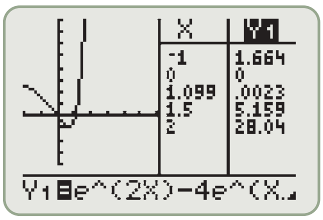
Simplify with the properties of logarithms. $\boxed{x = \ln 3 \quad \text{or} \quad x = \ln 1}$

Approximate or evaluate exactly the right sides. $\boxed{x \approx 1.0986}$ or $\boxed{x = 0}$

■ **YOUR TURN** Solve the equation $100^x - 10^x - 2 = 0$ exactly and then approximate the answer to four decimal places.

Technology Tip

The graph of the function $e^{2x} - 4e^x + 3$ is shown. The x-intercepts are $x \approx 1.10$ and $x = 0$.



■ **Answer:** $x = \log 2 \approx 0.3010$

Solving Logarithmic Equations

We can solve simple logarithmic equations using the property of one-to-one functions. For more complicated logarithmic equations, we can employ properties of logarithms and properties of inverses. **Solutions must be checked to eliminate extraneous solutions.**

STRATEGIES FOR SOLVING LOGARITHMIC EQUATIONS

TYPE OF EQUATION	STRATEGY	EXAMPLE
Simple	1. Combine logarithms on each side of the equation using properties. 2. Use the one-to-one property to equate the arguments. 3. Solve for the variable. 4. Check the results and eliminate any extraneous solutions.	$\log(x-3) + \log x = \log 4$ $\log x(x-3) = \log 4$ $x(x-3) = 4$ $x^2 - 3x - 4 = 0$ $(x-4)(x+1) = 0$ $x = -1, 4$ Eliminate $x = -1$ because $\log(-1)$ is undefined. $\boxed{x = 4}$
Complicated	1. Combine and isolate the logarithmic expressions. 2. Rewrite the equation in exponential form. 3. Solve for the variable. 4. Check the results and eliminate any extraneous solutions.	$\log_5(x+2) - \log_5 x = 2$ $\log_5\left(\frac{x+2}{x}\right) = 2$ $\frac{x+2}{x} = 5^2$ $x + 2 = 25x$ $24x = 2$ $\boxed{x = \frac{1}{12}}$ $\log_5\left(\frac{1}{12} + 2\right) - \log_5\left(\frac{1}{12}\right)$ $= \log_5\left(\frac{25}{12}\right) - \log_5\left(\frac{1}{12}\right)$ $= \log_5\left[\frac{25/12}{1/12}\right] = \log_5[25] = 2$ ✓

EXAMPLE 5 Solving a Simple Logarithmic Equation

Solve the equation $\log_4(2x-3) = \log_4(x) + \log_4(x-2)$.

Solution:

Apply the product property (5) on the right side. $\quad \log_4(2x-3) = \log_4[x(x-2)]$

Apply the property of one-to-one functions. $\quad 2x - 3 = x(x-2)$

Distribute and simplify. $\quad x^2 - 4x + 3 = 0$

Factor. $\quad (x-3)(x-1) = 0$

Solve for x. $\quad x = 3 \quad \text{or} \quad x = 1$

The possible solution $x = 1$ must be eliminated because it is not in the domain of two of the logarithmic functions.

$$x = 1: \overbrace{\log_4(-1)}^{\text{undefined}} \stackrel{?}{=} \log_4(1) + \overbrace{\log_4(-1)}^{\text{undefined}}$$

$$\boxed{x = 3}$$

YOUR TURN Solve the equation $\ln(x+8) = \ln(x) + \ln(x+3)$.

Study Tip

Solutions should be checked in the original equation to eliminate extraneous solutions.

Answer: $x = 2$

EXAMPLE 6 Solving a More Complicated Logarithmic Equation

Solve the equation $\log_3(9x) - \log_3(x - 8) = 4$.

Solution:

Employ the quotient property (6) on the left side. $\log_3\left(\dfrac{9x}{x-8}\right) = 4$

Write in exponential form. $\log_b x = y \Rightarrow x = b^y$ $\dfrac{9x}{x-8} = 3^4$

Simplify the right side. $\dfrac{9x}{x-8} = 81$

Multiply the equation by the LCD, $x - 8$. $9x = 81(x - 8)$

Eliminate parentheses. $9x = 81x - 648$

Solve for x. $-72x = -648$

$\boxed{x = 9}$

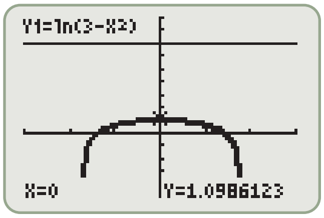
Check: $\log_3[9 \cdot 9] - \log_3[9 - 8] = \log_3[81] - \log_3 1 = 4 - 0 = 4$

■ **Answer:** $x = 2$

■ **YOUR TURN** Solve the equation $\log_2(4x) - \log_2(2) = 2$.

Technology Tip

Graphs of $y_1 = \ln(3 - x^2)$ and $y_2 = 7$ are shown.



No intersection of the graphs indicates no solution.

EXAMPLE 7 Solving a Logarithmic Equation with No Solution

Solve the equation $\ln(3 - x^2) = 7$.

Solution:

Write in exponential form. $3 - x^2 = e^7$

Simplify. $x^2 = 3 - e^7$

$3 - e^7$ is negative. $x^2 =$ negative real number

There are no real numbers that when squared yield a negative real number. Therefore, there is $\boxed{\text{no real solution}}$.

Applications

Archaeologists determine the age of a fossil by how much carbon 14 is present at the time of discovery. The number of grams of carbon 14 based on the radioactive decay of the isotope is given by

$$A = A_0 e^{-0.000124t}$$

where A is the number of grams of carbon 14 at the present time, A_0 is the number of grams of carbon 14 while alive, and t is the number of years since death. Using the inverse properties, we can isolate t.

Words	Math
Divide by A_0.	$\dfrac{A}{A_0} = e^{-0.000124t}$
Take the natural logarithm of both sides.	$\ln\left(\dfrac{A}{A_0}\right) = \ln\left(e^{-0.000124t}\right)$
Simplify the right side utilizing properties of inverses.	$\ln\left(\dfrac{A}{A_0}\right) = -0.000124t$
Solve for t.	$t = -\dfrac{1}{0.000124}\ln\left(\dfrac{A}{A_0}\right)$

Let's assume that animals have approximately 1000 mg of carbon 14 in their bodies when they are alive. If a fossil has 200 mg of carbon 14, approximately how old is the fossil? Substituting $A = 200$ and $A_0 = 1000$ into our equation for t, we find

$$t = -\frac{1}{0.000124}\ln\left(\frac{1}{5}\right) \approx 12{,}979.338$$

The fossil is approximately 13,000 years old.

EXAMPLE 8 Calculating How Many Years It Will Take for Money to Double

You save \$1000 from a summer job and put it in a CD earning 5% compounding continuously. How many years will it take for your money to double? Round to the nearest year.

Solution:

Recall the compound continuous interest formula.	$A = Pe^{rt}$
Substitute $P = 1000$, $A = 2000$, and $r = 0.05$.	$2000 = 1000e^{0.05t}$
Divide by 1000.	$2 = e^{0.05t}$
Take the natural logarithm of both sides.	$\ln 2 = \ln\left(e^{0.05t}\right)$
Simplify with the property $\ln e^x = x$.	$\ln 2 = 0.05t$
Solve for t.	$t = \frac{\ln 2}{0.05} \approx 13.8629$

It will take almost $\boxed{\text{14 years}}$ for your money to double.

■ **YOUR TURN** How long will it take \$1000 to triple (become \$3000) in a savings account earning 10% a year compounding continuously? Round your answer to the nearest year.

■ **Answer:** approximately 11 years

When an investment is compounded continuously, how long will it take for that investment to double?

WORDS	MATH
Write the interest formula for compounding continuously.	$A = Pe^{rt}$
Let $A = 2P$ (investment doubles).	$2P = Pe^{rt}$
Divide both sides of the equation by P.	$2 = e^{rt}$
Take the natural log of both sides of the equation.	$\ln 2 = \ln e^{rt}$
Simplify the right side by applying the property $\ln e^x = x$.	$\ln 2 = rt$
Divide both sides by r to get the exact value for t.	$t = \frac{\ln 2}{r}$
Now, approximate $\ln 2 \approx 0.7$.	$t \approx \frac{0.7}{r}$
Multiply the numerator and denominator by 100.	$t \approx \frac{70}{100r}$

This is the "rule of 70."

If we divide 70 by the interest rate (compounding continuously), we get the approximate time for an investment to double. In Example 8, the interest rate (compounding continuously) is 5%. Dividing 70 by 5 yields 14 years.

SECTION 3.4 SUMMARY

Strategy for Solving Exponential Equations

Type of Equation	Strategy
Simple	1. Rewrite both sides of the equation in terms of the same base. 2. Use the one-to-one property to equate the exponents. 3. Solve for the variable.
Complicated	1. Isolate the exponential expression. 2. Take the same logarithm* of both sides. 3. Simplify using the inverse properties. 4. Solve for the variable.

*Take the logarithm with the base that is equal to the base of the exponent and use the property $\log_b b^x = x$ or take the natural logarithm and use the property $\ln M^p = p \ln M$.

Strategy for Solving Logarithmic Equations

Type of Equation	Strategy
Simple	1. Combine logarithms on each side of the equation using properties. 2. Use the one-to-one property to equate the arguments. 3. Solve for the variable. 4. Check the results and eliminate any extraneous solutions.
Complicated	1. Combine and isolate the logarithmic expressions. 2. Rewrite the equation in exponential form. 3. Solve for the variable. 4. Check the results and eliminate any extraneous solutions.

SECTION 3.4 EXERCISES

SKILLS

In Exercises 1–14, solve the exponential equations exactly for *x*.

1. $2^{x^2} = 16$ **2.** $169^x = 13$ **3.** $\left(\frac{2}{3}\right)^{x+1} = \frac{27}{8}$ **4.** $\left(\frac{3}{5}\right)^{x+1} = \frac{25}{9}$ **5.** $e^{2x+3} = 1$

6. $10^{x^2-1} = 1$ **7.** $7^{2x-5} = 7^{3x-4}$ **8.** $125^x = 5^{2x-3}$ **9.** $2^{x^2+12} = 2^{7x}$ **10.** $5^{x^2-3} = 5^{2x}$

11. $9^x = 3^{x^2-4x}$ **12.** $16^{x-1} = 2^{x^2}$ **13.** $e^{5x-1} = e^{x^2+3}$ **14.** $10^{x^2-8} = 100^x$

In Exercises 15–40, solve the exponential equations exactly and then approximate your answers to three decimal places.

15. $27 = 2^{3x-1}$ **16.** $15 = 7^{3-2x}$ **17.** $3e^x - 8 = 7$ **18.** $5e^x + 12 = 27$

19. $9 - 2e^{0.1x} = 1$ **20.** $21 - 4e^{0.1x} = 5$ **21.** $2(3^x) - 11 = 9$ **22.** $3(2^x) + 8 = 35$

23. $e^{3x+4} = 22$ **24.** $e^{x^2} = 73$ **25.** $3e^{2x} = 18$ **26.** $4(10^{3x}) = 20$

27. $4e^{2x+1} = 17$ **28.** $5\left(10^{x^2+2x+1}\right) = 13$ **29.** $3\left(4^{x^2-4}\right) = 16$ **30.** $7 \cdot \left(\frac{1}{4}\right)^{6-5x} = 3$

31. $e^{2x} + 7e^x - 3 = 0$ **32.** $e^{2x} - 4e^x - 5 = 0$ **33.** $(3^x - 3^{-x})^2 = 0$ **34.** $(3^x - 3^{-x})(3^x + 3^{-x}) = 0$

35. $\dfrac{2}{e^x - 5} = 1$ **36.** $\dfrac{17}{e^x + 4} = 2$ **37.** $\dfrac{20}{6 - e^{2x}} = 4$ **38.** $\dfrac{4}{3 - e^{3x}} = 8$

39. $\dfrac{4}{10^{2x} - 7} = 2$ **40.** $\dfrac{28}{10^x + 3} = 4$

In Exercises 41–58, solve the logarithmic equations exactly.

41. $\log_3(2x + 1) = 4$

42. $\log_2(3x - 1) = 3$

43. $\log_2(4x - 1) = -3$

44. $\log_4(5 - 2x) = -2$

45. $\ln x^2 - \ln 9 = 0$

46. $\log x^2 + \log x = 3$

47. $\log_5(x - 4) + \log_5 x = 1$

48. $\log_2(x - 1) + \log_2(x - 3) = 3$

49. $\log(x - 3) + \log(x + 2) = \log(4x)$

50. $\log_2(x + 1) + \log_2(4 - x) = \log_2(6x)$

51. $\log(4 - x) + \log(x + 2) = \log(3 - 2x)$

52. $\log(3 - x) + \log(x + 3) = \log(1 - 2x)$

53. $\log_4(4x) - \log_4\left(\frac{x}{4}\right) = 3$

54. $\log_3(7 - 2x) - \log_3(x + 2) = 2$

55. $\log(2x - 5) - \log(x - 3) = 1$

56. $\log_3(10 - x) - \log_3(x + 2) = 1$

57. $\log_4(x^2 + 5x + 4) - 2\log_4(x + 1) = 2$

58. $\log_2(x + 1) + \log_2(x + 5) - \log_2(2x + 5) = 2$

In Exercises 59–72, solve the logarithmic equations exactly and then approximate your answers, if possible, to three decimal places.

59. $\log(2x + 5) = 2$

60. $\ln(4x - 7) = 3$

61. $\ln(x^2 + 1) = 4$

62. $\log(x^2 + 4) = 2$

63. $\ln(2x + 3) = -2$

64. $\log(3x - 5) = -1$

65. $\log(2 - 3x) + \log(3 - 2x) = 1.5$

66. $\log_2(3 - x) + \log_2(1 - 2x) = 5$

67. $\ln(x) + \ln(x - 2) = 4$

68. $\ln(4x) + \ln(2 + x) = 2$

69. $\log_7(1 - x) - \log_7(x + 2) = \log_7 x$

70. $\log_5(x + 1) - \log_5(x - 1) = \log_5 x$

71. $\ln\sqrt{x + 4} - \ln\sqrt{x - 2} = \ln\sqrt{x + 1}$

72. $\log(\sqrt{1 - x}) - \log(\sqrt{x + 2}) = \log x$

APPLICATIONS

73. Health. After strenuous exercise, Sandy's heart rate R (beats per minute) can be modeled by

$$R(t) = 151e^{-0.055t}, \quad 0 \le t \le 15$$

where t is the number of minutes that have elapsed after she stops exercising.

a. Find Sandy's heart rate at the end of exercising (when she stops at time $t = 0$).

b. Determine how many minutes it takes after Sandy stops exercising for her heart rate to drop to 100 beats per minute. Round to the nearest minute.

c. Find Sandy's heart rate 15 minutes after she had stopped exercising.

74. Business. A local business purchased a new company van for \$45,000. After 2 years the book value of the van is \$30,000.

a. Find an exponential model for the value of the van using $V(t) = V_0e^{kt}$ where V is the value of the van in dollars and t is time in years.

b. Approximately how many years will it take for the book value of the van to drop to \$20,000?

75. Money. If money is invested in a savings account earning 3.5% interest compounded yearly, how many years will pass until the money triples?

76. Money. If money is invested in a savings account earning 3.5% interest compounded monthly, how many years will pass until the money triples?

77. Money. If \$7500 is invested in a savings account earning 5% interest compounded quarterly, how many years will pass until there is \$20,000?

78. Money. If \$9000 is invested in a savings account earning 6% interest compounded continuously, how many years will pass until there is \$15,000?

For Exercises 79, 80, and 87, refer to the following:

$$\textbf{Richter scale: } M = \frac{2}{3}\log\left(\frac{E}{E_0}\right) \qquad E_0 = 10^{4.4} \textbf{ joules}$$

79. Earthquakes. On September 25, 2003, an earthquake that measured 7.4 on the Richter scale shook Hokkaido, Japan. How much energy (joules) did the earthquake emit?

80. Earthquakes. Again, on that same day (September 25, 2003), a second earthquake that measured 8.3 on the Richter scale shook Hokkaido, Japan. How much energy (joules) did the earthquake emit?

For Exercises 81, 82, and 88, refer to the following:

$$\textbf{Decibel: } D = 10\log\left(\frac{I}{I_T}\right) \qquad I_T = 1 \times 10^{-12}\ \mathrm{W/m^2}$$

81. Sound. Matt likes to drive around campus in his classic Mustang with the stereo blaring. If his boom stereo has a sound intensity of 120 decibels, how many watts per square meter does the stereo emit?

82. Sound. The New York Philharmonic has a sound intensity of 100 decibels. How many watts per square meter does the orchestra emit?

83. Anesthesia. When a person has a cavity filled, the dentist typically administers a local anesthetic. After leaving the dentist's office, one's mouth often remains numb for several more hours. If a shot of anesthesia is injected into the bloodstream at the time of the procedure ($t = 0$), and the amount of anesthesia still in the bloodstream t hours after the initial injection is given by $A = A_0 e^{-0.5t}$, in how many hours will only 10% of the original anesthetic still be in the bloodstream?

84. Investments. Money invested in an account that compounds interest continuously at a rate of 3% a year is modeled by $A = A_0 e^{0.03t}$, where A is the amount in the investment after t years and A_0 is the initial investment. How long will it take the initial investment to double?

85. Biology. The U.S. Fish and Wildlife Service is releasing a population of the endangered Mexican gray wolf in a protected area along the New Mexico and Arizona border. They estimate the population of the Mexican gray wolf to be approximated by

$$P(t) = \frac{200}{1 + 24e^{-0.2t}}$$

How many years will it take for the population to reach 100 wolves?

86. Introducing a New Car Model. If the number of new model Honda Accord hybrids purchased in North America is given by $N = \dfrac{100{,}000}{1 + 10e^{-2t}}$, where t is the number of weeks after Honda releases the new model, how many weeks will it take after the release until there are 50,000 Honda hybrids from that batch on the road?

87. Earthquakes. A P wave measures 6.2 on the Richter scale, and an S wave measures 3.3 on the Richter scale. What is their combined measure on the Richter scale?

88. Sound. You and a friend get front row seats to a rock concert. The music level is 100 decibels, and your normal conversation is 60 decibels. If your friend is telling you something during the concert, how many decibels are you subjecting yourself to?

CATCH THE MISTAKE

In Exercises 89–92, explain the mistake that is made.

89. Solve the equation: $4e^x = 9$.

Solution:

Take the natural log of both sides. $\ln(4e^x) = \ln 9$

Apply the property of inverses. $4x = \ln 9$

Solve for x. $x = \dfrac{\ln 9}{4} \approx 0.55$

This is incorrect. What mistake was made?

90. Solve the equation: $\log(x) + \log(3) = 1$.

Solution:

Apply the product property (5). $\log(3x) = 1$

Exponentiate (base 10). $10^{\log(3x)} = 1$

Apply the properties of inverses. $3x = 1$

Solve for x. $x = \dfrac{1}{3}$

This is incorrect. What mistake was made?

91. Solve the equation: $\log(x) + \log(x + 3) = 1$ for x.

Solution:

Apply the product property (5). $\log(x^2 + 3x) = 1$

Exponentiate both sides (base 10). $10^{\log(x^2+3x)} = 10^1$

Apply the property of inverses. $x^2 + 3x = 10$

Factor. $(x + 5)(x - 2) = 0$

Solve for x. $x = -5$ and $x = 2$

This is incorrect. What mistake was made?

92. Solve the equation: $\log x + \log 2 = \log 5$.

Solution:

Combine the logarithms on the left. $\log(x + 2) = \log 5$

Apply the property of one-to-one functions. $x + 2 = 5$

Solve for x. $x = 3$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 93–98, determine whether each statement is true or false.

93. The sum of logarithms with the same base is equal to the logarithm of the product.

94. A logarithm squared is equal to two times the logarithm.

95. $e^{\log x} = x$

96. $e^x = -2$ has no solution.

97. $\log_3(x^2 + x - 6) = 1$ has two solutions.

98. The division of two logarithms with the same base is equal to the logarithm of the subtraction.

CHALLENGE

99. Solve for x in terms of b:

$$\frac{1}{3}\log_b(x^3) + \frac{1}{2}\log_b(x^2 - 2x + 1) = 2$$

100. Solve exactly:

$$2\log_b(x) + 2\log_b(1 - x) = 4$$

101. Solve $y = \dfrac{3000}{1 + 2e^{-0.2t}}$ for t in terms of y.

102. State the range of values of x that the following identity holds: $e^{\ln(x^2-a)} = x^2 - a$.

103. A function called the hyperbolic cosine is defined as the average of exponential growth and exponential decay by $f(x) = \dfrac{e^x + e^{-x}}{2}$. If we restrict the domain of f to $[0, \infty)$, find its inverse.

104. A function called the hyperbolic sine is defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$. Find its inverse.

TECHNOLOGY

105. Solve the equation $\ln 3x = \ln(x^2 + 1)$. Using a graphing calculator, plot the graphs $y = \ln(3x)$ and $y = \ln(x^2 + 1)$ in the same viewing rectangle. Zoom in on the point where the graphs intersect. Does this agree with your solution?

106. Solve the equation $10^{x^2} = 0.001^x$. Using a graphing calculator, plot the graphs $y = 10^{x^2}$ and $y = 0.001^x$ in the same viewing rectangle. Does this confirm your solution?

107. Use a graphing utility to help solve $3^x = 5x + 2$.

108. Use a graphing utility to help solve $\log x^2 = \ln(x - 3) + 2$.

109. Use a graphing utility to graph $y = \dfrac{e^x + e^{-x}}{2}$. State the domain. Determine whether there are any symmetry and asymptote.

110. Use a graphing utility to graph $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$. State the domain. Determine whether there are any symmetry and asymptote.

PREVIEW TO CALCULUS

111. The hyperbolic sine function is defined by $\sinh x = \dfrac{e^x - e^{-x}}{2}$. Find its inverse function $\sinh^{-1} x$.

112. The hyperbolic tangent is defined by $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. Find its inverse function $\tanh^{-1} x$.

In Exercises 113–114, refer to the following:

In calculus, to find the derivative of a function of the form $y = k^x$, where k is a constant, we apply logarithmic differentiation. The first step in this process consists of writing $y = k^x$ in an equivalent form using the natural logarithm. Use the properties of this section to write an equivalent form of the following implicitly defined functions.

113. $y = 2^x$

114. $y = 4^x \cdot 3^{x+1}$

SECTION
3.5 EXPONENTIAL AND LOGARITHMIC MODELS

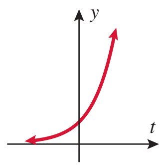
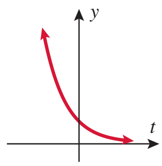
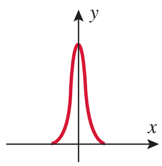
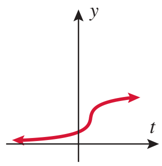
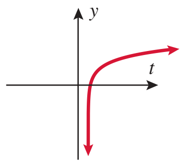
SKILLS OBJECTIVES

- Apply exponential growth and exponential decay models to biological, demographic, and economic phenomena.
- Represent distributions by means of a Gaussian model.
- Use logistic growth models to represent phenomena involving limits to growth.
- Solve problems such as species populations, credit card payoff, and wearoff of anesthesia through logarithmic models.

CONCEPTUAL OBJECTIVE

- Recognize exponential growth, exponential decay, Gaussian distributions, logistic growth, and logarithmic models.

The following table summarizes the five primary models that involve exponential and logarithmic functions:

NAME	MODEL	GRAPH	APPLICATIONS
Exponential growth	$f(t) = ce^{kt} \quad k > 0$		World populations, bacteria growth, appreciation, global spread of the HIV virus
Exponential decay	$f(t) = ce^{-kt} \quad k > 0$		Radioactive decay, carbon dating, depreciation
Gaussian (normal) distribution	$f(x) = ce^{-(x-a)^2/k}$		Bell curve (grade distribution), life expectancy, height/weight charts, intensity of a laser beam, IQ tests
Logistic growth	$f(t) = \dfrac{a}{1 + ce^{-kt}}$		Conservation biology, learning curve, spread of virus on an island, carrying capacity
Logarithmic	$f(t) = a + c \log t$ $f(t) = a + c \ln t$		Population of species, anesthesia wearing off, time to pay off credit cards

Exponential Growth Models

Quite often one will hear that something "grows exponentially," meaning that it grows very fast and at increasing speed. In mathematics, the precise meaning of **exponential growth** is a *growth rate of a function that is proportional to its current size*. Let's assume you get a 5% raise every year in a government job. If your annual starting salary out of college is \$40,000, then your first raise will be \$2000. Fifteen years later your annual salary will be approximately \$83,000 and your next 5% raise will be around \$4150. The raise is always 5% of the current salary, so the larger the current salary, the larger the raise.

In Section 3.1, we saw that interest that is compounded continuously is modeled by $A = Pe^{rt}$. Here A stands for amount and P stands for principal. There are similar models for populations; these take the form $N(t) = N_0e^{rt}$, where N_0 represents the number of people at time $t = 0$, r is the annual growth rate, t is time in years, and N represents the number of people at time t. In general, any model of the form $f(x) = ce^{kx}$, $k > 0$, models exponential growth.

EXAMPLE 1 World Population Projections

The world population is the total number of humans on Earth at a given time. In 2000 the world population was 6.1 billion and in 2005 the world population was 6.5 billion. Find the annual growth rate and determine what year the population will reach 9 billion.

Solution:

Assume an exponential growth model. $N(t) = N_0e^{rt}$

Let $t = 0$ correspond to 2000. $N(0) = N_0 = 6.1$

In 2005, $t = 5$, the population was 6.5 billion. $6.5 = 6.1e^{5r}$

Solve for r.

$$\frac{6.5}{6.1} = e^{5r}$$

$$\ln\left(\frac{6.5}{6.1}\right) = \ln\left(e^{5r}\right)$$

$$\ln\left(\frac{6.5}{6.1}\right) = 5r$$

$$r \approx 0.012702681$$

The annual growth rate is approximately $\boxed{1.3\%}$ per year.

Assuming the growth rate stays the same, write a population model. $N(t) = 6.1e^{0.013t}$

Let $N(t) = 9$. $9 = 6.1e^{0.013t}$

Solve for t.

$$e^{0.013t} = \frac{9}{6.1}$$

$$\ln\left(e^{0.013t}\right) = \ln\left(\frac{9}{6.1}\right)$$

$$0.013t = \ln\left(\frac{9}{6.1}\right)$$

$$t \approx 29.91813894$$

In $\boxed{2030}$ the world population will reach 9 billion if the same growth rate is maintained.

■ **YOUR TURN** The population of North America (United States and Canada) was 300 million in 1995, and in 2005 the North American population was 332 million. Find the annual growth rate (round to the nearest percent) and use that rounded growth rate to determine what year the population will reach 1 billion.

■ **Answer:** 1% per year; 2115

Exponential Decay Models

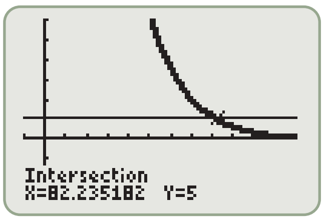
We mentioned radioactive decay briefly in Section 3.1. Radioactive decay is the process in which a radioactive isotope of an element (atoms) loses energy by emitting radiation in the form of particles. This results in loss of mass of the isotope, which we measure as a reduction in the rate of radioactive emission. This process is random, but given a large number of atoms, the decay rate is directly proportional to the mass of the radioactive substance. Since the mass is decreasing, we say this represents *exponential decay*, $m = m_0e^{-rt}$, where m_0 represents the initial mass at time $t = 0$, r is the decay rate, t is time, and m represents the mass at time t. In general, any model of the form $f(x) = ce^{-kx}$, $k > 0$, models **exponential decay**.

Typically, the decay rate r is expressed in terms of the half-life h. Recall (Section 3.1) that half-life is the time it takes for a quantity to decrease by half.

Words	Math
Write the radioactive decay model.	$m = m_0e^{-rt}$
Divide both sides by m_0.	$\dfrac{m}{m_0} = e^{-rt}$
The remaining mass of the radioactive isotope is half of the initial mass when $t = h$.	$\dfrac{1}{2} = e^{-rh}$
Solve for r.	
Take the natural logarithm of both sides.	$\ln\left(\dfrac{1}{2}\right) = \ln\left(e^{-rh}\right)$
Simplify.	$\underbrace{\ln 1}_{0} - \ln 2 = -rh$
	$rh = \ln 2$
	$\boxed{r = \dfrac{\ln 2}{h}}$

Technology Tip

Graphs of $Y_1 = 500e^{-0.56x}$ with $x = t$ and $Y_2 = 5$ are shown.



EXAMPLE 2 Radioactive Decay

The radioactive isotope of potassium ^{42}K, which is vital in the diagnosis of brain tumors, has a half-life of 12.36 hours.

a. Determine the exponential decay model that represents the mass of ^{42}K.

b. If 500 milligrams of potassium-42 are taken, how many milligrams of this isotope will remain after 48 hours?

c. How long will it take for the original 500-milligram sample to decay to a mass of 5 milligrams?

Solution (a):

Write the relationship between rate of decay and half-life. $\quad r = \dfrac{\ln 2}{h}$

Let $h = 12.36$. $\quad r \approx 0.056$

Write the exponential decay model for the mass of ^{42}K. $\quad \boxed{m = m_0e^{-0.056t}}$

Solution (b):

Let $m_0 = 500$ and $t = 48$. $\qquad m = 500e^{-(0.056)(48)} \approx 34.00841855$

There are approximately $\boxed{34 \text{ milligrams}}$ of ^{42}K still in the body after 48 hours.

Note: Had we used the full value of $r = 0.056079868$, the resulting mass would have been $m = 33.8782897$, which is approximately 34 milligrams.

Solution (c):

Write the exponential decay model for the mass of ^{42}K. $\qquad m = m_0 e^{-0.056t}$

Let $m = 5$ and $m_0 = 500$. $\qquad 5 = 500e^{-0.056t}$

Solve for t.

Divide by 500. $\qquad e^{-0.056t} = \frac{5}{500} = \frac{1}{100}$

Take the natural logarithm of both sides. $\qquad \ln\left(e^{-0.056t}\right) = \ln\left(\frac{1}{100}\right)$

Simplify. $\qquad -0.056t = \ln\left(\frac{1}{100}\right)$

Divide by -0.056 and approximate with a calculator. $\qquad t \approx 82.2352$

It will take approximately $\boxed{82 \text{ hours}}$ for the original 500-milligram substance to decay to a mass of 5 milligrams.

YOUR TURN The radioactive element radon-222 has a half-life of 3.8 days.

a. Determine the exponential decay model that represents the mass of radon-222.

b. How much of a 64-gram sample of radon-222 will remain after 7 days? Round to the nearest gram.

c. How long will it take for the original 64-gram sample to decay to a mass of 4 grams? Round to the nearest day.

Answer: **a.** $m = m_0 e^{-0.1824t}$
b. 18 g
c. 15 days

Gaussian (Normal) Distribution Models

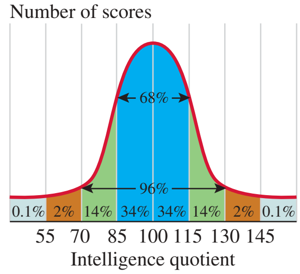
If your instructor plots the grades from the last test, typically you will see a **Gaussian (normal) distribution** of scores, otherwise known as the *bell-shaped curve*. Other examples of phenomena that tend to follow a Gaussian distribution are SAT scores, height distributions of adults, and standardized tests like IQ assessments.

The graph to the right represents a Gaussian distribution of IQ scores. The average score, which for IQ is 100, is the x-value at which the maximum occurs. The typical probability distribution is

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

where μ is the average or mean value and the variance is σ^2.

Any model of the form $f(x) = ce^{-(x-a)^2/k}$ is classified as a **Gaussian model**.



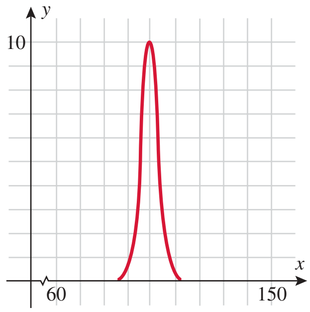
EXAMPLE 3 **Weight Distributions**

Suppose each member of a Little League football team is weighed and the weight distribution follows the Gaussian model $f(x) = 10e^{-(x-100)^2/25}$.

a. Graph the weight distribution.
b. What is the average weight of a member of this team?
c. Approximately how many boys weigh 95 pounds?

Solution:

a.



b. $\boxed{\text{100 pounds}}$

c. $f(95) = 10e^{-(95-100)^2/25}$

$= 10e^{-25/25}$

$= 10e^{-1}$

≈ 3.6788

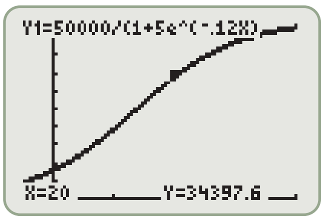
Approximately $\boxed{\text{4 boys}}$ weigh 95 pounds.

Logistic Growth Models

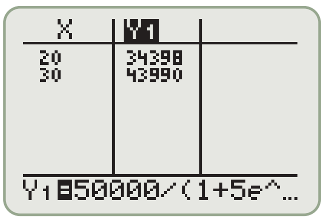
Earlier in this section, we discussed exponential growth models for populations that experience uninhibited growth. Now we will turn our attention to *logistic growth*, which models population growth when there are factors that impact the ability to grow, such as food and space. For example, if 10 rabbits are dropped off on an uninhabited island, they will reproduce and the population of rabbits on that island will experience rapid growth. The population will continue to increase rapidly until the rabbits start running out of space or food on the island. In other words, under favorable conditions the growth is not restricted, while under less favorable conditions the growth becomes restricted. This type of growth is represented by **logistic growth models**, $f(x) = \dfrac{a}{1 + ce^{-kx}}$. Ultimately, the population of rabbits reaches the island's *carrying capacity*, a.

Technology Tip

Plot the graph of $Y_1 = \dfrac{500000}{1 + 5e^{-0.12x}}$ with $x = t$. Use the $\boxed{\text{TRACE}}$ key to find $f(20)$ and $f(30)$.

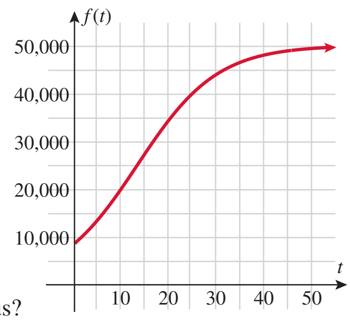


A table of values supports the solution at $t = 20$ and $t = 30$.



EXAMPLE 4 **Number of Students on a College Campus**

In 2008 the University of Central Florida was the sixth largest university in the country. The number of students can be modeled by the function $f(t) = \dfrac{50{,}000}{1 + 5e^{-0.12t}}$, where t is time in years and $t = 0$ corresponds to 1970.



a. How many students attended UCF in 1990? Round to the nearest thousand.
b. How many students attended UCF in 2000?
c. What is the carrying capacity of the UCF main campus?

Round all answers to the nearest thousand.

Solution (a): Let $t = 20$. $f(20) = \dfrac{50{,}000}{1 + 5e^{-0.12(20)}} \approx \boxed{34{,}000}$

Solution (b): Let $t = 30$. $f(30) = \dfrac{50{,}000}{1 + 5e^{-0.12(30)}} \approx \boxed{44{,}000}$

Solution (c): As t increases, the UCF student population approaches $\boxed{50{,}000}$.

Logarithmic Models

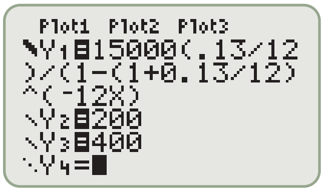
Homeowners typically ask the question, "If I increase my payment, how long will it take to pay off my current mortgage?" In general, a loan over t years with an annual interest rate r with n periods per year corresponds to an interest rate per period of $i = \frac{r}{n}$. Typically, loans are paid in equal payments consisting of the principal P plus total interest divided by the total number of periods over the life of the loan nt. The periodic payment R is given by

$$R = P\frac{i}{1 - (1 + i)^{-nt}}$$

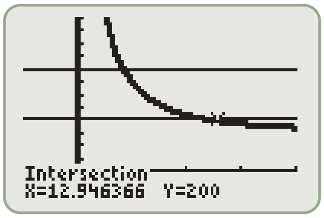
We can find the time (in years) it will take to pay off the loan as a function of periodic payment by solving for t.

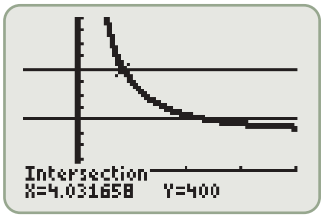
Words	**Math**
Multiply both sides by $1 - (1 + i)^{-nt}$.	$R[1 - (1 + i)^{-nt}] = Pi$
Eliminate the brackets.	$R - R(1 + i)^{-nt} = Pi$
Subtract R.	$-R(1 + i)^{-nt} = Pi - R$
Divide by $-R$.	$(1 + i)^{-nt} = 1 - \dfrac{Pi}{R}$
Take the natural log of both sides.	$\ln(1 + i)^{-nt} = \ln\left(1 - \dfrac{Pi}{R}\right)$
Use the power property for logarithms.	$-nt\ln(1 + i) = \ln\left(1 - \dfrac{Pi}{R}\right)$
Isolate t.	$t = -\dfrac{\ln\left(1 - \dfrac{Pi}{R}\right)}{n\ln(1 + i)}$
Let $i = \dfrac{r}{n}$.	$\boxed{t = -\dfrac{\ln\left(1 - \dfrac{Pr}{nR}\right)}{n\ln\left(1 + \dfrac{r}{n}\right)}}$

Technology Tip



Use the keystrokes [2nd] [Calc] [5:Intersect] to find the points of intersection.





EXAMPLE 5 Paying Off Credit Cards

James owes \$15,000 on his credit card. The annual interest rate is 13% compounded monthly.

a. Find the time it will take to pay off his credit card if he makes payments of \$200 per month.
b. Find the time it will take to pay off his credit card if he makes payments of \$400 per month.

Let $P = 15{,}000$, $r = 0.13$, and $n = 12$.

$$t = -\frac{\ln\left(1 - \frac{15{,}000(0.13)}{12R}\right)}{12\ln\left(1 + \frac{0.13}{12}\right)}$$

Solution (a): Let $R = 200$.

$$t = -\frac{\ln\left(1 - \frac{15{,}000(0.13)}{12(200)}\right)}{12\ln\left(1 + \frac{0.13}{12}\right)} \approx 13$$

\$200 monthly payments will allow James to pay off his credit card in about $\boxed{13 \text{ years}}$.

Solution (b): Let $R = 400$.

$$t = -\frac{\ln\left(1 - \frac{15{,}000(0.13)}{12(400)}\right)}{12\ln\left(1 + \frac{0.13}{12}\right)} \approx 4$$

\$400 monthly payments will allow James to pay off the balance in approximately $\boxed{4 \text{ years}}$. It is important to note that doubling the payment reduced the time to pay off the balance by less than a third.

SECTION 3.5 SUMMARY

In this section, we discussed five main types of models that involve exponential and logarithmic functions.

Name	Model	Applications
Exponential growth	$f(t) = ce^{kt}$, $k > 0$	Uninhibited growth (populations/inflation)
Exponential decay	$f(t) = ce^{-kt}$, $k > 0$	Carbon dating, depreciation
Gaussian (normal) distributions	$f(x) = ce^{-(x-a)^2/k}$	Bell curves (standardized tests, height/weight charts, distribution of power flux of laser beams)
Logistic growth	$f(t) = \dfrac{a}{1 + ce^{-kt}}$	Conservation biology (growth limited by factors like food and space), learning curve
Logarithmic	$f(t) = a + c\log t$ $f(t) = a + c\ln t$ or quotients of logarithmic functions	Time to pay off credit cards, annuity planning

SECTION 3.5 EXERCISES

SKILLS

In Exercises 1–6, match the function with the graph (a to f) and the model name (i to v).

1. $f(t) = 5e^{2t}$

2. $N(t) = 28e^{-t/2}$

3. $T(x) = 4e^{-(x-80)^2/10}$

4. $P(t) = \dfrac{200}{1 + 5e^{-0.4t}}$

5. $D(x) = 4 + \log(x - 1)$

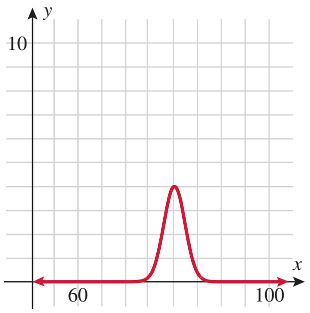
6. $h(t) = 2 + \ln(t + 3)$

Model Name

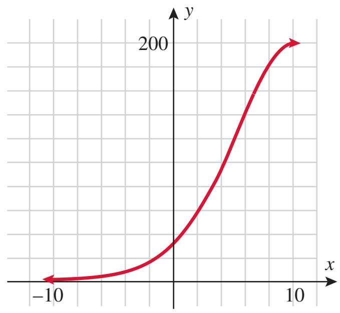
i. Logarithmic **ii.** Logistic **iii.** Gaussian **iv.** Exponential growth **v.** Exponential decay

Graphs

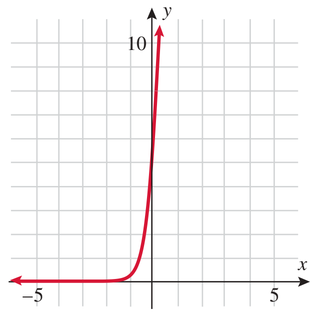
a.



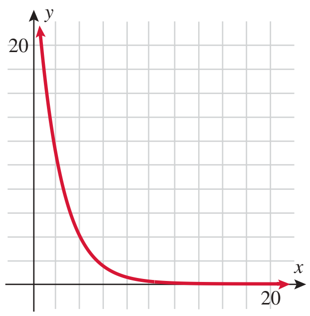
b.



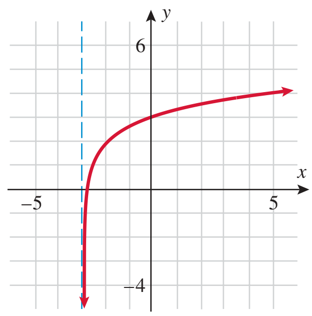
c.



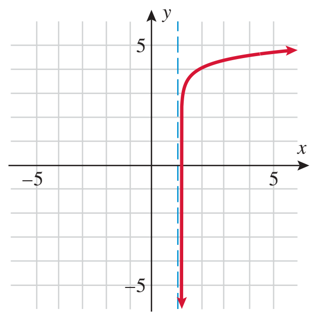
d.



e.



f.



APPLICATIONS

7. Population Growth. The population of the Philippines in 2003 was 80 million. It increases 2.36% per year. What was the expected population of the Philippines in 2010? Apply the formula $N = N_0e^{rt}$, where N represents the number of people.

8. Population Growth. China's urban population is growing at 2.5% a year, compounding continuously. If there were 13.7 million people in Shanghai in 1996, approximately how many people will there be in 2016? Apply the formula $N = N_0e^{rt}$, where N represents the number of people.

9. Population Growth. Port St. Lucie, Florida, had the United States' fastest growth rate among cities with a population of 100,000 or more between 2003 and 2004. In 2003 the population was 103,800 and increasing at a rate of 12% per year. In what year should the population reach 200,000? (Let $t = 0$ correspond to 2003.) Apply the formula $N = N_0e^{rt}$, where N represents the number of people.

10. Population Growth. San Francisco's population has been declining since the "dot com" bubble burst. In 2002 the population was 776,000. If the population is declining at a rate of 1.5% per year, in what year will the population be 700,000? (Let $t = 0$ correspond to 2002.) Apply the formula $N = N_0e^{-rt}$, where N represents the number of people.

11. **Cellular Phone Plans.** The number of cell phones in China is exploding. In 2007 there were 487.4 million cell phone subscribers, and the number is increasing at a rate of 16.5% per year. How many cell phone subscribers were there in 2010 according to this model? Use the formula $N = N_0 e^{rt}$, where N represents the number of cell phone subscribers. Let $t = 0$ correspond to 2007.

12. **Bacteria Growth.** A colony of bacteria is growing exponentially. Initially, 500 bacteria were in the colony. The growth rate is 20% per hour. (a) How many bacteria should be in the colony in 12 hours? (b) How many in 1 day? Use the formula $N = N_0 e^{rt}$, where N represents the number of bacteria.

13. **Real Estate Appreciation.** In 2004 the average house in Birmingham, AL cost \$185,000, and real estate prices were increasing at an amazing rate of 30% per year. What was the expected cost of an average house in Birmingham in 2007? Use the formula $N = N_0 e^{rt}$, where N represents the average cost of a home. Round to the nearest thousand.

14. **Real Estate Appreciation.** The average cost of a single family home in Seattle, WA in 2004 was \$230,000. In 2005 the average cost was \$252,000. If this trend continued, what was the expected cost in 2007? Use the formula $N = N_0 e^{rt}$, where N represents the average cost of a home. Round to the nearest thousand.

15. **Oceanography (Growth of Phytoplankton).** Phytoplankton are microscopic plants that live in the ocean. Phytoplankton grow abundantly in oceans around the world and are the foundation of the marine food chain. One variety of phytoplankton growing in tropical waters is increasing at a rate of 20% per month. If it is estimated that there are 100 million in the water, how many will there be in 6 months? Utilize formula $N = N_0 e^{rt}$, where N represents the population of phytoplankton.

16. **Oceanography (Growth of Phytoplankton).** In Arctic waters there are an estimated 50,000,000 phytoplankton. The growth rate is 12% per month. How many phytoplankton will there be in 3 months? Utilize formula $N = N_0 e^{rt}$, where N represents the population of phytoplankton.

17. **HIV/AIDS.** In 2003 an estimated 1 million people had been infected with HIV in the United States. If the infection rate increases at an annual rate of 2.5% a year compounding continuously, how many Americans will be infected with the HIV virus by 2015?

18. **HIV/AIDS.** In 2003 there were an estimated 25 million people who have been infected with HIV in sub-Saharan Africa. If the infection rate increases at an annual rate of 9% a year compounding continuously, how many Africans will be infected with the HIV virus by 2015?

19. **Anesthesia.** When a person has a cavity filled, the dentist typically gives a local anesthetic. After leaving the dentist's office, one's mouth often is numb for several more hours. If 100 milliliters of anesthesia is injected into the local tissue at the time of the procedure ($t = 0$), and the amount of anesthesia still in the local tissue t hours after the initial injection is given by $A = 100e^{-0.5t}$, how much remains in the local tissue 4 hours later?

20. **Anesthesia.** When a person has a cavity filled, the dentist typically gives a local anesthetic. After leaving the dentist's office, one's mouth often is numb for several more hours. If 100 milliliters of anesthesia is injected into the local tissue at the time of the procedure ($t = 0$), and the amount of anesthesia still in the local tissue t hours after the initial injection is given by $A = 100e^{-0.5t}$, how much remains in the local tissue 12 hours later?

21. **Business.** The sales S (in thousands of units) of a new mp3 player after it has been on the market for t years can be modeled by

$$S(t) = 750\left(1 - e^{-kt}\right)$$

   a. If 350,000 units of the mp3 player were sold in the first year, find k to four decimal places.
   b. Use the model found in part (a) to estimate the sales of the mp3 player after it has been on the market for 3 years.

22. **Business.** During an economic downturn the annual profits of a company dropped from \$850,000 in 2008 to \$525,000 in 2010. Assume the exponential model $P(t) = P_0 e^{kt}$ for the annual profit where P is profit in thousands of dollars, and t is time in years.

   a. Find the exponential model for the annual profit.
   b. Assuming the exponential model was applicable in the year 2012, estimate the profit (to the nearest thousand dollars) for the year 2012.

23. **Radioactive Decay.** Carbon-14 has a half-life of 5730 years. How long will it take 5 grams of carbon-14 to be reduced to 2 grams?

24. **Radioactive Decay.** Radium-226 has a half-life of 1600 years. How long will it take 5 grams of radium-226 to be reduced to 2 grams?

25. **Radioactive Decay.** The half-life of uranium-238 is 4.5 billion years. If 98% of uranium-238 remains in a fossil, how old is the fossil?

26. **Decay Levels in the Body.** A drug has a half-life of 12 hours. If the initial dosage is 5 milligrams, how many milligrams will be in the patient's body in 16 hours?

In Excercises 27–30, use the following formula for Newton's Law of Cooling:

If you take a hot dinner out of the oven and place it on the kitchen countertop, the dinner cools until it reaches the temperature of the kitchen. Likewise, a glass of ice set on a table in a room eventually melts into a glass of water at that room temperature. The rate at which the hot dinner cools or the ice in the glass melts at any given time is proportional to the difference between its temperature and the temperature of its surroundings (in this case, the room). This is called **Newton's law of cooling** (or warming) and is modeled by

$$T = T_S + (T_0 - T_S)e^{-kt}$$

where T is the temperature of an object at time t, T_s is the temperature of the surrounding medium, T_0 is the temperature of the object at time $t = 0$, t is the time, and k is a constant.

27. Newton's Law of Cooling. An apple pie is taken out of the oven with an internal temperature of 325°F. It is placed on a rack in a room with a temperature of 72°F. After 10 minutes, the temperature of the pie is 200°F. What will the temperature of the pie be 30 minutes after coming out of the oven?

28. Newton's Law of Cooling. A cold drink is taken out of an ice chest with a temperature of 38°F and placed on a picnic table with a surrounding temperature of 75°F. After 5 minutes, the temperature of the drink is 45°F. What will the temperature of the drink be 20 minutes after it is taken out of the chest?

29. Forensic Science (Time of Death). A body is discovered in a hotel room. At 7:00 A.M. a police detective found the body's temperature to be 85°F. At 8:30 A.M. a medical examiner measures the body's temperature to be 82°F. Assuming the room in which the body was found had a constant temperature of 74°F, how long has the victim been dead? (Normal body temperature is 98.6°F.)

30. Forensic Science (Time of Death). At 4 A.M. a body is found in a park. The police measure the body's temperature to be 90°F. At 5 A.M. the medical examiner arrives and determines the temperature to be 86°F. Assuming the temperature of the park was constant at 60°F, how long has the victim been dead?

31. Depreciation of Automobile. A new Lexus IS250 has a book value of \$38,000, and after 1 year has a book value of \$32,000. What is the car's value in 4 years? Apply the formula $N = N_0e^{-rt}$, where N represents the value of the car. Round to the nearest hundred.

32. Depreciation of Automobile. A new Hyundai Triburon has a book value of \$22,000, and after 2 years a book value of \$14,000. What is the car's value in 4 years? Apply the formula $N = N_0e^{-rt}$, where N represents the value of the car. Round to the nearest hundred.

33. Automotive. A new model BMW convertible coupe is designed and produced in time to appear in North America in the fall. BMW Corporation has a limited number of new models available. The number of new model BMW convertible coupes purchased in North America is given by $N = \dfrac{100{,}000}{1 + 10e^{-2t}}$, where t is the number of weeks after the BMW is released.

a. How many new model BMW convertible coupes will have been purchased 2 weeks after the new model becomes available?
b. How many after 30 weeks?
c. What is the maximum number of new model BMW convertible coupes that will be sold in North America?

34. iPhone. The number of iPhones purchased is given by $N = \dfrac{2{,}000{,}000}{1 + 2e^{-4t}}$, where t is the time in weeks after they are made available for purchase.

a. How many iPhones are purchased within the first 2 weeks?
b. How many iPhones are purchased within the first month?

35. Spread of a Disease. The number of MRSA (methicillin-resistant *Staphylococcus aureus*) cases has been rising sharply in England and Wales since 1997. In 1997, 2422 cases were reported. The number of cases reported in 2003 was 7684. How many cases might be expected in 2010? (Let $t = 0$ correspond to 1997.) Use the formula $N = N_0e^{-rt}$, where N represents the number of cases reported.

36. Spread of a Virus. Dengue fever, an illness carried by mosquitoes, is occurring in one of the worst outbreaks in decades across Latin America and the Caribbean. In 2004, 300,000 cases were reported, and 630,000 cases in 2007. How many cases in 2010? (Let $t = 0$ be 2004.) Use the formula $N = N_0e^{-rt}$, where N represents the number of cases.

37. Carrying Capacity. The Virginia Department of Fish and Game stock a mountain lake with 500 trout. Officials believe the lake can support no more than 10,000 trout. The number of trout is given by $N = \dfrac{10{,}000}{1 + 19e^{-1.56t}}$, where t is time in years. How many years will it take for the trout population to reach 5000?

38. Carrying Capacity. The World Wildlife Fund has placed 1000 rare pygmy elephants in a conservation area in Borneo. They believe 1600 pygmy elephants can be supported in this environment. The number of elephants is given by $N = \dfrac{1600}{1 + 0.6e^{-0.14t}}$, where t is time in years. How many years will it take the herd to reach 1200 elephants?

39. Lasers. The intensity of a laser beam is given by the ratio of power to area. A particular laser beam has an intensity function given by $I = e^{-r^2}$ mW/cm^2, where r is the radius off the center axis given in centimeters. Where is the beam brightest (largest intensity)?

40. Lasers. The intensity of a laser beam is given by the ratio of power to area. A particular laser beam has an intensity function given by $I = e^{-r^2}$ mW/cm^2, where r is the radius off the center axis given in centimeters. What percentage of the on-axis intensity ($r = 0$) corresponds to $r = 2$ cm?

41. Grade Distribution. Suppose the first test in this class has a normal, or bell-shaped, grade distribution of test scores, with an average score of 75. An approximate function that models your class's grades on test 1 is $N(x) = 10e^{-(x-75)^2/25^2}$, where N represents the number of students who received the score x.

a. Graph this function.
b. What is the average grade?
c. Approximately how many students scored a 50?
d. Approximately how many students scored 100?

42. Grade Distribution. Suppose the final exam in this class has a normal, or bell-shaped, grade distribution of exam scores, with an average score of 80. An approximate function that models your class's grades on the exam is $N(x) = 10e^{-(x-80)^2/16^2}$, where N represents the number of students who received the score x.

a. Graph this function.
b. What is the average grade?
c. Approximately how many students scored a 60?
d. Approximately how many students scored 100?

43. Time to Pay Off Debt. Diana just graduated from medical school owing $80,000 in student loans. The annual interest rate is 9%.

a. Approximately how many years will it take to pay off her student loan if she makes a monthly payment of $750?
b. Approximately how many years will it take to pay off her loan if she makes a monthly payment of $1000?

44. Time to Pay Off Debt. Victor owes $20,000 on his credit card. The annual interest rate is 17%.

a. Approximately how many years will it take him to pay off this credit card if he makes a monthly payment of $300?
b. Approximately how many years will it take him to pay off this credit card if he makes a monthly payment of $400?

For Exercises 45 and 46, refer to the following:

A local business borrows $200,000 to purchase property. The loan has an annual interest rate of 8% compounded monthly and a minimum monthly payment of $1467.

45. Time to Pay Off Debt/Business.

a. Approximately how many years will it take the business to pay off the loans if only the minimum payment is made?
b. How much interest will the business pay over the life of the loan if only the minimum payment is made?

46. Time to Pay Off Debt/Business.

a. Approximately how many years will it take the business to pay off the loan if the minimum payment is doubled?
b. How much interest will the business pay over the life of the loan if the minimum payment is doubled?
c. How much in interest will the business save by doubling the minimum payment (see Exercise 45, part b)?

CATCH THE MISTAKE

In Exercises 47 and 48, explain the mistake that is made.

47. The city of Orlando, Florida, has a population that is growing at 7% a year, compounding continuously. If there were 1.1 million people in greater Orlando in 2006, approximately how many people will there be in 2016? Apply the formula $N = N_0e^{rt}$, where N represents the number of people.

Solution:

Use the population growth model.	$N = N_0e^{rt}$
Let $N_0 = 1.1$, $r = 7$, and $t = 10$.	$N = 1.1e^{(7)(10)}$
Approximate with a calculator.	2.8×10^{30}

This is incorrect. What mistake was made?

48. The city of San Antonio, Texas, has a population that is growing at 5% a year, compounding continuously. If there were 1.3 million people in the greater San Antonio area in 2006, approximately how many people will there be in 2016? Apply the formula $N = N_0e^{rt}$, where N represents the number of people.

Solution:

Use the population growth model.	$N = N_0e^{rt}$
Let $N_0 = 1.3$, $r = 5$, and $t = 10$.	$N = 1.3e^{(5)(10)}$
Approximate with a calculator.	6.7×10^{21}

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 49–52, determine whether each statement is true or false.

49. When a species gets placed on an endangered species list, the species begins to grow rapidly, and then reaches a carrying capacity. This can be modeled by logistic growth.

50. A professor has 400 students one semester. The number of names (of her students) she is able to memorize can be modeled by a logarithmic function.

51. The spread of lice at an elementary school can be modeled by exponential growth.

52. If you purchase a laptop computer this year ($t = 0$), then the value of the computer can be modeled with exponential decay.

CHALLENGE

In Exercises 53 and 54, refer to the logistic model $f(t) = \dfrac{a}{1 + ce^{-kt}}$, where a is the carrying capacity.

53. As c increases, does the model reach the carrying capacity in less time or more time?

54. As k increases, does the model reach the carrying capacity in less time or more time?

55. A culture of 100 bacteria grows at a rate of 20% every day. Two days later, 60 of the same type of bacteria are placed in a culture that allows a 30% daily growth rate. After how many days do both cultures have the same population?

56. Consider the quotient $Q = \dfrac{P_1 e^{r_1 t}}{P_2 e^{r_2 t}}$ of two models of exponential growth.

a. If $r_1 > r_2$, what can you say about Q?

b. If $r_1 < r_2$, what can you say about Q?

57. Consider the models of exponential decay $f(t) = (2 + c)e^{-k_1 t}$ and $g(t) = ce^{-k_2 t}$. Suppose that $f(1) = g(1)$, what is the relationship between k_1 and k_2?

58. Suppose that both logistic growth models $f(t) = \dfrac{a_1}{1 + c_1 e^{-k_1 t}}$ and $g(t) = \dfrac{a_2}{1 + c_2 e^{-k_2 t}}$ have horizontal asymptote $y = 100$. What can you say about the corresponding carrying capacities?

TECHNOLOGY

59. Wing Shan just graduated from dental school owing \$80,000 in student loans. The annual interest is 6%. Her time t to pay off the loan is given by

$$t = -\frac{\ln\left[1 - \dfrac{80{,}000(0.06)}{nR}\right]}{n\ln\left(1 + \dfrac{0.06}{n}\right)}$$

where n is the number of payment periods per year and R is the periodic payment.

a. Use a graphing utility to graph

$$t_1 = -\frac{\ln\left[1 - \dfrac{80{,}000(0.06)}{12x}\right]}{12\ln\left(1 + \dfrac{0.06}{12}\right)} \text{ as } Y_1 \text{ and}$$

$$t_2 = -\frac{\ln\left[1 - \dfrac{80{,}000(0.06)}{26x}\right]}{26\ln\left(1 + \dfrac{0.06}{26}\right)} \text{ as } Y_2.$$

Explain the difference in the two graphs.

b. Use the $\boxed{\text{TRACE}}$ key to estimate the number of years that it will take Wing Shan to pay off her student loan if she can afford a monthly payment of \$800.

c. If she can make a biweekly payment of \$400, estimate the number of years that it will take her to pay off the loan.

d. If she adds \$200 more to her monthly or \$100 more to her biweekly payment, estimate the number of years that it will take her to pay off the loan.

60. Amy has a credit card debt in the amount of \$12,000. The annual interest is 18%. Her time t to pay off the loan is given by

$$t = -\frac{\ln\left[1 - \frac{12{,}000(0.18)}{nR}\right]}{n\ln\left(1 + \frac{0.18}{n}\right)}$$

where n is the number of payment periods per year and R is the periodic payment.

a. Use a graphing utility to graph

$$t_1 = -\frac{\ln\left[1 - \frac{12{,}000(0.18)}{12x}\right]}{12\ln\left(1 + \frac{0.18}{12}\right)} \text{ as } Y_1 \text{ and}$$

$$t_2 = -\frac{\ln\left[1 - \frac{12{,}000(0.18)}{26x}\right]}{26\ln\left(1 + \frac{0.18}{26}\right)} \text{ as } Y_2.$$

Explain the difference in the two graphs.

b. Use the $\boxed{\text{TRACE}}$ key to estimate the number of years that it will take Amy to pay off her credit card if she can afford a monthly payment of \$300.

c. If she can make a biweekly payment of \$150, estimate the number of years that it will take her to pay off the credit card.

d. If Amy adds \$100 more to her monthly or \$50 more to her biweekly payment, estimate the number of years that it will take her to pay off the credit card.

PREVIEW TO CALCULUS

In Exercises 61–64, refer to the following:

In calculus, we find the derivative, $f'(x)$, of a function $f(x)$ by allowing h to approach 0 in the difference quotient $\frac{f(x+h) - f(x)}{h}$ of functions involving exponential functions.

61. Find the difference quotient of the exponential growth model $f(x) = Pe^{kx}$, where P and k are positive constants.

62. Find the difference quotient of the exponential decay model $f(x) = Pe^{-kx}$, where P and k are positive constants.

63. Use the fact that $\frac{e^h - 1}{h} = 1$ when h is close to zero to find the derivative of $f(x) = e^x + x$.

64. Find the difference quotient of $f(x) = \cosh x$ and use it to prove that $(\cosh x)' = \sinh x$.

CHAPTER 3 INQUIRY-BASED LEARNING PROJECT

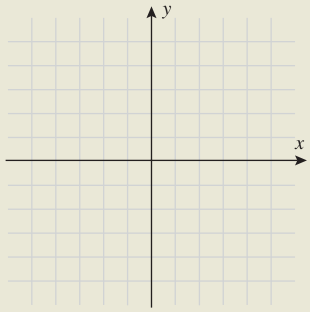
Among other ideas, in Chapters 1 and 2 you studied functions and their inverses. For instance, you worked with this familiar quadratic function: $y = x^2$. In words, this means "squaring x equals y." The equation of its inverse function can be written $x = y^2$; "squaring y equals x." Of course, we call y the "square root of x." In order to write this relationship with y in terms of x, mathematicians devised the symbol for square root, and so we write $y = \sqrt{x}$.

Keep these ideas in mind as you look now at an exponential function and the need to define a new function and new symbol for its inverse.

1. Lef f be the base 10 exponential function, $f(x) = 10^x$.

a. Graph the exponential function $y = 10^x$ by plotting points.

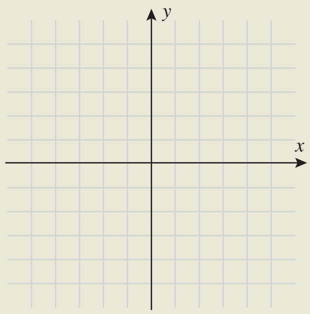
x	*y*
−3	
−2	
−1	
0	
1	
2	
3	



b. Discuss whether or not $f(x) = 10^x$ has an inverse function. How did you determine this?

c. Using the definition of inverse function, complete the table below for the function $y = f^{-1}(x)$. Then plot the points to make a graph.

x	*y*



d. In part (a) $y = 10^x$, so we could say, "10 to the power of x equals y." Write a similar statement about the inverse relationship between x and y in part (d). How would you write this as an equation?

2. If you wanted to obtain the graph of $y = f^{-1}(x)$ using your graphing calculator, you would need to solve for y in the equation you wrote in part (d) above. To this end, we need a new symbol to represent the relationship $x = 10^y$. We will call y the "base 10 logarithm of x" and write $y = \log_{10}(x)$.

a. The base 10 logarithm is also called the common logarithm. Use the "log" key on your graphing calculator to graph $y = \log_{10}(x)$. How does this graph differ from the one you sketched in part 1(c)?

b. What are the domain and range of $y = \log_{10}(x)$?

MODELING OUR WORLD

The following table summarizes the average yearly temperature in degrees Fahrenheit (°F) and carbon dioxide emissions in parts per million (ppm) for **Mauna Loa, Hawaii**.

YEAR	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
TEMPERATURE	44.45	43.29	43.61	43.35	46.66	45.71	45.53	47.53	45.86	46.23
CO_2 EMISSIONS (PPM)	316.9	320.0	325.7	331.1	338.7	345.9	354.2	360.6	369.4	379.7

In the Modeling Our World in Chapters 1 and 2, the temperature and carbon emissions were modeled with *linear functions* and *polynomial functions*, respectively. Now, let us model these same data using *exponential* and *logarithmic functions*.

1. Plot the temperature data, with time on the horizontal axis and temperature on the vertical axis. Let $t = 1$ correspond to 1960.

2. Find a *logarithmic function* with base e, $f(t) = A \ln(Bt)$, that models the temperature in Mauna Loa.

a. Apply data from 1965 and 2005.
b. Apply data from 2000 and 2005.
c. Apply regression and all data given.

3. Predict what the temperature will be in Mauna Loa in 2020.

a. Use the line found in Exercise 2(a).
b. Use the line found in Exercise 2(b).
c. Use the line found in Exercise 2(c).

4. Predict what the temperature will be in Mauna Loa in 2100.

a. Use the line found in Exercise 2(a).
b. Use the line found in Exercise 2(b).
c. Use the line found in Exercise 2(c).

5. Do your models support the claim of "global warming"? Explain. Do these logarithmic models give similar predictions to the linear models found in Chapter 1 and the polynomial models found in Chapter 2?

6. Plot the carbon dioxide emissions data, with time on the horizontal axis and carbon dioxide emissions on the vertical axis. Let $t = 0$ correspond to 1960.

7. Find an *exponential function* with base e, $f(t) = Ae^{bt}$, that models the CO_2 emissions (ppm) in Mauna Loa.

a. Apply data from 1960 and 2005.
b. Apply data from 1960 and 2000.
c. Apply regression and all data given.

8. Predict the expected CO_2 levels in Mauna Loa in 2020.

a. Use the line found in Exercise 7(a).
b. Use the line found in Exercise 7(b).
c. Use the line found in Exercise 7(c).

9. Predict the expected CO_2 levels in Mauna Loa in 2100.

a. Use the line found in Exercise 7(a).
b. Use the line found in Exercise 7(b).
c. Use the line found in Exercise 7(c).

10. Do your models support the claim of "global warming"? Explain. Do these exponential models give similar predictions to the linear models found in Chapter 1 or the polynomial models found in Chapter 2?

11. Comparing the models developed in Chapters 1 and 2, do you believe that global temperatures are best modeled with a linear, polynomial, or logarithmic function?

12. Comparing the models developed in Chapters 1 and 2, do you believe that CO_2 emissions are best modeled by linear, polynomial, or exponential functions?

SECTION	CONCEPT	KEY IDEAS/FORMULAS
3.1	**Exponential functions and their graphs**	
	Evaluating exponential functions	$f(x) = b^x \quad b > 0, b \neq 1$
	Graphs of exponential functions	y-intercept $(0, 1)$ Horizontal asymptote: $y = 0$; the points $(1, b)$ and $(-1, 1/b)$
	The natural base e	$f(x) = e^x$
	Applications of exponential functions	Doubling time: $P = P_0 2^{t/d}$ Compound interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compounded continuously: $A = Pe^{rt}$
3.2	**Logarithmic functions and their graphs**	$y = \log_b x \quad x > 0$ $b > 0, b \neq 1$
	Evaluating logarithms	$y = \log_b x$ and $x = b^y$
	Common and natural logarithms	$y = \log x$ Common (base 10) $y = \ln x$ Natural (base e)
	Graphs of logarithmic functions	x-intercept $(1, 0)$ Vertical asymptote: $x = 0$; the points $(b, 1)$ and $(1/b, -1)$
	Applications of logarithms	Decibel scale: $D = 10 \log\left(\frac{I}{I_T}\right) \quad I_T = 1 \times 10^{-12}\ \text{W/m}^2$ Richter scale: $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right) \quad E_0 = 10^{4.4}$ joules
3.3	**Properties of logarithms**	
	Properties of logarithms	1. $\log_b 1 = 0$ 2. $\log_b b = 1$ 3. $\log_b b^x = x$ 4. $b^{\log_b x} = x \quad x > 0$ Product property: 5. $\log_b MN = \log_b M + \log_b N$ Quotient property: 6. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$ Power property: 7. $\log_b M^p = p \log_b M$
	Change-of-base formula	$\log_b M = \frac{\log M}{\log b}$ or $\log_b M = \frac{\ln M}{\ln b}$

Section	Concept	Key Ideas/Formulas
3.4	**Exponential and logarithmic equations**	
	Solving exponential equations	*Simple exponential equations* 1. Rewrite both sides of the equation in terms of the same base. 2. Use the one-to-one property to equate the exponents. 3. Solve for the variable. *Complicated exponential equations* 1. Isolate the exponential expression. 2. Take the same logarithm of both sides. 3. Simplify using the inverse properties. 4. Solve for the variable.
	Solving logarithmic equations	*Simple logarithmic equations* 1. Combine logarithms on each side of the equation using properties. 2. Use the one-to-one property to equate the exponents. 3. Solve for the variable. 4. Check the results and eliminate any extraneous solutions. *Complicated logarithmic equations* 1. Combine and isolate the logarithmic expressions. 2. Rewrite the equation in exponential form. 3. Solve for the variable. 4. Check the results and eliminate any extraneous solutions.
3.5	**Exponential and logarithmic models**	
	Exponential growth models	$f(x) = ce^{kx} \quad k > 0$
	Exponential decay models	$f(x) = ce^{-kx} \quad k > 0$
	Gaussian (normal) distribution models	$f(x) = ce^{-(x-a)^2/k}$
	Logistic growth models	$f(x) = \dfrac{a}{1 + ce^{-kx}}$
	Logarithmic models	$f(x) = a + c \log x$ $f(x) = a + c \ln x$

3.1 Exponential Functions and Their Graphs

Approximate each number using a calculator and round your answer to two decimal places.

1. $8^{4.7}$ **2.** $\pi^{2/5}$ **3.** $4 \cdot 5^{0.2}$ **4.** $1.2^{1.2}$

Approximate each number using a calculator and round your answer to two decimal places.

5. $e^{3.2}$ **6.** e^{π} **7.** $e^{\sqrt{\pi}}$ **8.** $e^{-2.5\sqrt{3}}$

Evaluate each exponential function for the given values.

9. $f(x) = 2^{4-x}$ $f(-2.2)$

10. $f(x) = -2^{x+4}$ $f(1.3)$

11. $f(x) = \left(\frac{2}{5}\right)^{1-6x}$ $f\left(\frac{1}{2}\right)$

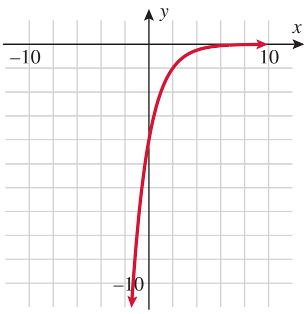
12. $f(x) = \left(\frac{4}{7}\right)^{5x+1}$ $f\left(\frac{1}{5}\right)$

Match the graph with the function.

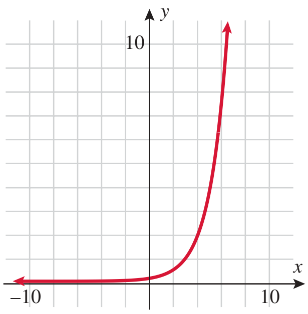
13. $y = 2^{x-2}$ **14.** $y = -2^{2-x}$

15. $y = 2 + 3^{x+2}$ **16.** $y = -2 - 3^{2-x}$

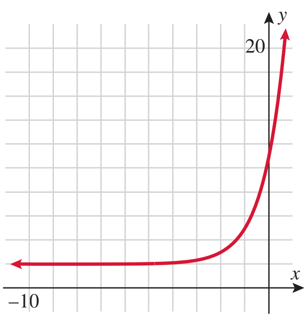
a.



b.



c.



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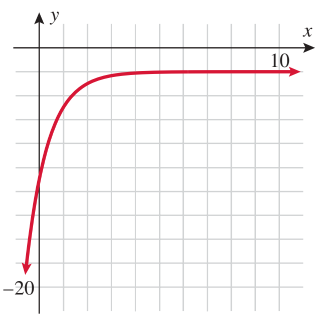


State the *y*-intercept and the horizontal asymptote, and graph the exponential function.

17. $y = -6^{-x}$ **18.** $y = 4 - 3^{x}$

19. $y = 1 + 10^{-2x}$ **20.** $y = 4^{x} - 4$

State the *y*-intercept and horizontal asymptote, and graph the exponential function.

21. $y = e^{-2x}$ **22.** $y = e^{x-1}$

23. $y = 3.2e^{x/3}$ **24.** $y = 2 - e^{1-x}$

Applications

25. Compound Interest. If \$4500 is deposited into an account paying 4.5% compounding semiannually, how much will you have in the account in 7 years?

26. Compound Interest. How much money should be put in a savings account now that earns 4.0% a year compounded quarterly if you want \$25,000 in 8 years?

27. Compound Interest. If \$13,450 is put in a money market account that pays 3.6% a year compounded continuously, how much will be in the account in 15 years?

28. Compound Interest. How much money should be invested today in a money market account that pays 2.5% a year compounded continuously if you desire \$15,000 in 10 years?

3.2 Logarithmic Functions and Their Graphs

Write each logarithmic equation in its equivalent exponential form.

29. $\log_4 64 = 3$ **30.** $\log_4 2 = \frac{1}{2}$

31. $\log\left(\frac{1}{100}\right) = -2$ **32.** $\log_{16} 4 = \frac{1}{2}$

Write each exponential equation in its equivalent logarithmic form.

33. $6^3 = 216$ **34.** $10^{-4} = 0.0001$

35. $\frac{4}{169} = \left(\frac{2}{13}\right)^2$ **36.** $\sqrt[3]{512} = 8$

Evaluate the logarithms exactly.

37. $\log_7 1$ **38.** $\log_4 256$

39. $\log_{1/6} 1296$ **40.** $\log 10^{12}$

Approximate the common and natural logarithms utilizing a calculator. Round to two decimal places.

41. log 32 **42.** ln 32

43. ln 0.125 **44.** log 0.125

State the domain of the logarithmic function in interval notation.

45. $f(x) = \log_3(x + 2)$ **46.** $f(x) = \log_2(2 - x)$

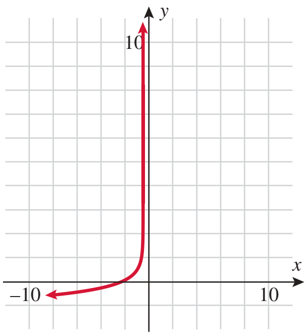
47. $f(x) = \log(x^2 + 3)$ **48.** $f(x) = \log(3 - x^2)$

Match the graph with the function.

49. $y = \log_7 x$ **50.** $y = -\log_7(-x)$

51. $y = \log_7(x + 1) - 3$ **52.** $y = -\log_7(1 - x) + 3$

a.



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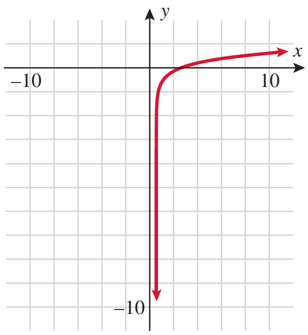

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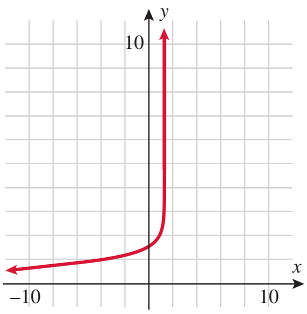

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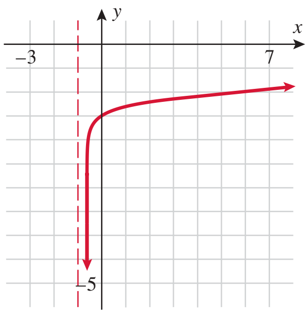


Graph the logarithmic function with transformation techniques.

53. $f(x) = \log_4(x - 4) + 2$ **54.** $f(x) = \log_4(x + 4) - 3$

55. $f(x) = -\log_4(x) - 6$ **56.** $f(x) = -2\log_4(-x) + 4$

Applications

57. Chemistry. Calculate the pH value of milk, assuming it has a concentration of hydrogen ions given by $H^+ = 3.16 \times 10^{-7}$.

58. Chemistry. Calculate the pH value of Coca-Cola, assuming it has a concentration of hydrogen ions given by $H^+ = 2.0 \times 10^{-3}$.

59. Sound. Calculate the decibels associated with a teacher speaking to a medium-sized class if the sound intensity is 1×10^{-7} W/m^2.

60. Sound. Calculate the decibels associated with an alarm clock if the sound intensity is 1×10^{-4} W/m^2.

3.3 Properties of Logarithms

Use the properties of logarithms to simplify each expression.

61. $\log_{2.5} 2.5$ **62.** $\log_2 \sqrt{16}$ **63.** $2.5^{\log_{2.5} 6}$ **64.** $e^{-3\ln 6}$

Write each expression as a sum or difference of logarithms.

65. $\log_c x^a y^b$ **66.** $\log_3 x^2 y^{-3}$

67. $\log_j\left(\dfrac{rs}{t^3}\right)$ **68.** $\log x^c \sqrt{x + 5}$

69. $\log\left[\dfrac{a^{1/2}}{b^{3/2}c^{2/5}}\right]$ **70.** $\log_7\left[\dfrac{c^3 d^{1/3}}{e^6}\right]^{1/3}$

Evaluate the logarithms using the change-of-base formula.

71. $\log_8 3$ **72.** $\log_5 \frac{1}{2}$ **73.** $\log_\pi 1.4$ **74.** $\log_{\sqrt{3}} 2.5$

3.4 Exponential and Logarithmic Equations

Solve the exponential equations exactly for *x*.

75. $4^x = \frac{1}{256}$ **76.** $3^{x^2} = 81$

77. $e^{3x-4} = 1$ **78.** $e^{\sqrt{x}} = e^{4.8}$

79. $\left(\frac{1}{3}\right)^{x+2} = 81$ **80.** $100^{x^2-3} = 10$

Solve the exponential equation. Round your answer to three decimal places.

81. $e^{2x+3} - 3 = 10$ **82.** $2^{2x-1} + 3 = 17$

83. $e^{2x} + 6e^x + 5 = 0$ **84.** $4e^{0.1x} = 64$

85. $(2^x - 2^{-x})(2^x + 2^{-x}) = 0$ **86.** $5(2^x) = 25$

Solve the logarithmic equations exactly.

87. $\log(3x) = 2$

88. $\log_3(x + 2) = 4$

89. $\log_4 x + \log_4 2x = 8$

90. $\log_6 x + \log_6(2x - 1) = \log_6 3$

Solve the logarithmic equations. Round your answers to three decimal places.

91. $\ln x^2 = 2.2$

92. $\ln(3x - 4) = 7$

93. $\log_3(2 - x) - \log_3(x + 3) = \log_3 x$

94. $4\log(x + 1) - 2\log(x + 1) = 1$

3.5 Exponential and Logarithmic Models

95. **Compound Interest.** If Tania needs \$30,000 a year from now for a down payment on a new house, how much should she put in a 1-year CD earning 5% a year compounding continuously so that she will have exactly \$30,000 a year from now?

96. **Stock Prices.** Jeremy is tracking the stock value of Best Buy (BBY on the NYSE). In 2003 he purchased 100 shares at \$28 a share. The stock did not pay dividends because the company reinvested all earnings. In 2005 Jeremy cashed out and sold the stock for \$4000. What was the annual rate of return on BBY?

97. **Compound Interest.** Money is invested in a savings account earning 4.2% interest compounded quarterly. How many years will pass until the money doubles?

98. **Compound Interest.** If \$9000 is invested in an investment earning 8% interest compounded continuously, how many years will pass until there is \$22,500?

99. **Population.** Nevada has the fastest-growing population according to the U.S. Census Bureau. In 2004 the population of Nevada was 2.62 million and increasing at an annual rate of 3.5%. What is the expected population in 2014? (Let $t = 0$ be 2004.) Apply the formula $N = N_0e^{rt}$, where N is the population.

100. **Population.** The Hispanic population in the United States is the fastest growing of any ethnic group. In 1996 there were an estimated 28.3 million Hispanics in the United States, and in 2000 there were an estimated 32.5 million. What is the expected population of Hispanics in the United States in 2014? (Let $t = 0$ be 1996.) Apply the formula $N = N_0e^{rt}$, where N is the population.

101. **Bacteria Growth.** Bacteria are growing exponentially. Initially, there were 1000 bacteria; after 3 hours there were 2500. How many bacteria should be expected in 6 hours? Apply the formula $N = N_0e^{rt}$, where N is the number of bacteria.

102. **Population.** In 2003 the population of Phoenix, Arizona, was 1,388,215. In 2004 the population was 1,418,041. What is the expected population in 2014? (Let $t = 0$ be 2003.) Apply the formula $N = N_0e^{rt}$, where N is the population.

103. **Radioactive Decay.** Strontium-90 has a half-life of 28 years. How long will it take for 20 grams of this to decay to 5 grams? Apply the formula $N = N_0e^{-rt}$, where N is the number of grams.

104. **Radioactive Decay.** Plutonium-239 has a half-life of 25,000 years. How long will it take for 100 grams to decay to 20 grams? Apply the formula $N = N_0e^{-rt}$, where N is the number of grams.

105. **Wild Life Population.** The *Boston Globe* reports that the fish population of the Essex River in Massachusetts is declining. In 2003 it was estimated there were 5600 fish in the river, and in 2004 there were only 2420 fish. How many fish should there have been in 2010 if this trend continued? Apply the formula $N = N_0e^{-rt}$, where N is the number of fish.

106. **Car Depreciation.** A new Acura TSX costs \$28,200. In 2 years the value will be \$24,500. What is the expected value in 6 years? Apply the formula $N = N_0e^{-rt}$, where N is the value of the car.

107. **Carrying Capacity.** The carrying capacity of a species of beach mice in St. Croix is given by $M = 1000\left(1 - e^{-0.035t}\right)$, where M is the number of mice and t is time in years ($t = 0$ corresponds to 1998). How many mice should there have been in 2010?

108. **Population.** The city of Brandon, Florida, had 50,000 residents in 1970, and since the crosstown expressway was built, its population has increased 2.3% per year. If the growth continues at the same rate, how many residents will Brandon have in 2030?

Technology Exercies

Section 3.1

109. Use a graphing utility to graph the function $f(x) = \left(1 + \frac{\sqrt{2}}{x}\right)^x$. Determine the horizontal asymptote as x increases.

110. Use a graphing utility to graph the functions $y = e^{-x+2}$ and $y = 3^x + 1$ in the same viewing screen. Estimate the coordinates of the point of intersection. Round your answers to three decimal places.

Section 3.2

111. Use a graphing utility to graph the functions $y = \log_{2.4}(3x - 1)$ and $y = \log_{0.8}(x - 1) + 3.5$ in the same viewing screen. Estimate the coordinates of the point of intersection. Round your answers to three decimal places.

112. Use a graphing utility to graph the functions $y = \log_{2.5}(x - 1) + 2$ and $y = 3.5^{x-2}$ in the same viewing screen. Estimate the coordinates of the point(s) of intersection. Round your answers to three decimal places.

Section 3.3

113. Use a graphing utility to graph $f(x) = \log_2\left(\frac{x^3}{x^2 - 1}\right)$ and $g(x) = 3\log_2 x - \log_2(x + 1) - \log_2(x - 1)$ in the same viewing screen. Determine the domain where the two functions give the same graph.

114. Use a graphing utility to graph $f(x) = \ln\left(\frac{9 - x^2}{x^2 - 1}\right)$ and $g(x) = \ln(3 - x) + \ln(3 + x) - \ln(x + 1) - \ln(x - 1)$ in the same viewing screen. Determine the domain where the two functions give the same graph.

Section 3.4

115. Use a graphing utility to graph $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. State the domain. Determine if there are any symmetry and asymptote.

116. Use a graphing utility to graph $y = \frac{1}{e^x - e^{-x}}$. State the domain. Determine if there are any symmetry and asymptote.

Section 3.5

117. A drug with initial dosage of 4 milligrams has a half-life of 18 hours. Let (0, 4) and (18, 2) be two points.

a. Determine the equation of the dosage.
b. Use $\boxed{\text{STAT}}$ $\boxed{\text{CALC}}$ $\boxed{\text{ExpReg}}$ to model the equation of the dosage.
c. Are the equations in (a) and (b) the same?

118. In Exercise 105, let $t = 0$ be 2003 and (0, 5600) and (1, 2420) be the two points.

a. Use $\boxed{\text{STAT}}$ $\boxed{\text{CALC}}$ $\boxed{\text{ExpReg}}$ to model the equation for the fish population.
b. Using the equation found in (a), how many fish should be expected in 2010?
c. Does the answer in (b) agree with the answer in Exercise 105?

1. Simplify $\log 10^{x^3}$.

2. Use a calculator to evaluate $\log_5 326$ (round to two decimal places).

3. Find the exact value of $\log_{1/3} 81$.

4. Rewrite the expression $\ln\left[\dfrac{e^{5x}}{x(x^4+1)}\right]$ in a form with no logarithms of products, quotients, or powers.

In Exercises 5–20, solve for x, exactly if possible. If an approximation is required, round your answer to three decimal places.

5. $e^{x^2-1} = 42$

6. $e^{2x} - 5e^x + 6 = 0$

7. $27e^{0.2x+1} = 300$

8. $3^{2x-1} = 15$

9. $3\ln(x-4) = 6$

10. $\log(6x+5) - \log 3 = \log 2 - \log x$

11. $\ln(\ln x) = 1$

12. $\log_2(3x-1) - \log_2(x-1) = \log_2(x+1)$

13. $\log_6 x + \log_6(x-5) = 2$

14. $\ln(x+2) - \ln(x-3) = 2$

15. $\ln x + \ln(x+3) = 1$

16. $\log_2\left(\dfrac{2x+3}{x-1}\right) = 3$

17. $\dfrac{12}{1+2e^x} = 6$

18. $\ln x + \ln(x-3) = 2$

19. State the domain of the function $f(x) = \log\left(\dfrac{x}{x^2-1}\right)$.

20. State the range of x values for which the following is true: $10^{\log(4x-a)} = 4x - a$.

In Exercises 21–24, find all intercepts and asymptotes, and graph.

21. $f(x) = 3^{-x} + 1$

22. $f(x) = \left(\frac{1}{2}\right)^x - 3$

23. $f(x) = \ln(2x-3) + 1$

24. $f(x) = \log(1-x) + 2$

25. **Interest.** If \$5000 is invested at a rate of 6% a year, compounded quarterly, what is the amount in the account after 8 years?

26. **Interest.** If \$10,000 is invested at a rate of 5%, compounded continuously, what is the amount in the account after 10 years?

27. **Sound.** A lawn mower's sound intensity is approximately 1×10^{-3} W/m^2. Assuming your threshold of hearing is 1×10^{-12} W/m^2, calculate the decibels associated with the lawn mower.

28. **Population.** The population in Seattle, Washington, has been increasing at a rate of 5% a year. If the population continues to grow at that rate, and in 2004 there are 800,000 residents, how many residents will there be in 2014? *Hint:* $N = N_0 e^{rt}$.

29. **Earthquake.** An earthquake is considered moderate if it is between 5 and 6 on the Richter scale. What is the energy range in joules for a moderate earthquake?

30. **Radioactive Decay.** The mass $m(t)$ remaining after t hours from a 50-gram sample of a radioactive substance is given by the equation $m(t) = 50e^{-0.0578t}$. After how long will only 30 grams of the substance remain? Round your answer to the nearest hour.

31. **Bacteria Growth.** The number of bacteria in a culture is increasing exponentially. Initially, there were 200 in the culture. After 2 hours there are 500. How many should be expected in 8 hours? Round your answer to the nearest hundred.

32. **Carbon Decay.** Carbon-14 has a half-life of 5730 years. How long will it take for 100 grams to decay to 40 grams?

33. **Spread of a Virus.** The number of people infected by a virus is given by $N = \dfrac{2000}{1+3e^{-0.4t}}$, where t is time in days. In how many days will 1000 people be infected?

34. **Oil Consumtion.** The world consumption of oil was 76 million barrels per day in 2002. In 2004 the consumption was 83 million barrels per day. How many barrels are expected to be consumed in 2014?

35. Use a graphing utility to graph $y = \dfrac{e^x - e^{-x}}{2}$. State the domain. Determine if there are any symmetry and asymptote.

36. Use a graphing utility to help solve the equation $4^{3-x} = 2x - 1$. Round your answer to two decimal places.

1. Find the domain and range of the function $f(x) = \dfrac{3}{\sqrt{x^2 - 9}}$.

2. If $f(x) = 1 + 3x$ and $g(x) = x^2 - 1$, find

   a. $f + g$ **b.** $f - g$ **c.** $f \cdot g$ **d.** $\dfrac{f}{g}$

   and state the domain of each.

3. Write the function below as a composite of two functions f and g. (More than one answer is correct.)

$$f(g(x)) = \frac{1 - e^{2x}}{1 + e^{2x}}$$

4. Determine whether $f(x) = \sqrt[5]{x^3 + 1}$ is one-to-one. If f is one-to-one, find its inverse f^{-1}.

5. Find the quadratic function whose vertex is $(-2, 3)$ and goes through the point $(1, -1)$.

6. Write the polynomial $f(x) = 3x^3 + 6x^2 - 15x - 18$ as a product of linear factors.

7. Solve the equation $e^x + \sqrt{e^x} - 12 = 0$. Round your answer to three decimal places.

8. Using the function $f(x) = 4x - x^2$, evaluate the difference quotient $\dfrac{f(x + h) - f(x)}{h}$.

9. Given the piecewise-defined function

$$f(x) = \begin{cases} 5 & -2 < x \leq 0 \\ 2 - \sqrt{x} & 0 < x < 4 \\ x - 3 & x \geq 4 \end{cases}$$

   find

   a. $f(4)$ **b.** $f(0)$ **c.** $f(1)$ **d.** $f(-4)$

   e. State the domain and range in interval notation.

   f. Determine the intervals where the function is increasing, decreasing, or constant.

10. Sketch the graph of the function $y = \sqrt{1 - x}$ and identify all transformations.

11. Determine whether the function $f(x) = \sqrt{x - 4}$ is one-to-one.

12. The volume of a cylinder with circular base is 400 cubic inches. Its height is 10 inches. Find its radius. Round your answer to three decimal places.

13. Find the vertex of the parabola associated with the quadratic function $f(x) = -4x^2 + 8x - 5$.

14. Find a polynomial of minimum degree (there are many) that has the zeros $x = -5$ (multiplicity 2) and $x = 9$ (multiplicity 4).

15. Use synthetic division to find the quotient $Q(x)$ and remainder $r(x)$ of $(3x^2 - 4x^3 - x^4 + 7x - 20) \div (x + 4)$.

16. Given the zero $x = 2 + i$ of the polynomial $P(x) = x^4 - 7x^3 + 13x^2 + x - 20$, determine all the other zeros and write the polynomial as the product of linear factors.

17. Find the vertical and slant asymptotes of $f(x) = \dfrac{x^2 + 7}{x - 3}$.

18. Graph the rational function $f(x) = \dfrac{3x}{x + 1}$. Give all asymptotes.

19. Graph the function $f(x) = 5x^2(7 - x)^2(x + 3)$.

20. If \$5400 is invested at 2.75% compounded monthly, how much is in the account after 4 years?

21. Give the exact value of $\log_3 243$.

22. Write the expression $\frac{1}{2}\ln(x + 5) - 2\ln(x + 1) - \ln(3x)$ as a single logarithm.

23. Solve the logarithmic equation exactly: $10^{2\log(4x+9)} = 121$.

24. Give an exact solution to the exponential equation $5^{x^2} = 625$.

25. If \$8500 is invested at 4% compounded continuously, how many years will pass until there is \$12,000?

26. Use a graphing utility to help solve the equation $e^{3-2x} = 2^{x-1}$. Round your answer to two decimal places.

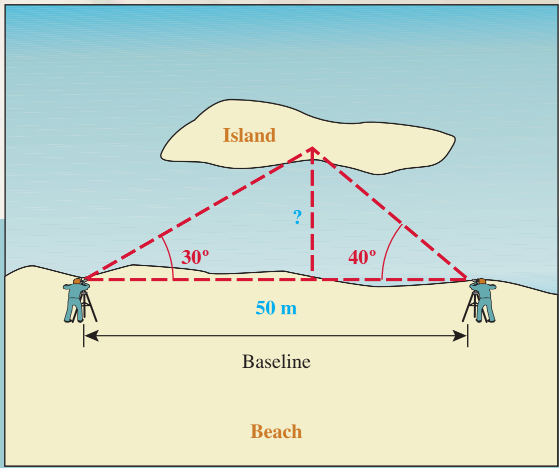
27. Strontium-90 with an initial amount of 6 grams has a half-life of 28 years.

   a. Use $\boxed{\text{STAT}}$ $\boxed{\text{CALC}}$ $\boxed{\text{ExpReg}}$ to model the equation of the amount remaining.

   b. How many grams will remain after 32 years? Round your answer to two decimal places.

4

Trigonometric Functions of Angles



Surveyors use trigonometry to indirectly measure distances. Since angles are easier to measure than distances, surveyors set up a baseline between two stations and measure the distance between the two stations and the angles made by the baseline and some third station.

If there are two stations along the shoreline and the distance along the beach between the two stations is 50 meters and the angles between the baseline (beach) and the line of sight to the island are 30° and 40°, then the Law of Sines can be used to find the shortest distance from the beach to the island.*

*See Section 4.4, Exercises 47 and 48.

IN THIS CHAPTER angle measure will be defined in terms of both degrees and radians. Then the six trigonometric functions will be defined for acute angles in terms of right triangle ratios. Next the trigonometric functions will be defined for nonacute angles. Lastly, the Law of Sines and the Law of Cosines will be used to solve oblique triangles.

TRIGONOMETRIC FUNCTIONS OF ANGLES

4.1 Angle Measure

- Angles and Their Measure
- Coterminal Angles
- Arc Length
- Area of a Circular Sector
- Linear and Angular Speeds

4.2 Right Triangle Trigonometry

- Right Triangle Ratios
- Evaluating Trigonometric Functions Exactly for Special Angle Measures
- Solving Right Triangles

4.3 Trigonometric Functions of Angles

- Trigonometric Functions: The Cartesian Plane
- Ranges of the Trigonometric Functions
- Reference Angles and Reference Right Triangles
- Evaluating Trigonometric Functions for Nonacute Angles

4.4 The Law of Sines

- Solving Oblique Triangles

4.5 The Law of Cosines

- Solving Oblique Triangles Using the Law of Cosines
- The Area of a Triangle

LEARNING OBJECTIVES

- Understand angle measures in both degrees and radians, and convert between the two.
- Find trigonometric function values for acute angles.
- Find trigonometric function values for any (acute or nonacute) angle.
- Use the Law of Sines to solve oblique triangles.
- Use the Law of Cosines to solve oblique triangles.

SECTION 4.1 ANGLE MEASURE

SKILLS OBJECTIVES

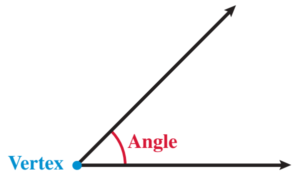
- Convert angle measure between degrees and radians.
- Find the complement or supplement of an angle.
- Identify coterminal angles.
- Calculate the length of an arc along a circle.
- Calculate the area of a circular sector.

CONCEPTUAL OBJECTIVES

- Understand that degrees and radians are both angle measures.
- Realize that radians are unitless (dimensionless).
- Understand the relationship between linear speed and angular speed.

Angles and Their Measure

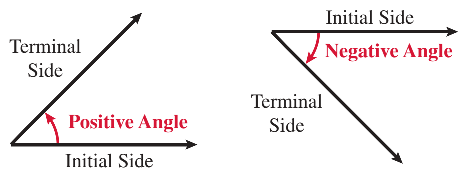
An **angle** is formed when a ray is rotated around its endpoint. The common endpoint is called the **vertex**.



The ray in its original position is called the **initial ray** or the **initial side** of an angle. In the Cartesian plane, we assume the initial side of an angle is the positive *x*-axis. The ray after it is rotated is called the **terminal ray** or the **terminal side** of an angle. Rotation in a counterclockwise direction corresponds to a **positive angle**, whereas rotation in a clockwise direction corresponds to a **negative angle**.

Study Tip

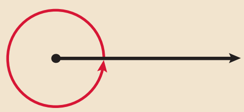
Positive angle: counterclockwise
Negative angle: clockwise



Lengths, or distances, can be measured in different units: feet, miles, and meters are three common units. In order to compare angles of different sizes, we need a standard unit of measure. One way to measure the size of an angle is with **degree measure**.

DEFINITION Degree Measure of Angles

An angle formed by one complete counterclockwise rotation has **measure 360 degrees**, denoted 360°.



One complete counterclockwise revolution = 360º

WORDS	MATH
360° represents 1 complete counterclockwise rotation.	$\frac{360°}{360°} = 1$
180° represents a $\frac{1}{2}$ counterclockwise rotation.	$\frac{180°}{360°} = \frac{1}{2}$
90° represents a $\frac{1}{4}$ counterclockwise rotation.	$\frac{90°}{360°} = \frac{1}{4}$
1° represents a $\frac{1}{360}$ counterclockwise rotation.	$\frac{1°}{360°} = \frac{1}{360}$

The Greek letter θ (theta) is the most common name for an angle in mathematics. Other common names of angles are α (alpha), β (beta), and γ (gamma).

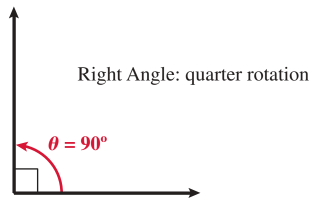
Study Tip

Greek letters are often used to denote angles in trigonometry.

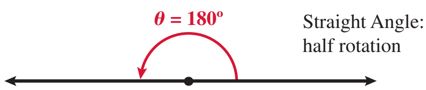
WORDS | **MATH**

An angle measuring exactly 90° is called a **right angle**.

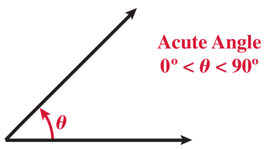
A right angle is often represented by the adjacent sides of a rectangle, indicating that the two rays are ***perpendicular***.



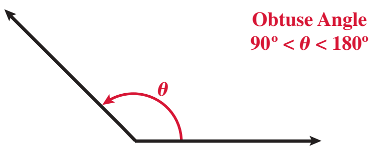
An angle measuring exactly 180° is called a **straight angle**.



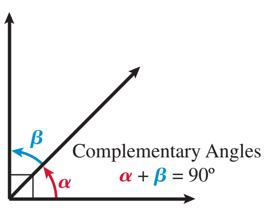
An angle measuring greater than 0°, but less than 90°, is called an **acute angle**.



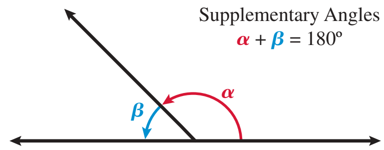
An angle measuring greater than 90°, but less than 180°, is called an **obtuse angle**.



If the sum of the measures of two positive angles is 90°, the angles are called **complementary**. We say that α is the **complement** of β (and vice versa).



If the sum of the measures of two positive angles is 180°, the angles are called **supplementary**. We say that α is the **supplement** of β (and vice versa).



EXAMPLE 1 Finding Measures of Complementary and Supplementary Angles

Find the measure of each angle.

a. Find the complement of 50°.

b. Find the supplement of 110°.

c. Represent the complement of α in terms of α.

d. Find two supplementary angles such that the first angle is twice as large as the second angle.

Solution:

a. The sum of complementary angles is 90°. $\quad \theta + 50° = 90°$

Solve for θ. $\quad \theta = \boxed{40°}$

b. The sum of supplementary angles is 180°. $\quad \theta + 110° = 180°$

Solve for θ. $\quad \theta = \boxed{70°}$

c. Let β be the complement of α.

The sum of complementary angles is 90°. $\quad \alpha + \beta = 90°$

Solve for β. $\quad \beta = \boxed{90° - \alpha}$

d. The sum of supplementary angles is 180°. $\quad \alpha + \beta = 180°$

Let $\beta = 2\alpha$. $\quad \alpha + 2\alpha = 180°$

Solve for α. $\quad 3\alpha = 180°$

$\alpha = 60°$

Substitute $\alpha = 60°$ into $\beta = 2\alpha$. $\quad \beta = 120°$

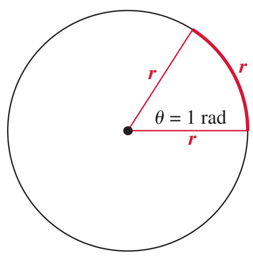
The angles have measures $\boxed{60°}$ and $\boxed{120°}$.

YOUR TURN Find two supplementary angles such that the first angle is 3 times as large as the second angle.

Answer: The angles have measures 45° and 135°.

It is important not to confuse an angle with its measure. In Example 1(d), angle α is a rotation and the measure of that rotation is 60°.

In geometry and most everyday applications, angles are measured in degrees. However, in calculus a more natural angle measure is *radian measure*. Using radian measure allows us to write trigonometric functions as functions of not only angles but also real numbers in general.



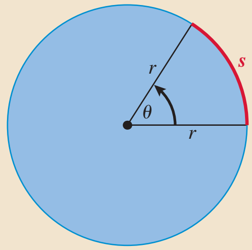
Now we think of the angle in the context of a circle. A **central angle** is an angle that has its vertex at the center of a circle. When the intercepted arc's length is equal to the radius, the measure of the central angle is 1 **radian**.

DEFINITION **Radian Measure**

If a central angle θ in a circle with radius r intercepts an arc on the circle of length s (**arc length**), then the measure of θ, in **radians**, is given by

$$\theta(\text{in radians}) = \frac{s}{r}$$

Note: The formula is valid only if s (arc length) and r (radius) are expressed in the same units.



CAUTION

To correctly calculate radians from the formula $\theta = \frac{s}{r}$, the radius and arc length must be expressed in the same units.

Note that both s and r are measured in units of length. When both are given in the same units, the units cancel, giving the number of radians as a *dimensionless* (unitless) real number. One full rotation corresponds to an arc length equal to the circumference $2\pi r$ of the circle with radius r. We see then that one full rotation is equal to 2π radians.

$$\theta_{\text{full rotation}} = \frac{2\pi r}{r} = 2\pi$$

EXAMPLE 2 Finding the Radian Measure of an Angle

What is the measure (in radians) of a central angle θ that intercepts an arc of length 6 centimeters on a circle with radius 2 meters?

COMMON MISTAKE

A common mistake is to forget to first put the radius and arc length in the same units.

CORRECT

Write the formula relating radian measure to arc length and radius.

$$\theta\,(\text{in radians}) = \frac{s}{r}$$

Substitute $s = 6\text{ cm}$ and $r = 2\text{ m}$ into the radian expression.

$$\theta = \frac{6\text{ cm}}{2\text{ m}}$$

Convert the radius (2) meters to centimeters: $2\text{ m} = 200\text{ cm}$.

$$\theta = \frac{6\text{ cm}}{200\text{ cm}}$$

The units, cm, cancel and the result is a unitless real number.

$$\theta = 0.03\text{ rad}$$

INCORRECT

Substitute $s = 6\text{ cm}$ and $r = 2\text{ m}$ into the radian expression.

$$\theta = \frac{6\text{ cm}}{2\text{ m}}$$

Simplify. $\theta = 3\text{ rad}$

ERROR

CAUTION

Units for arc length and radius must be the same to use $\theta = \frac{s}{r}$.

YOUR TURN What is the measure (in radians) of a central angle θ that intercepts an arc of length 12 millimeters on a circle with radius 4 centimeters?

Answer: 0.3 rad

In the above example, the units, cm, canceled, therefore correctly giving *radians* as a unitless real number. Because radians are unitless, the word radians (or rad) is often omitted. If an angle measure is given simply as a real number, then radians are implied.

Words	Math
The measure of θ is 4 degrees.	$\theta = 4°$
The measure of θ is 4 radians.	$\theta = 4$

Study Tip

If an angle measure is given as a real number, then radians are implied.

Study Tip

If we let

- θ_d = angle measure in degrees
- θ_R = angle measure in radians

then

- $\theta_R = \theta_d \cdot \dfrac{\pi}{180°}$
- $\theta_d = \theta_R \cdot \dfrac{180°}{\pi}$

Converting Between Degrees and Radians

An angle corresponding to one full rotation is said to have measure 360° or 2π radians. Therefore, $180° = \pi$ rad.

- To convert degrees to radians, multiply the degree measure by $\dfrac{\pi}{180°}$.
- To convert radians to degrees, multiply the radian measure by $\dfrac{180°}{\pi}$.

EXAMPLE 3 Converting Between Degrees and Radians

Convert:

a. 45° to radians **b.** 472° to radians **c.** $\dfrac{2\pi}{3}$ to degrees

Solution (a):

Multiply 45° by $\dfrac{\pi}{180°}$. $(45°)\left(\dfrac{\pi}{180°}\right) = \dfrac{45°\pi}{180°}$

Simplify. $= \dfrac{\pi}{4}$ radians

Note: $\dfrac{\pi}{4}$ is the exact value. A calculator can be used to approximate this expression. Scientific and graphing calculators have a π button (on most scientific calculators, it requires using a shift or second command). The decimal approximation rounded to three decimal places is 0.785.

Exact value: $\boxed{\dfrac{\pi}{4}}$

Approximate value: $\boxed{0.785}$

Solution (b):

Multiply 472° by $\dfrac{\pi}{180°}$. $472°\left(\dfrac{\pi}{180°}\right)$

Simplify (factor out the common 4). $= \boxed{\dfrac{118}{45}\pi}$

Approximate with a calculator. $\boxed{\approx 8.238}$

Solution (c):

Multiply $\dfrac{2\pi}{3}$ by $\dfrac{180°}{\pi}$. $\dfrac{2\pi}{3} \cdot \dfrac{180°}{\pi}$

Simplify. $= \boxed{120°}$

YOUR TURN Convert:

a. 60° to radians **b.** 460° to radians **c.** $\dfrac{3\pi}{2}$ to degrees

Answer:

a. $\dfrac{\pi}{3}$ or approximately 1.047

b. $\dfrac{23}{9}\pi$ or approximately 8.029

c. 270°

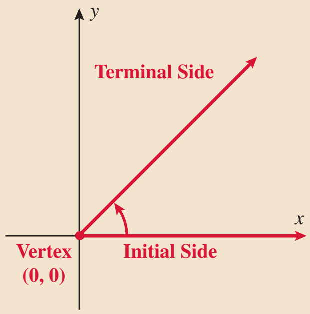
Coterminal Angles

Angles in Standard Position

If the *initial side* of an angle is aligned along the *positive x-axis* and the *vertex* of the angle is positioned at the *origin*, then the angle is said to be in *standard position.*

DEFINITION **Standard Position**

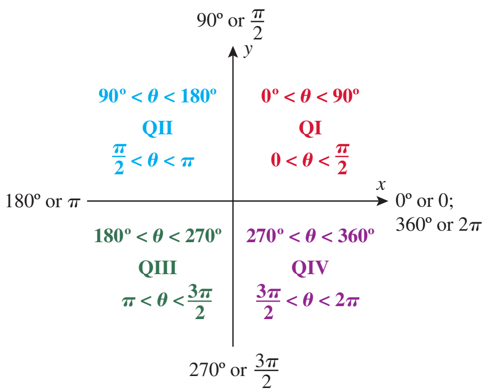
An angle is said to be in **standard position** if its initial side is along the positive x-axis and its vertex is at the origin.



Study Tip

Both the initial side (initial ray) and the terminal side (terminal ray) of an angle are rays.

We say that an angle lies in the quadrant in which its terminal side lies. Angles in standard position with terminal sides along the x-axis or y-axis (90°, 180°, 270°, 360°, etc.) are called **quadrantal angles**.

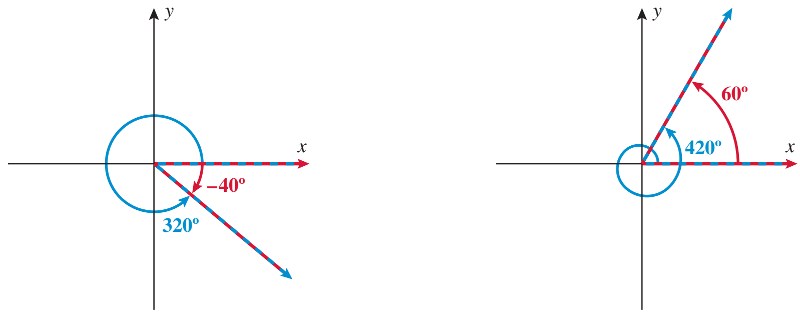


Coterminal Angles

DEFINITION **Coterminal Angles**

Two angles in standard position with the same terminal side are called **coterminal angles**.

For example, **−40°** and **320°** are measures of coterminal angles; their terminal rays are identical even though they are formed by rotations in opposite directions. The angles **60°** and **420°** are also coterminal; angles larger than 360° or less than −360° are generated by continuing the rotation beyond one full circle. Thus, all coterminal angles have the same initial side (positive x-axis) and the same terminal side, just different amounts and/or direction of rotation.



To find the measure of the smallest nonnegative coterminal angle of a given angle measured in degrees follow this procedure:

- If the given angle is positive, subtract 360° (repeatedly until the result is a positive angle less than or equal to 360°).
- If the given angle is negative, add 360° (repeatedly until the result is a positive angle less than or equal to 360°).

Similarly, if the angle is measured in radians, subtract or add equivalently 2π until your result is a positive angle less than or equal to 2π.

EXAMPLE 4 Finding Measures of Coterminal Angles

Determine the angle with the smallest possible positive measure that is coterminal with each of the following angles:

a. 830° **b.** $-520°$ **c.** $\frac{11\pi}{3}$

Solution (a):

Since 830° is positive, subtract 360°. $830° - 360° = 470°$

Subtract 360° again. $470° - 360° = \boxed{110°}$

Solution (b):

Since $-520°$ is negative, add 360°. $-520° + 360° = -160°$

Add 360° again. $-160° + 360° = \boxed{200°}$

Solution (c):

Since $\frac{11\pi}{3}$ is positive, subtract 2π. $\frac{11\pi}{3} - 2\pi = \boxed{\frac{5\pi}{3}}$

Answer: **a.** 180° **b.** 290° **c.** $\frac{11\pi}{6}$

YOUR TURN Determine the angle with the smallest possible positive measure that is coterminal with each of the following angles:

a. 900° **b.** $-430°$ **c.** $-\frac{13\pi}{6}$

Applications of Radian Measure

We now look at applications of radian measure that involve calculating *arc lengths*, *areas of circular sectors*, and *angular and linear speeds*. All of these applications are related to the definition of radian measure.

Arc Length

DEFINITION **Arc Length**

If a central angle θ in a circle with radius r intercepts an arc on the circle of length s, then the **arc length** s is given by

$$s = r\theta \qquad \theta \text{ is given in radians}$$

Study Tip

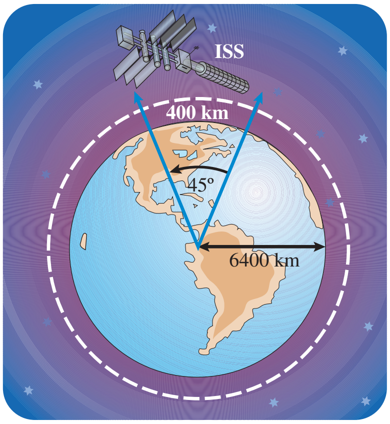
To use the relationship

$$s = r\theta$$

the angle θ must be in radians.

EXAMPLE 5 **Finding Arc Length When the Angle Has Degree Measure**

The International Space Station (ISS) is in an approximately circular orbit 400 kilometers above the surface of the Earth. If the ground station tracks the space station when it is within a 45° central angle of this circular orbit above the tracking antenna, how many kilometers does the ISS cover while it is being tracked by the ground station? Assume that the radius of the Earth is 6400 kilometers. Round to the nearest kilometer.



Solution:

Write the formula for arc length when the angle has degree measure. $\quad s = r\theta_d\left(\dfrac{\pi}{180°}\right)$

Substitute $r = 6400 + 400 = 6800$ km and $\theta_d = 45°$. $\quad s = (6800\text{ km})(45°)\left(\dfrac{\pi}{180°}\right)$

Evaluate with a calculator. $\quad s \approx 5340.708$ km

Round to the nearest kilometer. $\quad s \approx \boxed{5341\text{ km}}$

The ISS travels approximately 5341 kilometers during the ground station tracking.

■ **YOUR TURN** If the ground station in Example 5 could track the ISS within a 60° central angle, how far would the ISS travel during the tracking?

Technology Tip

Use the TI to evaluate the expression for s.

$$s = (6800\text{ km})(45°)\left(\frac{\pi}{180°}\right)$$

Press [2nd] [∧] for π. Type [6800] [x] [45] [x] [2nd] [∧] [÷] [180] [ENTER].

```
6800*45*π/180
      5340.707511
```

Study Tip

When the angle is given in degrees, the arc length formula becomes

$$s = r \cdot \theta_d\left(\frac{\pi}{180°}\right)$$

■ **Answer:** 7121 km

Area of a Circular Sector

DEFINITION **Area of a Circular Sector**

The **area of a sector of a circle** with radius r and central angle θ is given by

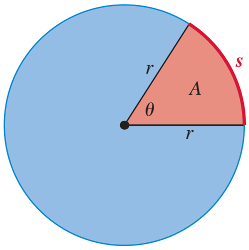
$$A = \frac{1}{2}r^2\theta \qquad \theta \text{ is given in radians}$$

Study Tip

To use the relationship

$$A = \frac{1}{2}r^2\theta$$

the angle θ must be in radians.



WORDS	**MATH**
Write the ratio of the area of the sector to the area of the entire circle.	$\dfrac{A}{\pi r^2}$
Write the ratio of the central angle θ to the measure of one full rotation.	$\dfrac{\theta}{2\pi}$
The ratios must be equal (proportionality of sector to circle).	$\dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi}$
Multiply both sides of the equation by πr^2.	$\pi r^2 \cdot \dfrac{A}{\pi r^2} = \dfrac{\theta}{2\pi} \cdot \pi r^2$
Simplify.	$A = \dfrac{1}{2} r^2 \theta$

Study Tip

When the angle is given in degrees, the area of a circular sector becomes

$$A = \frac{1}{2} r^2 \theta_d \left(\frac{\pi}{180°}\right)$$

EXAMPLE 6 Finding the Area of a Sector When the Angle Has Degree Measure

Sprinkler heads come in all different sizes depending on the angle of rotation desired. If a sprinkler head rotates 90° and has enough pressure to keep a constant 25-foot spray, what is the area of the sector of the lawn that gets watered? Round to the nearest square foot.

Solution:

Write the formula for circular sector area in degrees.	$A = \dfrac{1}{2} r^2 \theta_d \left(\dfrac{\pi}{180°}\right)$
Substitute $r = 25$ ft and $\theta_d = 90°$ into the area equation.	$A = \dfrac{1}{2}(25 \text{ ft})^2(90°)\left(\dfrac{\pi}{180°}\right)$
Simplify.	$A = \left(\dfrac{625\pi}{4}\right) \text{ft}^2 \approx 490.87 \text{ ft}^2$
Round to the nearest square foot.	$A \approx \boxed{491 \text{ ft}^2}$

■ **YOUR TURN** If a sprinkler head rotates 180° and has enough pressure to keep a constant 30-foot spray, what is the area of the sector of the lawn it can water? Round to the nearest square foot.

■ **Answer:** $450\pi \text{ ft}^2 \approx 1414 \text{ ft}^2$

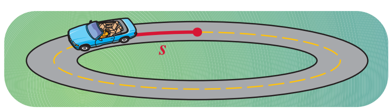
Linear and Angular Speeds

Recall the relationship between distance, rate (assumed to be constant), and time: $d = rt$. Rate is speed, and in words this formula can be rewritten as

$$\text{distance} = \text{speed} \cdot \text{time} \quad \text{or} \quad \text{speed} = \frac{\text{distance}}{\text{time}}$$

It is important to note that we assume speed is constant. If we think of a car driving around a circular track, the distance it travels is the arc length s; and if we let v represent speed and t represent time, we have the formula for speed along a circular path (*linear speed*):

$$v = \frac{s}{t}$$



DEFINITION **Linear Speed**

If a point P moves along the circumference of a circle at a constant speed, then the **linear speed** v is given by

$$v = \frac{s}{t}$$

where s is the arc length and t is the time.

EXAMPLE 7 **Linear Speed**

A car travels at a constant speed around a circular track with circumference equal to 2 miles. If the car records a time of 15 minutes for 9 laps, what is the linear speed of the car in miles per hour?

Solution:

Calculate the distance traveled around the circular track.

$$s = (9 \text{ laps})\left(\frac{2 \text{ mi}}{\text{lap}}\right) = 18 \text{ mi}$$

Substitute $t = 15$ min and $s = 18$ mi into $v = \frac{s}{t}$.

$$v = \frac{18 \text{ mi}}{15 \text{ min}}$$

Convert the linear speed from miles per minute to miles per hour.

$$v = \left(\frac{18 \text{ mi}}{15 \text{ min}}\right)\left(\frac{60 \text{ min}}{1 \text{ hr}}\right)$$

Simplify.

$$v = \boxed{72 \text{ mph}}$$

YOUR TURN A car travels at a constant speed around a circular track with circumference equal to 3 miles. If the car records a time of 12 minutes for 7 laps, what is the linear speed of the car in miles per hour?

Answer: 105 mph

To calculate linear speed, we find how fast a position along the circumference of a circle is changing. To calculate *angular speed*, we find how fast the central angle is changing.

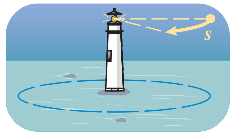
DEFINITION **Angular Speed**

If a point P moves along the circumference of a circle at a constant speed, then the central angle θ that is formed with the terminal side passing through point P also changes over some time t at a constant speed. The **angular speed** ω (omega) is given by

$$\omega = \frac{\theta}{t} \qquad \text{where } \theta \text{ is given in radians}$$

EXAMPLE 8 Angular Speed

A lighthouse in the middle of a channel rotates its light in a circular motion with constant speed. If the beacon of light completes 1 rotation every 10 seconds, what is the angular speed of the beacon in radians per minute?



Solution:

Calculate the angle measure in radians associated with 1 rotation. $\theta = 2\pi$

Substitute $\theta = 2\pi$ and $t = 10$ sec into $\omega = \dfrac{\theta}{t}$. $\omega = \dfrac{2\pi \text{ (rad)}}{10 \text{ sec}}$

Convert the angular speed from radians per second to radians per minute. $\omega = \dfrac{2\pi \text{ (rad)}}{10 \text{ sec}} \cdot \dfrac{60 \text{ sec}}{1 \text{ min}}$

Simplify. $\omega = \boxed{12\pi \text{ rad/min}}$

■ **YOUR TURN** If the lighthouse in Example 8 is adjusted so that the beacon rotates 1 time every 40 seconds, what is the angular speed of the beacon in radians per minute?

■ **Answer:** $\omega = 3\pi$ rad/min

Relationship Between Linear and Angular Speeds

Angular speed and *linear speed* are related through the *radius.*

WORDS	MATH
Write the definition of radian measure.	$\theta = \dfrac{s}{r}$
Write the definition of arc length (θ in radians).	$s = r\theta$
Divide both sides by t.	$\dfrac{s}{t} = \dfrac{r\theta}{t}$
Rewrite the right side of the equation.	$\dfrac{s}{t} = r\dfrac{\theta}{t}$
Recall the definitions of **linear** and **angular** speeds.	$v = \dfrac{s}{t}$ and $\omega = \dfrac{\theta}{t}$
Substitute $v = \dfrac{s}{t}$ and $\omega = \dfrac{\theta}{t}$ into $\dfrac{s}{t} = r\dfrac{\theta}{t}$.	$v = r\omega$

Study Tip

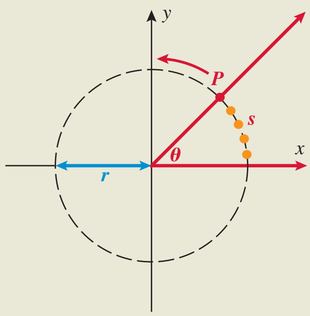
This relationship between linear and angular speed assumes the angle is given in radians.

RELATING LINEAR AND ANGULAR SPEEDS

If a point P moves at a constant speed along the circumference of a circle with radius r, then the **linear speed** v and the **angular speed** ω are related by

$$v = r\omega \qquad \text{or} \qquad \omega = \frac{v}{r}$$

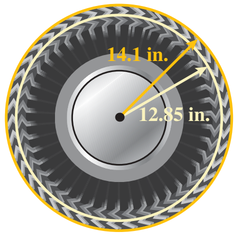
Note: This relationship is true only when θ is given in radians.



Notice that tires of two different radii with the same angular speed have different linear speeds. The larger tire has the faster linear speed.

EXAMPLE 9 Relating Linear and Angular Speeds

A Ford F-150 truck comes standard with tires that have a diameter of 25.7 inches (17″ rims). If the owner decides to upgrade to tires with a diameter of 28.2 inches (19″ rims) without having the onboard computer updated, how fast will the truck *actually* be traveling when the speedometer reads 75 miles per hour?



Solution:

The computer onboard the F-150 "thinks" the tires are 25.7 inches in diameter and knows the angular speed. Use the programmed tire diameter and speedometer reading to calculate the angular speed. Then use that angular speed and the upgraded tire diameter to get the actual speed (linear speed).

STEP 1 *Calculate the angular speed of the tires.*

Write the formula for the angular speed. $\omega = \dfrac{v}{r}$

Substitute $v = 75$ mph and $r = \dfrac{25.7}{2} = 12.85$ in. into the formula. $\omega = \dfrac{75 \text{ mi/hr}}{12.85 \text{ in.}}$

$1 \text{ mi} = 5280 \text{ ft} = 63{,}360 \text{ in.}$ $\omega = \dfrac{75(63{,}360) \text{ in./hr}}{12.85 \text{ in.}}$

Simplify. $\omega \approx 369{,}805 \text{ rad/hr}$

STEP 2 *Calculate the actual linear speed of the truck.*

Write the linear speed formula. $v = r\omega$

Substitute $r = \dfrac{28.2}{2} = 14.1$ in.

and $\omega \approx 369{,}805$ rad/hr. $v = (14.1 \text{ in.})(369{,}805 \text{ rad/hr})$

Simplify. $v \approx 5{,}214{,}251 \text{ in./hr}$

$1 \text{ mi} = 5280 \text{ ft} = 63{,}360 \text{ in.}$ $v \approx \dfrac{5{,}214{,}251 \text{ in./hr}}{63{,}360 \text{ in./mi}}$

$v \approx \boxed{82.296 \text{ mph}}$

Although the speedometer indicates a speed of 75 miles per hour, the actual speed is approximately $\boxed{\text{82 miles per hour}}$.

Study Tip

We could have solved Example 9 the following way:

$$\frac{75 \text{ mph}}{25.7} = \frac{x}{28.2}$$

$$x = \left(\frac{28.2}{25.7}\right)(75)$$

$$\approx 82.296 \text{ mph}$$

YOUR TURN Suppose the owner of the Ford F-150 truck in Example 9 decides to downsize the tires from their original 25.7-inch diameter to a 24.4-inch diameter. If the speedometer indicates a speed of 65 miles per hour, what is the actual speed of the truck?

Answer: approximately 62 mph

SECTION 4.1 SUMMARY

Angle measures can be converted between degrees and radians in the following way:

- To convert degrees to radians, multiply the degree measure by $\frac{\pi}{180°}$.
- To convert radians to degrees, multiply the radian measure by $\frac{180°}{\pi}$.

(Remember that $\pi = 180°$.)

Coterminal angles in standard position have terminal sides that coincide.

The length of a circular arc is given by $s = r\theta$, where θ is the central angle given in radians and r is the radius of the circle.

The area of a circular sector is given by $A = \frac{1}{2}r^2\theta$, where θ is the central angle given in radians and r is the radius of the circle.

Linear speed, $v = \frac{s}{t}$, and angular speed, $\omega = \frac{\theta}{t}$, are related through the radius: $v = r\omega$.

SECTION 4.1 EXERCISES

SKILLS

In Exercises 1–6, find (*a*) the complement and (*b*) the supplement of the given angles.

1. 18° **2.** 39° **3.** 42° **4.** 57° **5.** 89° **6.** 75°

In Exercises 7–12, find the measure (in radians) of a central angle θ that intercepts an arc of length s on a circle with radius r.

7. $r = 22$ in., $s = 4$ in. **8.** $r = 6$ in., $s = 1$ in. **9.** $r = 100$ cm, $s = 20$ mm

10. $r = 1$ m, $s = 2$ cm **11.** $r = \frac{1}{4}$ in., $s = \frac{1}{32}$ in. **12.** $r = \frac{3}{4}$ cm, $s = \frac{3}{14}$ cm

In Exercises 13–28, convert from degrees to radians. Leave the answers in terms of π.

13. 30° **14.** 60° **15.** 45° **16.** 90° **17.** 315° **18.** 270°

19. 75° **20.** 100° **21.** 170° **22.** 340° **23.** 780° **24.** 540°

25. −210° **26.** −320° **27.** −3600° **28.** 1800°

In Exercises 29–42, convert from radians to degrees.

29. $\frac{\pi}{6}$ **30.** $\frac{\pi}{4}$ **31.** $\frac{3\pi}{4}$ **32.** $\frac{7\pi}{6}$ **33.** $\frac{3\pi}{8}$ **34.** $\frac{11\pi}{9}$ **35.** $\frac{5\pi}{12}$

36. $\frac{7\pi}{3}$ **37.** 9π **38.** -6π **39.** $\frac{19\pi}{20}$ **40.** $\frac{13\pi}{36}$ **41.** $-\frac{7\pi}{15}$ **42.** $-\frac{8\pi}{9}$

In Exercises 43–50, convert from radians to degrees. Round your answers to the nearest hundredth of a degree.

43. 4 **44.** 3 **45.** 0.85 **46.** 3.27

47. −2.7989 **48.** −5.9841 **49.** $2\sqrt{3}$ **50.** $5\sqrt{7}$

In Exercises 51–56, convert from degrees to radians. Round your answers to three significant digits.

51. 47° **52.** 65° **53.** 112° **54.** 172° **55.** 56.5° **56.** 298.7°

In Exercises 57–68, state in which quadrant or on which axis each angle with the given measure in standard position would lie.

57. 145° **58.** 175° **59.** 270° **60.** 180° **61.** −540° **62.** −450°

63. $\frac{2\pi}{5}$ **64.** $\frac{4\pi}{7}$ **65.** $\frac{13\pi}{4}$ **66.** $\frac{18\pi}{11}$ **67.** 2.5 **68.** 11.4

In Exercises 69–80, determine the angle of the smallest possible positive measure that is coterminal with each of the angles whose measure is given. Use degree or radian measures accordingly.

69. $412°$ **70.** $379°$ **71.** $-92°$ **72.** $-187°$ **73.** $-390°$ **74.** $945°$

75. $\frac{29\pi}{3}$ **76.** $\frac{47\pi}{7}$ **77.** $-\frac{313\pi}{9}$ **78.** $-\frac{217\pi}{4}$ **79.** -30 **80.** 42

In Exercises 81–88, find the exact length of the arc made by the indicated central angle and radius of each circle.

81. $\theta = \frac{\pi}{12}$, $r = 8$ ft **82.** $\theta = \frac{\pi}{8}$, $r = 6$ yd **83.** $\theta = \frac{1}{2}$, $r = 5$ in. **84.** $\theta = \frac{3}{4}$, $r = 20$ m

85. $\theta = 22°$, $r = 18\ \mu$m **86.** $\theta = 14°$, $r = 15\ \mu$m **87.** $\theta = 8°$, $r = 1500$ km **88.** $\theta = 3°$, $r = 1800$ km

In Exercises 89–94, find the area of the circular sector given the indicated radius and central angle. Round your answers to three significant digits.

89. $\theta = \frac{3\pi}{8}$, $r = 2.2$ km **90.** $\theta = \frac{5\pi}{6}$, $r = 13$ mi **91.** $\theta = 56°$, $r = 4.2$ cm

92. $\theta = 27°$, $r = 2.5$ mm **93.** $\theta = 1.2°$, $r = 1.5$ ft **94.** $\theta = 14°$, $r = 3.0$ ft

In Exercises 95–98, find the linear speed of a point that moves with constant speed in a circular motion if the point travels along the circle of arc length s in time t.

95. $s = 2$ m, $t = 5$ sec **96.** $s = 12$ ft, $t = 3$ min **97.** $s = 68{,}000$ km, $t = 250$ hr **98.** $s = 7524$ mi, $t = 12$ days

In Exercises 99–102, find the distance traveled (arc length) of a point that moves with constant speed v along a circle in time t.

99. $v = 2.8$ m/sec, $t = 3.5$ sec **100.** $v = 6.2$ km/hr, $t = 4.5$ hr

101. $v = 4.5$ mi/hr, $t = 20$ min **102.** $v = 5.6$ ft/sec, $t = 2$ min

In Exercises 103–106, find the angular speed (radians/second) associated with rotating a central angle θ in time t.

103. $\theta = 25\pi$, $t = 10$ sec **104.** $\theta = \frac{3\pi}{4}$, $t = \frac{1}{6}$ sec **105.** $\theta = 200°$, $t = 5$ sec **106.** $\theta = 60°$, $t = 0.2$ sec

In Exercises 107–110, find the linear speed of a point traveling at a constant speed along the circumference of a circle with radius r and angular speed ω.

107. $\omega = \frac{2\pi \text{ rad}}{3 \text{ sec}}$, $r = 9$ in. **108.** $\omega = \frac{3\pi \text{ rad}}{4 \text{ sec}}$, $r = 8$ cm **109.** $\omega = \frac{\pi \text{ rad}}{20 \text{ sec}}$, $r = 5$ mm **110.** $\omega = \frac{5\pi \text{ rad}}{16 \text{ sec}}$, $r = 24$ ft

In Exercises 111–114, find the distance a point travels along a circle over a time t, given the angular speed ω and radius r of the circle. Round your answers to three significant digits.

111. $r = 5$ cm, $\omega = \frac{\pi \text{ rad}}{6 \text{ sec}}$, $t = 10$ sec **112.** $r = 2$ mm, $\omega = 6\pi \frac{\text{rad}}{\text{sec}}$, $t = 11$ sec

113. $r = 5.2$ in., $\omega = \frac{\pi \text{ rad}}{15 \text{ sec}}$, $t = 10$ min **114.** $r = 3.2$ ft, $\omega = \frac{\pi \text{ rad}}{4 \text{ sec}}$, $t = 3$ min

APPLICATIONS

For Exercises 115 and 116, refer to the following:

A common school locker combination lock is shown. The lock has a dial with 40 calibration marks numbered 0 to 39. A combination consists of three of these numbers (e.g., 5-35-20). To open the lock, the following steps are taken:

Tacojim/iStockphoto

- Turn the dial clockwise two full turns.
- Continue turning clockwise until the first number of the combination.
- Turn the dial counterclockwise one full turn.
- Continue turning counterclockwise until the second number is reached.
- Turn the dial clockwise again until the third number is reached.
- Pull the shank and the lock will open.

115. Combination Lock. Given that the initial position of the dial is at zero (shown in the illustration), how many degrees is the dial rotated in total (sum of clockwise and counterclockwise rotations) in opening the lock if the combination is 35-5-20?

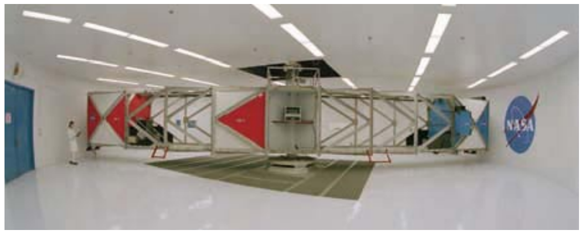
116. Combination Lock. Given that the initial position of the dial is at zero (shown in the illustration), how many degrees is the dial rotated in total (sum of clockwise and counterclockwise rotations) in opening the lock if the combination is 20-15-5?

117. Tires. A car owner decides to upgrade from tires with a diameter of 24.3 inches to tires with a diameter of 26.1 inches. If she doesn't update the onboard computer, how fast will she actually be traveling when the speedometer reads 65 miles per hour? Round to the nearest miles per hour.

118. Tires. A car owner decides to upgrade from tires with a diameter of 24.8 inches to tires with a diameter of 27.0 inches. If she doesn't update the onboard computer, how fast will she actually be traveling when the speedometer reads 70 miles per hour? Round to the nearest miles per hour.

For Exercises 119 and 120, refer to the following:

NASA explores artificial gravity as a way to counter the physiologic effects of extended weightlessness for future space exploration. NASA's centrifuge has a 58-foot-diameter arm.



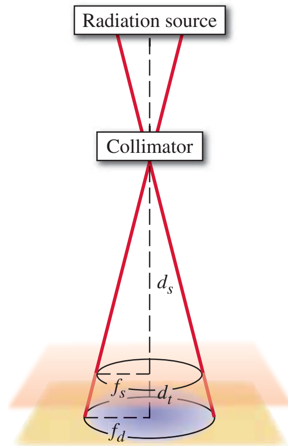
Courtesy NASA

119. NASA. If two humans are on opposite (red and blue) ends of the centrifuge and their linear speed is 200 miles per hour, how fast is the arm rotating? Express the answer in radians per second to two significant digits.

120. NASA. If two humans are on opposite (red and blue) ends of the centrifuge and they rotate one full rotation every second, what is their linear speed in feet per second?

For Exercises 121–124, refer to the following:

A collimator is a device used in radiation treatment that narrows beams or waves, causing the waves to be more aligned in a specific direction. The use of a collimator facilitates the focusing of radiation to treat an affected region of tissue beneath the skin. In the figure, d_s is the distance from the radiation source to the skin and d_t is the distance from the outer layer of skin to the targeted tissue. The field size on the skin (diameter of the circular treated skin) is $2f_s$, and $2f_d$ is the targeted field size at depth d_t (the diameter of the targeted tissue at the specified depth beneath the skin surface).



121. Health/Medicine. Radiation treatment is applied to a field size of 8 centimeters at a depth 2.5 centimeters below the skin surface. If the treatment head is positioned 80 centimeters from the skin, find the targeted field size to the nearest millimeter.

122. Health/Medicine. Radiation treatment is applied to a field size of 4 centimeters lying at a depth of 3.5 centimeters below the skin surface. If the field size on the skin is required to be 3.8 centimeters, find the distance from the skin that the radiation source must be located to the nearest millimeter.

123. Health/Medicine. Radiation treatment is applied to a field size on the skin of 3.75 centimeters to reach an affected region of tissue with field size of 4 centimeters at some depth below the skin. If the treatment head is positioned 60 centimeters from the skin surface, find the desired depth below the skin to the target area to the nearest millimeter.

124. Health/Medicine. Radiation treatment is applied to a field size on the skin of 4.15 centimeters to reach an affected area lying 4.5 centimeters below the skin surface. If the treatment head is positioned 60 centimeters from the skin surface, find the field size of the targeted area to the nearest millimeter.

For Exercises 125 and 126, refer to the following:

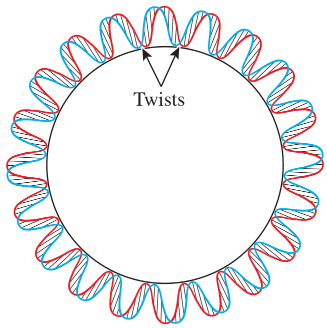
Sniffers outside a chemical munitions disposal site monitor the atmosphere surrounding the site to detect any toxic gases. In the event that there is an accidental release of toxic fumes, the data provided by the sniffers make it possible to determine both the distance d that the fumes reach as well as the angle of spread θ that sweep out a circular sector.

125. Environment. If the maximum angle of spread is 105° and the maximum distance at which the toxic fumes were detected was 9 miles from the site, find the area of the circular sector affected by the accidental release.

126. Environment. To protect the public from the fumes, officials must secure the perimeter of this area. Find the perimeter of the circular sector in Exercise 125.

For Exercises 127 and 128, refer to the following:

The structure of human DNA is a *linear* double helix formed of nucleotide base pairs (two nucleotides) that are stacked with spacing of 3.4 angstroms (3.4×10^{-12}m), and each base pair is rotated 36° with respect to an adjacent pair and has 10 base pairs per helical turn. The DNA of a virus or a bacterium, however, is a *circular* double helix (see the figure below) with the structure varying among species.



(*Source*: http://www.biophysics.org/Portals/1/PDFs/Education/Vologodskii.pdf.)

127. Biology. If the circular DNA of a virus has 10 twists (or turns) per circle and an inner diameter of 4.5 nanometers, find the arc length between consecutive twists of the DNA.

128. Biology. If the circular DNA of a virus has 40 twists (or turns) per circle and an inner diameter of 2.0 nanometers, find the arc length between consecutive twists of the DNA.

CATCH THE MISTAKE

In Exercises 129 and 130, explain the mistake that is made.

129. If the radius of a set of tires on a car is 15 inches and the tires rotate 180° per second, how fast is the car traveling (linear speed) in miles per hour?

Solution:

Write the formula for linear speed. $v = r\omega$

Let $r = 15$ in. and $\omega = 180°$ per sec. $v = (15\text{ in.})\left(\dfrac{180°}{\text{sec}}\right)$

Simplify. $v = 2700\,\dfrac{\text{in.}}{\text{sec}}$

Let 1 mi = 5280 ft = 63,360 in. and 1 hr = 3600 sec. $v = \left(\dfrac{2700 \cdot 3600}{63{,}360}\right)\text{mph}$

Simplify. $v \approx 153.4$ mph

This is incorrect. The correct answer is approximately 2.7 miles per hour. What mistake was made?

130. If a bicycle has tires with radius 10 inches and the tires rotate 90° per $\frac{1}{2}$ second, how fast is the bicycle traveling (linear speed) in miles per hour?

Solution:

Write the formula for linear speed. $v = r\omega$

Let $r = 10$ in. and $\omega = 180°$ per sec. $v = (10\text{ in.})\left(\dfrac{180°}{\text{sec}}\right)$

Simplify. $v = \dfrac{1800\text{ in.}}{\text{sec}}$

Let 1 mi = 5280 ft = 63,360 in. and 1 hr = 3600 sec. $v = \left(\dfrac{1800 \cdot 3600}{63{,}360}\right)\text{mph}$

Simplify. $v \approx 102.3$ mph

This is incorrect. The correct answer is approximately 1.8 miles per hour. What mistake was made?

CONCEPTUAL

In Exercises 131–134, determine whether each statement is true or false.

131. If the radius of a circle doubles, then the arc length (associated with a fixed central angle) doubles.

132. If the radius of a circle doubles, then the area of the sector (associated with a fixed central angle) doubles.

133. If the angular speed doubles, then the number of revolutions doubles.

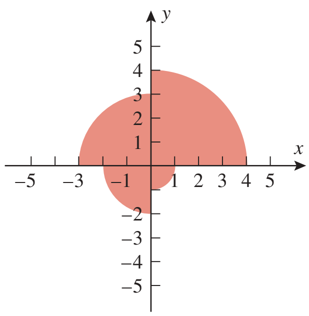
134. If the central angle of a sector doubles, then the area corresponding to the sector is double the area of the original sector.

CHALLENGE

135. What is the measure (in degrees) of the smaller angle the hour and minute hands make when the time is 12:20?

136. What is the measure (in degrees) of the smaller angle the hour and minute hands make when the time is 9:10?

137. Find the area of the shaded region below:



138. Find the perimeter of the shaded region in Exercise 137.

TECHNOLOGY

In Exercises 139 and 140, find the measure in degrees of a central angle θ that intercepts an arc on a circle with indicated radius r and arc length s.

139. $r = 78.6$ cm, $s = 94.4$ cm

140. $r = 14.2$ in., $s = 23.8$ in.

PREVIEW TO CALCULUS

In calculus we work with real numbers; thus, the measure of an angle must be in radians.

141. What is the measure (in radians) of a central angle θ that intercepts an arc of length 2π centimeters on a circle of radius 10 centimeters?

142. Determine the angle of the smallest possible positive measure (in radians) that is coterminal with the angle 750°.

143. The area of a sector of a circle with radius 3 inches and central angle θ is $\frac{3\pi}{2}$ in.^{2}. What is the radian measure of θ?

144. An object is rotating at 600° per second, find the central angle θ, in radians, when $t = 3$ sec.

SECTION 4.2 RIGHT TRIANGLE TRIGONOMETRY

SKILLS OBJECTIVES

- Learn the trigonometric functions as ratios of sides of a right triangle.
- Evaluate trigonometric functions exactly for special angles.
- Evaluate trigonometric functions using a calculator.

CONCEPTUAL OBJECTIVES

- Understand that right triangle ratios are based on the properties of similar triangles.
- Understand the difference between evaluating trigonometric functions exactly and using a calculator.

The word **trigonometry** stems from the Greek words *trigonon*, which means triangle, and *metrein*, which means to measure. Trigonometry began as a branch of geometry and was utilized extensively by early Greek mathematicians to determine unknown distances. The major *trigonometric functions*, including *sine*, *cosine*, and *tangent*, were first defined as ratios of sides in a right triangle. This is the way we will define them in this section. Since the two angles, besides the right angle, in a right triangle have to be acute, a second kind of definition was needed to extend the domain of trigonometric functions to nonacute angles in the Cartesian plane (Section 4.3). Starting in the eighteenth century, broader definitions of the trigonometric functions came into use, under which the functions are associated with points along the unit circle (Section 5.1).

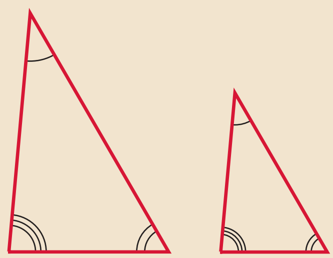
Right Triangle Ratios

Similar Triangles

The word *similar* in mathematics means identical in shape, although not necessarily the same size. It is important to note that two triangles can have the exact same shape (same angles) but have different sizes.

DEFINITION **Similar Triangles**

Similar triangles are triangles with equal corresponding angle measures (equal angles).

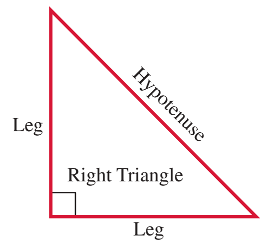


Study Tip

Although an angle and its measure are fundamentally different, out of convenience when "equal angles" is stated, this implies "equal angle measures."

Right Triangles

A **right triangle** is a triangle in which one of the angles is a right angle 90°. Since one angle is 90°, the other two angles must be complementary (sum to 90°), so that the sum of all three angles is 180°. The longest side of a right triangle, called the **hypotenuse**, is opposite the right angle. The other two sides are called the **legs** of the right triangle.

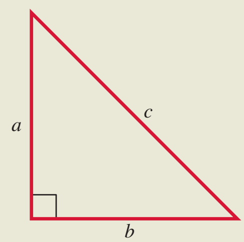


The *Pythagorean theorem* relates the sides of a right triangle. It says that the sum of the squares of the lengths of the two legs is equal to the square of the length of the hypotenuse. It is important to note that length (a synonym of distance) is always positive.

PYTHAGOREAN THEOREM

In any right triangle, the square of the length of the longest side (hypotenuse) is equal to the sum of the squares of the lengths of the other two sides (legs).

$$a^2 + b^2 = c^2$$



Study Tip

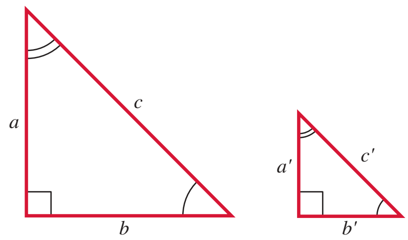
The Pythagorean theorem applies only to *right* triangles.

It is important to note that the Pythagorean theorem applies *only* to right triangles. It is also important to note that it does not matter which side is called a or b, as long as the square of the longest side is equal to the sum of the squares of the shorter sides.

Right-triangle trigonometry relies on the properties of similar triangles. Since similar triangles have the same shape (equal corresponding angles), the sides opposite the corresponding angles must be proportional.

Right Triangle Ratios

The concept of similar triangles, one of the basic insights in trigonometry, allows us to determine the length of a side of one triangle if we know the length of certain sides of a similar triangle. Consider the following similar triangles:



In similar triangles, the sides opposite corresponding angles must be proportional so the following ratios hold true:

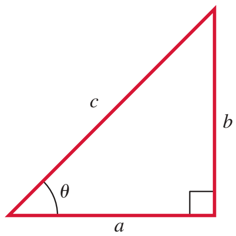
$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Separate the common ratios into three equations:

$$\frac{a}{a'} = \frac{b}{b'} \qquad \frac{b}{b'} = \frac{c}{c'} \qquad \frac{a}{a'} = \frac{c}{c'}$$

For any right triangle, there are six possible ratios of sides that can be calculated for each acute angle θ:

$$\frac{b}{c} \quad \frac{a}{c} \quad \frac{b}{a} \quad \frac{c}{b} \quad \frac{c}{a} \quad \frac{a}{b}$$



These ratios are referred to as **trigonometric ratios** or **trigonometric functions**, since they depend on the angle θ, and each is given a name:

Function Name	Abbreviation
Sine	sin
Cosine	cos
Tangent	tan
Cosecant	csc
Secant	sec
Cotangent	cot

Words	Math
The sine of θ	$\sin\theta$
The cosine of θ	$\cos\theta$
The tangent of θ	$\tan\theta$
The cosecant of θ	$\csc\theta$
The secant of θ	$\sec\theta$
The cotangent of θ	$\cot\theta$

Sine, cosine, tangent, cotangent, secant, and cosecant are names given to specific ratios of lengths of sides of right triangles.

Study Tip

Notice that

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

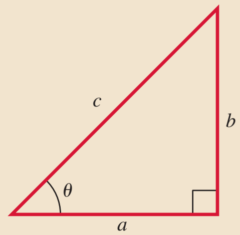
$$\cot\theta = \frac{1}{\tan\theta}$$

DEFINITION **Trigonometric Functions**

Let θ be an acute angle in a right triangle, then

$$\sin\theta = \frac{b}{c} \qquad \cos\theta = \frac{a}{c} \qquad \tan\theta = \frac{b}{a}$$

$$\csc\theta = \frac{c}{b} \qquad \sec\theta = \frac{c}{a} \qquad \cot\theta = \frac{a}{b}$$



The following terminology will be used throughout this text (refer to the right triangle above):

- The **hypotenuse** is always opposite the right angle.
- One leg (b) is **opposite** the angle θ.
- One leg (a) is **adjacent** to the angle θ.

Also notice that since $\sin\theta = \frac{b}{c}$ and $\cos\theta = \frac{a}{c}$, then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{a}$.

Using this terminology, we arrive at an alternative definition that is easier to remember.

Study Tip

Trigonometric functions are functions of a specified angle. Always specify the angle. "Sin" alone means nothing. Sin θ specifies the angle dependency. The same is true for the other five trigonometric functions.

DEFINITION **Trigonometric Functions (Alternate Form)**

For an acute angle θ in a right triangle:

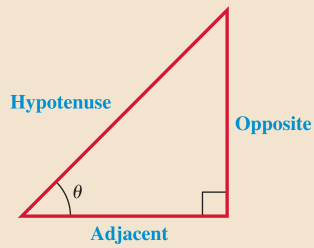
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

and their reciprocals:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}}$$



Study Tip

SOHCAHTOA

$$\text{SOH: } \text{Sin}\,\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\text{CAH: } \text{Cos}\,\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{TOA: } \text{Tan}\,\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Reciprocal Identities

The three main trigonometric functions should be learned in terms of the following ratios:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Study Tip

You need to learn only the three main trigonometric ratios: $\sin\theta$, $\cos\theta$, and $\tan\theta$. The other three can always be calculated as reciprocals of these main three for an acute angle θ.

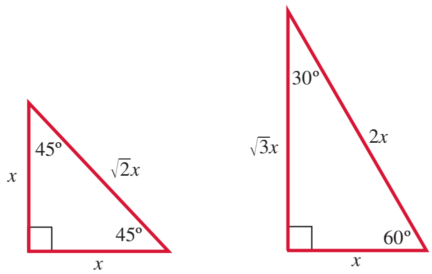
The remaining three trigonometric functions can be derived from $\sin\theta$, $\cos\theta$, and $\tan\theta$ using the *reciprocal identities*. Recall that the **reciprocal** of x is $\frac{1}{x}$ for $x \neq 0$.

RECIPROCAL IDENTITIES

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

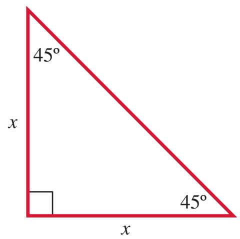
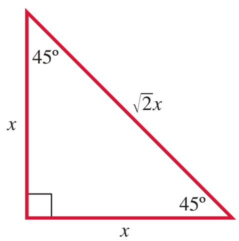
Evaluating Trigonometric Functions Exactly for Special Angle Measures

There are three special acute angles that are very important in trigonometry: 30°, 45°, and 60°. We can combine the relationships governing their side lengths

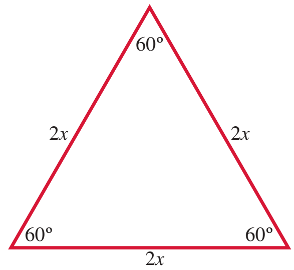
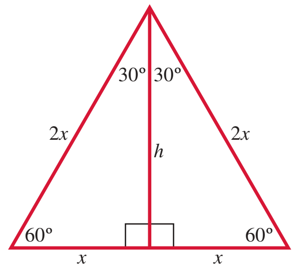
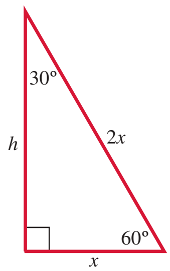
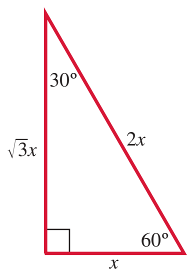


with the trigonometric ratios developed in this section to evaluate the trigonometric functions for the special angle measures of 30°, 45°, and 60°.

Consider a 45°-45°-90° triangle.

WORDS	MATH
A 45°-45°-90° triangle is an isosceles (two legs are equal) right triangle.	
Apply the Pythagorean theorem.	$x^2 + x^2 = \text{hypotenuse}^2$
Simplify the left side of the equation.	$2x^2 = \text{hypotenuse}^2$
Solve for the hypotenuse.	$\text{hypotenuse} = \pm\sqrt{2x^2} = \pm\sqrt{2}\,\lvert x\rvert$
x and the hypotenuse are lengths and must be positive.	$\text{hypotenuse} = \sqrt{2}\,x$
The hypotenuse of a 45°-45°-90° triangle is $\sqrt{2}$ times the length of either leg.	

Let us now determine the relationship of the sides of a 30°-60°-90° triangle. We start with an equilateral triangle (equal sides and equal angles of measure 60°).

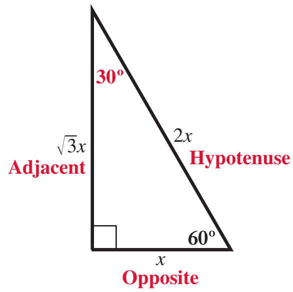
WORDS	MATH
Draw an equilateral triangle with sides $2x$.	
Draw a line segment from one vertex that is perpendicular to the opposite side; this line segment represents the height of the triangle, h, and bisects the base. There are now two identical 30°-60°-90° triangles.	
Notice that in each triangle the hypotenuse is twice the shortest leg, which is opposite the 30° angle.	
To find the length h, use the Pythagorean theorem.	$h^2 + x^2 = (2x)^2$
	$h^2 + x^2 = 4x^2$
Solve for h.	$h^2 = 3x^2$
	$h = \pm\sqrt{3x^2} = \pm\sqrt{3}\lvert x\rvert$
h and x are lengths and must be positive.	$h = \sqrt{3}x$
The hypotenuse of a 30°-60°-90° is twice the length of the leg opposite the 30° angle, the shortest leg. The leg opposite the 60° angle is $\sqrt{3}$ times the length of the leg opposite the 30° angle, the shortest leg.	

EXAMPLE 1 Evaluating the Trigonometric Functions Exactly for 30°

Evaluate the six trigonometric functions for an angle that measures 30°.

Solution:

Label the sides (opposite, adjacent, and hypotenuse) of the 30°-60°-90° triangle with respect to the **30°** angle.



Use the right triangle ratio definitions of sine, cosine, and tangent.

$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{2x} = \frac{1}{2}$$

$$\cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{3}x} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Use the reciprocal identities to obtain the value of the cosecant, secant, and cotangent functions.

$$\csc 30° = \frac{1}{\sin 30°} = \frac{1}{\frac{1}{2}} = 2$$

$$\sec 30° = \frac{1}{\cos 30°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot 30° = \frac{1}{\tan 30°} = \frac{1}{\frac{\sqrt{3}}{3}} = \frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}$$

The six trigonometric functions evaluated for an angle measuring 30° are

$$\boxed{\sin 30° = \frac{1}{2}} \qquad \boxed{\cos 30° = \frac{\sqrt{3}}{2}} \qquad \boxed{\tan 30° = \frac{\sqrt{3}}{3}}$$

$$\boxed{\csc 30° = 2} \qquad \boxed{\sec 30° = \frac{2\sqrt{3}}{3}} \qquad \boxed{\cot 30° = \sqrt{3}}$$

■ **YOUR TURN** Evaluate the six trigonometric functions for an angle that measures 60°.

■ **Answer:**

$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$

$\tan 60° = \sqrt{3}$ $\csc 60° = \frac{2\sqrt{3}}{3}$

$\sec 60° = 2$ $\cot 60° = \frac{\sqrt{3}}{3}$

In comparing our answers in Example 1 and Your Turn, we see that the following cofunction relationships are true. We call these *cofunction* relationships.

$$\sin 30° = \cos 60° \qquad \sec 30° = \csc 60° \qquad \tan 30° = \cot 60°$$

$$\sin 60° = \cos 30° \qquad \sec 60° = \csc 30° \qquad \tan 60° = \cot 30°$$

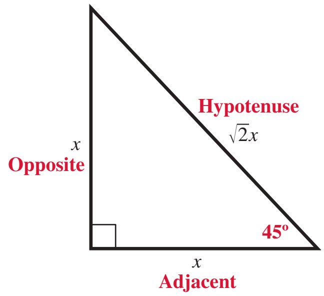
Notice that 30° and 60° are complementary angles.

EXAMPLE 2 Evaluating the Trigonometric Functions Exactly for 45°

Evaluate the six trigonometric functions for an angle that measures 45°.

Solution:

Label the sides of the 45°-45°-90° triangle as opposite, adjacent, or hypotenuse with respect to one of the **45°** angles.



Use the right triangle ratio definitions of sine, cosine, and tangent.

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{x} = 1$$

Use the reciprocal identities to obtain the values of the cosecant, secant, and cotangent functions.

$$\csc 45° = \frac{1}{\sin 45°} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

$$\sec 45° = \frac{1}{\cos 45°} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

$$\cot 45° = \frac{1}{\tan 45°} = \frac{1}{1} = 1$$

The six trigonometric functions evaluated for an angle measuring 45° are

$$\boxed{\sin 45° = \frac{\sqrt{2}}{2}} \quad \boxed{\cos 45° = \frac{\sqrt{2}}{2}} \quad \boxed{\tan 45° = 1}$$

$$\boxed{\csc 45° = \sqrt{2}} \quad \boxed{\sec 45° = \sqrt{2}} \quad \boxed{\cot 45° = 1}$$

We see that the following *cofunction* relationships are true

$$\sin 45° = \cos 45° \qquad \sec 45° = \csc 45° \qquad \tan 45° = \cot 45°$$

since 45° and 45° are complementary angles.

Study Tip

$\sin 45° = \frac{\sqrt{2}}{2}$ is exact, whereas if we evaluate with a calculator, we get an approximation:

$$\sin 45° \approx 0.7071$$

The trigonometric function values for the three special angle measures, 30°, 45°, and 60°, are summarized in the following table.

Trigonometric Function Values for Special Angles

θ Degrees	θ Radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$

It is important to **learn** the special values in **red** for sine and cosine. All other values in the table can be found through reciprocals or quotients of these two functions. Remember that the tangent function is the ratio of the sine to cosine functions.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\dfrac{\text{opposite}}{\text{hypotenuse}}}{\dfrac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{adjacent}}$$

Study Tip

If you memorize the values for sine and cosine for the angles given in the table, then the other trigonometric function values in the table can be found using the quotient and reciprocal identities.

Using Calculators to Evaluate (Approximate) Trigonometric Functions

We now turn our attention to using calculators to evaluate trigonometric functions, which often results in an approximation. Scientific and graphing calculators have buttons for sine (sin), cosine (cos), and tangent (tan) functions.

EXAMPLE 3 Evaluating Trigonometric Functions with a Calculator

Use a calculator to find the values of

a. $\sin 75°$ **b.** $\tan 67°$ **c.** $\sec 52°$ **d.** $\cos\left(\frac{\pi}{6}\right)$ **e.** $\tan\left(\frac{\pi}{8}\right)$

Round your answers to four decimal places.

Solution:

a. 0.965925826 $\boxed{\approx 0.9659}$

b. 2.355852366 $\boxed{\approx 2.3559}$

c. $\cos 52° \approx 0.615661475$ $1/x$ (or x^{-1}) 1.624269245 $\boxed{\approx 1.6243}$

d. 0.866025403 $\boxed{\approx 0.8660}$

e. 0.414213562 $\boxed{\approx 0.4142}$

Note: We know $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.8660$.

■ **YOUR TURN** Use a calculator to find the values of

a. $\cos 22°$ **b.** $\tan 81°$ **c.** $\csc 37°$ **d.** $\sin\left(\frac{\pi}{4}\right)$ **e.** $\cot\left(\frac{\pi}{12}\right)$

Round your answers to four decimal places.

Study Tip

In calculating secant, cosecant, and cotangent function values with a calculator, it is important not to round the number until after using the reciprocal function key $1/x$.

■ **Answer: a.** 0.9272 **b.** 6.3138 **c.** 1.6616 **d.** 0.7071 **e.** 3.7321

When calculating secant, cosecant, and cotangent function values with a calculator, it is important not to round the number until after using the reciprocal function key $1/x$ or x^{-1} in order to be as accurate as possible.

Study Tip

To solve a right triangle means to find the lengths of the sides and the angle measures.

Solving Right Triangles

A triangle has three angles and three sides. To *solve a triangle* means to find the length of all three sides and the measures of all three angles.

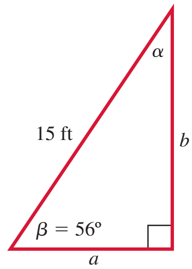
Technology Tip

```
15cos(56)
        8.387893552
```

```
15sin(56)
        12.43556359
```

EXAMPLE 4 Solving a Right Triangle Given an Angle and a Side

Solve the right triangle—find a, b, and α.



Solution:

STEP 1 Solve for α.

The two acute angles in a right triangle are complementary. $\alpha + 56° = 90°$

Solve for α. $\boxed{\alpha = 34°}$

STEP 2 Solve for a.

Cosine of an angle is equal to the adjacent side over the hypotenuse. $\cos 56° = \dfrac{a}{15}$

Solve for a. $a = 15\cos 56°$

Evaluate the right side of the expression using a calculator. $a \approx 8.38789$

Round a to two significant digits. $\boxed{a \approx 8.4 \text{ ft}}$

STEP 3 Solve for b.

Notice that there are two ways to solve for b: trigonometric functions or the Pythagorean theorem. Although it is tempting to use the Pythagorean theorem, it is better to use the given information with trigonometric functions than to use a value that has already been rounded, which could make results less accurate.

Sine of an angle is equal to the opposite side over the hypotenuse. $\sin 56° = \dfrac{b}{15}$

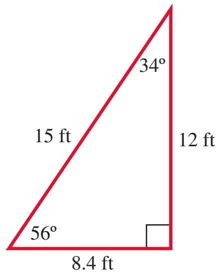
Solve for b. $b = 15\sin 56°$

Evaluate the right side of the expression using a calculator. $b \approx 12.43556$

Round b to two significant digits. $\boxed{b \approx 12 \text{ ft}}$

STEP 4 Check.

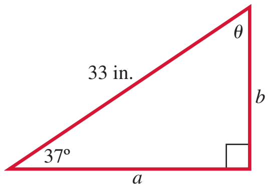
Check the trigonometric values of the specific angles by calculating the trigonometric ratios.



$\sin 34° \stackrel{?}{=} \dfrac{8.4}{15}$ $\qquad$ $\cos 34° \stackrel{?}{=} \dfrac{12}{15}$ $\qquad$ $\tan 34° \stackrel{?}{=} \dfrac{8.4}{12}$

$0.5592 \approx 0.56$ $\qquad$ $0.8290 \approx 0.80$ $\qquad$ $0.6745 \approx 0.70$

■ **YOUR TURN** Solve the right triangle—find a, b, and θ.

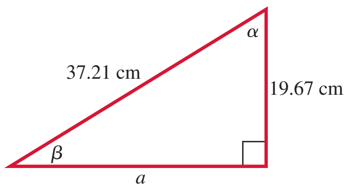


■ **Answer:** $\theta = 53°$, $a \approx 26$ in., $b \approx 20$ in.

Sometimes in solving a right triangle, the side lengths are given and we need to find the angles. To do this, we can work backwards from the table of known values. For example, if $\cos\theta = \frac{1}{2}$, what is θ? We see from the table of exact values that $\theta = 60°$ or $\dfrac{\pi}{3}$. To find an angle that is not listed in the table, we use the inverse cosine function, $\cos^{-1}$, key on a calculator. In a later section, we will discuss inverse trigonometric functions, but for now we will simply use the inverse trigonometric function keys on a calculator to determine the unknown angle.

EXAMPLE 5 Solving a Right Triangle Given Two Sides

Solve the right triangle—find a, α, and β.



Solution:

STEP 1 Solve for α.

Cosine of an angle is equal to the adjacent side over hypotenuse. $\qquad \cos\alpha = \dfrac{19.67 \text{ cm}}{37.21 \text{ cm}}$

Evaluate the right side using a calculator. $\qquad \cos\alpha \approx 0.528621338$

Write the angle α in terms of the inverse cosine function. $\qquad \alpha \approx \cos^{-1} 0.528621338$

Use a calculator to evaluate the inverse cosine function. $\qquad \alpha \approx 58.08764854°$

Round α to the nearest hundredth of a degree. $\qquad \boxed{\alpha \approx 58.09°}$

STEP 2 Solve for β.

The two acute angles in a right triangle are complementary. $\qquad \alpha + \beta = 90°$

Substitute $\alpha \approx 58.09°$. $\qquad 58.09 + \beta \approx 90°$

Solve for β. $\qquad \boxed{\beta \approx 31.91°}$

The answer is already rounded to the nearest hundredth of a degree.

Technology Tip

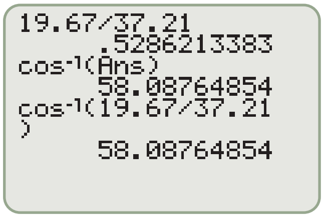
To find $\cos^{-1}\left(\dfrac{19.67}{37.21}\right)$, first press [19.67] [÷] [37.21] [ENTER].

Next, press [2nd] [COS] for [COS^{-1}] and [2nd] [(−)] for the answer, [ANS]

[COS^{-1}] [ANS] [)] [ENTER].

```
19.67/37.21
        .5286213383
cos-1(Ans)
        58.08764854
```

Or enter the entire expression as $\cos^{-1}\left(\dfrac{19.67}{37.21}\right)$. Press [2nd] [COS] [19.67] [÷] [37.21] [)] [ENTER].



Using a scientific calculator, press [cos⁻¹] [(] [19.67] [÷] [37.21] [)] [ENTER].

Step 3 **Solve for *a*.**

Use the Pythagorean theorem. $a^2 + b^2 = c^2$

Substitute the given values for b and c. $a^2 + 19.67^2 = 37.21^2$

Solve for a. $a \approx 31.5859969$

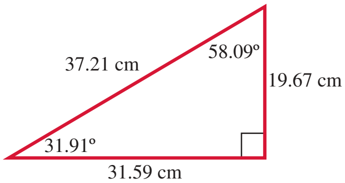
Round a to four significant digits. $\boxed{a \approx 31.59 \text{ cm}}$

Step 4 **Check.**

Check the trigonometric values of the specific angles by calculating the trigonometric ratio.

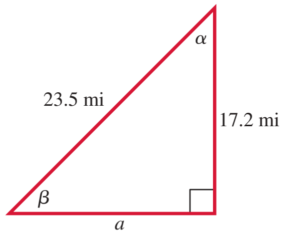
$$\sin 31.91° \stackrel{?}{=} \frac{19.67}{37.21} \qquad \sin 58.09° \stackrel{?}{=} \frac{31.59}{37.21}$$

$$0.5286 \approx 0.5286 \qquad 0.8489 \approx 0.8490$$



■ **Answer:** $a \approx 16.0$ mi, $\alpha \approx 43.0°$, $\beta \approx 47.0°$

■ **YOUR TURN** Solve the right triangle—find a, α, and β.



Study Tip

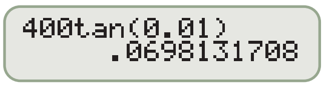
To find a length in a right triangle, use the sine, cosine, or tangent function. To find an angle measure in a right triangle, given the proper ratio of side lengths, use the inverse sine, inverse cosine, or inverse tangent function.

Applications

Suppose NASA wants to talk with the International Space Station (ISS), which is traveling at a speed of 17,700 miles per hour (7900 meters per second), 400 kilometers (250 miles) above the surface of Earth. If the antennas at the ground station in Houston have a pointing error of even 1/100 of a degree, that is, 0.01°, the ground station will miss the chance to talk with the astronauts.

Technology Tip

To calculate 400 tan 0.01°, press

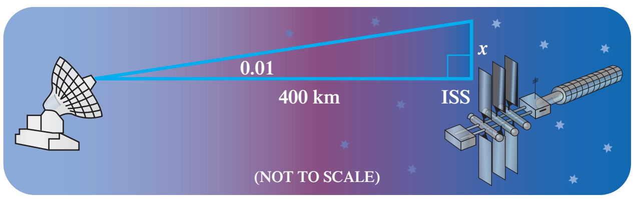


EXAMPLE 6 Pointing Error

Assume that the ISS (which is 108 meters long and 73 meters wide) is in a 400-kilometer low Earth orbit. If the communications antennas have a 0.01° pointing error, how many meters off will the communications link be?

Solution:

Draw a right triangle that depicts this scenario.



Identify the tangent ratio. $\tan 0.01° = \dfrac{x}{400}$

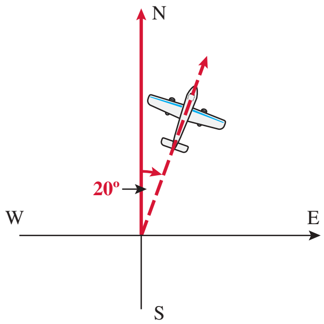
Solve for x. $x = (400 \text{ km}) \tan 0.01°$

Evaluate the expression on the right. $x \approx 0.06981317$ km

400 kilometers is accurate to three significant digits, so we express the answer to three significant digits.

The pointing error causes the signal to be off by $\boxed{69.8 \text{ meters}}$. Since the ISS is only 108 meters long, it is possible that the signal will be missed by the astronaut crew.

In navigation, the word **bearing** means the direction in which a vessel is pointed. **Heading** is the direction in which the vessel is actually traveling. Heading and bearing are only synonyms when there is no wind. Direction is often given as a bearing, which is the measure of an acute angle with respect to the north–south vertical line. "The plane has a bearing of N 20° E" means that the plane is pointed 20° to the east of due north.

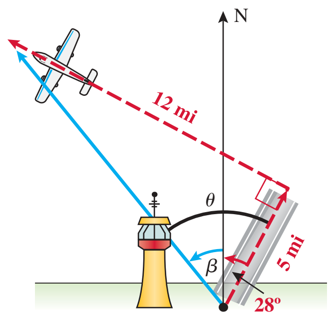


EXAMPLE 7 Bearing (Navigation)

A jet takes off bearing N 28° E and flies 5 miles, and then makes a left (90°) turn and flies 12 miles farther. If the control tower operator wants to locate the plane, what bearing should she use?

Solution:

Draw a picture that represents this scenario.



Identify the tangent ratio. $\tan\theta = \dfrac{12}{5}$

Use the inverse tangent function to solve for θ. $\theta = \tan^{-1}\left(\dfrac{12}{5}\right) \approx 67.4°$

Subtract 28° from θ to find the bearing, β. $\beta \approx 67.4° - 28° \approx 39.4°$

Round to the nearest degree. $\boxed{\beta \approx \text{N } 39° \text{ W}}$

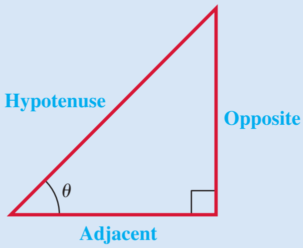
SECTION 4.2 SUMMARY

The trigonometric functions defined in terms of ratios of side lengths of right triangles are given by

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



And the remaining three trigonometric functions can be found using the reciprocal identities:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sin\theta \neq 0$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta \neq 0$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta \neq 0$$

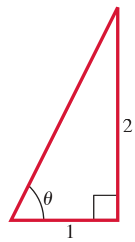
The following table lists the values of the sine and cosine functions for special acute angles:

θ (DEGREES)	θ (RADIANS)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

SECTION 4.2 EXERCISES

SKILLS

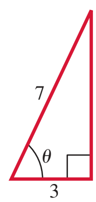
In Exercises 1–6, refer to the triangle in the drawing to find the indicated trigonometric function values. Rationalize any denominators containing radicals that you encounter in the answers.



1. $\cos\theta$ **2.** $\sin\theta$ **3.** $\sec\theta$

4. $\csc\theta$ **5.** $\tan\theta$ **6.** $\cot\theta$

For Exercises 7–12, refer to the triangle in the drawing to find the indicated trigonometric function values. Rationalize any denominators containing radicals that you encounter in the answers.



7. $\sin\theta$ **8.** $\cos\theta$ **9.** $\sec\theta$

10. $\csc\theta$ **11.** $\cot\theta$ **12.** $\tan\theta$

In Exercises 13–18, match the trigonometric function values.

a. $\frac{1}{2}$ **b.** $\frac{\sqrt{3}}{2}$ **c.** $\frac{\sqrt{2}}{2}$

13. $\sin 30°$ **14.** $\sin 60°$ **15.** $\cos\left(\frac{\pi}{6}\right)$ **16.** $\cos\left(\frac{\pi}{3}\right)$ **17.** $\sin 45°$ **18.** $\cos\left(\frac{\pi}{4}\right)$

In Exercises 19–21, use the results in Exercises 13–18 and the trigonometric quotient identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to calculate the following values:

19. $\tan 30°$ **20.** $\tan\left(\frac{\pi}{4}\right)$ **21.** $\tan 60°$

In Exercises 22–30, use the results in Exercises 13–21 and the reciprocal identities $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, and $\cot\theta = \dfrac{1}{\tan\theta}$ to calculate the following values:

22. $\csc 30°$ **23.** $\sec 30°$ **24.** $\cot\left(\frac{\pi}{6}\right)$ **25.** $\csc\left(\frac{\pi}{3}\right)$ **26.** $\sec 60°$ **27.** $\cot 60°$

28. $\csc 45°$ **29.** $\sec\left(\frac{\pi}{4}\right)$ **30.** $\cot\left(\frac{\pi}{4}\right)$

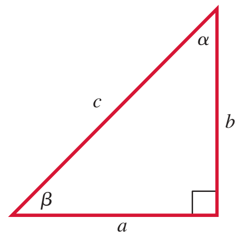
In Exercises 31–46, use a calculator to evaluate the trigonometric functions for the indicated angle values. Round your answers to four decimal places.

31. $\sin 37°$ **32.** $\sin 17.8°$ **33.** $\cos 82°$ **34.** $\cos 21.9°$ **35.** $\sin\left(\frac{\pi}{12}\right)$ **36.** $\sin\left(\frac{5\pi}{9}\right)$

37. $\cos\left(\frac{6\pi}{5}\right)$ **38.** $\cos\left(\frac{13\pi}{7}\right)$ **39.** $\tan 54°$ **40.** $\tan 43.2°$ **41.** $\tan\left(\frac{\pi}{8}\right)$ **42.** $\cot\left(\frac{3\pi}{5}\right)$

43. $\csc\left(\frac{10\pi}{19}\right)$ **44.** $\sec\left(\frac{4\pi}{9}\right)$ **45.** $\cot 55°$ **46.** $\cot 29°$

In Exercises 47–54, refer to the right triangle diagram and the given information to find the indicated measure. Write your answers for angle measures in decimal degrees.



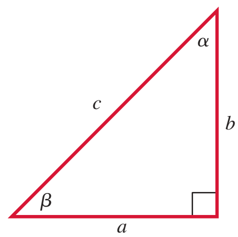
47. $\alpha = 55°$, $c = 22$ ft; find a. **48.** $\alpha = 55°$, $c = 22$ ft; find b.

49. $\alpha = 20.5°$, $b = 14.7$ mi; find a. **50.** $\beta = 69.3°$, $a = 0.752$ mi; find b.

51. $\beta = 25°$, $a = 11$ km; find c. **52.** $\beta = 75°$, $b = 26$ km; find c.

53. $b = 2.3$ m, $c = 4.9$ m; find α. **54.** $b = 7.8$ m, $c = 13$ m; find β.

In Exercises 55–66, refer to the right triangle diagram and the given information to solve the right triangle. Write your answers for angle measures in decimal degrees.



55. $\alpha = 32°$ and $c = 12$ ft **56.** $\alpha = 65°$ and $c = 37$ ft

57. $\beta = 72°$ and $c = 9.7$ mm **58.** $\beta = 45°$ and $c = 7.8$ mm

59. $\alpha = 54.2°$ and $a = 111$ mi **60.** $\beta = 47.2°$ and $a = 9.75$ mi

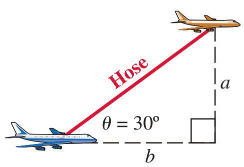
61. $a = 42.5$ ft and $b = 28.7$ ft **62.** $a = 19.8$ ft and $c = 48.7$ ft

63. $a = 35{,}236$ km and $c = 42{,}766$ km **64.** $b = 0.1245$ mm and $c = 0.8763$ mm

65. $\beta = 25.4°$ and $b = 11.6$ in. **66.** $\beta = 39.21°$ and $b = 6.3$ m

APPLICATIONS

Exercises 67 and 68 illustrate a mid-air refueling scenario that military aircraft often enact. Assume the elevation angle that the hose makes with the plane being fueled is $\theta = 30°$.

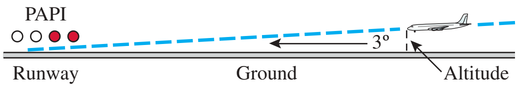


67. Mid-Air Refueling. If the hose is 150 feet long, what should be the altitude difference a between the two planes? Round to the nearest foot.

68. Mid-Air Refueling. If the smallest acceptable altitude difference a between the two planes is 100 feet, how long should the hose be? Round to the nearest foot.

Exercises 69–72 are based on the idea of a glide slope (the angle the flight path makes with the ground).

Precision Approach Path Indicator (PAPI) lights are used as a visual approach slope aid for pilots landing aircraft. A typical glide path for commercial jet airliners is 3°. The space shuttle has an outer glide approach of 18°–20°. PAPI lights are typically configured as a row of four lights. All four lights are on, but in different combinations of red or white. If all four lights are white, then the angle of descent is too high; if all four lights are red, then the angle of descent is too low; and if there are two white and two red, then the approach is perfect.



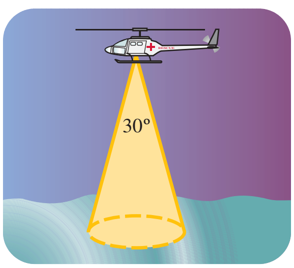
69. Glide Path of a Commercial Jet Airliner. If a commercial jetliner is 5000 feet (about 1 mile) ground distance from the runway, what should the altitude of the plane be to achieve two red and two white PAPI lights? (Assume this corresponds to a 3° glide path.)

70. Glide Path of a Commercial Jet Airliner. If a commercial jetliner is at an altitude of 450 feet when it is 5200 feet from the runway (approximately 1 mile ground distance), what is the glide slope angle? Will the pilot see white lights, red lights, or both?

71. Glide Path of the Space Shuttle *Orbiter*. If the pilot of the space shuttle *Orbiter* is at an altitude of 3000 feet when she is 15,500 feet (approximately 3 miles) from the shuttle landing facility (ground distance), what is her glide slope angle (round to the nearest degree)? Is she too high or too low?

72. Glide Path of the Space Shuttle *Orbiter*. If the same pilot in Exercise 71 raises the nose of the gliding shuttle so that she drops only 500 feet by the time she is 7800 feet from the shuttle landing strip (ground distance), what is her glide angle then (round to the nearest degree)? Is she within the specs to land the shuttle?

In Exercises 73 and 74, refer to the illustration below, which shows a search and rescue helicopter with a 30° field of view with a searchlight.



73. Search and Rescue. If the search and rescue helicopter is flying at an altitude of 150 feet above sea level, what is the diameter of the circle illuminated on the surface of the water?

74. Search and Rescue. If the search and rescue helicopter is flying at an altitude of 500 feet above sea level, what is the diameter of the circle illuminated on the surface of the water?

For Exercises 75–78, refer to the following:

Geostationary orbits are useful because they cause a satellite to appear stationary with respect to a fixed point on the rotating Earth. As a result, an antenna (dish TV) can point in a fixed direction and maintain a link with the satellite. The satellite orbits in the direction of Earth's rotation at an altitude of approximately 35,000 kilometers.

75. Dish TV. If your dish TV antenna has a pointing error of 0.000278°, how long would the satellite have to be in order to maintain a link? Round your answer to the nearest meter.

76. Dish TV. If your dish TV antenna has a pointing error of 0.000139°, how long would the satellite have to be in order to maintain a link? Round your answer to the nearest meter.

77. Dish TV. If the satellite in a geostationary orbit (at 35,000 kilometers) was only 10 meters long, about how accurate would the pointing of the dish have to be? Give the answer in degrees to two significant digits.

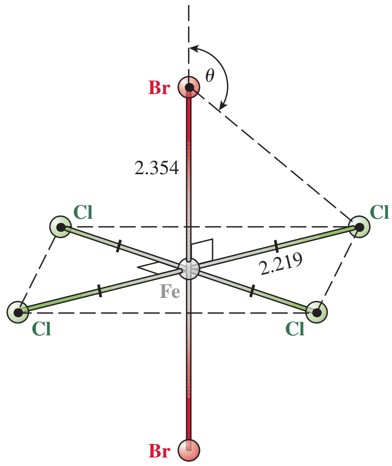
78. Dish TV. If the satellite in a geostationary orbit (at 35,000 kilometers) was only 30 meters long, about how accurate would the pointing of the dish have to be? Give the answer in degrees to two significant digits.

79. Angle of Inclination (Skiing). The angle of inclination of a mountain with triple black diamond ski trails is 65°. If a skier at the top of the mountain is at an elevation of 4000 feet, how long is the ski run from the top to the base of the mountain? Round to the nearest foot.

80. Bearing (Navigation). If a plane takes off bearing N 33° W and flies 6 miles and then makes a right (90°) turn and flies 10 miles further, what bearing will the traffic controller use to locate the plane?

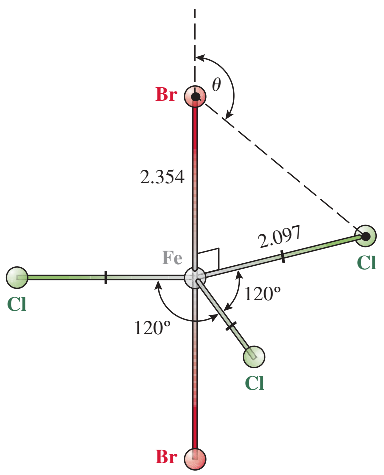
For Exercises 81 and 82, refer to the following:

The structure of molecules is critical to the study of materials science and organic chemistry, and has countless applications to a variety of interesting phenomena. Trigonometry plays a critical role in determining the bonding angles of molecules. For instance, the structure of the $(FeCl_4Br_2)^{-3}$ ion (dibromatetetrachlorideferrate III) is shown in the figure below.

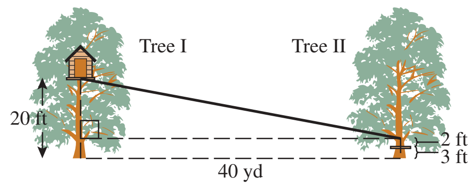


81. Chemistry. Determine the angle θ [i.e., the angle between the axis containing the apical bromide atom (Br) and the segment connecting Br to Cl].

82. Chemistry. Now, suppose one of the chlorides (Cl) is removed. The resulting structure is triagonal in nature, resulting in the following structure. Does the angle θ change? If so, what is its new value?



83. Construction. Two neighborhood kids are planning to build a treehouse in Tree I, and connect it with a zipline to Tree II that is 40 yards away. The base of the treehouse will be 20 feet above the ground, and a platform will be nailed into Tree II, 3 feet above the ground. The plan is to connect the base of the treehouse on Tree I to an anchor 2 feet above the platform on Tree II.



How much zipline (in feet) will they need? Round your answer to the nearest foot.

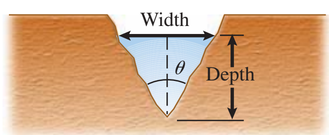
84. Construction. In Exercise 83, what is the angle of depression β that the zipline makes with Tree I? Express your answer in two significant digits.

85. Construction. A pool that measures 5 feet above the ground is to be placed between the trees directly in the path of the zipline in Exercise 83. Assuming that a rider of the zipline dangles at most 3 feet below the wire anywhere in route, what is the closest the edge of the pool can be placed to tree 2 so that the pool will not impede a rider's trip down the zipline?

86. Construction. If the treehouse is to be built so that its base is now 22 feet above the base of tree 1, where should the anchor on tree 2 (to which the zipline is connected) be placed in order to ensure the same angle of depression found in Exercise 84?

For Exercises 87 and 88, refer to the following:

A canal constructed by a water-users association can be approximated by an isosceles triangle (see the figure below). When the canal was originally constructed, the depth of the canal was 5.0 feet and the angle defining the shape of the canal was 60°.

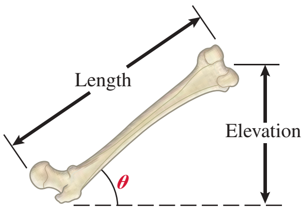


87. Environmental Science. If the width of the water surface today is 4.0 feet, find the depth of the water running through the canal.

88. Environmental Science. One year later a survey is performed to measure the effects of erosion on the canal. It is determined that when the water depth is 4.0 feet, the width of the water surface is 5.0 feet. Find the angle θ defining the shape of the canal to the nearest degree. Has erosion affected the shape of the canal? Explain.

For Exercises 89 and 90, refer to the following:

After breaking a femur, a patient is placed in traction. The end of a femur of length l is lifted to an elevation forming an angle θ with the horizontal (angle of elevation).



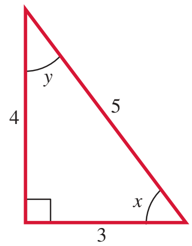
89. Health/Medicine. A femur 18 inches long is placed into traction, forming an angle of 15° with the horizontal. Find the height of elevation at the end of the femur.

90. Health/Medicine. A femur 18 inches long is placed in traction with an elevation of 6.2 inches. What is the angle of elevation of the femur?

CATCH THE MISTAKE

For Exercises 91–94, explain the mistake that is made.

For the triangle in the drawing, calculate the indicated trigonometric function values.



91. Calculate $\sin y$.

Solution:

Formulate sine in terms of trigonometric ratios. $\sin y = \dfrac{\text{opposite}}{\text{hypotenuse}}$

The opposite side is 4, and the hypotenuse is 5. $\sin y = \dfrac{4}{5}$

This is incorrect. What mistake was made?

92. Calculate $\tan x$.

Solution:

Formulate tangent in terms of trigonometric ratios. $\tan x = \dfrac{\text{adjacent}}{\text{opposite}}$

The adjacent side is 3, and the opposite side is 4. $\tan x = \dfrac{3}{4}$

This is incorrect. What mistake was made?

93. Calculate $\sec x$.

Solution:

Formulate sine in terms of trigonometric ratios. $\sin x = \dfrac{\text{opposite}}{\text{hypotenuse}}$

The opposite side is 4, and the hypotenuse is 5. $\sin x = \dfrac{4}{5}$

Write secant as the reciprocal of sine. $\sec x = \dfrac{1}{\sin x}$

Simplify. $\sec x = \dfrac{1}{\frac{4}{5}} = \dfrac{5}{4}$

This is incorrect. What mistake was made?

94. Calculate $\csc y$.

Solution:

Formulate cosine in terms of trigonometric ratios. $\cos y = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

The adjacent side is 4, and the hypotenuse is 5. $\cos y = \dfrac{4}{5}$

Write cosecant as the reciprocal of cosine. $\csc y = \dfrac{1}{\cos y}$

Simplify. $\csc y = \dfrac{1}{\frac{4}{5}} = \dfrac{5}{4}$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 95–98, determine whether each statement is true or false.

95. If you are given the measures of two sides of a right triangle, you can solve the right triangle.

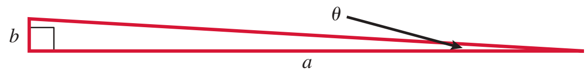
96. If you are given the measures of one side and one acute angle of a right triangle, you can solve the right triangle.

97. If you are given the two acute angles of a right triangle, you can solve the right triangle.

98. If you are given the hypotenuse of a right triangle and the angle opposite the hypotenuse, you can solve the right triangle.

In Exercises 99–102, use trignometric ratios and the assumption that *a* is much larger than *b*.

Thus far, in this text we have discussed trigonometric values only for acute angles, or for $0° < \theta < 90°$. How do we determine these values when θ is approximately 0° or 90°? We will formally consider these cases in the next section, but for now, draw and label a right triangle that has one angle very close to 0°, so that the opposite side is very small compared to the adjacent side. Then the hypotenuse and the adjacent side will be very close to the same length.



99. Approximate $\sin 0°$ without using a calculator.

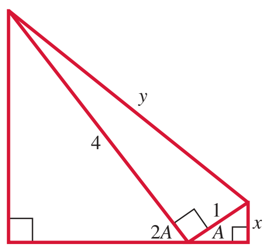
100. Approximate $\cos 0°$ without using a calculator.

101. Approximate $\cos 90°$ without using a calculator.

102. Approximate $\sin 90°$ without using a calculator.

CHALLENGE

For Exercises 103 and 104, consider the following diagram:



103. Determine x.

104. Determine y.

TECHNOLOGY

105. Calculate $\sec 70°$ in the following two ways:

a. Find $\cos 70°$ to three decimal places and then divide 1 by that number. Write that number to five decimal places.

b. With a calculator in degree mode, enter 70, cos, 1/x, and round the result to five decimal places.

106. Calculate $\csc 40°$ in the following two ways:

a. Find $\sin 40°$ to three decimal places and then divide 1 by that number. Write this last result to five decimal places.

b. With a calculator in degree mode, enter 40, sin, 1/x, and round the result to five decimal places.

107. Calculate $\cot 54.9°$ in the following two ways:

a. Find $\tan 54.9°$ to three decimal places and then divide 1 by that number. Write that number to five decimal places.

b. With a calculator in degree mode, enter 54.9, tan, 1/x, and round the result to five decimal places.

108. Calculate $\sec 18.6°$ in the following two ways:

a. Find $\cos 18.6°$ to three decimal places and then divide 1 by that number. Write that number to five decimal places.

b. With a calculator in degree mode, enter 18.6, cos, 1/x, and round the result to five decimal places.

PREVIEW TO CALCULUS

In calculus, the value of $F(b) - F(a)$ of a function $F(x)$ at $x = a$ and $x = b$ plays an important role in the calculation of definite integrals. In Exercises 109–112, find the exact value of $F(b) - F(a)$.

109. $F(x) = \sec x, a = \frac{\pi}{6}, b = \frac{\pi}{3}$

110. $F(x) = \sin^3 x, a = 0, b = \frac{\pi}{4}$

111. $F(x) = \tan x + 2\cos x, a = 0, b = \frac{\pi}{3}$

112. $F(x) = \dfrac{\cot x - 4\sin x}{\cos x}, a = \frac{\pi}{4}, b = \frac{\pi}{3}$

SECTION

4.3 TRIGONOMETRIC FUNCTIONS OF ANGLES

SKILLS OBJECTIVES

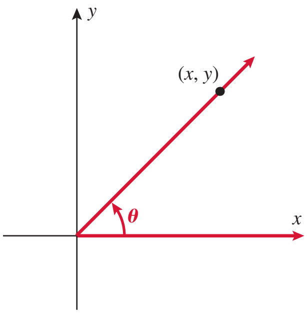
- Calculate trigonometric function values for nonacute angles.
- Determine the reference angle of a nonacute angle.
- Calculate the trigonometric function values for quadrantal angles.

CONCEPTUAL OBJECTIVES

- Understand that right triangle ratio definitions of trigonometric functions for acute angles are consistent with definitions of trigonometric functions for all angles in the Cartesian plane.
- Understand why some trigonometric functions are undefined for quadrantal angles.
- Understand that the ranges of the sine and cosine functions are bounded, whereas the ranges for the other four trigonometric functions are unbounded.

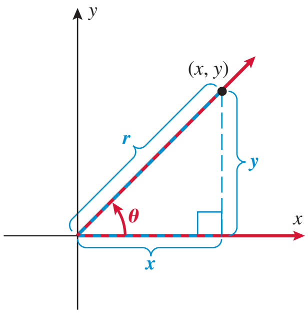
In Section 4.2, we defined trigonometric functions as ratios of side lengths of right triangles. This definition holds only for acute ($0° < \theta < 90°$) angles, since the two angles in a right triangle other than the right angle must be acute. We now define trigonometric functions as ratios of x- and y-coordinates and distances in the Cartesian plane, which for acute angles is consistent with right triangle trigonometry. However, this second approach also enables us to formulate trigonometric functions for quadrantal angles (whose terminal side lies along an axis) and nonacute angles.

Trigonometric Functions: The Cartesian Plane



To define the trigonometric functions in the Cartesian plane, let us start with an acute angle θ in standard position. Choose any point (x, y) on the terminal side of the angle as long as it is not the vertex (the origin).

A right triangle can be drawn so that the right angle is made when a perpendicular segment connects the point (x, y) to the x-axis. Notice that the side opposite $\boldsymbol{\theta}$ has length $\boldsymbol{y}$ and the other leg of the right triangle has length x.



The distance $\boldsymbol{r}$ from the origin $(0, 0)$ to the point (x, y) can be found using the distance formula.

$$r = \sqrt{(x - 0)^2 + (y - 0)^2}$$

$$r = \sqrt{x^2 + y^2}$$

Since r is a distance and x and y are not both zero, r is always positive. $\qquad r > 0$

Using our first definition of trigonometric functions in terms of right triangle ratios (Section 4.2), we know that $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$. From this picture, we see that sine can also be defined by the relation $\sin\theta = \dfrac{y}{r}$. Similar reasoning holds for all six trigonometric functions and leads us to the second definition of the trigonometric functions, in terms of ratios of coordinates of a point and distances in the Cartesian plane.

DEFINITION Trigonometric Functions

Let (x, y) be a point, other than the origin, on the terminal side of an angle θ in standard position. Let r be the distance from the point (x, y) to the origin. Then the six trigonometric functions are defined as

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

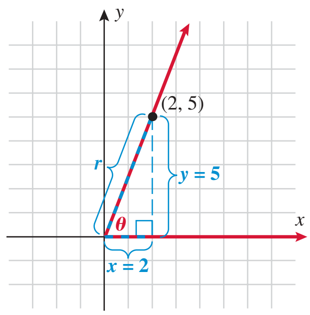
where $r = \sqrt{x^2 + y^2}$, or $x^2 + y^2 = r^2$. The distance r is positive: $r > 0$.

EXAMPLE 1 Calculating Trigonometric Function Values for Acute Angles

The terminal side of an angle θ in standard position passes through the point $(2, 5)$. Calculate the values of the six trigonometric functions for angle θ.

Solution:

STEP 1 Draw the angle and label the point (2, 5).



STEP 2 Calculate the distance *r*.

$$r = \sqrt{2^2 + 5^2} = \sqrt{29}$$

STEP 3 Formulate the trigonometric functions in terms of *x*, *y*, and *r*.

Let $x = 2, y = 5, r = \sqrt{29}$.

$$\sin\theta = \frac{y}{r} = \frac{5}{\sqrt{29}} \qquad \cos\theta = \frac{x}{r} = \frac{2}{\sqrt{29}} \qquad \tan\theta = \frac{y}{x} = \frac{5}{2}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{29}}{5} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{29}}{2} \qquad \cot\theta = \frac{x}{y} = \frac{2}{5}$$

STEP 4 Rationalize any denominators containing a radical.

$$\sin\theta = \frac{5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{5\sqrt{29}}{29} \qquad \cos\theta = \frac{2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}} = \frac{2\sqrt{29}}{29}$$

STEP 5 Write the values of the six trigonometric functions for θ.

$$\boxed{\sin\theta = \frac{5\sqrt{29}}{29}} \qquad \boxed{\cos\theta = \frac{2\sqrt{29}}{29}} \qquad \boxed{\tan\theta = \frac{5}{2}}$$

$$\boxed{\csc\theta = \frac{\sqrt{29}}{5}} \qquad \boxed{\sec\theta = \frac{\sqrt{29}}{2}} \qquad \boxed{\cot\theta = \frac{2}{5}}$$

Note: In Example 1, we could have used the values of the sine, cosine, and tangent functions along with the reciprocal identities to calculate the cosecant, secant, and cotangent function values.

Study Tip

There is no need to memorize definitions for secant, cosecant, and cotangent functions, since their values can be derived from the reciprocals of the sine, cosine, and tangent function values.

YOUR TURN The terminal side of an angle θ in standard position passes through the point $(3, 7)$. Calculate the values of the six trigonometric functions for angle θ.

Answer:

$\sin\theta = \frac{7\sqrt{58}}{58}$ $\quad\cos\theta = \frac{3\sqrt{58}}{58}$

$\tan\theta = \frac{7}{3}$ $\quad\csc\theta = \frac{\sqrt{58}}{7}$

$\sec\theta = \frac{\sqrt{58}}{3}$ $\quad\cot\theta = \frac{3}{7}$

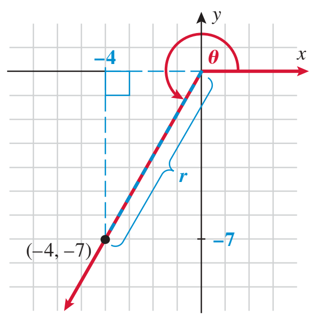
We can now find values for nonacute angles (angles with measure greater than or equal to $90°$) as well as negative angles.

EXAMPLE 2 Calculating Trigonometric Function Values for Nonacute Angles

The terminal side of an angle θ in standard position passes through the point $(-4, -7)$. Calculate the values of the six trigonometric functions for angle θ.

Solution:

STEP 1 Draw the angle and label the point $(-4, -7)$.



STEP 2 Calculate the distance r. $\quad r = \sqrt{(-4)^2 + (-7)^2} = \sqrt{65}$

STEP 3 Formulate the trigonometric functions in terms of x, y, and r.

Let $x = -4$, $y = -7$, and $r = \sqrt{65}$.

$$\sin\theta = \frac{y}{r} = \frac{-7}{\sqrt{65}} \qquad \cos\theta = \frac{x}{r} = \frac{-4}{\sqrt{65}} \qquad \tan\theta = \frac{y}{x} = \frac{-7}{-4} = \frac{7}{4}$$

$$\csc\theta = \frac{r}{y} = \frac{\sqrt{65}}{-7} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{65}}{-4} \qquad \cot\theta = \frac{x}{y} = \frac{-4}{-7} = \frac{4}{7}$$

STEP 4 **Rationalize the radical denominators in the sine and cosine functions.**

$$\sin\theta = \frac{y}{r} = \frac{-7}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}} = -\frac{7\sqrt{65}}{65}$$

$$\cos\theta = \frac{x}{r} = \frac{-4}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}} = -\frac{4\sqrt{65}}{65}$$

STEP 5 **Write the values of the six trigonometric functions for θ.**

$$\boxed{\sin\theta = -\frac{7\sqrt{65}}{65}} \qquad \boxed{\cos\theta = -\frac{4\sqrt{65}}{65}} \qquad \boxed{\tan\theta = \frac{7}{4}}$$

$$\boxed{\csc\theta = -\frac{\sqrt{65}}{7}} \qquad \boxed{\sec\theta = -\frac{\sqrt{65}}{4}} \qquad \boxed{\cot\theta = \frac{4}{7}}$$

YOUR TURN The terminal side of an angle θ in standard position passes through the point $(-3, -5)$. Calculate the values of the six trigonometric functions for angle θ.

Answer:

$\sin\theta = -\frac{5\sqrt{34}}{34}$ $\quad\cos\theta = -\frac{3\sqrt{34}}{34}$

$\tan\theta = \frac{5}{3}$ $\quad\csc\theta = -\frac{\sqrt{34}}{5}$

$\sec\theta = -\frac{\sqrt{34}}{3}$ $\quad\cot\theta = \frac{3}{5}$

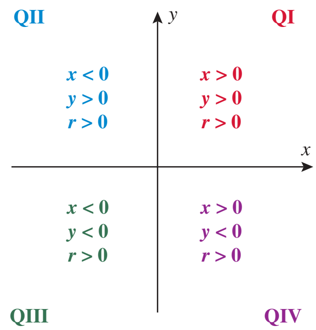
Algebraic Signs of Trigonometric Functions

We have defined trigonometric functions as ratios of x, y, and r. Since r is the distance from the origin to the point (x, y) and distance is never negative, r is always taken as the positive solution to $r^2 = x^2 + y^2$, so $r = \sqrt{x^2 + y^2}$.

The x-coordinate is positive in quadrants **I** and **IV** and negative in quadrants **II** and **III**.

The y-coordinate is positive in quadrants **I** and **II** and negative in quadrants **III** and **IV**.

Recall the definition of the six trigonometric functions in the Cartesian plane:



$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

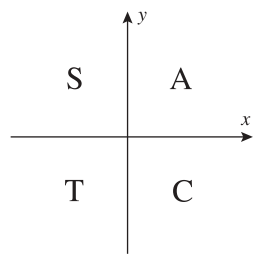
$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

Therefore, the algebraic sign, $+$ or $-$, of each trigonometric function will depend on which quadrant contains the terminal side of angle θ. Let us look at the three main trigonometric functions: sine, cosine, and tangent. In quadrant I, all three functions are positive since x, y, and r are all positive. However, in quadrant II, only sine is positive since y and r are both positive. In quadrant III, only tangent is positive, and in quadrant IV, only cosine is positive. The expression "**A**ll **S**tudents **T**ake **C**alculus" helps us remember which of the three main trigonometric functions are positive in each quadrant.

PHRASE	QUADRANT	POSITIVE TRIGONOMETRIC FUNCTION
All	I	**A**ll three: sine, cosine, and tangent
Students	II	**S**ine
Take	III	**T**angent
Calculus	IV	**C**osine

Study Tip

All **S**tudents **T**ake **C**alculus is an expression that helps us remember which of the three (sine, cosine, tangent) functions are positive in quadrants I, II, III, and IV.



Positivity Chart

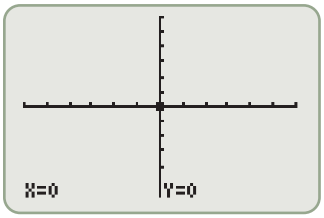
The following table indicates the algebraic sign of all six trigonometric functions according to the quadrant in which the terminal side of an angle θ lies. Notice that the reciprocal functions have the same sign.

Terminal Side of θ in Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

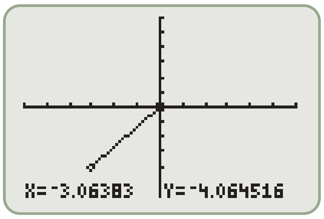
Technology Tip

To draw the terminal side of the angle θ in quadrant III through the point $(-3, -4)$, press GRAPH 2nd: DRAW ▼ 2:Line(ENTER.

Now use arrows to move the cursor to the origin and press ENTER.



Use ◀ to move the cursor to the left to about $x = -3$ and press ENTER. Use ▼ to move the cursor down to about $y = -4$ and press ENTER.



■ **Answer:** $\cos\theta = -\dfrac{\sqrt{7}}{4}$

EXAMPLE 3 Evaluating a Trigonometric Function When One Trigonometric Function Value and the Quadrant of the Terminal Side Is Known

If $\cos\theta = -\frac{3}{5}$ and the terminal side of angle θ lies in quadrant III, find $\sin\theta$.

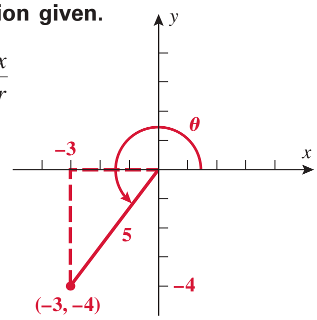
Solution:

Step 1 Draw some angle θ in quadrant III.

Step 2 Identify known quantities from the information given.

Recall that $\cos\theta = \dfrac{x}{r}$ and $r > 0$. $\qquad \cos\theta = -\dfrac{3}{5} = \dfrac{-3}{5} = \dfrac{x}{r}$

Identify x and r. $\qquad x = -3$ and $r = 5$



Step 3 Since x and r are known, find y.

Substitute $x = -3$ and $r = 5$ into $x^2 + y^2 = r^2$. $\qquad (-3)^2 + y^2 = 5^2$

Solve for y. $\qquad 9 + y^2 = 25$

$$y^2 = 16$$

$$y = \pm 4$$

Step 4 Select the sign of y based on quadrant information.

Since the terminal side of angle θ lies in quadrant III, $y < 0$. $\qquad y = -4$

Step 5 Find $\sin\theta$.

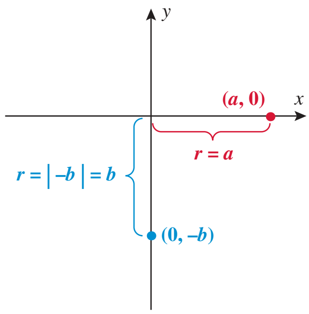
$$\sin\theta = \frac{y}{r} = \frac{-4}{5}$$

$$\boxed{\sin\theta = -\frac{4}{5}}$$

■ **YOUR TURN** If $\sin\theta = -\frac{3}{4}$ and the terminal side of angle θ lies in quadrant III, find $\cos\theta$.

We can also make a table showing the values of the trigonometric functions when the terminal side of angle θ lies along each axis (i.e., when θ is any of the quadrantal angles).

When the terminal side lies along the x-axis, then $y = 0$. When $y = 0$, notice that $r = \sqrt{x^2 + y^2} = \sqrt{x^2} = |x|$. When the terminal side lies along the positive x-axis, $x > 0$; and when the terminal side lies along the negative x-axis, $x < 0$. Therefore, when the terminal side of the angle lies on the positive x-axis, then $y = 0$, $x > 0$, and $r = x$; and when the terminal side lies along the negative x-axis, then $y = 0$, $x < 0$, and $r = |x|$. A similar argument can be made when the terminal side lies along the y-axis, which results in $r = |y|$.



Terminal Side of θ Lies Along the	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
Positive x-axis (e.g., $0°$ or $360°$ or 0 or 2π)	0	1	0	undefined	1	undefined
Positive y-axis $\left(\text{e.g., } 90° \text{ or } \frac{\pi}{2}\right)$	1	0	undefined	0	undefined	1
Negative x-axis (e.g., $180°$ or π)	0	-1	0	undefined	-1	undefined
Negative y-axis $\left(\text{e.g., } 270° \text{ or } \frac{3\pi}{2}\right)$	-1	0	undefined	0	undefined	-1

EXAMPLE 4 Working with Values of the Trigonometric Functions for Quadrantal Angles

Evaluate each of the following expressions, if possible:

a. $\cos 540° + \sin 270°$ **b.** $\cot\left(\frac{\pi}{2}\right) + \tan\left(-\frac{\pi}{2}\right)$

Solution (a):

The terminal side of an angle with measure 540° lies along the negative x-axis. $540° - 360° = 180°$

Evaluate cosine of an angle whose terminal side lies along the negative x-axis. $\cos 540° = -1$

Evaluate sine of an angle whose terminal side lies along the negative y-axis. $\sin 270° = -1$

Sum the sine and cosine values. $\cos 540° + \sin 270° = -1 + (-1)$

$$\cos 540° + \sin 270° = \boxed{-2}$$

Check: Evaluate this expression with a calculator.

Solution (b):

Evaluate cotangent of an angle whose terminal side lies along the positive y-axis. $\cot\left(\frac{\pi}{2}\right) = 0$

The terminal side of an angle with measure $-\frac{\pi}{2}$ lies along the negative y-axis. $\tan\left(-\frac{\pi}{2}\right) = \tan\left(\frac{3\pi}{2}\right)$

The tangent function is undefined for an angle whose terminal side lies along the negative y-axis. $\tan\left(-\frac{\pi}{2}\right)$ is $\boxed{\text{undefined}}$

Even though $\cot\left(\frac{\pi}{2}\right)$ is defined, since $\tan\left(-\frac{\pi}{2}\right)$ is undefined, the sum of the two expressions is also undefined.

■ **YOUR TURN** Evaluate each of the following expressions, if possible:

a. $\csc\left(\frac{\pi}{2}\right) + \sec \pi$ **b.** $\csc(-630°) + \sec(-630°)$

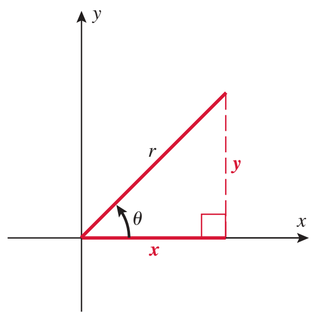
■ **Answer: a.** 0 **b.** undefined

Ranges of the Trigonometric Functions

Thus far, we have discussed what the algebraic sign of a trigonometric function value for an angle in a particular quadrant, but we haven't discussed how to find actual values of the trigonometric functions for nonacute angles. We will need to define *reference angles* and reference right triangles. However, before we proceed, let's get a feel for the ranges (set of values of the functions) we will expect.

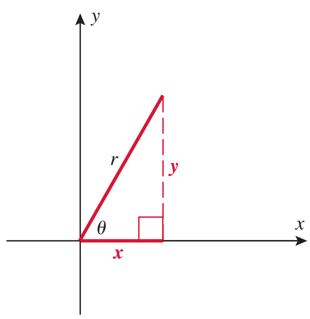
Let us start with an angle θ in quadrant I and the sine function defined as the ratio $\sin\theta = \frac{y}{r}$.

$$\sin\theta = \frac{y}{r}$$



If we keep the value of r constant, then as the measure of θ increases toward 90° or $\frac{\pi}{2}$, y increases. Notice that the value of y approaches the value of r until they are equal when $\theta = 90°\left(\text{or } \frac{\pi}{2}\right)$, and y can never be larger than r.

$$y \le r$$



A similar analysis can be conducted in quadrant IV as θ approaches $-90°$ from $0°$ (note that y is negative in quadrant IV). A result that is valid in all four quadrants is $|y| \leq r$.

Words	Math
Write the absolute value inequality as a double inequality.	$-r \leq y \leq r$
Divide both sides by r.	$-1 \leq \dfrac{y}{r} \leq 1$
Let $\sin\theta = \dfrac{y}{r}$.	$-1 \leq \sin\theta \leq 1$

Similarly, by allowing θ to approach $0°$ and $180°$, we can show that $|x| \leq r$, which leads to the range of the cosine function: $-1 \leq \cos\theta \leq 1$. Sine and cosine values range between -1 and 1 and since secant and cosecant are reciprocals of the cosine and sine functions, respectively, their ranges are stated as

$$\sec\theta \leq -1 \text{ or } \sec\theta \geq 1 \qquad \csc\theta \leq -1 \text{ or } \csc\theta \geq 1$$

Since $\tan\theta = \dfrac{y}{x}$ and $\cot\theta = \dfrac{x}{y}$ and since $x < y$, $x = y$, and $x > y$ are all possible, the values of the tangent and cotangent functions can be any real numbers (positive, negative, or zero). The following box summarizes the ranges of the trigonometric functions.

Ranges of the Trigonometric Functions

For any angle θ for which the trigonometric functions are defined, the six trigonometric functions have the following ranges:

- $-1 \leq \sin\theta \leq 1$
- $-1 \leq \cos\theta \leq 1$
- $\sec\theta \leq -1$ or $\sec\theta \geq 1$
- $\csc\theta \leq -1$ or $\csc\theta \geq 1$
- $\tan\theta$ and $\cot\theta$ can equal any real number.

EXAMPLE 5 Determining Whether a Value Is Within the Range of a Trigonometric Function

Determine whether each statement is possible or not.

a. $\cos\theta = 1.001$

b. $\cot\theta = 0$

c. $\sec\theta = \dfrac{\sqrt{3}}{2}$

Solution (a): Not possible, because $1.001 > 1$.

Solution (b): Possible, because $\cot 90° = 0$.

Solution (c): Not possible, because $\dfrac{\sqrt{3}}{2} \approx 0.866 < 1$.

YOUR TURN Determine whether each statement is possible or not.

a. $\sin\theta = -1.1$ **b.** $\tan\theta = 2$ **c.** $\csc\theta = \sqrt{3}$

Answer: **a.** not possible **b.** possible **c.** possible

Reference Angles and Reference Right Triangles

Now that we know the trigonometric function ranges and their algebraic signs in each of the four quadrants, we can evaluate the trigonometric functions of nonacute angles. Before we do that, however, we first must discuss *reference angles* and *reference right triangles*.

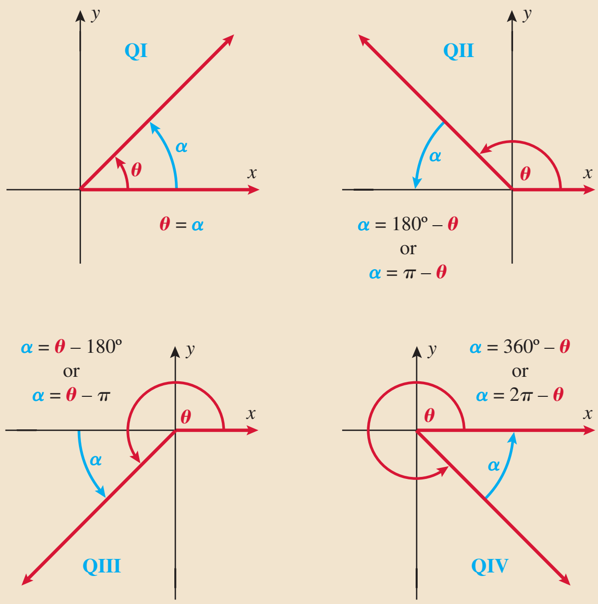
Every nonquadrantal angle in standard position has a corresponding *reference angle* and *reference right triangle*. We have already calculated the trigonometric function values for quadrantal angles.

Study Tip

The reference angle is the acute angle that the terminal side makes with the x-axis, not the y-axis.

DEFINITION Reference Angle

For angle θ, $0° < \theta < 360°$ or $0 < \theta < 2\pi$, in standard position whose terminal side lies in one of the four quadrants, there exists a **reference angle** α, which is the acute angle formed by the terminal side of angle θ and the x-axis.



The reference angle is the positive, acute angle that the terminal side makes with the x-axis.

EXAMPLE 6 Finding Reference Angles

Find the reference angle for each angle given.

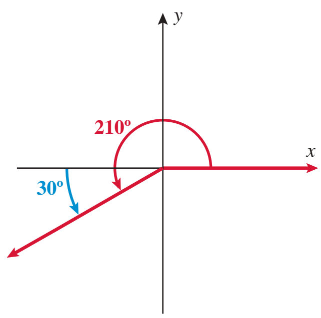
a. $210°$ **b.** $\frac{3\pi}{4}$ **c.** $422°$

Solution (a):

The terminal side of angle θ lies in quadrant III.

The reference angle is formed by the terminal side and the negative x-axis.

$210° - 180° = \boxed{30°}$

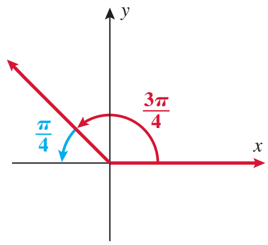


Solution (b):

The terminal side of angle θ lies in quadrant II.

The reference angle is formed by the terminal side and the negative x-axis.

$$\pi - \frac{3\pi}{4} = \boxed{\frac{\pi}{4}}$$

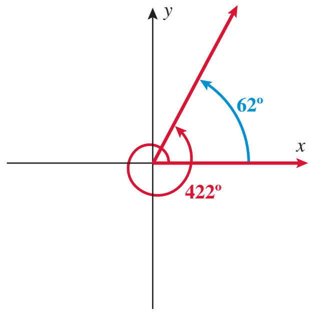


Solution (c):

The terminal side of angle θ lies in quadrant I.

The reference angle is formed by the terminal side and the positive x-axis.

$$422° - 360° = \boxed{62°}$$



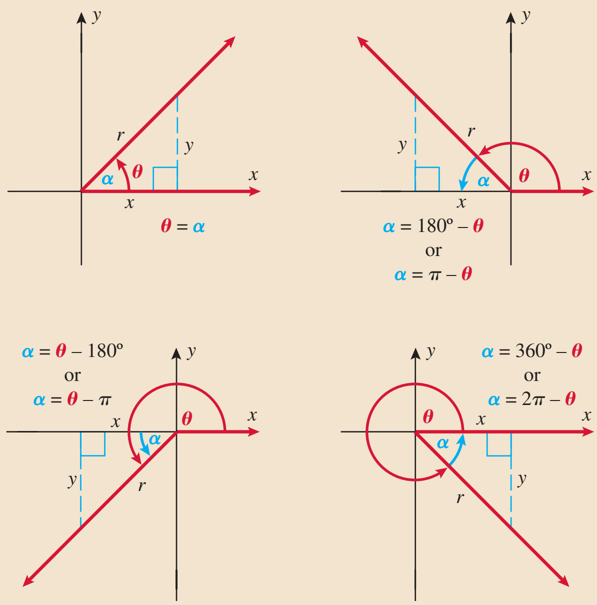
■ **YOUR TURN** Find the reference angle for each angle given.

a. 160° **b.** $\frac{7\pi}{4}$ **c.** 600°

■ **Answer:**
a. 20° **b.** $\frac{\pi}{4}$ **c.** 60°

DEFINITION Reference Right Triangle

To form a **reference right triangle** for angle θ, where $0° < \theta < 360°$ or $0 < \theta < 2\pi$, drop a perpendicular line from the terminal side of the angle to the x-axis. The right triangle now has reference angle α as one of its acute angles.



In Section 4.2, we first defined the trigonometric functions of an acute angle as ratios of lengths of sides of a right triangle. For example, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$. The lengths of the sides of triangles are always positive.

In this section, we defined the sine function of any angle as $\sin\theta = \dfrac{y}{r}$. Notice in the above box that for a nonacute angle θ, $\sin\theta = \dfrac{y}{r}$; and for the acute reference angle α, $\sin\alpha = \dfrac{|y|}{r}$. The only difference between these two expressions is the algebraic sign, since r is always positive and y is positive or negative depending on the quadrant.

Therefore, to calculate the trigonometric function values for a nonacute angle, simply find the trigonometric values for the reference angle and determine the correct algebraic sign according to the quadrant in which the terminal side lies.

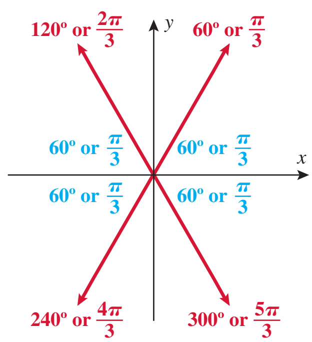
Study Tip

To find the trigonometric function values of nonacute angles, first find the trigonometric values of the reference angle and then use the quadrant information to determine the algebraic sign.

Evaluating Trigonometric Functions for Nonacute Angles

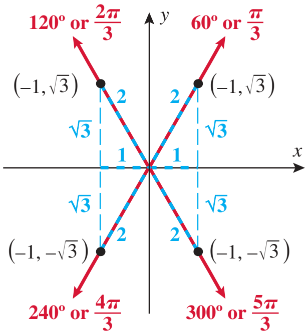
Let's look at a specific example before we generalize a procedure for evaluating trigonometric function values for nonacute angles.

Suppose we have the angles in standard position with measure **60°**, **120°**, **240°**, and **300°** or $\dfrac{\pi}{3}$, $\dfrac{2\pi}{3}$, $\dfrac{4\pi}{3}$, or $\dfrac{5\pi}{3}$, respectively. Notice that the reference angle for all of these angles is **60°** or $\dfrac{\pi}{3}$.



If we draw reference triangles and let the shortest leg have length 1, we find that the other leg has length $\sqrt{3}$ and the hypotenuse has length 2. (Recall the relationships for side lengths of a 30°–60°–90° triangle.)

Notice that the legs of the triangles have lengths (always positive) 1 and $\sqrt{3}$; however, the coordinates are $(\pm 1, \pm\sqrt{3})$. Therefore, when we calculate the trigonometric functions for any of the angles, $60°\left(\dfrac{\pi}{3}\right)$, $120°\left(\dfrac{2\pi}{3}\right)$, $240°\left(\dfrac{4\pi}{3}\right)$, and $300°\left(\dfrac{5\pi}{3}\right)$, we can simply calculate the trigonometric functions for the reference angle, $60°\left(\dfrac{\pi}{3}\right)$, and determine the algebraic sign (+ or −) for the particular trigonometric function and quadrant.



Study Tip

The value of a trigonometric function of an angle is the same as the trigonometric value of its reference angle, except there may be an algebraic sign (+ or −) difference between the two values.

To find the value of $\cos 120^\circ$, we first recognize that the terminal side of an angle with 120° measure lies in quadrant II. We also know that cosine is negative in quadrant II. We then calculate the cosine of the reference angle, **60°**.

$$\cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2}$$

Since we know $\cos 120^\circ$ is negative because it lies in quadrant II, we know that

$$\cos 120^\circ = -\frac{1}{2}$$

Similarly, we know that $\cos 240^\circ = -\frac{1}{2}$ and $\cos 300^\circ = \frac{1}{2}$.

For any angle whose terminal side lies along one of the axes, we consult the table in this section for the values of the trigonometric functions for quadrantal angles. If the terminal side lies in one of the four quadrants, then the angle is said to be nonquadrantal and the following procedure can be used.

Procedure for Evaluating Function Values for Any Nonquadrantal Angle θ

Step 1:
- If $0^\circ < \theta < 360^\circ$ or $0 < \theta < 2\pi$, proceed to Step 2.
- If $\theta < 0^\circ$, add 360° as many times as needed to get a coterminal angle with measure between 0° and 360°. Similarly, if $\theta < 0$, add 2π as many times as needed to get a coterminal angle with measure between 0 and 2π.
- If $\theta > 360^\circ$, subtract 360° as many times as needed to get a coterminal angle with measure between 0° and 360°. Similarly, if $\theta > 2\pi$, subtract 2π as many times as needed to get a coterminal angle with measure between 0 and 2π.

Step 2: Find the quadrant in which the terminal side of the angle in Step 1 lies.
Step 3: Find the reference angle α of the angle found in Step 1.
Step 4: Find the trigonometric function values for the reference angle α.
Step 5: Determine the correct algebraic signs (+ or −) for the trigonometric function values based on the quadrant identified in Step 2.
Step 6: Combine the trigonometric values found in Step 4 with the algebraic signs in Step 5 to give the trigonometric function values of θ.

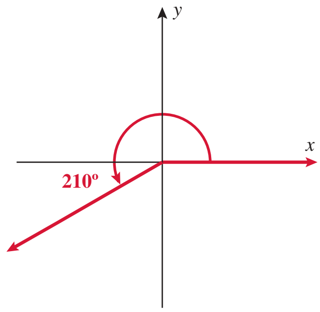
We follow the above procedure for all angles except when we get to Step 4. In Step 4, we evaluate exactly if possible the special angles $\left(30^\circ, 45^\circ, 60^\circ \text{ or } \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right)$; otherwise, we use a calculator to approximate.

EXAMPLE 7 Evaluating the Cosine Function of a Special Angle Exactly

Find the exact value of $\cos 210°$.

Solution:

The terminal side of $\theta = 210°$ lies in quadrant III.



Find the reference angle for $\theta = 210°$. $210° - 180° = 30°$

Find the value of the cosine of the reference angle. $\cos 30° = \dfrac{\sqrt{3}}{2}$

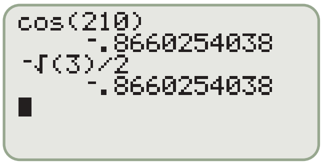
Determine the algebraic sign for the cosine in quadrant III. Negative $(-)$

Combine the algebraic sign of the cosine in quadrant III with the value of the cosine of the reference angle. $\cos 210° = \boxed{-\dfrac{\sqrt{3}}{2}}$

YOUR TURN Find the exact value of $\sin 330°$.

Technology Tip

Use a calculator to check the answer.



Answer: $-\frac{1}{2}$

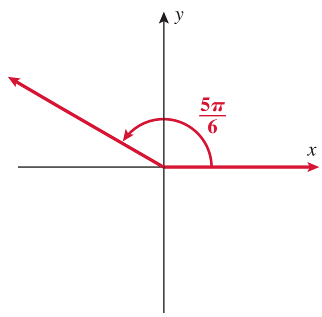
EXAMPLE 8 Evaluating the Cosecant Function of a Special Angle Exactly

Find the exact value of $\csc\left(-\dfrac{7\pi}{6}\right)$.

Solution:

Add 2π to get a coterminal angle between 0 and 2π. $-\dfrac{7\pi}{6} + 2\pi = \dfrac{5\pi}{6}$

The terminal side of the angle lies in quadrant II.



Find the reference angle for the angle with measure $\dfrac{5\pi}{6}$. $\pi - \dfrac{5\pi}{6} = \dfrac{\pi}{6}$

Find the value of the cosecant of the reference angle. $\csc\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sin\left(\dfrac{\pi}{6}\right)} = \dfrac{1}{\frac{1}{2}} = 2$

Determine the algebraic sign for the cosecant in quadrant II. — Positive $(+)$

Combine the algebraic sign of the cosecant in quadrant II with the value of the cosecant of the reference angle. — $\csc\left(-\frac{7\pi}{6}\right) = \boxed{2}$

■ **YOUR TURN** Find the exact value of $\sec\left(-\frac{11\pi}{6}\right)$.

■ **Answer:** $\frac{2\sqrt{3}}{3}$

EXAMPLE 9 Finding Exact Angle Measures Given Trigonometric Function Values

Find all values of θ where $0° \le \theta \le 360°$, when $\sin\theta = -\frac{\sqrt{3}}{2}$.

Solution:

Determine in which quadrants sine is negative. — QIII and QIV

Since the absolute value of $\sin\theta$ is $\frac{\sqrt{3}}{2}$, the reference angle has measure 60°. — $\sin 60° = \frac{\sqrt{3}}{2}$

Determine the angles between 180° and 360° in quadrants III and IV with reference angle 60°.

Quadrant III: $180° + 60° = 240°$

Quadrant IV: $360° - 60° = 300°$

The two angles are $\boxed{240°}$ and $\boxed{300°}$.

■ **YOUR TURN** Find all values of θ where $0° \le \theta \le 360°$, when $\cos\theta = -\frac{\sqrt{3}}{2}$.

■ **Answer:** 150° and 210°

EXAMPLE 10 Finding Approximate Angle Measures Given Trigonometric Function Values

Find the measure of an angle θ (rounded to the nearest degree) if $\sin\theta = -0.6293$ and the terminal side of θ (in standard position) lies in quadrant III, where $0° \le \theta \le 360°$.

Solution:

The sine of the reference angle is 0.6293. — $\sin\alpha = 0.6293$

Find the reference angle. — $\alpha = \sin^{-1}(0.6293) \approx 38.998°$

Round the reference angle to the nearest degree. — $\alpha \approx 39°$

Find θ, which lies in quadrant III. — $\theta \approx 180° + 39° \approx 219°$

$\boxed{\theta \approx 219°}$

Check with a calculator. — $\sin 219° \approx -0.6293$

Technology Tip

When $\sin\theta = -0.6293$, use the $\boxed{\sin^{-1}}$ key and the absolute value to find the reference angle.

```
sin⁻¹(0.6293)
        38.99849667
```

■ **YOUR TURN** Find the measure of θ, the smallest positive angle (rounded to the nearest degree), if $\cos\theta = -0.5299$ and the terminal side of θ (in standard position) lies in quadrant II.

■ **Answer:** 122°

SECTION 4.3 SUMMARY

The trigonometric functions are defined in the Cartesian plane for any angle as follows:

Let (x, y) be a point, other than the origin, on the terminal side of an angle θ in standard position. Let r be the distance from the point (x, y) to the origin. Then the sine, cosine, and tangent functions are defined as

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

The range of the sine and cosine functions is $[-1, 1]$, whereas the range of the secant and cosecant functions is $(-\infty, -1] \cup [1, \infty)$.

Reference angles and reference right triangles can be used to evaluate trigonometric functions for nonacute angles.

SECTION 4.3 EXERCISES

SKILLS

In Exercises 1–14, the terminal side of an angle θ in standard position passes through the indicated point. Calculate the values of the six trigonometric functions for angle θ.

1. $(3, 6)$ **2.** $(8, 4)$ **3.** $\left(\frac{1}{2}, \frac{2}{5}\right)$ **4.** $\left(\frac{4}{7}, \frac{2}{3}\right)$ **5.** $(-2, 4)$

6. $(-1, 3)$ **7.** $(-4, -7)$ **8.** $(-9, -5)$ **9.** $\left(-\sqrt{2}, \sqrt{3}\right)$ **10.** $\left(-\sqrt{3}, \sqrt{2}\right)$

11. $\left(-\sqrt{5}, -\sqrt{3}\right)$ **12.** $\left(-\sqrt{6}, -\sqrt{5}\right)$ **13.** $\left(-\frac{10}{3}, -\frac{4}{3}\right)$ **14.** $\left(-\frac{2}{9}, -\frac{1}{3}\right)$

In Exercises 15–24, indicate the quadrant in which the terminal side of θ must lie in order for the information to be true.

15. $\cos\theta$ is positive and $\sin\theta$ is negative.

16. $\cos\theta$ is negative and $\sin\theta$ is positive.

17. $\tan\theta$ is negative and $\sin\theta$ is positive.

18. $\tan\theta$ is positive and $\cos\theta$ is negative.

19. $\sec\theta$ and $\csc\theta$ are both positive.

20. $\sec\theta$ and $\csc\theta$ are both negative.

21. $\cot\theta$ and $\cos\theta$ are both positive.

22. $\cot\theta$ and $\sin\theta$ are both negative.

23. $\tan\theta$ is positive and $\sec\theta$ is negative.

24. $\cot\theta$ is negative and $\csc\theta$ is positive.

In Exercises 25–36, find the indicated trigonometric function values.

25. If $\cos\theta = -\frac{3}{5}$, and the terminal side of θ lies in quadrant III, find $\sin\theta$.

26. If $\tan\theta = -\frac{5}{12}$, and the terminal side of θ lies in quadrant II, find $\cos\theta$.

27. If $\sin\theta = \frac{60}{61}$, and the terminal side of θ lies in quadrant II, find $\tan\theta$.

28. If $\cos\theta = \frac{40}{41}$, and the terminal side of θ lies in quadrant IV, find $\tan\theta$.

29. If $\tan\theta = \frac{84}{13}$, and the terminal side of θ lies in quadrant III, find $\sin\theta$.

30. If $\sin\theta = -\frac{7}{25}$, and the terminal side of θ lies in quadrant IV, find $\cos\theta$.

31. If $\sec\theta = -2$, and the terminal side of θ lies in quadrant III, find $\tan\theta$.

32. If $\cot\theta = 1$, and the terminal side of θ lies in quadrant I, find $\sin\theta$.

33. If $\csc\theta = \frac{2}{\sqrt{3}}$ and the terminal side of θ lies in quadrant II, find $\cot\theta$.

34. If $\sec\theta = -\frac{13}{5}$ and the terminal side of θ lies in quadrant II, find $\csc\theta$.

35. If $\cot\theta = -\sqrt{3}$ and the terminal side of θ lies in quadrant IV, find $\sec\theta$.

36. If $\cot\theta = -\frac{13}{84}$ and the terminal side of θ lies in quadrant II, find $\csc\theta$.

In Exercises 37–46, evaluate each expression, if possible.

37. $\cos(-270°) + \sin 450°$

38. $\sin(-270°) + \cos 450°$

39. $\sin 630° + \tan(-540°)$

40. $\cos(-720°) + \tan 720°$

41. $\cos(3\pi) - \sec(-3\pi)$

42. $\sin\left(-\frac{5\pi}{2}\right) + \csc\left(\frac{3\pi}{2}\right)$

43. $\csc\left(-\frac{7\pi}{2}\right) - \cot\left(\frac{7\pi}{2}\right)$

44. $\sec(-3\pi) + \tan(3\pi)$

45. $\tan 720° + \sec 720°$

46. $\cot 450° - \cos(-450°)$

In Exercises 47–56, determine whether each statement is possible or not.

47. $\sin\theta = -0.999$

48. $\cos\theta = 1.0001$

49. $\cos\theta = \frac{2\sqrt{6}}{3}$

50. $\sin\theta = \frac{\sqrt{2}}{10}$

51. $\tan\theta = 4\sqrt{5}$

52. $\cot\theta = -\frac{\sqrt{6}}{7}$

53. $\sec\theta = -\frac{4}{\sqrt{7}}$

54. $\csc\theta = \frac{\pi}{2}$

55. $\cot\theta = 500$

56. $\sec\theta = 0.9996$

In Exercises 57–68, evaluate the following expressions *exactly*:

57. $\cos 240°$

58. $\cos 120°$

59. $\sin\left(\frac{5\pi}{3}\right)$

60. $\sin\left(\frac{7\pi}{4}\right)$

61. $\tan 210°$

62. $\sec 135°$

63. $\tan(-315°)$

64. $\sec(-330°)$

65. $\csc\left(\frac{11\pi}{6}\right)$

66. $\csc\left(-\frac{4\pi}{3}\right)$

67. $\cot(-315°)$

68. $\cot 150°$

In Exercises 69–76, find all possible values of θ, where $0° \le \theta \le 360°$.

69. $\cos\theta = \frac{\sqrt{3}}{2}$

70. $\sin\theta = \frac{\sqrt{3}}{2}$

71. $\sin\theta = -\frac{1}{2}$

72. $\cos\theta = -\frac{1}{2}$

73. $\cos\theta = 0$

74. $\sin\theta = 0$

75. $\sin\theta = -1$

76. $\cos\theta = -1$

In Exercises 77–90, find the smallest positive measure of θ (rounded to the nearest degree) if the indicated information is true.

77. $\sin\theta = 0.9397$ and the terminal side of θ lies in quadrant II.

78. $\cos\theta = 0.7071$ and the terminal side of θ lies in quadrant IV.

79. $\cos\theta = -0.7986$ and the terminal side of θ lies in quadrant II.

80. $\sin\theta = -0.1746$ and the terminal side of θ lies in quadrant III.

81. $\tan\theta = -0.7813$ and the terminal side of θ lies in quadrant IV.

82. $\cos\theta = -0.3420$ and the terminal side of θ lies in quadrant III.

83. $\tan\theta = -0.8391$ and the terminal side of θ lies in quadrant II.

84. $\tan\theta = 11.4301$ and the terminal side of θ lies in quadrant III.

85. $\sin\theta = -0.3420$ and the terminal side of θ lies in quadrant IV.

86. $\sin\theta = -0.4226$ and the terminal side of θ lies in quadrant III.

87. $\sec \theta = 1.0001$ and the terminal side of θ lies in quadrant I.

88. $\sec \theta = -3.1421$ and the terminal side of θ lies in quadrant II.

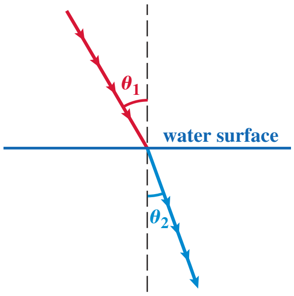
89. $\csc \theta = -2.3604$ and the terminal side of θ lies in quadrant IV.

90. $\csc \theta = -1.0001$ and the terminal side of θ lies in quadrant III.

APPLICATIONS

In Exercises 91–94, refer to the following:

When light passes from one substance to another, such as from air to water, its path bends. This is called refraction and is what is seen in eyeglass lenses, camera lenses, and gems. The rule governing the change in the path is called *Snell's law*, named after a Dutch astronomer: $n_1 \sin \theta_1 = n_2 \sin \theta_2$, where n_1 and n_2 are the indices of refraction of the different substances and θ_1 and θ_2 are the respective angles that light makes with a line perpendicular to the surface at the boundary between substances. The figure shows the path of light rays going from air to water. Assume that the index of refraction in air is 1.



91. If light rays hit the water's surface at an angle of 30° from the perpendicular and are refracted to an angle of 22° from the perpendicular, then what is the refraction index for water? Round the answer to two significant digits.

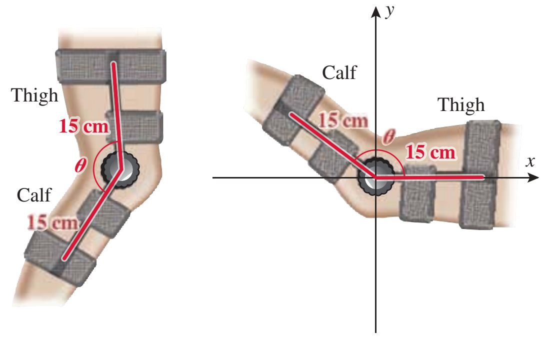
92. If light rays hit a glass surface at an angle of 30° from the perpendicular and are refracted to an angle of 18° from the perpendicular, then what is the refraction index for that glass? Round the answer to two significant digits.

93. If the refraction index for a diamond is 2.4, then to what angle is light refracted if it enters the diamond at an angle of 30°? Round the answer to two significant digits.

94. If the refraction index for a rhinestone is 1.9, then to what angle is light refracted if it enters the rhinestone at an angle of 30°? Round the answer to two significant digits.

For Exercises 95 and 96, refer to the following:

An orthotic knee brace can be used to treat knee injuries by locking the knee at an angle θ chosen to facilitate healing. The angle θ is measured from the metal bar on the side of the brace on the thigh to the metal bar on the side of the brace on the calf (see the figure on the left). To make working with the brace more convenient, rotate the image such that the thigh brace aligns with the positive x-axis (see the figure on the right below).

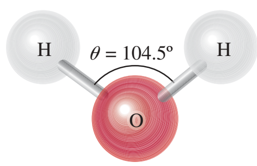


95. Health/Medicine. If $\theta = 165°$, find the measure of the reference angle. What is the physical meaning of the reference angle?

96. Health/Medicine. If $\theta = 160°$, find the measure of the reference angle. What would an angle greater than 180° represent?

For Exercises 97 and 98, refer to the following:

Water covers two-thirds of the Earth's surface and every living thing is dependent on it. For example, the human body is made up of over 70% water. The water molecule is composed of one oxygen atom and two hydrogen atoms and exhibits a bent shape with the oxygen molecule at the center. The angle θ between the O-H bonds is 104.5°.



(*Source:* http://www.wiley.com/college/boyer/0470003790/reviews/pH/ph_water.htm.)

97. Chemistry. Sketch the water molecule in the xy-coordinate system in a convenient manner for illustrating angles. Find the reference angle. Illustrate both the angle θ and the reference angle on the sketch.

98. Chemistry. Find cos(104.5°) using the reference angle found in Exercise 97.

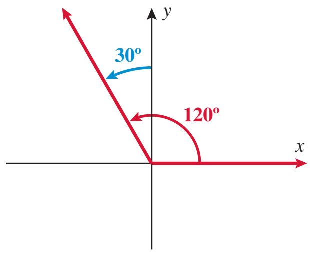
CATCH THE MISTAKE

In Exercises 99 and 100, explain the mistake that is made.

99. Evaluate the expression $\sec 120°$ exactly.

Solution:

120° lies in quadrant II.
The reference angle is 30°.



Find the cosine of the reference angle. $\cos 30° = \dfrac{\sqrt{3}}{2}$

Cosine is negative in quadrant II. $\cos 120° = -\dfrac{\sqrt{3}}{2}$

Secant is the reciprocal of cosine. $\sec 120° = -\dfrac{2}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{3}$

This is incorrect. What mistake was made?

100. Find the measure of the smallest positive angle θ (rounded to the nearest degree) if $\cos\theta = -0.2388$ and the terminal side of θ (in standard position) lies in quadrant III.

Solution:

Evaluate with a calculator. $\theta = \cos^{-1}(-0.2388) = 103.8157°$

Approximate to the nearest degree. $\theta \approx 104°$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 101–108, determine whether each statement is true or false.

101. It is possible for all six trigonometric functions of the same angle to have positive values.

102. It is possible for all six trigonometric functions of the same angle to have negative values.

103. The trigonometric function value for any angle with negative measure must be negative.

104. The trigonometric function value for any angle with positive measure must be positive.

105. $\sec^2\theta - 1$ can be negative for some value of θ.

106. $(\sec\theta)(\csc\theta)$ is negative only when the terminal side of θ lies in quadrant II or IV.

107. $\cos\theta = \cos(\theta + 360°n)$, where n is an integer.

108. $\sin\theta = \sin(\theta + 2\pi n)$, where n is an integer.

CHALLENGE

109. If the terminal side of angle θ passes through the point $(-3a, 4a)$, find $\cos\theta$. Assume $a > 0$.

110. If the terminal side of angle θ passes through the point $(-3a, 4a)$, find $\sin\theta$. Assume $a > 0$.

111. Find the equation of the line with negative slope that passes through the point $(a, 0)$ and makes an acute angle θ with the x-axis. The equation of the line will be in terms of x, a, and a trigonometric function of θ. Assume $a > 0$.

112. Find the equation of the line with positive slope that passes through the point $(a, 0)$ and makes an acute angle θ with the x-axis. The equation of the line will be in terms of x, a, and a trigonometric function of θ. Assume $a > 0$.

113. If $\tan\theta = \dfrac{a}{b}$, where a and b are positive, and if θ lies in quadrant III, find $\sin\theta$.

114. If $\tan\theta = -\dfrac{a}{b}$, where a and b are positive, and if θ lies in quadrant II, find $\cos\theta$.

115. If $\csc\theta = -\dfrac{a}{b}$, where a and b are positive, and if θ lies in quadrant IV, find $\cot\theta$.

116. If $\sec\theta = -\dfrac{a}{b}$, where a and b are positive, and if θ lies in quadrant III, find $\tan\theta$.

TECHNOLOGY

In Exercises 117–124, use a calculator to evaluate the following expressions. If you get an error, explain why.

117. $\cos 270°$

118. $\tan 270°$

119. $\cot 270°$

120. $\sin(-270°)$

121. $\cos(-270°)$

122. $\csc(-270°)$

123. $\sec(-270°)$

124. $\sec 270°$

PREVIEW TO CALCULUS

In calculus, the value $F(b) - F(a)$ of a function $F(x)$ at $x = a$ and $x = b$ plays an important role in the calculation of definite integrals.

In Exercises 125–128, find the exact value of $F(b) - F(a)$.

125. $F(x) = 2\tan x + \cos x, a = -\frac{\pi}{6}, b = \frac{\pi}{4}$

126. $F(x) = \sin^2 x + \cos^2 x, a = \frac{3\pi}{4}, b = \frac{7\pi}{6}$

127. $F(x) = \sec^2 x + 1, a = \frac{5\pi}{6}, b = \frac{4\pi}{3}$

128. $F(x) = \cot x - \csc^2 x, a = \frac{7\pi}{6}, b = \frac{7\pi}{4}$

SECTION 4.4 THE LAW OF SINES

SKILLS OBJECTIVES

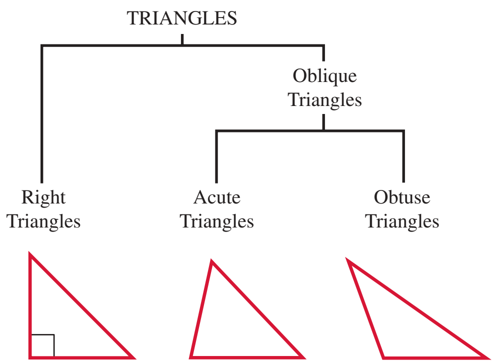
- Solve AAS or ASA triangle cases.
- Solve ambiguous SSA triangle cases.

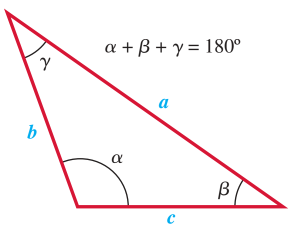
CONCEPTUAL OBJECTIVES

- Understand the derivation of the Law of Sines.
- Understand that the ambiguous case can yield no triangle, one triangle, or two triangles.
- Understand why an AAA case cannot be solved.

Solving Oblique Triangles

Thus far we have discussed only *right* triangles. There are, however, two types of triangles, right and *oblique*. An **oblique triangle** is any triangle that does not have a right angle. An oblique triangle is either an **acute triangle**, having three acute angles, or an **obtuse triangle**, having one obtuse (between 90° and 180°) angle.





It is customary to label oblique triangles in the following way:

- angle α (alpha) opposite side a
- angle β (beta) opposite side b
- angle γ (gamma) opposite side c

Remember that the sum of the three angles of any triangle must equal 180°. In Section 4.3, we solved right triangles. In this section, we solve oblique triangles, which means we find the lengths of all three sides and the measures of all three angles.

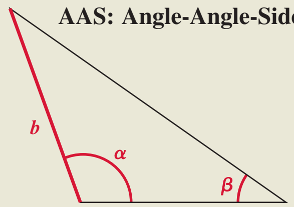
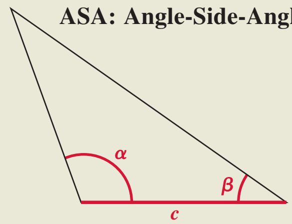
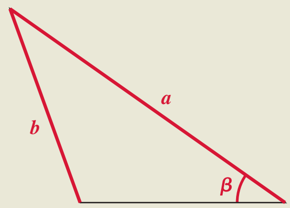
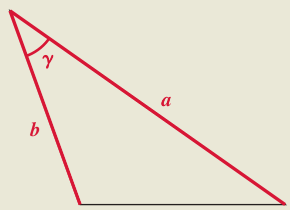
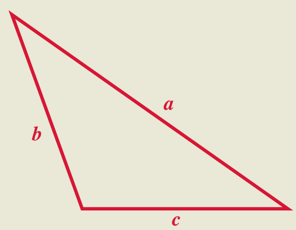
Four Cases

To solve an oblique triangle, *we need to know the length of one side* and one of the following three:

- two angles
- one angle and another side
- the other two sides

This requirement leads to the following four possible cases to consider:

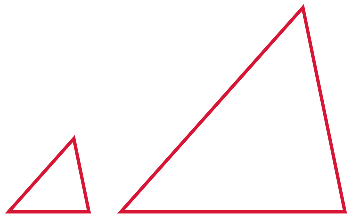
Required information to solve oblique triangles

Case	What's Given	Examples/Names
Case 1	Measures of one side and two angles	 **AAS: Angle-Angle-Side**  **ASA: Angle-Side-Angle**
Case 2	Measures of two sides and the angle opposite one of them	 **SSA: Side-Side-Angle**
Case 3	Measures of two sides and the angle between them	 **SAS: Side-Angle-Side**
Case 4	Measures of three sides	 **SSS: Side-Side-Side**

Study Tip

To solve triangles, at least one side must be known.

Notice that there is **no AAA case**, because two similar triangles can have the same angle measures but different side lengths.

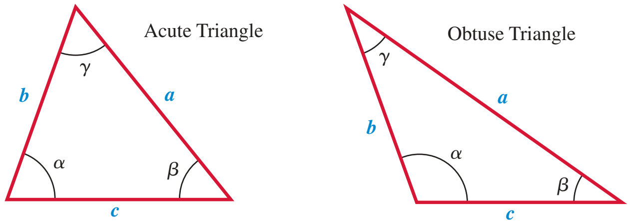


That is why at least the length of one side must be known.

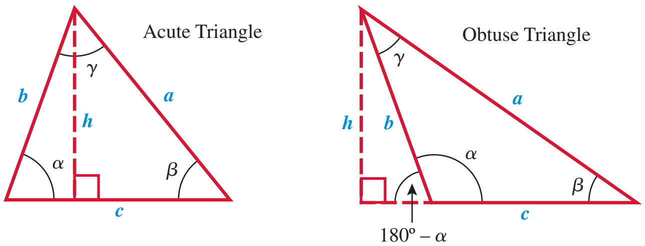
In this section, we will derive the Law of Sines, which will enable us to solve Case 1 and Case 2 problems. In the next section, we will derive the Law of Cosines, which will enable us to solve Case 3 and Case 4 problems.

The Law of Sines

Let us start with two oblique triangles, an acute triangle and an obtuse triangle.



The following discussion applies to both triangles. First, construct an altitude (perpendicular) h from the vertex at angle γ to the side (or its extension) opposite γ.



Words	**Math**
Formulate sine ratios for the acute triangle.	$\sin\alpha = \dfrac{h}{b}$ and $\sin\beta = \dfrac{h}{a}$
Formulate sine ratios for the obtuse triangle.	$\sin(180° - \alpha) = \dfrac{h}{b}$ and $\sin\beta = \dfrac{h}{a}$
For the obtuse triangle, apply the sine difference identity.*	$\begin{aligned}\sin(180° - \alpha) &= \sin 180° \cos\alpha - \cos 180° \sin\alpha \\ &= 0 \cdot \cos\alpha - (-1)\sin\alpha \\ &= \sin\alpha\end{aligned}$
Therefore, in either triangle we find the same equation.	$\sin\alpha = \dfrac{h}{b}$ and $\sin\beta = \dfrac{h}{a}$
Solve for h in both equations.	$h = b\sin\alpha$ and $h = a\sin\beta$
Since h is equal to itself, equate the expressions for h.	$b\sin\alpha = a\sin\beta$

*The sine difference identify, $\sin(x - y) = \sin x \cdot \cos y - \cos x \cdot \sin y$ is derived in Section 6.2.

Divide both sides by ab. $\dfrac{b\sin\alpha}{ab} = \dfrac{a\sin\beta}{ab}$

Divide out common factors. $\boxed{\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}}$

In a similar manner, we can extend an altitude (perpendicular) from angle α, and we will find that $\boxed{\dfrac{\sin\gamma}{c} = \dfrac{\sin\beta}{b}}$. Equating these two expressions leads us to the third ratio of the *Law of Sines*: $\boxed{\dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}}$.

THE LAW OF SINES

For a triangle with sides of lengths a, b, and c, and opposite angles of measures α, β, and γ, the following is true:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

In other words, the ratio of the sine of an angle in a triangle to its opposite side is equal to the ratios of the sines of the other two angles to their opposite sides.

Study Tip

When an angle and the side opposite that angle are known, the Law of Sines can be used provided one other piece of information (side length/angle measure) is known.

Notice that in both cases 1 and 2 where an angle and the side opposite that angle are known, the Law of Sines can be used as long as one other piece of information is known (i.e., side length or angle measure).

Some things to note before we begin solving oblique triangles are:

- The angles and sides share the same progression of magnitude:
  - The longest side of a triangle is opposite the largest angle.
  - The shortest side of a triangle is opposite the smallest angle.
- Draw the triangle and label the angles and sides.
- If two angle measures are known, start by determining the third angle.
- Whenever possible, in successive steps always return to given values rather than refer to calculated (approximate) values.

Keeping these pointers in mind will help you determine whether your answers are reasonable.

Study Tip

Remember that the longest side is opposite the largest angle; the shortest side is opposite the smallest angle.

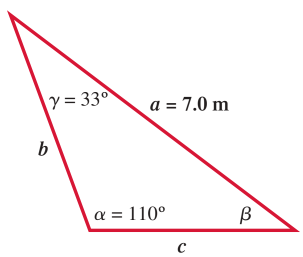
Study Tip

Always use given values rather than calculated (approximated) values for better accuracy.

Case 1: Two Angles and One Side (AAS or ASA)

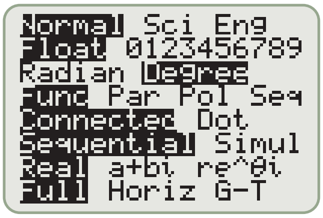
EXAMPLE 1 Using the Law of Sines to Solve a Triangle (AAS)

Solve the triangle.



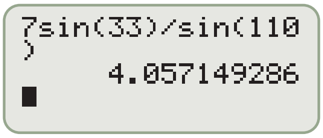
Technology Tip

Step 1: Set the calculator to degree mode by typing MODE.



Step 2: Use the calculator to find $b = \frac{7\sin 37^\circ}{\sin 110^\circ}$.

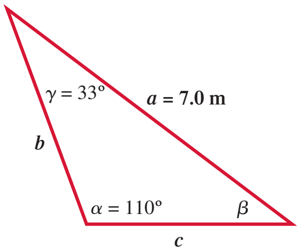
Step 3: Use the calculator to find $c = \frac{7\sin 33^\circ}{\sin 110^\circ}$.



Study Tip

Notice in Step 3 that we used a that is given, as opposed to b that has been calculated (approximated).

Solution:



This is an AAS (angle-angle-side) case because two angles and a side are given and the side is opposite one of the angles.

STEP 1 Find β.

The sum of the measures of the angles in a triangle is 180°. $\alpha + \beta + \gamma = 180^\circ$

Let $\alpha = 110^\circ$ and $\gamma = 33^\circ$. $110^\circ + \beta + 33^\circ = 180^\circ$

Solve for β. $\boxed{\beta = 37^\circ}$

STEP 2 Find b.

Use the Law of Sines with the known side a. $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$

Isolate b. $b = \frac{a\sin\beta}{\sin\alpha}$

Let $\alpha = 110^\circ$, $\beta = 37^\circ$, and $a = 7$ m. $b = \frac{7\sin 37^\circ}{\sin 110^\circ}$

Use a calculator to approximate b. $b \approx 4.483067$ m

Round b to two significant digits. $\boxed{b \approx 4.5 \text{ m}}$

STEP 3 Find c.

Use the Law of Sines with the known side a. $\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$

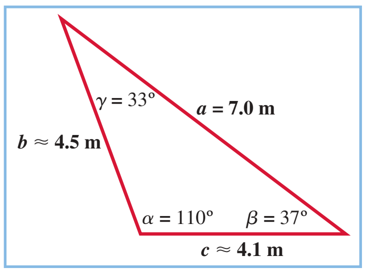
Isolate c. $c = \frac{a\sin\gamma}{\sin\alpha}$

Let $\alpha = 110^\circ$, $\gamma = 33^\circ$, and $a = 7$ m. $c = \frac{(7 \text{ m})\sin 33^\circ}{\sin 110^\circ}$

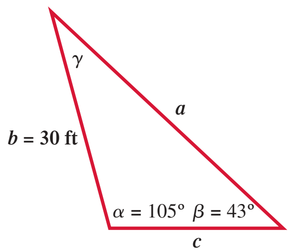
Use a calculator to approximate c. $c \approx 4.057149$ m

Round c to two significant digits. $\boxed{c \approx 4.1 \text{ m}}$

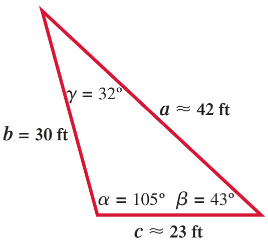
STEP 4 Draw and label the triangle.



■ **YOUR TURN** Solve the triangle.

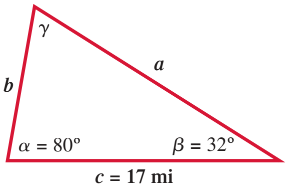


■ **Answer:** $\gamma = 32^\circ$, $a \approx 42$ ft, $c \approx 23$ ft



EXAMPLE 2 Using the Law of Sines to Solve a Triangle (ASA)

Solve the triangle.



Solution:

This is an ASA (angle-side-angle) case because two angles and a side are given and the side is not opposite one of the angles.

STEP 1 Find γ.

The sum of the measures of the angles in a triangle is 180°. $\alpha + \beta + \gamma = 180°$

Let $\alpha = 80°$ and $\beta = 32°$. $80° + 32° + \gamma = 180°$

Solve for γ. $\boxed{\gamma = 68°}$

STEP 2 Find b.

Write the Law of Sines to include the known side c. $\dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$

Isolate b. $b = \dfrac{c\sin\beta}{\sin\gamma}$

Let $\beta = 32°$, $\gamma = 68°$, and $c = 17$ mi. $b = \dfrac{(17\text{ mi})\sin 32°}{\sin 68°}$

Use a calculator to approximate b. $b \approx 9.7161177$ mi

Round b to two significant digits. $\boxed{b \approx 9.7\text{ mi}}$

STEP 3 Find a.

Write the Law of Sines again incorporating the known side c. $\dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}$

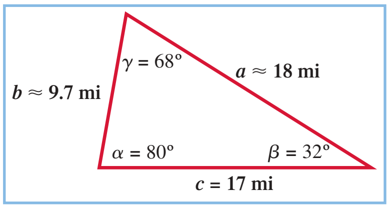
Isolate a. $a = \dfrac{c\sin\alpha}{\sin\gamma}$

Let $\alpha = 80°$, $\gamma = 68°$, and $c = 17$ mi. $a = \dfrac{(17\text{ mi})\sin 80°}{\sin 68°}$

Use a calculator to approximate a. $a \approx 18.056539$ mi

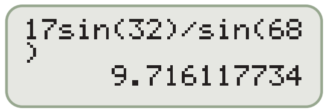
Round a to two significant digits. $\boxed{a \approx 18\text{ mi}}$

STEP 4 Draw and label the triangle.

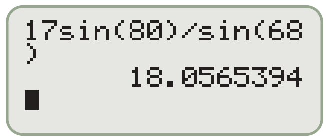


Technology Tip

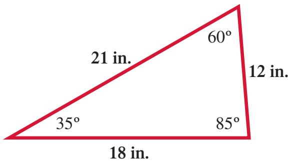
Step 2: Use the calculator to find $b = \dfrac{17\sin 32°}{\sin 68°}$.



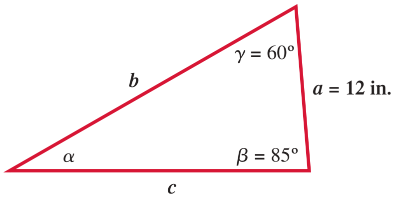
Step 3: Use the calculator to find $a = \dfrac{17\sin 80°}{\sin 68°}$.



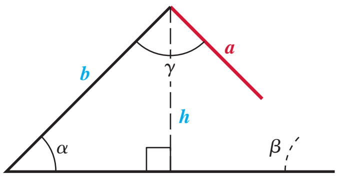
■ **Answer:** $\alpha = 35°$, $b \approx 21$ in., $c \approx 18$ in.



■ **YOUR TURN** Solve the triangle.



Case 2 (Ambiguous Case): Two Sides and One Angle (SSA)



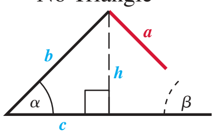
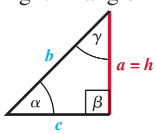
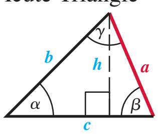
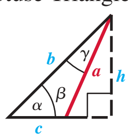
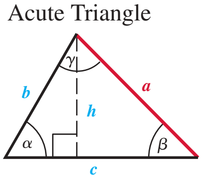
If we are given the measures of two sides and an angle opposite one of the sides, then we call that Case 2, SSA (side-side-angle). This case is called the ambiguous case, because the given information by itself can represent one triangle, two triangles, or no triangle at all. If the angle given is acute, then the possibilities are zero, one, or two triangles. If the angle given is obtuse, then the possibilities are zero or one triangle. The possibilities come from the fact that $\sin\alpha = k$, where $0 < k < 1$, has two solutions for α: one in quadrant I (acute angle) and one in quadrant II (obtuse angle).

In the figure on the left, note that

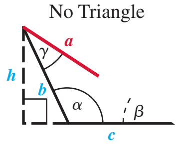
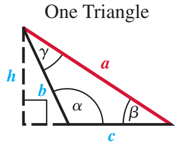
- $h = b\sin\alpha$ by the definition of the sine ratio,
- a may turn out to be smaller than, equal to, or larger than h.

Since $0 < \sin\alpha < 1$, then $h < b$.

Given Angle (α) Is Acute

CONDITION	PICTURE	NUMBER OF TRIANGLES
$0 < a < h$, in this case, $\sin\beta > 1$ (impossible)	No Triangle 	0
$a = h$, in this case, $\sin\beta = 1$	Right Triangle 	1
$h < a < b$, in this case, $0 < \sin\beta < 1$	Acute Triangle  Obtuse Triangle 	2
$a \geq b$, in this case, $0 < \sin\beta < 1$	Acute Triangle 	1

Given Angle (α) Is Obtuse

CONDITION	PICTURE	NUMBER OF TRIANGLES
$a \leq b$, in this case, $\sin\beta \geq 1$ (impossible)	No Triangle 	0
$a > b$, in this case, $0 < \sin\beta < 1$	One Triangle 	1

Study Tip

If an angle given is obtuse, then the side opposite that angle must be longer than the other sides (longest side opposite largest angle).

EXAMPLE 3 Solving the Ambiguous Case (SSA)—One Triangle

Solve the triangle $a = 23$ ft, $b = 11$ ft, and $\alpha = 122°$.

Solution:

This is the ambiguous case because the measures of two sides and an angle opposite one of those sides are given. Since the given angle α is obtuse and $a > b$, we expect one triangle.

STEP 1 Find β.

Use the Law of Sines. $$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

Isolate $\sin\beta$. $$\sin\beta = \frac{b\sin\alpha}{a}$$

Let $a = 23$ ft, $b = 11$ ft, and $\alpha = 122°$. $$\sin\beta = \frac{(11\text{ ft})\sin 122°}{23\text{ ft}}$$

Use a calculator to evaluate the right side. $$\sin\beta \approx 0.40558822$$

Use a calculator to approximate β. $$\beta \approx \sin^{-1}(0.40558822) \approx \boxed{24°}$$

STEP 2 Find γ.

The measures of angles in a triangle sum to 180°. $$\alpha + \beta + \gamma = 180°$$

Substitute $\alpha = 122°$ and $\beta \approx 24°$. $$122° + 24° + \gamma \approx 180°$$

Solve for γ. $$\boxed{\gamma \approx 34°}$$

STEP 3 Find c.

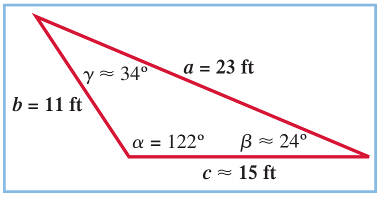
Use the Law of Sines. $$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c}$$

Isolate c. $$c = \frac{a\sin\gamma}{\sin\alpha}$$

Substitute $a = 23$ ft, $\alpha = 122°$, and $\gamma \approx 34°$. $$c \approx \frac{(23\text{ ft})\sin 34°}{\sin 122°}$$

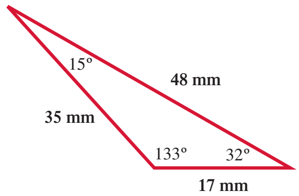
Use a calculator to evaluate c. $$\boxed{c \approx 15\text{ ft}}$$

STEP 4 Draw and label the triangle.



■ **YOUR TURN** Solve the triangle $\alpha = 133°$, $a = 48$ mm, and $c = 17$ mm.

■ **Answer:** $\beta \approx 32°$, $\gamma \approx 15°$, $b \approx 35$ mm



EXAMPLE 4 Solving the Ambiguous Case (SSA)—Two Triangles

Solve the triangle $a = 8.1$ m, $b = 8.3$ m, and $\alpha = 72°$.

Solution:

This is the ambiguous case because the measures of two sides and an angle opposite one of those sides are given. Since the given angle α is acute and $a < b$, we expect two triangles.

STEP 1 Find β.

Write the Law of Sines for the given information. $\qquad \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

Isolate $\sin\beta$. $\qquad \sin\beta = \dfrac{b\sin\alpha}{a}$

Let $a = 8.1$ m, $b = 8.3$ m, and $\alpha = 72°$. $\qquad \sin\beta = \dfrac{(8.3\text{ m})\sin 72°}{8.1\text{ m}}$

Use a calculator to evaluate the right side. $\qquad \sin\beta \approx 0.974539393$

Use a calculator to approximate β. Note that β can be acute or obtuse. $\qquad \beta \approx \sin^{-1}(0.974539393) \approx 77°$

This is the quadrant I solution (β is acute). $\qquad \boxed{\beta_1 \approx 77°}$

The quadrant II solution (β is obtuse) is $\beta_2 = 180 - \beta_1$. $\qquad \boxed{\beta_2 \approx 103°}$

STEP 2 Find γ.

The measures of the angles in a triangle sum to 180°. $\qquad \alpha + \beta + \gamma = 180°$

Substitute $\alpha = 72°$ and $\beta_1 \approx 77°$. $\qquad 72° + 77° + \gamma_1 \approx 180°$

Solve for γ_1. $\qquad \boxed{\gamma_1 \approx 31°}$

Substitute $\alpha = 72°$ and $\beta_2 \approx 103°$. $\qquad 72° + 103° + \gamma_2 \approx 180°$

Solve for γ_2. $\qquad \boxed{\gamma_2 \approx 5°}$

STEP 3 Find c.

Use the Law of Sines. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}$

Isolate c. $\quad c = \dfrac{a\sin\gamma}{\sin\alpha}$

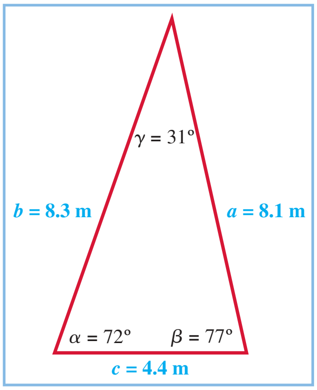
Substitute $a = 8.1$ m, $\alpha = 72°$, and $\gamma_1 \approx 31°$. $\quad c_1 \approx \dfrac{(8.1\text{ m})\sin 31°}{\sin 72°}$

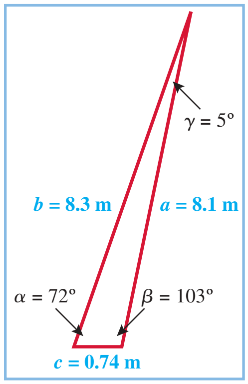
Use a calculator to evaluate c_1. $\quad \boxed{c_1 \approx 4.4\text{ m}}$

Substitute $a = 8.1$ m, $\alpha = 72°$, and $\gamma_2 \approx 5°$. $\quad c_2 \approx \dfrac{(8.1\text{ m})\sin 5°}{\sin 72°}$

Use a calculator to evaluate c_2. $\quad \boxed{c_2 \approx 0.74\text{ m}}$

STEP 4 Draw and label the two triangles.





Study Tip

Notice that when there are two solutions in the SSA case, one triangle will be obtuse.

EXAMPLE 5 Solving the Ambiguous Case (SSA)—No Triangle

Solve the triangle $\alpha = 107°$, $a = 6$, and $b = 8$.

Solution:

This is the ambiguous case because the measures of two sides and an angle opposite one of those sides are given. Since the given angle α is obtuse and $a < b$, we expect no triangle since the longer side is not opposite the largest angle.

Write the Law of Sines. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

Isolate $\sin\beta$. $\quad \sin\beta = \dfrac{b\sin\alpha}{a}$

Let $\alpha = 107°$, $a = 6$, and $b = 8$. $\quad \sin\beta = \dfrac{8\sin 107°}{6}$

Use a calculator to evaluate the right side. $\quad \sin\beta \approx 1.28 > 1$

Since the range of the sine function is $[-1, 1]$, there is no angle β such that $\sin\beta \approx 1.28$. Therefore, there is $\boxed{\text{no triangle}}$ with the given measurements.

Note: Had the geometric contradiction not been noticed, your work analytically will show a contradiction of $\sin\beta > 1$.

SECTION 4.4 SUMMARY

In this section, we solved oblique triangles. When given the measures of three parts of a triangle, we classify the triangle according to the given data (sides and angles). Four cases arise:

- one side and two angles (AAS or ASA)
- two sides and the angle opposite one of the sides (SSA)
- two sides and the angle between sides (SAS)
- three sides (SSS)

The Law of Sines

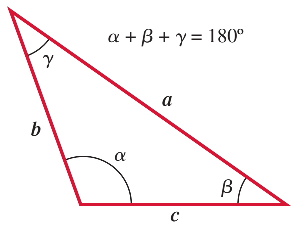
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

can be used to solve the first two cases (AAS or ASA, and SSA). It is important to note that the SSA case is called the ambiguous case because any one of three results is possible: no triangle, one triangle, or two triangles.

SECTION 4.4 EXERCISES

SKILLS

In Exercises 1–6, classify each triangle problem as cases AAS, ASA, SAS, SSA, or SSS on the basis of the given information.



1. c, a, and α

2. c, a, and γ

3. a, b, and c

4. a, b, and γ

5. α, β, and c

6. β, γ, and a

In Exercises 7–16, solve each of the following triangles with the given measures.

7. $\alpha = 45°, \beta = 60°, a = 10$ m

8. $\beta = 75°, \gamma = 60°, b = 25$ in.

9. $\alpha = 46°, \gamma = 72°, b = 200$ cm

10. $\gamma = 100°, \beta = 40°, a = 16$ ft

11. $\alpha = 16.3°, \gamma = 47.6°, c = 211$ yd

12. $\beta = 104.2°, \gamma = 33.6°, a = 26$ in.

13. $\alpha = 30°, \beta = 30°, c = 12$ m

14. $\alpha = 45°, \gamma = 75°, c = 9$ in.

15. $\beta = 26°, \gamma = 57°, c = 100$ yd

16. $\alpha = 80°, \gamma = 30°, b = 3$ ft

In Exercises 17–34, the measures of two sides and an angle are given. Determine whether a triangle (or two) exist, and if so, solve the triangle(s).

17. $a = 4, b = 5, \alpha = 16°$

18. $b = 30, c = 20, \beta = 70°$

19. $a = 12, c = 12, \gamma = 40°$

20. $b = 111, a = 80, \alpha = 25°$

21. $a = 21, b = 14, \beta = 100°$

22. $a = 13, b = 26, \alpha = 120°$

23. $\alpha = 30°, b = 18, a = 9$

24. $\alpha = 45°, b = \sqrt{2}, a = 1$

25. $\alpha = 34°, b = 7, a = 10$

26. $\alpha = 71°, b = 5.2, a = 5.2$

27. $\alpha = 21.3°, b = 6.18, a = 6.03$

28. $\alpha = 47.3°, b = 7.3, a = 5.32$

29. $\alpha = 116°, b = 4\sqrt{3}, a = 5\sqrt{2}$

30. $\alpha = 51°, b = 4\sqrt{3}, a = 4\sqrt{5}$

31. $b = 500, c = 330, \gamma = 40°$

32. $b = 16, a = 9, \beta = 137°$

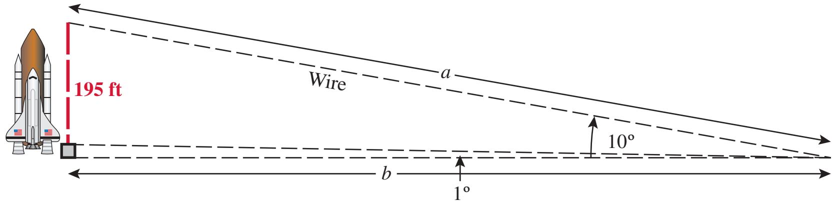
33. $a = \sqrt{2}, b = \sqrt{7}, \beta = 106°$

34. $b = 15.3, c = 27.2, \gamma = 11.6°$

APPLICATIONS

For Exercises 35 and 36, refer to the following:

NASA Kennedy Space Center

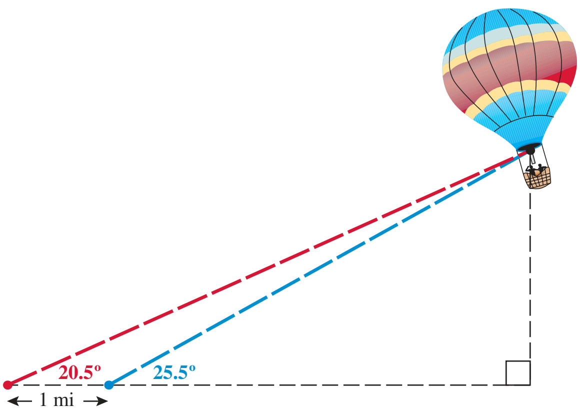


On the launch pad at Kennedy Space Center, there is an escape basket that can hold four astronauts. The basket slides down a wire that is attached 195 feet high, above the base of the launch pad. The angle of inclination measured from where the basket would touch the ground to the base of the launch pad is 1°, and the angle of inclination from that same point to where the wire is attached is 10°.

35. **NASA.** How long is the wire *a*?

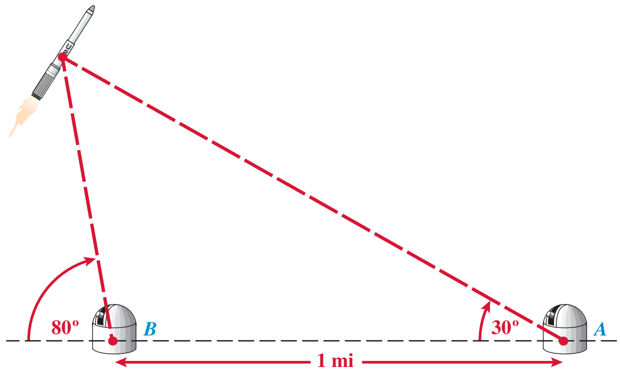
36. **NASA.** How far from the launch pad does the basket touch the ground? That is, find *b*.

37. **Hot-Air Balloon.** A hot-air balloon is sighted at the same time by two friends who are 1.0 mile apart on the same side of the balloon. The angles of elevation from the two friends are 20.5° and 25.5°. How high is the balloon?



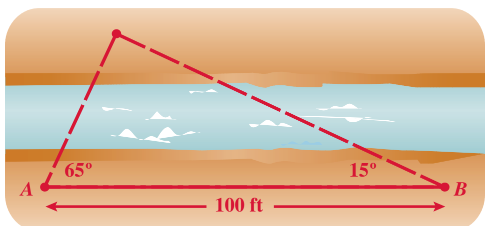
38. **Hot-Air Balloon.** A hot-air balloon is sighted at the same time by two friends who are 2 miles apart on the same side of the balloon. The angles of elevation from the two friends are 10° and 15°. How high is the balloon?

39. **Rocket Tracking.** A tracking station has two telescopes that are 1.0 mile apart. The telescopes can lock onto a rocket after it is launched and record the angles of elevation to the rocket. If the angles of elevation from telescopes *A* and *B* are 30° and 80°, respectively, then how far is the rocket from telescope *A*?



40. **Rocket Tracking.** Given the data in Exercise 39, how far is the rocket from telescope *B*?

41. **Distance Across River.** An engineer wants to construct a bridge across a fast-moving river. Using a straight-line segment between two points that are 100 feet apart along his side of the river, he measures the angles formed when sighting the point on the other side where he wants to have the bridge end. If the angles formed at points *A* and *B* are 65° and 15°, respectively, how far is it from point *A* to the point on the other side of the river? Round to the nearest foot.

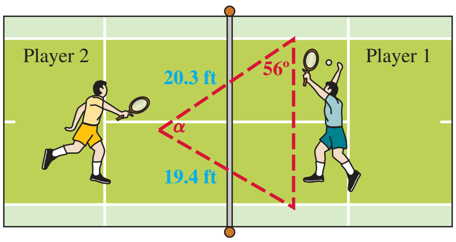


42. **Distance Across River.** Given the data in Exercise 41, how far is it from point *B* to the point on the other side of the river? Round to the nearest foot.

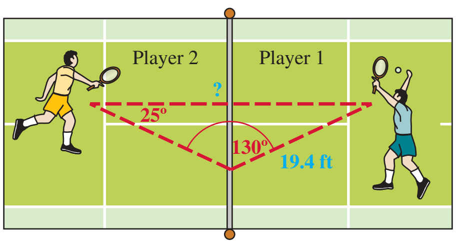
43. Lifeguard Posts. Two lifeguard chairs, labeled P and Q, are located 400 feet apart. A troubled swimmer is spotted by both lifeguards. If the lifeguard at P reports the swimmer at angle 35° (with respect to the line segment connecting P and Q) and the lifeguard at Q reports the swimmer at angle 41°, how far is the swimmer from P?

44. Rock Climbing. A rock climbing enthusiast is creating a climbing route rated as 5.8 level (i.e., medium difficulty) on the wall at the local rock gym. Given the difficulty of the route, he wants to avoid placing any two holds on the same vertical or horizontal line on the wall. If he places holds at P, Q, and R such that $\angle QPR = 40°$, $QR = 6$ feet, and $QP = 4.5$ feet, how far is the hold at P from the hold at R?

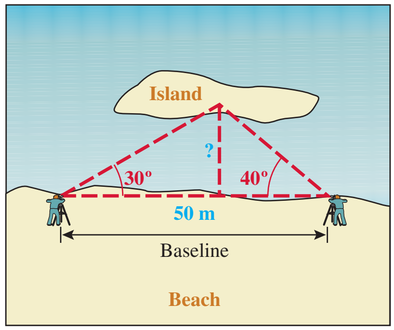
45. Tennis. After a long rally between two friends playing tennis, Player 2 lobs the ball into Player 1's court, enabling him to hit an overhead smash such that the angle between the racquet head (at the point of contact with the ball) and his body is 56°. The ball travels 20.3 feet to the other side of the court where Player 2 volleys the ball off the ground at an angle α such that the ball travels directly toward Player 1. The ball travels 19.4 feet during this return. Find angle α.



46. Tennis. Shocked by the move Player 1 made in Exercise 45, Player 2 is forced to quickly deflect the ball straight back to Player 1. Player 1 reaches behind himself and is able to contact the ball at the same height above the ground with which Player 1 initially hit it. If the angle between Player 1's racquet position at the end of the previous shot and its current position at the point of contact of this shot is 130°, and the angle with which it contacts the ball is 25°, how far has the ball traveled horizontally as a result of Player 2's hit?

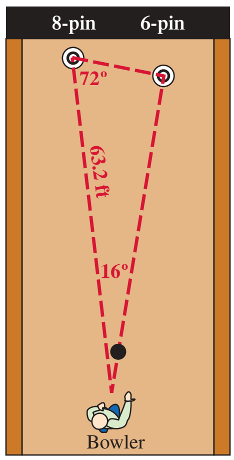


47. Surveying. There are two stations along the shoreline and the distance along the beach between the two stations is 50 meters. The angles between the baseline (beach) and the line of sight to the island are 30° and 40°. Find the shortest distance from the beach to the island. Round to the nearest meter.



48. Surveying. There are two stations along a shoreline and the distance along the beach between the two stations is 200 feet. The angles between the baseline (beach) and the line of sight to the island are 30° and 50°. Find the shortest distance from the beach to the island. Round to the nearest foot.

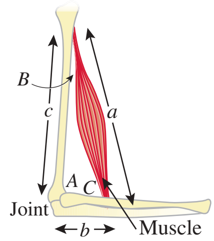
49. Bowling. The 6-8 split is common in bowling. To make this split, a bowler stands dead center and throws the ball hard and straight directly toward the right of the 6 pin. The distance from the ball at the point of release to the 8 pin is 63.2 feet. See the diagram. How far does the ball travel from the bowler to the 6 pin?



50. Bowling. A bowler is said to get a strike on the "Brooklyn side" of the head pin if he hits the head pin on the side opposite the pocket. (For a right-handed bowler, the pocket is to the right of the head pin.) There is a small range for the angle at which the ball must contact the head pin in order to convert all of the pins. If the measurements are as shown, how far does the ball travel (assuming it is thrown straight with no hook) before it contacts the head pin?

For Exercises 51 and 52, refer to the following:

To quantify the torque (rotational force) of the elbow joint of a human arm (see the figure to the right), it is necessary to identify angles A, B, and C as well as lengths a, b, and c. Measurements performed on an arm determine that the measure of angle C is 95°, the measure of angle A is 82°, and the length of the muscle a is 23 centimeters.



51. **Health/Medicine.** Find the length of the forearm from the elbow joint to the muscle attachment b.

52. **Health/Medicine.** Find the length of the upper arm from the muscle attachment to the elbow joint c.

CATCH THE MISTAKE

In Exercises 53 and 54, explain the mistake that is made.

53. Solve the triangle $\alpha = 120°, a = 7$, and $b = 9$.

Solution:

Use the Law of Sines to find β. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

Let $\alpha = 120°, a = 7$, and $b = 9$. $\quad \dfrac{\sin 120°}{7} = \dfrac{\sin\beta}{9}$

Solve for $\sin\beta$. $\quad \sin\beta = 1.113$

Solve for β. $\quad \beta = 42°$

Sum the angle measures to 180°. $\quad 120° + 42° + \gamma = 180°$

Solve for γ. $\quad \gamma = 18°$

Use the Law of Sines to find c. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}$

Let $\alpha = 120°, a = 7$, and $\gamma = 18°$. $\quad \dfrac{\sin 120°}{7} = \dfrac{\sin 18°}{c}$

Solve for c. $\quad c = 2.5$

$\alpha = 120°, \beta = 42°, \gamma = 18°, a = 7, b = 9$, and $c = 2.5$.

This is incorrect. The longest side is not opposite the longest angle. There is no triangle that makes the original measurements work. What mistake was made?

54. Solve the triangle $\alpha = 40°, a = 7$, and $b = 9$.

Solution:

Use the Law of Sines to find β. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

Let $\alpha = 40°, a = 7$, and $b = 9$. $\quad \dfrac{\sin 40°}{7} = \dfrac{\sin\beta}{9}$

Solve for $\sin\beta$. $\quad \sin\beta = 0.826441212$

Solve for β. $\quad \beta = 56°$

Find γ. $\quad 40° + 56° + \gamma = 180°$

$\gamma = 84°$

Use the Law of Sines to find c. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}$

Let $\alpha = 40°, a = 7$, and $\gamma = 84°$. $\quad \dfrac{\sin 40°}{7} = \dfrac{\sin 84°}{c}$

Solve for c. $\quad c = 11$

$\alpha = 40°, \beta = 56°, \gamma = 84°, a = 7, b = 9$ and $c = 11$.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 55–60, determine whether each statment is true or false.

55. The Law of Sines applies only to right triangles.

56. If you are given the measures of two sides and any angle, there is a unique solution for the triangle.

57. An acute triangle is an oblique triangle.

58. An obtuse triangle is an oblique triangle.

59. If you are given two sides that have the same length in a triangle, then there can be at most one triangle.

60. If α is obtuse and $\beta = \dfrac{\alpha}{2}$, then the situation is unambiguous.

CHALLENGE

The following identities are useful in Exercises 61 and 62, and will be derived in Chapter 6.

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(2x) = 2\sin x\cos x$$

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

61. Mollweide's Identity. For any triangle, the following identity is true. It is often used to check the solution of a triangle since all six pieces of information (three sides and three angles) are involved. Derive the identity using the Law of Sines.

$$(a+b)\sin\left(\tfrac{1}{2}\gamma\right) = c\cos\left[\tfrac{1}{2}(\alpha-\beta)\right]$$

62. The Law of Tangents. Use the Law of Sines and trigonometric identities to show that for any triangle, the following is true:

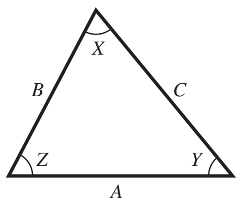
$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(\alpha-\beta)\right]}{\tan\left[\frac{1}{2}(\alpha+\beta)\right]}$$

63. Use the Law of Sines to prove that all angles in an equilateral triangle must have the same measure.

64. Suppose that you have a triangle with side lengths a, b, and c, and angles α, β, and γ, respectively, directly across from them. If it is known that $a = \frac{1}{\sqrt{2}}b$, $c = 2$, α is an acute angle, and $\beta = 2\alpha$, solve the triangle.

TECHNOLOGY

For Exercises 65–70, let A, B, and C be the lengths of the three sides with X, Y, and Z as the opposite corresponding angles. Write a program to solve the given triangle with a calculator.



65. $A = 10$, $Y = 40°$, and $Z = 72°$

66. $B = 42.8$, $X = 31.6°$, and $Y = 82.2°$

67. $A = 22$, $B = 17$, and $X = 105°$

68. $B = 16.5$, $C = 9.8$, and $Z = 79.2°$

69. $A = 25.7$, $C = 12.2$, and $X = 65°$

70. $A = 54.6$, $B = 12.9$, and $Y = 23°$

PREVIEW TO CALCULUS

In calculus, some applications of the derivative require the solution of triangles. In Exercises 71–74, solve each triangle using the Law of Sines.

71. In an oblique triangle ABC, $\beta = 45°$, $\gamma = 60°$, and $b = 20$ in. Find the length of a. Round your answer to the nearest unit.

72. In an oblique triangle ABC, $\beta = \frac{2\pi}{9}$, $\gamma = \frac{5\pi}{9}$, and $a = 200$ ft. Find the length of c. Round your answer to the nearest unit.

73. In an oblique triangle ABC, $b = 14$ m, $c = 14$ m, and $\alpha = \frac{4\pi}{7}$. Find the length of a. Round your answer to the nearest unit.

74. In an oblique triangle ABC, $b = 30$ cm, $c = 45$ cm, and $\gamma = 35°$. Find the length of a. Round your answer to the nearest unit.

SECTION 4.5 THE LAW OF COSINES

SKILLS OBJECTIVES

- Solve SAS triangles.
- Solve SSS triangles.
- Find the area of triangles in the SAS case.
- Find the area of triangles in the SSS case.

CONCEPTUAL OBJECTIVES

- Understand the derivation of the Law of Cosines.
- Develop a strategy for which angles (larger or smaller) and which method (the Law of Sines or the Law of Cosines) to select to solve oblique triangles.

Solving Oblique Triangles Using the Law of Cosines

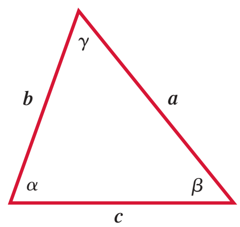
In Section 4.4, we learned that to solve oblique triangles means to find all three side lengths and angle measures, and that at least one side length must be known. We need two additional pieces of information to solve an oblique triangle (combinations of side lengths and/or angles). We found that there are four cases:

- Case 1: AAS or ASA (measures of two angles and a side are given)
- Case 2: SSA (measures of two sides and an angle opposite one of the sides are given)
- Case 3: SAS (measures of two sides and the angle between them are given)
- Case 4: SSS (measures of three sides are given)

We used the Law of Sines to solve Case 1 and Case 2 triangles. Now, we need the *Law of Cosines* to solve Case 3 and Case 4 triangles.

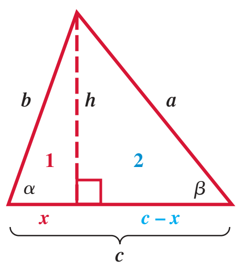
WORDS | **MATH**

Start with an oblique (acute) triangle.



Drop a perpendicular line segment from γ to side c with height h.

The result is two right triangles within the larger triangle.



Use the Pythagorean theorem to write the relationship between the side lengths in both right triangles.

Triangle 1: $x^2 + h^2 = b^2$

Triangle 2: $(c - x)^2 + h^2 = a^2$

Solve for h^2 in both equations.	
Triangle 1:	$h^2 = b^2 - x^2$
Triangle 2:	$h^2 = a^2 - (c - x)^2$
Since the segment of length h is shared, set $h^2 = h^2$, for the two triangles.	$b^2 - x^2 = a^2 - (c - x)^2$
Multiply out the squared binomial on the right.	$b^2 - x^2 = a^2 - \left(c^2 - 2cx + x^2\right)$
Eliminate the parentheses.	$b^2 - x^2 = a^2 - c^2 + 2cx - x^2$
Add x^2 to both sides.	$b^2 = a^2 - c^2 + 2cx$
Isolate a^2.	$a^2 = b^2 + c^2 - 2cx$
Notice that $\cos\alpha = \dfrac{x}{b}$. Let $x = b\cos\alpha$.	$\boxed{a^2 = b^2 + c^2 - 2bc\cos\alpha}$

Note: If we instead drop the perpendicular line segment with length h from the angle α or the angle β, we can derive the other two parts of the Law of Cosines:

$$\boxed{b^2 = a^2 + c^2 - 2ac\cos\beta} \quad \text{and} \quad \boxed{c^2 = a^2 + b^2 - 2ab\cos\gamma}$$

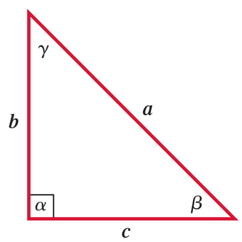
The Law of Cosines

For a triangle with sides of length a, b, and c, and opposite angle measures α, β, and γ, the following equations are true:

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



It is important to note that the Law of Cosines can be used to find side lengths or angles in any triangle in cases SAS or SSS, as long as three of the four variables in any of the equations are known, the fourth can be calculated.

Notice that in the special case of a right triangle (say, $\alpha = 90°$),

$$a^2 = b^2 + c^2 - 2bc\underbrace{\cos 90°}_{0}$$

one of the equations of the Law of Cosines reduces to the Pythagorean theorem:

$$\underbrace{a^2}_{\text{hyp}} = \underbrace{b^2}_{\text{leg}} + \underbrace{c^2}_{\text{leg}}$$

The Pythagorean theorem can thus be regarded as a special case of the Law of Cosines.

Case 3: Solving Oblique Triangles (SAS)

Study Tip

The Pythagorean theorem is a special case of the Law of Cosines.

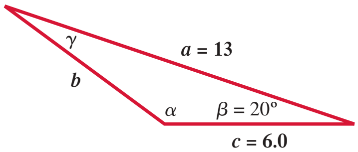
We now solve SAS triangle problems where the measures of two sides and the angle between them are given. We start by using the Law of Cosines to solve for the length of the side opposite the given angle. We then can apply either the Law of Sines or the Law of Cosines to find the second angle measure.

EXAMPLE 1 Using the Law of Cosines to Solve a Triangle (SAS)

Solve the triangle $a = 13$, $c = 6.0$, and $\beta = 20°$.

Solution:

The measures of two sides and the angle between them are given (SAS).



Notice that the Law of Sines can't be used, because it requires the measures of at least one angle and the side opposite that angle.

STEP 1 Find b.

Apply the Law of Cosines that involves β. $b^2 = a^2 + c^2 - 2ac\cos\beta$

Let $a = 13, c = 6.0$, and $\beta = 20°$. $b^2 = 13^2 + 6^2 - 2(13)(6)\cos 20°$

Evaluate the right side with a calculator. $b^2 \approx 58.40795$

Solve for b. $b \approx \pm 7.6425$

Round to two significant digits; b can only be positive. $\boxed{b \approx 7.6}$

STEP 2 Find γ.

Use the Law of Sines to find the smaller angle measure, γ. $\dfrac{\sin\gamma}{c} = \dfrac{\sin\beta}{b}$

Isolate $\sin\gamma$. $\sin\gamma = \dfrac{c\sin\beta}{b}$

Let $b \approx 7.6, c = 6.0$, and $\beta = 20°$. $\sin\gamma \approx \dfrac{6\sin 20°}{7.6}$

Apply the inverse sine function. $\gamma \approx \sin^{-1}\left(\dfrac{6\sin 20°}{7.6}\right)$

Evaluate the right side with a calculator. $\gamma \approx 15.66521°$

Round to the nearest degree. $\boxed{\gamma \approx 16°}$

STEP 3 Find α.

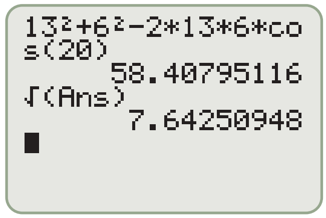
The angle measures must sum to 180°. $\alpha + 20° + 16° \approx 180°$

Solve for α. $\boxed{\alpha \approx 144°}$

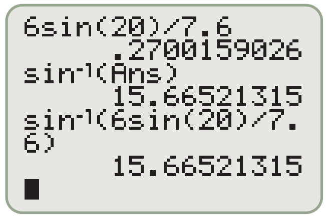
■ **YOUR TURN** Solve the triangle $b = 4.2$, $c = 1.8$, and $\alpha = 35°$.

Technology Tip

Step 1: Use the calculator to find the value of b.



Step 2: Use the calculator to find γ.



■ **Answer:** $a \approx 2.9$, $\gamma \approx 21°$, $\beta \approx 124°$

Notice the steps we took in solving an SAS triangle:

1. Find the length of the side opposite the given angle using the Law of Cosines.
2. Solve for the smaller angle using the Law of Sines.
3. Solve for the larger angle using the fact that angles of a triangle sum to 180°.

You may be thinking, "Would it matter if we had solved for α before solving for γ?" Yes, it does matter—in this problem you cannot solve for α by the Law of Sines before

Study Tip

Although the Law of Sines is sometimes ambiguous, the Law of Cosines is never ambiguous.

finding γ. The Law of Sines can be used only on the smaller angle (opposite the shortest side). If we had tried to use the Law of Sines with the obtuse angle α, the inverse sine would have resulted in $\alpha = 36°$. Since the sine function is positive in QI and QII, we would not know whether that angle was $\alpha = 36°$ or its supplementary angle $\alpha = 144°$. Notice that $c < a$; therefore, the angles opposite those sides must have the same relationship, $\gamma < \alpha$. We choose the smaller angle first. Alternatively, if we want to solve for the obtuse angle first, we can use the Law of Cosines to solve for α. If you use the Law of Cosines to find the second angle, you can choose either angle. The Law of Cosines can be used to find the measure of either acute or obtuse angles.

Case 4: Solving Oblique Triangles (SSS)

We now solve oblique triangles when all three side lengths are given (the SSS case). In this case, start by finding the largest angle (opposite the largest side) using the Law of Cosines. Then apply the Law of Sines to find either of the remaining two angles. Lastly, find the third angle with the triangle angle sum identity.

EXAMPLE 2 Using the Law of Cosines to Solve a Triangle (SSS)

Solve the triangle $a = 8$, $b = 6$, and $c = 7$.

Solution:

STEP 1 Identify the largest angle, which is α.

Write the equation of the Law of Cosines that involves α.	$a^2 = b^2 + c^2 - 2bc\cos\alpha$
Let $a = 8$, $b = 6$, and $c = 7$.	$8^2 = 6^2 + 7^2 - 2(6)(7)\cos\alpha$
Simplify and isolate $\cos\alpha$.	$\cos\alpha = \dfrac{6^2 + 7^2 - 8^2}{2(6)(7)} = 0.25$
Approximate with a calculator.	$\alpha = \cos^{-1}(0.25) \approx \boxed{75.5°}$

STEP 2 Find either of the remaining angles. We will solve for β.

Write the Law of Sines.	$\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$
Isolate $\sin\beta$.	$\sin\beta = \dfrac{b\sin\alpha}{a}$
Let $a = 8$, $b = 6$, and $\alpha = 75.5°$.	$\sin\beta \approx \dfrac{6\sin 75.5°}{8}$
Approximate with a calculator.	$\beta \approx \sin^{-1}\left(\dfrac{6\sin 75.5°}{8}\right) \approx \boxed{46.6°}$

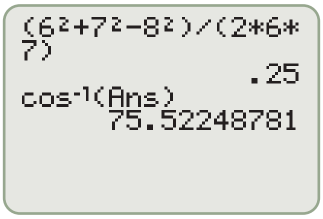
STEP 3 Find the third angle, γ.

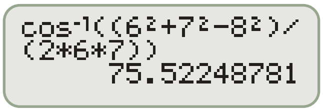
The sum of the angle measures is 180°.	$75.5° + 46.6° + \gamma \approx 180°$
Solve for γ.	$\boxed{\gamma \approx 57.9°}$

■ **YOUR TURN** Solve the triangle $a = 5$, $b = 7$, and $c = 8$.

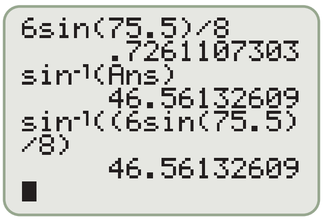
Technology Tip

Step 1: Use the calculator to find the value of α.



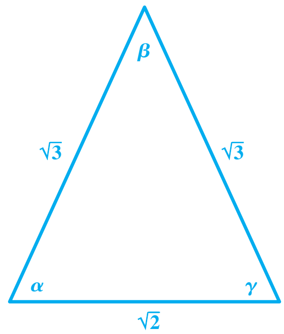


Step 2: Use the calculator to find β.

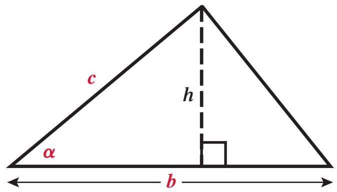


■ **Answer:** $\alpha \approx 38.2°$, $\beta \approx 60.0°$, $\gamma \approx 81.8°$

The Area of a Triangle



The general formula for the area of a triangle and the sine function together can be used to develop a formula for the area of a triangle when the measures of two sides and the angle between them are given.

Words	Math
Start with an acute triangle, given b, c, and α.	
Write the sine ratio in the right triangle for the acute angle α.	$\sin\alpha = \frac{h}{c}$
Solve for h.	$h = c\sin\alpha$
Write the formula for area of a triangle.	$A_{\text{triangle}} = \frac{1}{2}bh$
Substitute $h = c\sin\alpha$.	$\boxed{A_{\text{SAS}} = \frac{1}{2}bc\sin\alpha}$

Now we can calculate the area of this triangle with the given information (the measures of two sides and the angle between them: b, c, and α). Similarly, it can be shown that the other formulas for SAS triangles are

$$\boxed{A_{\text{SAS}} = \frac{1}{2}ab\sin\gamma} \quad \text{and} \quad \boxed{A_{\text{SAS}} = \frac{1}{2}ac\sin\beta}$$

Area of a Triangle (SAS)

For any triangle where the measures of two sides and the angle between them are known, the area for that triangle is given by one of the following formulas (depending on which angle and side measures are given):

$$A_{\text{SAS}} = \frac{1}{2}bc\sin\alpha \qquad \text{when } b, c, \text{ and } \alpha \text{ are known}$$

$$A_{\text{SAS}} = \frac{1}{2}ab\sin\gamma \qquad \text{when } a, b, \text{ and } \gamma \text{ are known}$$

$$A_{\text{SAS}} = \frac{1}{2}ac\sin\beta \qquad \text{when } a, c, \text{ and } \beta \text{ are known}$$

In other words, the area of a triangle equals one-half the product of two of its sides and the sine of the angle between them.

Technology Tip

Use the calculator to find A.

```
1/2*7*9.3*sin(86
)
         32.47070984
```

■ **Answer:** $6.2\ \text{m}^2$

EXAMPLE 3 Finding the Area of a Triangle (SAS Case)

Find the area of the triangle $a = 7.0$ ft, $b = 9.3$ ft, and $\gamma = 86°$.

Solution:

Apply the area formula where a, b, and γ are given. $A = \frac{1}{2}ab\sin\gamma$

Substitute $a = 7.0$ ft, $b = 9.3$ ft, and $\gamma = 86°$. $A = \frac{1}{2}(7.0\text{ ft})(9.3\text{ ft})\sin 86°$

Approximate with a calculator. $A \approx 32.47071\ \text{ft}^2$

Round to two significant digits. $\boxed{A \approx 32\ \text{ft}^2}$

■ **YOUR TURN** Find the area of the triangle $a = 3.2$ m, $c = 5.1$ m, and $\beta = 49°$.

Study Tip

The Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1$$

can be proven using any of the three trigonometric definitions:

$$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$$

The Law of Cosines can be used to develop a formula for the area of an SSS triangle, called **Heron's formula**.

WORDS	MATH
Start with any of the formulas for SAS triangles.	$A = \frac{1}{2}ab\sin\gamma$
Square both sides.	$A^2 = \frac{1}{4}a^2b^2\sin^2\gamma$
Isolate $\sin^2\gamma$.	$\frac{4A^2}{a^2b^2} = \sin^2\gamma$
Apply the Pythagorean identity.	$\frac{4A^2}{a^2b^2} = 1 - \cos^2\gamma$
Factor the difference of the two squares on the right.	$\frac{4A^2}{a^2b^2} = (1 - \cos\gamma)(1 + \cos\gamma)$
Solve the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos\gamma$, for $\cos\gamma$.	$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$
Substitute $\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$ into $\frac{4A^2}{a^2b^2} = (1 - \cos\gamma)(1 + \cos\gamma)$.	$\frac{4A^2}{a^2b^2} = \left[1 - \frac{a^2 + b^2 - c^2}{2ab}\right]\left[1 + \frac{a^2 + b^2 - c^2}{2ab}\right]$
Combine the expressions in brackets.	$\frac{4A^2}{a^2b^2} = \left[\frac{2ab - a^2 - b^2 + c^2}{2ab}\right]\left[\frac{2ab + a^2 + b^2 - c^2}{2ab}\right]$

Group the terms in the numerators on the right.
$$\frac{4A^2}{a^2b^2} = \left[\frac{-\left(a^2 - 2ab + b^2\right) + c^2}{2ab}\right]\left[\frac{\left(a^2 + 2ab + b^2\right) - c^2}{2ab}\right]$$

Write the numerators on the right as the difference of two squares.
$$\frac{4A^2}{a^2b^2} = \left[\frac{c^2 - (a - b)^2}{2ab}\right]\left[\frac{(a + b)^2 - c^2}{2ab}\right]$$

Factor the numerators on the right. Recall: $x^2 - y^2 = (x - y)(x + y)$.
$$\frac{4A^2}{a^2b^2} = \left[\frac{(c - [a - b])(c + [a - b])}{2ab}\right]\left[\frac{([a + b] - c)([a + b] + c)}{2ab}\right]$$

Simplify.
$$\frac{4A^2}{a^2b^2} = \left[\frac{(c - a + b)(c + a - b)}{2ab}\right]\left[\frac{(a + b - c)(a + b + c)}{2ab}\right]$$
$$\frac{4A^2}{a^2b^2} = \frac{(c - a + b)(c + a - b)(a + b - c)(a + b + c)}{4a^2b^2}$$

Solve for A^2 by multiplying both sides by $\frac{a^2b^2}{4}$.
$$A^2 = \frac{1}{16}(c - a + b)(c + a - b)(a + b - c)(a + b + c)$$

The semiperimeter s is half the perimeter of the triangle.
$$s = \frac{a + b + c}{2}$$

Manipulate each of the four factors:
$$c - a + b = a + b + c - 2a = 2s - 2a = 2(s - a)$$
$$c + a - b = a + b + c - 2b = 2s - 2b = 2(s - b)$$
$$a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c)$$
$$a + b + c = 2s$$

Substitute in these values for the four factors.
$$A^2 = \frac{1}{16}\cdot 2(s - a)\cdot 2(s - b)\cdot 2(s - c)\cdot 2s$$

Simplify.
$$A^2 = s(s - a)(s - b)(s - c)$$

Solve for A (area is always positive).
$$A = \boxed{\sqrt{s(s - a)(s - b)(s - c)}}$$

AREA OF A TRIANGLE (SSS CASE—HERON'S FORMULA)

For any triangle where the lengths of the three sides are known, the area for that triangle is given by the following formula:
$$A_{\text{SSS}} = \sqrt{s(s - a)(s - b)(s - c)}$$
where a, b, and c are the lengths of the sides of the triangle and s is half the perimeter of the triangle, called the semiperimeter.
$$s = \frac{a + b + c}{2}$$

Technology Tip

Use the calculator to find A.

```
√(10(10-5)(10-6)
(10-9))
         14.14213562
```

EXAMPLE 4 Finding the Area of a Triangle (SSS Case)

Find the area of the triangle $a = 5$, $b = 6$, and $c = 9$.

Solution:

Find the semiperimeter s.	$s = \dfrac{a+b+c}{2}$
Substitute $a = 5$, $b = 6$, and $c = 9$.	$s = \dfrac{5+6+9}{2}$
Simplify.	$s = 10$
Write the formula for the area of a triangle in the SSS case (Heron's formula).	$A = \sqrt{s(s-a)(s-b)(s-c)}$
Substitute $a = 5$, $b = 6$, $c = 9$, and $s = 10$.	$A = \sqrt{10(10-5)(10-6)(10-9)}$
Simplify the radicand.	$A = \sqrt{10 \cdot 5 \cdot 4 \cdot 1}$
Evaluate the radical.	$\boxed{A = 10\sqrt{2} \approx 14 \text{ sq units}}$

■ **YOUR TURN** Find the area of the triangle $a = 3$, $b = 5$, and $c = 6$.

■ **Answer:** $2\sqrt{14} \approx 7.5$ sq units

SECTION 4.5 SUMMARY

We can solve any triangle given three measures, as long as one of the measures is a side length. Depending on the information given, we either apply the **Law of Sines**

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

and the angle sum identity, or we apply a combination of the **Law of Cosines**,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad b^2 = a^2 + c^2 - 2ac\cos\beta \qquad c^2 = a^2 + b^2 - 2ab\cos\gamma$$

the Law of Sines, and the angle sum identity. The table below summarizes the strategies for solving oblique triangles covered in Sections 4.4 and 4.5.

Oblique Triangle	What's Known	Procedure for Solving
AAS or ASA	Two angles and a side	Step 1: Find the remaining angle with $\alpha + \beta + \gamma = 180°$. Step 2: Find the remaining sides with the Law of Sines.
SSA	Two sides and an angle opposite one of the sides	This is the ambiguous case, so there is either no triangle, one triangle, or two triangles. If the given angle is obtuse, then there is either one or no triangle. If the given angle is acute, then there is no triangle, one triangle, or two triangles. Step 1: Apply the Law of Sines to find one of the angles. Step 2: Find the remaining angle with $\alpha + \beta + \gamma = 180°$. Step 3: Find the remaining side with the Law of Sines. If two triangles exist, then the angle found in Step 1 can be either acute or obtuse, and Step 2 and Step 3 must be performed for each triangle.
SAS	Two sides and an angle between the sides	Step 1: Find the third side with the Law of Cosines. Step 2: Find the smaller angle with the Law of Sines. Step 3: Find the remaining angle with $\alpha + \beta + \gamma = 180°$.
SSS	Three sides	Step 1: Find the largest angle with the Law of Cosines. Step 2: Find either remaining angle with the Law of Sines. Step 3: Find the last remaining angle with $\alpha + \beta + \gamma = 180°$.

Formulas for calculating the areas of triangles (SAS and SSS cases) were derived. The three area formulas for the SAS case depend on which angles and sides are given.

$$A_{\text{SAS}} = \frac{1}{2}bc\sin\alpha \qquad A_{\text{SAS}} = \frac{1}{2}ab\sin\gamma \qquad A_{\text{SAS}} = \frac{1}{2}ac\sin\beta$$

The Law of Cosines was instrumental in developing a formula for the area of a triangle (SSS case) when all three sides are given.

$$\text{(Heron's formula)} \quad A_{\text{SSS}} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \frac{a+b+c}{2}$$

SECTION 4.5 EXERCISES

SKILLS

In Exercises 1–28, solve each triangle.

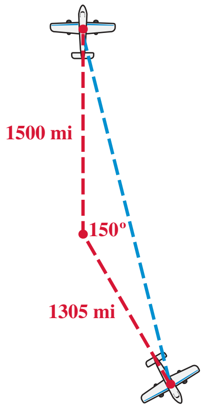
1. $a = 4,\ c = 3,\ \beta = 100°$
2. $a = 6,\ b = 10,\ \gamma = 80°$
3. $b = 7,\ c = 2,\ \alpha = 16°$
4. $b = 5,\ a = 6,\ \gamma = 170°$
5. $b = 5,\ c = 5,\ \alpha = 20°$
6. $a = 4.2,\ b = 7.3,\ \gamma = 25°$
7. $a = 9,\ c = 12,\ \beta = 23°$
8. $b = 6,\ c = 13,\ \alpha = 16°$
9. $a = 4,\ c = 8,\ \beta = 60°$
10. $b = 3, c = \sqrt{18}, \alpha = 45°$
11. $a = 8,\ b = 5,\ c = 6$
12. $a = 6,\ b = 9,\ c = 12$
13. $a = 4,\ b = 4,\ c = 5$
14. $a = 17,\ b = 20,\ c = 33$
15. $a = 8.2,\ b = 7.1,\ c = 6.3$
16. $a = 1492,\ b = 2001,\ c = 1776$
17. $a = 4,\ b = 5,\ c = 10$
18. $a = 1.3,\ b = 2.7,\ c = 4.2$
19. $a = 12,\ b = 5,\ c = 13$
20. $a = 4,\ b = 5,\ c = \sqrt{41}$
21. $\alpha = 40°, \beta = 35°, a = 6$
22. $b = 11.2, a = 19.0, \gamma = 13.3°$
23. $\alpha = 31°, b = 5, a = 12$
24. $a = 11, c = 12, \gamma = 60°$
25. $a = \sqrt{7}, b = \sqrt{8}, c = \sqrt{3}$
26. $\beta = 106°, \gamma = 43°, a = 1$
27. $b = 11, c = 2, \beta = 10°$
28. $\alpha = 25°, a = 6, c = 9$

In Exercises 29–50, find the area of each triangle with measures given.

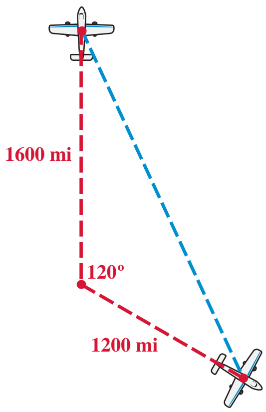
29. $a = 8, c = 16, \beta = 60°$
30. $b = 6, c = 4\sqrt{3}, \alpha = 30°$
31. $a = 1, b = \sqrt{2}, \alpha = 45°$
32. $b = 2\sqrt{2}, c = 4, \beta = 45°$
33. $a = 6, b = 8, \gamma = 80°$
34. $b = 9, c = 10, \alpha = 100°$
35. $a = 4, c = 7, \beta = 27°$
36. $a = 6.3, b = 4.8, \gamma = 17°$
37. $b = 100, c = 150, \alpha = 36°$
38. $c = 0.3, a = 0.7, \beta = 145°$
39. $a = 15,\ b = 15,\ c = 15$
40. $a = 1,\ b = 1,\ c = 1$
41. $a = 7, b = \sqrt{51}, c = 10$
42. $a = 9,\ b = 40,\ c = 41$
43. $a = 6,\ b = 10,\ c = 9$
44. $a = 40,\ b = 50,\ c = 60$
45. $a = 14.3,\ b = 15.7,\ c = 20.1$
46. $a = 146.5,\ b = 146.5,\ c = 100$
47. $a = 14{,}000,\ b = 16{,}500,\ c = 18{,}700$
48. $a = \sqrt{2}, b = \sqrt{3}, c = \sqrt{5}$
49. $a = 80,\ b = 75,\ c = 160$
50. $a = 19,\ b = 23,\ c = 3$

APPLICATIONS

51. Aviation. A plane flew due north at 500 miles per hour for 3 hours. A second plane, starting at the same point and at the same time, flew southeast at an angle 150° clockwise from due north at 435 miles per hour for 3 hours. At the end of the 3 hours, how far apart were the two planes? Round to the nearest mile.



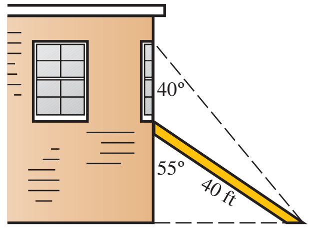
52. Aviation. A plane flew due north at 400 miles per hour for 4 hours. A second plane, starting at the same point and at the same time, flew southeast at an angle 120° clockwise from due north at 300 miles per hour for 4 hours. At the end of the 4 hours, how far apart were the two planes? Round to the nearest mile.



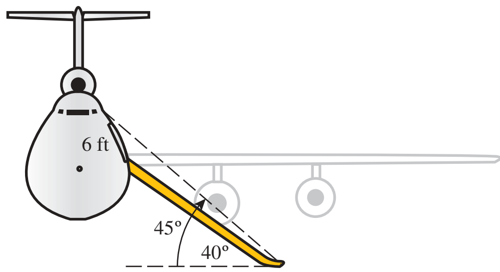
53. Aviation. A plane flew N 30° W at 350 miles per hour for 2.5 hours. A second plane, starting at the same point and at the same time, flew 35° at an angle clockwise from due north at 550 miles per hour for 2.5 hours. At the end of 2.5 hours, how far apart were the two planes? Round to the nearest mile.

54. Aviation. A plane flew N 30° W at 350 miles per hour for 3 hours. A second plane starts at the same point and takes off at the same time. It is known that after 3 hours, the two planes are 2100 miles apart. Find the original bearing of the second plane, to the nearest hundredth of a degree.

55. Sliding Board. A 40-foot slide leaning against the bottom of a building's window makes a 55° angle with the building. The angle formed with the building by the line of sight from the top of the window to the point on the ground where the slide ends is 40°. How tall is the window?

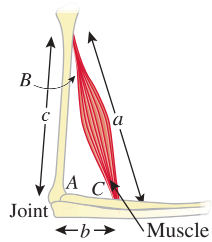


56. Airplane Slide. An airplane door is 6 feet high. If a slide attached to the bottom of the open door is at an angle of 40° with the ground, and the angle formed by the line of sight from where the slide touches the ground to the top of the door is 45°, how long is the slide?



For Exercises 57 and 58, refer to the following:

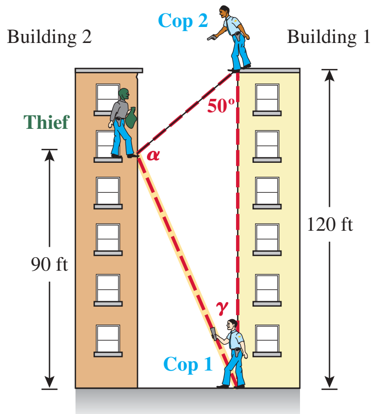
To quantify the torque (rotational force) of the elbow joint of a human arm (see the figure to the right), it is necessary to identify angles A, B, and C as well as lengths a, b, and c. Measurements performed on an arm determine that the measure of angle C is 105°, the length of the muscle a is 25.5 centimeters, and the length of the forearm from the elbow joint to the muscle attachment b is 1.76 centimeters.



57. Health/Medicine. Find the length of the upper arm from the muscle attachment to the elbow joint c.

58. Health/Medicine. Find the measure of angle B.

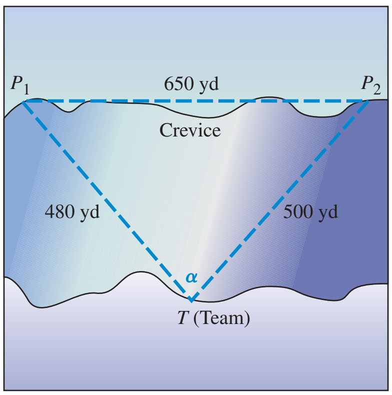
59. **Law Enforcement.** Two members of a SWAT team and the thief they are to apprehend are positioned as follows:



When the signal is given, Cop 2 shoots a zipline across to the window where the thief is spotted (at a 50° angle to Building 1) and Cop 1 shines a very bright light directly at the thief. Find the angle γ at which Cop 1 holds the light to shine it directly at the thief. Round to the nearest hundredth degree.

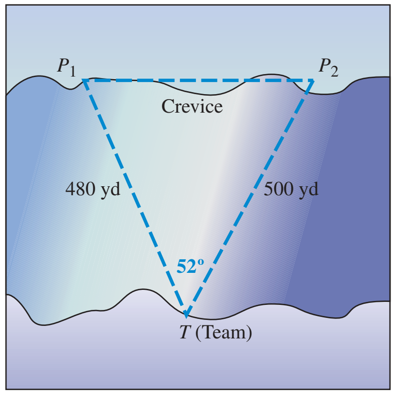
60. **Law Enforcement.** In reference to Exercise 59, what angle does the zipline make with respect to Building 2?

61. **Surveying.** A glaciologist needs to determine the length across a certain crevice on Mendenhall glacier in order to circumvent it with his team. He has the following measurements:



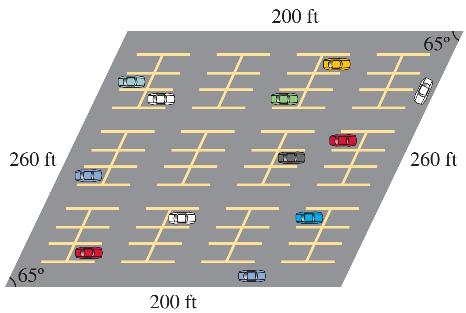
Find α.

62. **Surveying.** A glaciologist needs to determine the length across a certain crevice on Mendenhall glacier in order to circumvent it with her team. She has the following measurements:



Find the approximate length across the crevice.

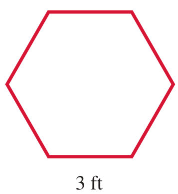
63. **Parking Lot.** A parking lot is to have the shape of a parallelogram that has adjacent sides measuring 200 feet and 260 feet. The acute angle between two adjacent sides is 65°. What is the area of the parking lot?

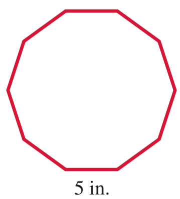


64. **Parking Lot.** A parking lot is to have the shape of a parallelogram that has adjacent sides measuring 250 feet and 300 feet. The acute angle between two adjacent sides is 55°. What is the area of the parking lot?

65. **Regular Hexagon.** A regular hexagon has sides measuring 3 feet. What is its area? Recall that the measure of an angle of a regular n-gon is given by the formula

$$\text{angle} = \frac{180°(n-2)}{n}.$$





66. **Regular Decagon.** A regular decagon has sides measuring 5 inches. What is its area?

67. **Geometry.** A quadrilateral $ABCD$ has sides of lengths $AB = 2$, $BC = 3$, $CD = 4$, and $DA = 5$. The angle between AB and BC is 135°. Find the area of $ABCD$.

68. **Geometry.** A quadrilateral $ABCD$ has sides of lengths $AB = 5$, $BC = 6$, $CD = 7$, and $DA = 8$. The angle between AB and BC is 135°. Find the area of $ABCD$.

CATCH THE MISTAKE

In Exercises 69 and 70, explain the mistake that is made.

69. Solve the triangle $b = 3$, $c = 4$, and $\alpha = 30°$.

Solution:

Step 1: Find a.

Apply the Law of Cosines. $\quad a^2 = b^2 + c^2 - 2bc\cos\alpha$

Let $b = 3$, $c = 4$, and $\alpha = 30°$. $\quad a^2 = 3^2 + 4^2 - 2(3)(4)\cos 30°$

Solve for a. $\quad a \approx 2.1$

Step 2: Find γ.

Apply the Law of Sines. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\gamma}{c}$

Solve for $\sin\gamma$. $\quad \sin\gamma = \dfrac{c\sin\alpha}{a}$

Solve for γ. $\quad \gamma = \sin^{-1}\left(\dfrac{c\sin\alpha}{a}\right)$

Let $a = 2.1$, $c = 4$, and $\alpha = 30°$. $\quad \gamma \approx 72°$

Step 3: Find β.

$\alpha + \beta + \gamma = 180° \qquad 30° + \beta + 72° = 180°$

Solve for β. $\quad \beta \approx 78°$

$a \approx 2.1$, $b = 3$, $c = 4$, $\alpha = 30°$, $\beta \approx 78°$, and $\gamma \approx 72°$

This is incorrect. The longest side is not opposite the largest angle. What mistake was made?

70. Solve the triangle $a = 6$, $b = 2$, and $c = 5$.

Solution:

Step 1: Find β.

Apply the Law of Cosines. $\quad b^2 = a^2 + c^2 - 2ac\cos\beta$

Solve for β. $\quad \beta = \cos^{-1}\left(\dfrac{a^2 + c^2 - b^2}{2ac}\right)$

Let $a = 6$, $b = 2$, $c = 5$. $\quad \beta \approx 18°$

Step 2: Find α.

Apply the Law of Sines. $\quad \dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$

Solve for α. $\quad \alpha = \sin^{-1}\left(\dfrac{a\sin\beta}{b}\right)$

Let $a = 6$, $b = 2$, and $\beta = 18°$. $\quad \alpha \approx 68°$

Step 3: Find γ. $\quad \alpha + \beta + \gamma = 180°$

$68° + 18° + \gamma = 180°$

$\gamma \approx 94°$

$a = 6$, $b = 2$, $c = 5$, $\alpha \approx 68°$, $\beta \approx 18°$, and $\gamma \approx 94°$

This is incorrect. The longest side is not opposite the largest angle. What mistake was made?

CONCEPTUAL

In Exercises 71–76, determine whether each statement is true or false.

71. Given the lengths of all three sides of a triangle, there is insufficient information to solve the triangle.

72. Given three angles of a triangle, there is insufficient information to solve the triangle.

73. The Pythagorean theorem is a special case of the Law of Cosines.

74. The Law of Cosines is a special case of the Pythagorean theorem.

75. If an obtuse triangle is isosceles, then knowing the measure of the obtuse angle and a side adjacent to it is sufficient to solve the triangle.

76. All acute triangles can be solved using the Law of Cosines.

CHALLENGE

77. Show that $\dfrac{\cos\alpha}{a} + \dfrac{\cos\beta}{b} + \dfrac{\cos\gamma}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$.

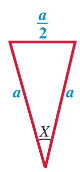
Hint: Use the Law of Cosines.

78. Show that $a = c\cos\beta + b\cos\gamma$. *Hint:* Use the Law of Cosines.

The following half-angle identities are useful in Exercises 79 and 80, and will be derived in Chapter 6.

$$\cos\left(\frac{x}{2}\right) = \sqrt{\frac{1 + \cos x}{2}} \qquad \tan\left(\frac{x}{2}\right) = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$$

79. Consider the following diagram and express $\cos\left(\dfrac{X}{2}\right)$ in terms of a.



80. Using the diagram in Exercise 77, express $\tan\left(\dfrac{X}{2}\right)$ in terms of a.

81. Show that the area for an SAA triangle is given by

$$A = \frac{a^2 \sin\beta \sin\gamma}{2\sin\alpha}$$

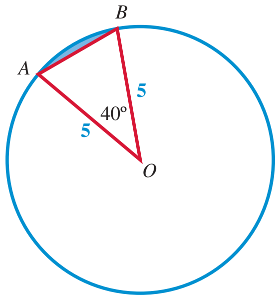
Assume that α, β, and a are given.

82. Show that the area of an isosceles triangle with equal sides of length s is given by

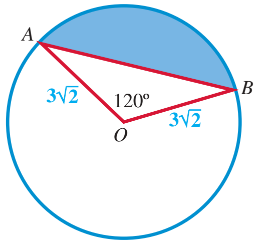
$$A_{\text{isosceles}} = \frac{1}{2}s^2 \sin\theta$$

where θ is the angle between the two equal sides.

83. Find the area of the shaded region.

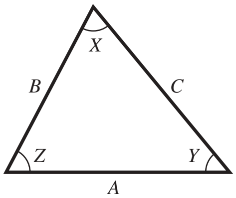


84. Find the area of the shaded region.



TECHNOLOGY

For Exercises 85–90, let A, B, and C be the lengths of the three sides with X, Y, and Z as the corresponding angle measures in a triangle. Write a program using a TI calculator to solve each triangle with the given measures.



85. $B = 45$, $C = 57$, and $X = 43°$

86. $B = 24.5$, $C = 31.6$, and $X = 81.5°$

87. $A = 29.8$, $B = 37.6$, and $C = 53.2$

88. $A = 100$, $B = 170$, and $C = 250$

89. $A = \sqrt{12}$, $B = \sqrt{21}$, and $Z = 62.8°$

90. $A = 1235$, $B = 987$, and $C = 1456$

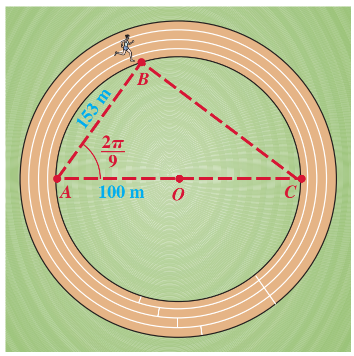
PREVIEW TO CALCULUS

In calculus, some applications of the derivative require the solution of triangles. In Exercises 91–94, solve each triangle using the Law of Cosines.

91. Two ships start moving from the same port at the same time. One moves north at 40 miles per hour, while the other moves southeast at 50 miles per hour. Find the distance between the ships 4 hours later. Round your answer to the nearest mile.

92. An airport radar detects two planes approaching. The distance between the planes is 80 miles; the closest plane is 60 miles from the airport and the other plane is 70 miles from the airport. What is the angle (in degrees) formed by the planes and the airport?

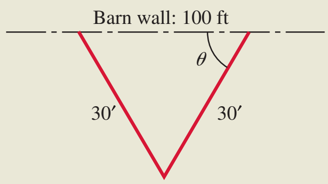
93. An athlete runs along a circular track, of radius 100 meters, runs from A to B and then decides to take a shortcut to go to C. If the measure of angle BAC is $\frac{2\pi}{9}$, find the distance covered by the athlete if the distance from A to B is 153 meters. Round your answer to the nearest integer.



94. A regular pentagon is inscribed in a circle of radius 10 feet. Find its perimeter. Round your answer to the nearest tenth.

CHAPTER 4 INQUIRY-BASED LEARNING PROJECT (A)

Dr. Parkinson has acquired two 30-foot sections of fence from her neighbor Mr. Wilson. She has decided to build a triangular corral for her animals. She plans to use a barn wall as the third side. (The barn wall is 100 feet long—see the diagram below.) As her contractor, you are expected to maximize the corral area. You have decided to approach this from a trigonometric viewpoint. Hence, you want to find the angle θ that maximizes the area. To do this, follow the steps below. (As a side note: This problem is an example of **optimization** and you will revisit this type of problem in precalculus and/or calculus courses. The outline below is designed to give you an understanding of how to set up and solve this type of problem. Realize that this triangle is an isosceles triangle—two equal side lengths—and that its perpendicular bisector can be used to find the height of the triangle.)



1. Write out the general formula for the area of a triangle.

2. To get an understanding of what happens to the area of the triangle as the angle θ changes, you will calculate the following dimensions using right triangle trigonometry. Do these calculations on a sheet of scratch paper. Since none of the triangles formed below are actually right triangles, you will need to construct a right triangle (using a perpendicular bisector) along with using the sine and cosine relationships to help you identify the base and height. (Use two decimals.)

θ	20°	40°	60°	80°
$b(\theta) =$ base				
$h(\theta) =$ height				
$A(\theta) =$ area				

You don't need to do every example in the chart by hand. However, do as many as you need to see what patterns emerge for calculating each base, height, and area. When you see the pattern, you will hopefully then be able to write a function for each piece of information. You can then use the table in your graphing calculator to list all the answers. However, you are encouraged to do at least two by hand before jumping to the function writing. Also be sure to check that the results you get on your table agree with the numbers you get by hand.

3. Write the base $b(\theta)$ of the triangle as a function of θ. (Show how you arrived at this answer.) Describe what happens to the base values as the θ values increase.

4. Write the height $h(\theta)$ of the triangle as a function of θ. (Show how you arrived at this answer.) Describe what happens to the height values as the θ values increase.
5. Write the area $A(\theta)$ as a function of θ using your results from 3 and 4. Describe what happens to your area values as θ increases.
6. Specify the **domain** for the area within the context of this problem. (**Vocabulary reminder:** The domain for this problem is the set of values of θ that makes sense from a physical standpoint; that is, you wouldn't build a corral using $\theta = -20°$.)
7. Use a graphing utility to graph your area function on its domain.
8. Estimate the maximum area and the θ that this corresponds to.
9. What can you conclude about the shape of the triangle that yields the maximum area in this example?

CHAPTER 4 INQUIRY-BASED LEARNING PROJECT (B)

When solving triangles using the Law of Sines, it is important to keep in mind the domain and range of the inverse sine function, because it can play a role in determining your answers. Make sure to check your answers for reasonableness, especially when solving an obtuse triangle, as you will see next.

For a triangle *ABC*, two side lengths and the measure of the angle between them are given below.

$$b = 24 \qquad c = 8 \qquad A = 15°$$

1. How many triangles are possible with these measurements? Explain.
2. Make a careful sketch of a triangle and label the given information. Also label the unknown side *a*, and the unknown angles opposite sides *b* and *c*, as *B* and *C*, respectively. What can you say about angle *B*?
3. Explain why the Law of Cosines is needed for this problem.
4. Find the length of side *a*. Round to three significant digits.
5. Suppose a fellow student now wants to find the measure of angle *B*, and decides to use the Law of Sines. Shown below are the steps he wrote out to illustrate his process for computing the measure of *B* and *C*. Look at the chart he filled in below.

$$B = \sin^{-1}\left[\frac{24}{16.4}\sin(15)\right] \approx 22.3°$$

$$C = 180° - (15° + 22.3°) = 142.7°$$

SIDES		ANGLES	
a	16.4	*A*	15°
b	24	*B*	22.3°
c	8	*C*	142.7°

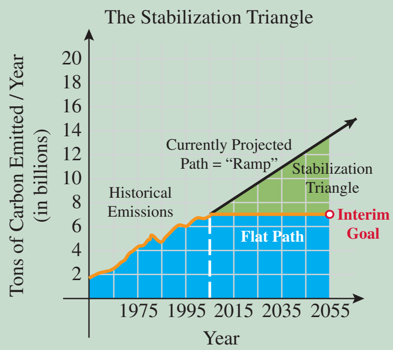
   What is wrong with the answers given by this student?

6. Because your calculator gives positive values of $\sin^{-1}x$ only between 0 and 90°, the answer this student got for the measure of angle *B* does not make sense. Show what he needs to do to correct his error.

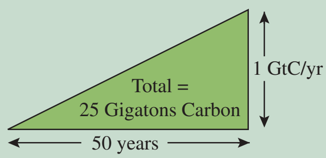
MODELING OUR WORLD

The Intergovernmental Panel on Climate Change (IPCC) claims that carbon dioxide (CO_2) production from increased industrial activity (such as fossil fuel burning and other human activities) has increased the CO_2 concentrations in the atmosphere. Because it is a greenhouse gas, elevated CO_2 levels will increase global mean (average) temperature. In this section, we will examine the increasing rate of carbon emissions on Earth.

In 1955 there were (globally) 2 billion tons of carbon emitted per year. In 2005 the carbon emission had more than tripled, reaching approximately 7 billion tons of carbon emitted per year. Currently, we are on the path to doubling our current carbon emissions in the next 50 years.



Two Princeton professors* (Stephen Pacala and Rob Socolow) introduced the Climate Carbon Wedge concept. A "wedge" is a strategy to reduce carbon emissions that grow in a 50-year time period from 0 to 1.0 GtC/yr (gigatons of carbon per year).



1. Consider eight scenarios (staying on path of one of the seven wedges). A check is done at the 10-year mark. What total GtC per year would we have to measure to correspond to the following projected paths?

a. Flat path (no increase) over 50 years (2005 to 2055)

b. Increase of 1 GtC over 50 years (2005 to 2055)

c. Increase of 2 GtC over 50 years (2005 to 2055)

d. Increase of 3 GtC over 50 years (2005 to 2055)

e. Increase of 4 GtC over 50 years (2005 to 2055)

f. Increase of 5 GtC over 50 years (2005 to 2055)

g. Increase of 6 GtC over 50 years (2005 to 2055)

h. Increase of 7 GtC over 50 years (2005 to 2055) (projected path)

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*S. Pacala and R. Socolow, "Stabilization Wedges: Solving the Climate Problem for the Next 50 Years with Current Technologies," *Science*, Vol. 305 (2004).

2. Consider the angle θ in each wedge. For each of the seven wedges (and the flat path), find the GtC/yr rate in terms of $\tan\theta$.

a. Flat path

b. Increase of 1 GtC/50 years

c. Increase of 2 GtC/50 years

d. Increase of 3 GtC/50 years

e. Increase of 4 GtC/50 years

f. Increase of 5 GtC/50 years

g. Increase of 6 GtC/50 years

h. Increase of 7 GtC/50 years (projected path)

3. Research the "climate carbon wedge" concept and discuss the types of changes (transportation efficiency, transportation conservation, building efficiency, efficiency in electricity production, alternate energies, etc.) the world would have to make that would correspond to each of the seven wedges.

a. Flat path

b. Wedge 1

c. Wedge 2

d. Wedge 3

e. Wedge 4

f. Wedge 5

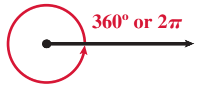
g. Wedge 6

h. Wedge 7

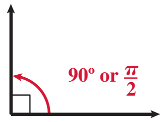
SECTION	CONCEPT	KEY IDEAS/FORMULAS
4.1	**Angle measure**	

Angles and their measure

Degrees and Radians



One complete counterclockwise rotation



Right Angle

Converting between degrees and radians (Remember that $\pi = 180°$.)

- Degrees to radians: Multiply by $\dfrac{\pi}{180°}$
- Radians to degrees: Multiply by $\dfrac{180°}{\pi}$

CONCEPT	KEY IDEAS/FORMULAS
Coterminal angles	Two angles in standard position with the same terminal side
Arc length	$s = r\theta$ $\quad$ θ is in radians.
Area of a circular sector	$A = \frac{1}{2}r^2\theta$ $\quad$ θ is in radians.

Linear and angular speeds

Linear speed v is given by

$$v = \frac{s}{t}$$

where s is the arc length (or distance along the arc) and t is time.

Angular speed ω is given by

$$\omega = \frac{\theta}{t}$$

where θ is given in radians.

Linear and angular speeds are related through the radius of the circle:

$$v = r\omega \quad \text{or} \quad \omega = \frac{v}{r}$$

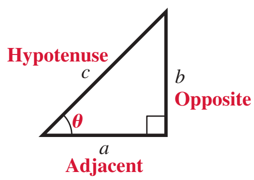
SECTION	CONCEPT	KEY IDEAS/FORMULAS
4.2	**Right triangle trigonometry**	

Right triangle ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{(SOH)}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{(CAH)}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \quad \text{(TOA)}$$



SECTION	CONCEPT	KEY IDEAS/FORMULAS

Reciprocal identities

$$\cot\theta = \frac{1}{\tan\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Evaluating trigonometric functions exactly for special angle measures

θ	sin θ	cos θ
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

The other trigonometric functions can be found for these values using $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and the reciprocal identities.

Solving right triangles

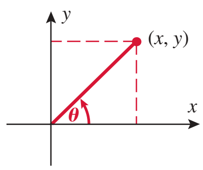
4.3 Trigonometric functions of angles

Trigonometric functions: The Cartesian plane

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

where $x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$



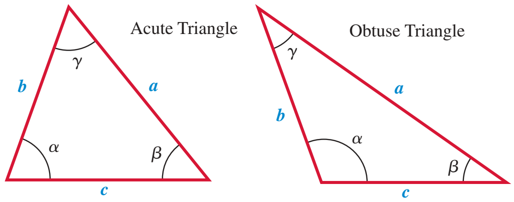
The distance r is positive: $r > 0$.

Algebraic signs of trigonometric functions

θ	QI	QII	QIII	QIV
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−

Trigonometric function values for quadrantal angles.

θ	0°	90°	180°	270°
$\sin\theta$	0	1	0	−1
$\cos\theta$	1	0	−1	0
$\tan\theta$	0	undefined	0	undefined
$\cot\theta$	undefined	0	undefined	0
$\sec\theta$	1	undefined	−1	undefined
$\csc\theta$	undefined	1	undefined	−1

Section	Concept	Key Ideas/Formulas
	Ranges of the trigonometric functions	$\sin\theta$ and $\cos\theta$: $[-1, 1]$ $\tan\theta$ and $\cot\theta$: $(-\infty, \infty)$ $\sec\theta$ and $\csc\theta$: $(-\infty, -1] \cup [1, \infty)$
	Reference angles and reference right triangles	The reference angle α for angle θ (between $0°$ and $360°$) is given by
		■ QI: $\alpha = \theta$ ■ QII: $\alpha = 180° - \theta$ or $\pi - \theta$ ■ QIII: $\alpha = \theta - 180°$ or $\theta - \pi$ ■ QIV: $\alpha = 360° - \theta$ or $2\pi - \theta$
	Evaluating trigonometric functions for nonacute angles	
4.4	**The Law of Sines**	
	Solving oblique triangles	Oblique (Nonright) Triangles  **The Law of Sines** $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$ Use for: ■ AAS (or ASA) triangles ■ SSA triangles (ambiguous case)
4.5	**The Law of Cosines**	
	Solving oblique triangles using the Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$ Use for: ■ SAS triangles ■ SSS triangles
	The area of a triangle	**The area of a triangle (SAS case)** $A_{SAS} = \frac{1}{2}bc\sin\alpha$ when b, c, and α are known. $A_{SAS} = \frac{1}{2}ab\sin\gamma$ when a, b, and γ are known. $A_{SAS} = \frac{1}{2}ac\sin\beta$ when a, c, and β are known. **The area of a triangle (SSS case)** Use Heron's formula for the SSS case: $A_{SSS} = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b, and c are the lengths of the sides of the triangle and s is half the perimeter of the triangle, called the semiperimeter. $s = \dfrac{a+b+c}{2}$

4.1 Angle Measure

Find (a) the complement and (b) the supplement of the given angles.

1. 28° **2.** 17° **3.** 35°

4. 78° **5.** 89.01° **6.** 0.013°

Convert from degrees to radians. Leave your answers in terms of π.

7. 135° **8.** 240° **9.** 330° **10.** 180°

11. 216° **12.** 108° **13.** 1620° **14.** 900°

Convert from radians to degrees.

15. $\frac{\pi}{3}$ **16.** $\frac{11\pi}{6}$ **17.** $\frac{5\pi}{4}$ **18.** $\frac{2\pi}{3}$

19. $\frac{5\pi}{9}$ **20.** $\frac{17\pi}{10}$ **21.** 10π **22.** $\frac{31\pi}{2}$

Applications

23. Clock. What is the measure (in degrees) of the angle that the minute hand sweeps in exactly 25 minutes?

24. Clock. What is the measure (in degrees) of the angle that the second hand sweeps in exactly 15 seconds?

25. A ladybug is clinging to the outer edge of a child's spinning disk. The disk is 4 inches in diameter and is spinning at 60 revolutions per minute. How fast is the ladybug traveling in inches/minute?

26. How fast is a motorcyclist traveling in miles per hour if his tires are 30 inches in diameter and the angular speed of the tire is 10π radians per second?

4.2 Right Triangle Trigonometry

Use the following triangle to find the indicated trigonometric functions. Rationalize any denominators that you encounter in the answers.

27. $\cos\theta$

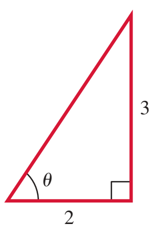
28. $\sin\theta$

29. $\sec\theta$

30. $\csc\theta$

31. $\tan\theta$

32. $\cot\theta$



Label each trigonometric function value with the corresponding value (a–c).

a. $\frac{\sqrt{3}}{2}$ **b.** $\frac{1}{2}$ **c.** $\frac{\sqrt{2}}{2}$

33. $\sin 30°$ **34.** $\cos 30°$ **35.** $\cos 60°$

36. $\sin 60°$ **37.** $\sin 45°$ **38.** $\cos 45°$

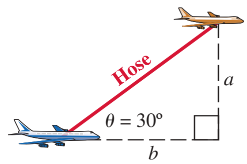
Use a calculator to approximate the following trigonometric function values. Round the answers to four decimal places.

39. $\sin 42°$ **40.** $\cos 57°$ **41.** $\cos 17.3°$ **42.** $\tan 25.2°$

43. $\cot 33°$ **44.** $\sec 16.8°$ **45.** $\csc 40.25°$ **46.** $\cot 19.76°$

The following exercises illustrate a mid-air refueling scenario that U.S. military aircraft often use. Assume the elevation angle that the hose makes with the plane being fueled is $\theta = 30°$.

47. Mid-Air Refueling. If the hose is 150 feet long, what should the altitude difference a be between the two planes?



48. Mid-Air Refueling. If the smallest acceptable altitude difference, a, between the two planes is 100 feet, how long should the hose be?

4.3 Trigonometric Functions of Angles

In the following exercises, the terminal side of an angle θ in standard position passes through the indicated point. Calculate the values of the six trigonometric functions for angle θ.

49. $(6, -8)$ **50.** $(-24, -7)$ **51.** $(-6, 2)$ **52.** $(-40, 9)$

53. $(\sqrt{3}, 1)$ **54.** $(-9, -9)$ **55.** $\left(\frac{1}{2}, -\frac{1}{4}\right)$ **56.** $\left(-\frac{3}{4}, \frac{5}{6}\right)$

57. $(-1.2, -2.4)$ **58.** $(0.8, -2.4)$

Evaluate the following expressions exactly:

59. $\sin 330°$ **60.** $\cos(-300°)$ **61.** $\tan 150°$

62. $\cot 315°$ **63.** $\sec(-150°)$ **64.** $\csc 210°$

65. $\sin\left(\frac{7\pi}{4}\right)$ **66.** $\cos\left(\frac{7\pi}{6}\right)$ **67.** $\tan\left(-\frac{2\pi}{3}\right)$

68. $\cot\left(\frac{4\pi}{3}\right)$ **69.** $\sec\left(\frac{5\pi}{4}\right)$ **70.** $\csc\left(-\frac{8\pi}{3}\right)$

71. $\sec\left(\frac{5\pi}{6}\right)$ **72.** $\cos\left(-\frac{11\pi}{6}\right)$

4.4 The Law of Sines

Solve the given triangles.

73. $\alpha = 10°, \beta = 20°, a = 4$

74. $\beta = 40°, \gamma = 60°, b = 10$

75. $\alpha = 5°, \beta = 45°, c = 10$

76. $\beta = 60°, \gamma = 70°, a = 20$

77. $\gamma = 11°, \alpha = 11°, c = 11$

78. $\beta = 20°, \gamma = 50°, b = 8$

79. $\alpha = 45°, \gamma = 45°, b = 2$

80. $\alpha = 60°, \beta = 20°, c = 17$

81. $\alpha = 12°, \gamma = 22°, a = 99$

82. $\beta = 102°, \gamma = 27°, a = 24$

Two sides and an angle are given. Determine whether a triangle (or two) exist and, if so, solve the triangle.

83. $a = 7, b = 9, \alpha = 20°$

84. $b = 24, c = 30, \beta = 16°$

85. $a = 10, c = 12, \alpha = 24°$

86. $b = 100, c = 116, \beta = 12°$

87. $a = 40, b = 30, \beta = 150°$

88. $b = 2, c = 3, \gamma = 165°$

89. $a = 4, b = 6, \alpha = 10°$

90. $c = 25, a = 37, \gamma = 4°$

4.5 The Law of Cosines

Solve each triangle.

91. $a = 40, b = 60, \gamma = 50°$

92. $b = 15, c = 12, \alpha = 140°$

93. $a = 24, b = 25, c = 30$

94. $a = 6, b = 6, c = 8$

95. $a = \sqrt{11}, b = \sqrt{14}, c = 5$

96. $a = 22, b = 120, c = 122$

97. $b = 7, c = 10, \alpha = 14°$

98. $a = 6, b = 12, \gamma = 80°$

99. $b = 10, c = 4, \alpha = 90°$

100. $a = 4, b = 5, \gamma = 75°$

101. $a = 10, b = 11, c = 12$

102. $a = 22, b = 24, c = 25$

103. $b = 16, c = 18, \alpha = 100°$

104. $a = 25, c = 25, \beta = 9°$

105. $b = 12, c = 40, \alpha = 10°$

106. $a = 26, b = 20, c = 10$

107. $a = 26, b = 40, c = 13$

108. $a = 1, b = 2, c = 3$

109. $a = 6.3, b = 4.2, \alpha = 15°$

110. $b = 5, c = 6, \beta = 35°$

Find the area of each triangle described.

111. $b = 16, c = 18, \alpha = 100°$

112. $a = 25, c = 25, \beta = 9°$

113. $a = 10, b = 11, c = 12$

114. $a = 22, b = 24, c = 25$

115. $a = 26, b = 20, c = 10$

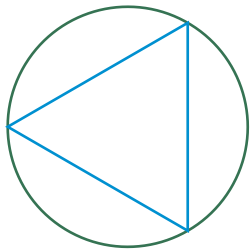
116. $a = 24, b = 32, c = 40$

117. $b = 12, c = 40, \alpha = 10°$

118. $a = 21, c = 75, \beta = 60°$

Applications

119. Area of Inscribed Triangle. The area of a triangle inscribed in a circle can be found if you know the lengths of the sides of the triangle and the radius of the circle: $A = \frac{abc}{4r}$. Find the radius of the circle that circumscribes the triangle if all the sides of the triangle measure 9.0 inches and the area of the triangle is 35 square inches.



120. Area of Inscribed Triangle. The area of a triangle inscribed in a circle can be found if you know the lengths of the sides of the triangle and the radius of the circle: $A = \frac{abc}{4r}$. Find the radius of the circle that circumscribes the triangle if the sides of the triangle measure 9, 12, and 15 inches and the area of the triangle is 54 square inches.

1. A 5-foot girl is standing *in* the Grand Canyon, and she wants to estimate the depth of the canyon. The sun casts her shadow 6 inches along the ground. To measure the shadow cast by the top of the canyon, she walks the length of the shadow. She takes 200 steps and estimates that each step is roughly 3 feet. Approximately how tall is the Grand Canyon?

2. Fill in the values in the table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°						
45°						
60°						

3. What is the difference between $\cos\theta = \frac{2}{3}$ and $\cos\theta \approx 0.6\overline{6}$?

4. Fill in the table with exact values for the quadrantal angles and the algebraic signs for the quadrants.

	0°	QI	90°	QII	180°	QIII	270°	QIV	360°
$\sin\theta$									
$\cos\theta$									

5. If $\cot\theta < 0$ and $\sec\theta > 0$, in which quadrant does the terminal side of θ lie?

6. Evaluate $\sin 210°$ exactly.

7. Convert $\dfrac{13\pi}{4}$ to degree measure.

8. Convert 260° to radian measure. Leave the answer in terms of π.

9. What is the area of the sector swept by the second hand of a clock in 25 seconds? Assume the radius of the sector is 3 inches.

10. What is the measure in radians of the smaller angle between the hour and minute hands at 10:10?

Solve the triangles if possible.

11. $\alpha = 30°, \beta = 40°, b = 10$

12. $\alpha = 47°, \beta = 98°, \gamma = 35°$

13. $a = 7, b = 9, c = 12$

14. $\alpha = 45°, a = 8, b = 10$

15. $a = 1, b = 1, c = 2$

16. $a = \dfrac{23}{7}, c = \dfrac{5}{7}, \beta = 61.2°$

17. $\alpha = 110°, \beta = 20°, a = 5$

18. $b = \dfrac{\sqrt{5}}{2}, c = 3\sqrt{5}, \alpha = 45°$

In Exercises 19 and 20, find the areas of the given triangles.

19. $\gamma = 72°, a = 10, b = 12$

20. $a = 7, b = 10, c = 13$

CHAPTERS 1–4 CUMULATIVE TEST

1. Find the average rate of change for $f(x) = \frac{5}{x}$ from $x = 2$ to $x = 4$.

2. Use interval notation to express the domain of the function $f(x) = \sqrt{x^2 - 25}$.

3. Using the function $f(x) = 5 - x^2$, evaluate the difference quotient $\frac{f(x+h) - f(x)}{h}$.

4. Given the piecewise-defined function

$$f(x) = \begin{cases} x^2 & x < 0 \\ 2x - 1 & 0 \le x < 5 \\ 5 - x & x \ge 5 \end{cases}$$

find:

a. $f(0)$ **b.** $f(4)$ **c.** $f(5)$ **d.** $f(-4)$

e. State the domain and range in interval notation.

f. Determine the intervals where the function is increasing, decreasing, or constant.

5. Evaluate $g(f(-1))$ for $f(x) = \sqrt[3]{x - 7}$ and $g(x) = \frac{5}{3 - x}$.

6. Find the inverse of the function $f(x) = \frac{5x + 2}{x - 3}$.

7. Find the quadratic function that has the vertex $(0, 7)$ and goes through the point $(2, -1)$.

8. Find all of the real zeros and state the multiplicity of each for the function $f(x) = \frac{1}{7}x^5 + \frac{2}{9}x^3$.

9. Graph the rational function $f(x) = \frac{x^2 + 3}{x - 2}$. Give all asymptotes.

10. Factor the polynomial $P(x) = 4x^4 - 4x^3 + 13x^2 + 18x + 5$ as a product of linear factors.

11. How much money should be put in a savings account now that earns 5.5% a year compounded continuously, if you want to have \$85,000 in 15 years?

12. Evaluate $\log_{4.7} 8.9$ using the change-of-base formula. Round the answer to three decimal places.

13. Solve the equation $5(10^{2x}) = 37$ for x. Round the answer to three decimal places.

14. Solve for x: $\ln\sqrt{6 - 3x} - \frac{1}{2}\ln(x + 2) = \ln(x)$.

15. In a 45º-45º-90º triangle, if the two legs have a length of 15 feet, how long is the hypotenuse?

16. **Height of a tree.** The shadow of a tree measures $15\frac{1}{3}$ feet. At the same time of day the shadow of a 6-foot pole measures 2.3 feet. How tall is the tree?

17. Convert 432° to radians.

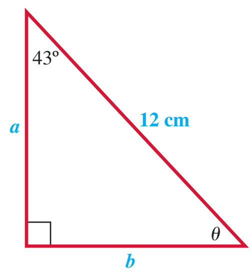
18. Convert $\frac{5\pi}{9}$ to degrees.

19. Find the exact value of $\tan\left(\frac{4\pi}{3}\right)$.

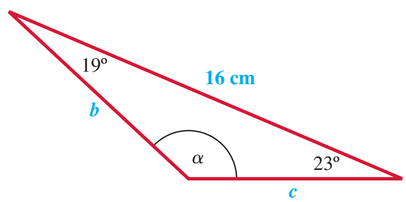
20. Find the exact value of $\sec\left(-\frac{7\pi}{6}\right)$.

21. Use a calculator to find the value of csc 37°. Round your answer to four decimal places.

22. In the right triangle below, find a, b, and θ. Round each to the nearest tenth.



23. Solve the triangle below. Round the side lengths to the nearest centimeter.



24. Solve the triangle $a = 2$, $b = 4$, and $c = 5$. Round your answer to the nearest degree.

5

Trigonometric Functions of Real Numbers

GIPhotostock/Photo Researchers, Inc.

An oscilloscope displays voltage (vertical axis) as a function of time (horizontal axis) of an electronic signal. The electronic signal is an electric representation of some periodic process that occurs in the real world, such as a human pulse or a sound wave. Oscilloscopes are used in medicine, the sciences, and engineering, and allow the shape of a signal to be displayed, which allows the amplitude and frequency of the repetitive signal to then be determined. The oscilloscope above displays a *sine* wave.

IN THIS CHAPTER we will use the unit circle approach to define trigonometric functions. We will graph the sine and cosine functions and find periods, amplitudes, and phase shifts. Applications such as harmonic motion will be discussed. Combinations of sinusoidal functions will be discussed through a technique called the addition of ordinates. Lastly, we will discuss the graphs of the other trigonometric functions (tangent, cotangent, secant, and cosecant).

TRIGONOMETRIC FUNCTIONS OF REAL NUMBERS

5.1 Trigonometric Functions: The Unit Circle Approach

- Trigonometric Functions and the Unit Circle
- Properties of Trigonometric (Circular) Functions

5.2 Graphs of Sine and Cosine Functions

- The Graphs of Sinusoidal Functions
- Graphing a Shifted Sinusoidal Function: $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$
- Harmonic Motion
- Graphing Sums of Functions: Addition of Ordinates

5.3 Graphs of Other Trigonometric Functions

- Graphing the Tangent, Cotangent, Secant, and Cosecant Functions
- Translations of Trigonometric Functions

LEARNING OBJECTIVES

- Define trigonometric functions using the unit circle approach.
- Graph a sinusoidal function and determine its amplitude, period, and phase shift.
- Graph tangent, cotangent, secant, and cosecant functions.

SECTION 5.1 TRIGONOMETRIC FUNCTIONS: THE UNIT CIRCLE APPROACH

SKILLS OBJECTIVES

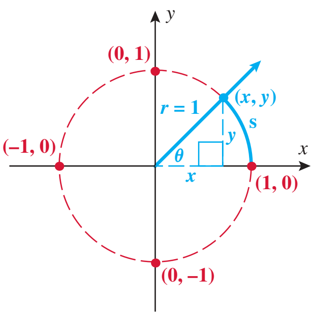
- Draw the unit circle showing the special angles, and label cosine and sine values.
- Determine the domain and range of trigonometric (circular) functions.
- Classify trigonometric functions as even or odd.

CONCEPTUAL OBJECTIVES

- Understand that the definition of trigonometric functions using the unit circle approach is consistent with both of the previous definitions (right triangle trigonometry and trigonometric functions of nonacute angles in the Cartesian plane).
- Relate x-coordinates and y-coordinates of points on the unit circle to the values of cosine and sine functions.
- Visualize the periodic properties of trigonometric (circular) functions.

Recall that the first definition of trigonometric functions we developed was in terms of ratios of sides of right triangles (Section 4.2). Then in Section 4.3 we superimposed right triangles on the Cartesian plane, which led to a second definition of trigonometric functions (for any angle) in terms of ratios of x- and y-coordinates of a point and the distance from the origin to that point. In this section, we inscribe the right triangles into the unit circle in the Cartesian plane, which will yield a third definition of trigonometric functions. It is important to note that all three definitions are consistent with one another.

Trigonometric Functions and the Unit Circle



Note: In radians, $\theta = \frac{s}{r}$, and since $r = 1$, we know that $\theta = s$.

Recall that the equation for the **unit circle** centered at the origin is given by $x^2 + y^2 = 1$. The term *circular function* is often used as a synonym for trigonometric function, but it is important to note that a circle is not a function (it does not pass the vertical line test).

If we form a central angle θ in the unit circle such that the terminal side lies in quadrant I, we can use the previous two definitions of the sine and cosine functions when $r = 1$ (i.e., in the unit circle).

Trigonometric Function	Right Triangle Trigonometry	Cartesian Plane
$\sin\theta$	$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{1} = y$	$\frac{y}{r} = \frac{y}{1} = y$
$\cos\theta$	$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{1} = x$	$\frac{x}{r} = \frac{x}{1} = x$

Notice that the point (x, y) on the unit circle can be written as $(\cos\theta, \sin\theta)$. We can now summarize the exact values for **sine** and **cosine** in the illustration on the following page.

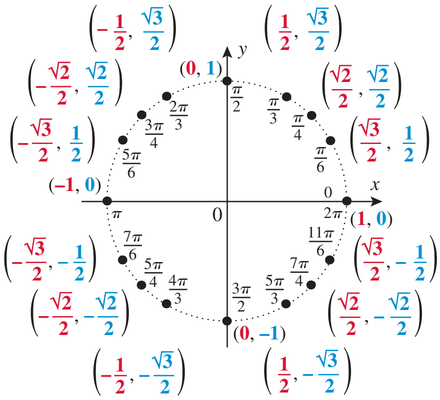
The following observations are consistent with properties of trigonometric functions we've studied already:

- $\sin\theta > 0$ in QI and QII.
- $\cos\theta > 0$ in QI and QIV.
- The unit circle equation $x^2 + y^2 = 1$ leads to the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$.

Study Tip

$(\cos\theta, \sin\theta)$ represents a unique point (x, y) on the unit circle.

$(x, y) = (\cos\theta, \sin\theta)$, where θ is the central angle whose terminal side intersects the unit circle at (x, y).



Trigonometric (Circular) Functions

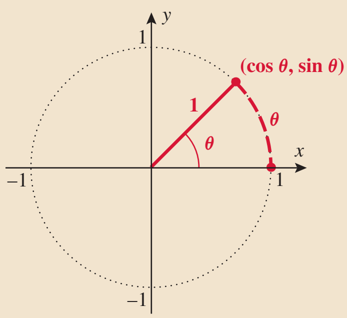
Using the unit circle relationship, $(x, y) = (\cos\theta, \sin\theta)$, where θ is the central angle whose terminal side intersects the unit circle at the point (x, y), we can now define the remaining trigonometric functions using this unit circle approach and the quotient and reciprocal identities. Because the trigonometric functions are defined in terms of the unit *circle*, the trigonometric functions are often called **circular functions**.

DEFINITION Trigonometric Functions

Unit Circle Approach

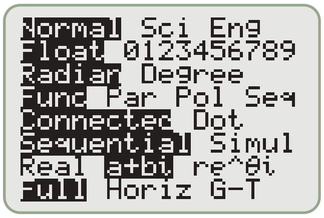
Let (x, y) be any point on the unit circle. If θ is a real number that represents the distance from the point $(1, 0)$ along the circumference to the point (x, y), then

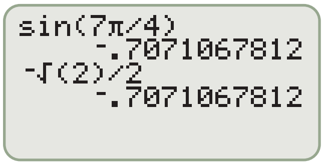
$\sin\theta = y$	$\cos\theta = x$	$\tan\theta = \dfrac{y}{x} \quad x \neq 0$
$\csc\theta = \dfrac{1}{y} \quad y \neq 0$	$\sec\theta = \dfrac{1}{x} \quad x \neq 0$	$\cot\theta = \dfrac{x}{y} \quad y \neq 0$

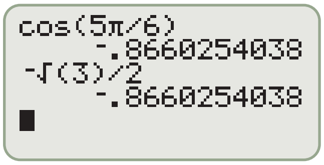


Technology Tip

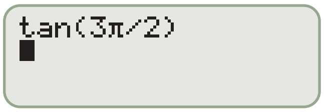
Use a TI/scientific calculator to check the values for $\sin\left(\frac{7\pi}{4}\right)$, $\cos\left(\frac{5\pi}{6}\right)$, and $\tan\left(\frac{3\pi}{2}\right)$. Be sure to set the TI in radian mode.







Since $\tan\left(\frac{3\pi}{2}\right)$ is undefined, TI will display an error message.



EXAMPLE 1 Finding Exact Trigonometric (Circular) Function Values

Find the exact values for

a. $\sin\left(\frac{7\pi}{4}\right)$ **b.** $\cos\left(\frac{5\pi}{6}\right)$ **c.** $\tan\left(\frac{3\pi}{2}\right)$

Solution (a):

The angle $\frac{7\pi}{4}$ corresponds to the coordinates $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ on the unit circle.

The value of the sine function is the y-coordinate. $\boxed{\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}}$

Solution (b):

The angle $\frac{5\pi}{6}$ corresponds to the coordinate $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ on the unit circle.

The value of the cosine function is the x-coordinate. $\boxed{\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}}$

Solution (c):

The angle $\frac{3\pi}{2}$ corresponds to the coordinate $(0, -1)$ on the unit circle.

The value of the cosine function is the x-coordinate. $\cos\left(\frac{3\pi}{2}\right) = 0$

The value of the sine function is the y-coordinate. $\sin\left(\frac{3\pi}{2}\right) = -1$

Tangent is the ratio of sine to cosine. $\tan\left(\frac{3\pi}{2}\right) = \dfrac{\sin\left(\frac{3\pi}{2}\right)}{\cos\left(\frac{3\pi}{2}\right)}$

Let $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$. $\tan\left(\frac{3\pi}{2}\right) = \frac{-1}{0}$

$\boxed{\tan\left(\frac{3\pi}{2}\right) \text{ is undefined.}}$

YOUR TURN Find the exact values for

a. $\sin\left(\frac{5\pi}{6}\right)$ **b.** $\cos\left(\frac{7\pi}{4}\right)$ **c.** $\tan\left(\frac{2\pi}{3}\right)$

Answer: a. $\frac{1}{2}$ **b.** $\frac{\sqrt{2}}{2}$ **c.** $-\sqrt{3}$

EXAMPLE 2 **Solving Equations Involving Trigonometric (Circular) Functions**

Use the unit circle to find all values of θ, $0 \leq \theta \leq 2\pi$, for which $\sin\theta = -\frac{1}{2}$.

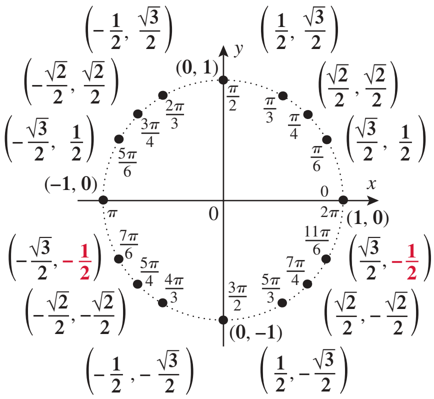
Solution:

The value of sine is the y-coordinate.

Since the value of sine is negative, θ must lie in quadrant III or quadrant IV.

There are two values for θ that are greater than or equal to zero and less than or equal to 2π that correspond to $\sin\theta = -\frac{1}{2}$.

$$\boxed{\theta = \frac{7\pi}{6}, \frac{11\pi}{6}}$$



■ **YOUR TURN** Find all values of θ, $0 \leq \theta \leq 2\pi$, for which $\cos\theta = -\frac{1}{2}$.

■ **Answer:** $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Properties of Trigonometric (Circular) Functions

Words	Math
For a point (x, y) that lies on the unit circle, $x^2 + y^2 = 1$.	$-1 \leq x \leq 1$ and $-1 \leq y \leq 1$
Since $(x, y) = (\cos\theta, \sin\theta)$, the following holds.	$-1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$
State the **domain and range of the cosine and sine functions**.	Domain: $(-\infty, \infty)$ Range: $[-1, 1]$
Since $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\csc\theta = \dfrac{1}{\sin\theta}$, the values for θ that make $\sin\theta = 0$ must be eliminated from the **domain of the cotangent and cosecant functions**. (The integer multiples of π, i.e., $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ can be written as $n\pi$.)	Domain: $\theta \neq n\pi$, n an integer
Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$, the values for θ that make $\cos\theta = 0$ must be eliminated from the **domain of the tangent and secant functions**.	Domain: $\theta \neq \dfrac{(2n+1)\pi}{2}$, n an integer

The following box summarizes the domains and ranges of the trigonometric functions.

DOMAINS AND RANGES OF THE TRIGONOMETRIC (CIRCULAR) FUNCTIONS

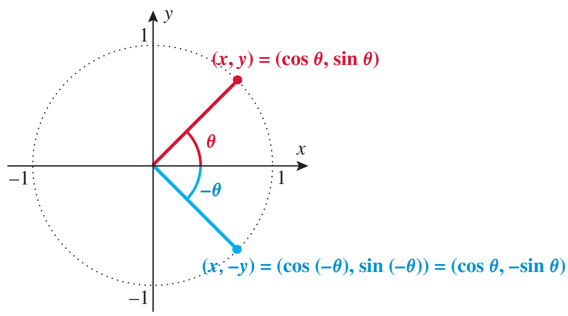
For any real number θ and integer n:

FUNCTION	DOMAIN	RANGE
$\sin\theta$	$(-\infty, \infty)$	$[-1, 1]$
$\cos\theta$	$(-\infty, \infty)$	$[-1, 1]$
$\tan\theta$	all real numbers such that $\theta \neq \dfrac{(2n+1)\pi}{2}$	$(-\infty, \infty)$
$\cot\theta$	all real numbers such that $\theta \neq n\pi$	$(-\infty, \infty)$
$\sec\theta$	all real numbers such that $\theta \neq \dfrac{(2n+1)\pi}{2}$	$(-\infty, -1] \cup [1, \infty)$
$\csc\theta$	all real numbers such that $\theta \neq n\pi$	$(-\infty, -1] \cup [1, \infty)$

Study Tip

- The sine function is an odd function.
- The cosine function is an even function.

Recall from algebra that **even functions** are functions for which $f(-x) = f(x)$ and **odd functions** are functions for which $f(-x) = -f(x)$.



The **cosine function is an even function**. $\boxed{\cos\theta = \cos(-\theta)}$

The **sine function is an odd function**. $\boxed{\sin(-\theta) = -\sin\theta}$

Technology Tip

Use a TI/scientific calculator to check the value of $\cos\left(-\dfrac{5\pi}{6}\right)$.

```
cos(-5π/6)
     -.8660254038
-√(3)/2
     -.8660254038
```

EXAMPLE 3 Using Properties of Trigonometric (Circular) Functions

Evaluate $\cos\left(-\dfrac{5\pi}{6}\right)$.

Solution:

The cosine function is an even function. $\cos\left(-\dfrac{5\pi}{6}\right) = \cos\left(\dfrac{5\pi}{6}\right)$

Use the unit circle to evaluate cosine. $\cos\left(\dfrac{5\pi}{6}\right) = -\dfrac{\sqrt{3}}{2}$

$$\cos\left(-\frac{5\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}}$$

■ **YOUR TURN** Evaluate $\sin\left(-\dfrac{5\pi}{6}\right)$.

■ **Answer:** $-\frac{1}{2}$

It is important to note that although trigonometric functions can be evaluated exactly for some special angles, a calculator can be used to approximate trigonometric functions for any angle. It is important to set the calculator to radian mode first, since θ is a real number.

EXAMPLE 4 Evaluating Trigonometric (Circular) Functions with a Calculator

Use a calculator to evaluate $\sin\left(\frac{7\pi}{12}\right)$. Round the answer to four decimal places.

CORRECT

Evaluate with a calculator.

0.965925826

Round to four decimal places.

$$\sin\left(\frac{7\pi}{12}\right) \approx \boxed{0.9659}$$

INCORRECT

Evaluate with a calculator.

0.031979376 **ERROR**

(Calculator in degree mode)

Many calculators automatically reset to degree mode after every calculation, so make sure to always check what mode the calculator indicates.

■ **YOUR TURN** Use a calculator to evaluate $\tan\left(\frac{9\pi}{5}\right)$. Round the answer to four decimal places.

EXAMPLE 5 Even and Odd Trigonometric (Circular) Functions

Show that the secant function is an even function.

Solution:

Show that $\sec(-\theta) = \sec\theta$.

Secant is the reciprocal of cosine. $\quad \sec(-\theta) = \dfrac{1}{\cos(-\theta)}$

Cosine is an even function, $\cos(-\theta) = \cos\theta$. $\quad \sec(-\theta) = \dfrac{1}{\cos\theta}$

Secant is the reciprocal of cosine, $\sec\theta = \dfrac{1}{\cos\theta}$. $\quad \sec(-\theta) = \dfrac{1}{\cos\theta} = \sec\theta$

Since $\sec(-\theta) = \sec\theta$, $\boxed{\text{the secant function is an even function}}$.

SECTION 5.1 SUMMARY

In this section, we have defined trigonometric functions as circular functions. Any point (x, y) that lies on the unit circle satisfies the equation $x^2 + y^2 = 1$. The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ can also be represented on the unit circle where $(x, y) = (\cos\theta, \sin\theta)$, and where θ is the central angle whose terminal side intersects the unit circle at the point (x, y). The cosine function is an even function, $\cos(-\theta) = \cos\theta$; the sine function is an odd function, $\sin(-\theta) = -\sin\theta$.

Study Tip

Set the calculator in radian mode before evaluating trigonometric functions in radians. Alternatively, convert the radian measure to degrees before evaluating the trigonometric function value.

Technology Tip

Use the TI/scientific calculator to evaluate $\sin\left(\frac{7\pi}{12}\right)$. Press 2nd ^ for π.

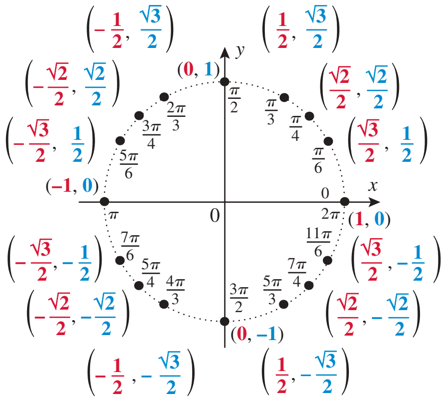
```
sin(7π/12)
      .9659258263
```

■ **Answer:** -0.7265

SECTION 5.1 EXERCISES

SKILLS

In Exercises 1–14, find the *exact* values of the indicated trigonometric functions using the unit circle.



1. $\sin\left(\frac{5\pi}{3}\right)$ **2.** $\cos\left(\frac{5\pi}{3}\right)$ **3.** $\cos\left(\frac{7\pi}{6}\right)$ **4.** $\sin\left(\frac{7\pi}{6}\right)$

5. $\sin\left(\frac{3\pi}{4}\right)$ **6.** $\cos\left(\frac{3\pi}{4}\right)$ **7.** $\tan\left(\frac{7\pi}{4}\right)$ **8.** $\cot\left(\frac{7\pi}{4}\right)$

9. $\sec\left(\frac{5\pi}{4}\right)$ **10.** $\csc\left(\frac{5\pi}{3}\right)$ **11.** $\tan\left(\frac{4\pi}{3}\right)$ **12.** $\cot\left(\frac{11\pi}{6}\right)$

13. $\csc\left(\frac{5\pi}{6}\right)$ **14.** $\cot\left(\frac{2\pi}{3}\right)$

In Exercises 15–34, use the unit circle and the fact that sine is an odd function and cosine is an even function to find the *exact* values of the indicated functions.

15. $\sin\left(-\frac{2\pi}{3}\right)$ **16.** $\sin\left(-\frac{5\pi}{4}\right)$ **17.** $\sin\left(-\frac{\pi}{3}\right)$ **18.** $\sin\left(-\frac{7\pi}{6}\right)$ **19.** $\cos\left(-\frac{3\pi}{4}\right)$

20. $\cos\left(-\frac{5\pi}{3}\right)$ **21.** $\cos\left(-\frac{5\pi}{6}\right)$ **22.** $\cos\left(-\frac{7\pi}{4}\right)$ **23.** $\sin\left(-\frac{5\pi}{4}\right)$ **24.** $\sin(-\pi)$

25. $\sin\left(-\frac{3\pi}{2}\right)$ **26.** $\sin\left(-\frac{\pi}{3}\right)$ **27.** $\cos\left(-\frac{\pi}{4}\right)$ **28.** $\cos\left(-\frac{3\pi}{4}\right)$ **29.** $\cos\left(-\frac{\pi}{2}\right)$

30. $\cos\left(-\frac{7\pi}{6}\right)$ **31.** $\csc\left(-\frac{5\pi}{6}\right)$ **32.** $\sec\left(-\frac{7\pi}{4}\right)$ **33.** $\tan\left(-\frac{11\pi}{6}\right)$ **34.** $\cot\left(-\frac{11\pi}{6}\right)$

In Exercises 35–54, use the unit circle to find all of the exact values of θ that make the equation true in the indicated interval.

35. $\cos\theta = \frac{\sqrt{3}}{2},\ 0 \le \theta \le 2\pi$ **36.** $\cos\theta = -\frac{\sqrt{3}}{2},\ 0 \le \theta \le 2\pi$ **37.** $\sin\theta = -\frac{\sqrt{3}}{2},\ 0 \le \theta \le 2\pi$

38. $\sin\theta = \frac{\sqrt{3}}{2},\ 0 \le \theta \le 2\pi$ **39.** $\sin\theta = 0,\ 0 \le \theta \le 4\pi$ **40.** $\sin\theta = -1, 0 \le \theta \le 4\pi$

41. $\cos\theta = -1,\ 0 \le \theta \le 4\pi$ **42.** $\cos\theta = 0,\ 0 \le \theta \le 4\pi$ **43.** $\tan\theta = -1,\ 0 \le \theta \le 2\pi$

44. $\cot\theta = 1,\ 0 \le \theta \le 2\pi$ **45.** $\sec\theta = -\sqrt{2},\ 0 \le \theta \le 2\pi$ **46.** $\csc\theta = \sqrt{2},\ 0 \le \theta \le 2\pi$

47. $\csc\theta$ is undefined, $0 \le \theta \le 2\pi$ **48.** $\sec\theta$ is undefined, $0 \le \theta \le 2\pi$ **49.** $\tan\theta$ is undefined, $0 \le \theta \le 2\pi$

50. $\cot\theta$ is undefined, $0 \le \theta \le 2\pi$ **51.** $\csc\theta = -2, 0 \le \theta \le 2\pi$ **52.** $\cot\theta = -\sqrt{3}, 0 \le \theta \le 2\pi$

53. $\sec\theta = \frac{2\sqrt{3}}{3}, 0 \le \theta \le 2\pi$ **54.** $\tan\theta = \frac{\sqrt{3}}{3},\ 0 \le \theta \le 2\pi$

APPLICATIONS

For Exercises 55 and 56, refer to the following:

The average daily temperature in Peoria, Illinois, can be predicted by the formula $T = 50 - 28\cos\left[\frac{2\pi(x - 31)}{365}\right]$, where x is the number of the day in the year (January 1 = 1, February 1 = 32, etc.) and T is in degrees Fahrenheit.

55. Atmospheric Temperature. What is the expected temperature on February 15?

56. Atmospheric Temperature. What is the expected temperature on August 15? (Assume it is not a leap year.)

For Exercises 57 and 58, refer to the following:

The human body temperature normally fluctuates during the day. A person's body temperature can be predicted by the formula $T = 99.1 - 0.5\sin\left(x + \frac{\pi}{12}\right)$, where x is the number of hours since midnight and T is in degrees Fahrenheit.

57. Body Temperature. What is the person's temperature at 6:00 A.M.?

58. Body Temperature. What is the person's temperature at 9:00 P.M.?

For Exercises 59 and 60, refer to the following:

The height of the water in a harbor changes with the tides. The height of the water at a particular hour during the day can be determined by the formula $h(x) = 5 + 4.8\sin\left[\frac{\pi}{6}(x + 4)\right]$, where x is the number of hours since midnight and h is the height of the tide in feet.

Bill Brooks/Alamy

Bill Brooks/Alamy

59. Tides. What is the height of the tide at 3:00 P.M.?

60. Tides. What is the height of the tide at 5:00 A.M.?

61. Yo-Yo Dieting. A woman has been yo-yo dieting for years. Her weight changes throughout the year as she gains and loses weight. Her weight in a particular month can be determined by the formula $w(x) = 145 + 10\cos\left(\frac{\pi}{6}x\right)$, where x is the month and w is in pounds. If $x = 1$ corresponds to January, how much does she weigh in June?

62. Yo-Yo Dieting. How much does the woman in Exercise 61 weigh in December?

63. Seasonal Sales. The average number of guests visiting the Magic Kingdom at Walt Disney World per day is given by $n(x) = 30{,}000 + 20{,}000\sin\left[\frac{\pi}{2}(x + 1)\right]$, where n is the number of guests and x is the month. If January corresponds to $x = 1$, how many people on average are visiting the Magic Kingdom per day in February?

64. Seasonal Sales. How many guests are visiting the Magic Kingdom in Exercise 63 in December?

For Exercises 65 and 66, refer to the following:

During the course of treatment of an illness, the concentration of a drug in the bloodstream in micrograms per microliter fluctuates during the dosing period of 8 hours according to the model

$$C(t) = 15.4 - 4.7\sin\left(\frac{\pi}{4}t + \frac{\pi}{2}\right), \quad 0 \leq t \leq 8$$

Note: This model does not apply to the first dose of the medication.

65. Health/Medicine. Find the concentration of the drug in the bloodstream at the beginning of a dosing period.

66. Health/Medicine. Find the concentration of the drug in the bloodstream 6 hours after taking a dose of the drug.

In Exercises 67 and 68, refer to the following:

By analyzing available empirical data, it has been determined that the body temperature of a particular species fluctuates during a 24-hour day according to the model

$$T(t) = 36.3 - 1.4\cos\left[\frac{\pi}{12}(t - 2)\right], \quad 0 \leq t \leq 24$$

where T represents temperature in degrees Celsius and t represents time in hours measured from 12:00 A.M. (midnight).

67. Biology. Find the approximate body temperature at midnight. Round your answer to the nearest degree.

68. Biology. Find the approximate body temperature at 2:45 P.M. Round your answer to the nearest degree.

CATCH THE MISTAKE

In Exercises 69 and 70, explain the mistake that is made.

69. Use the unit circle to evaluate $\tan\left(\frac{5\pi}{6}\right)$ exactly.

Solution:

Tangent is the ratio of sine to cosine. $\tan\left(\frac{5\pi}{6}\right) = \frac{\sin\left(\frac{5\pi}{6}\right)}{\cos\left(\frac{5\pi}{6}\right)}$

Use the unit circle to identify sine and cosine. $\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\cos\left(\frac{5\pi}{6}\right) = \frac{1}{2}$

Substitute values for sine and cosine. $\tan\left(\frac{5\pi}{6}\right) = \frac{-(\sqrt{3}/2)}{1/2}$

Simplify. $\tan\left(\frac{5\pi}{6}\right) = -\sqrt{3}$

This is incorrect. What mistake was made?

70. Use the unit circle to evaluate $\sec\left(\frac{11\pi}{6}\right)$ exactly.

Solution:

Secant is the reciprocal of cosine. $\sec\left(\frac{11\pi}{6}\right) = \frac{1}{\cos\left(\frac{11\pi}{6}\right)}$

Use the unit circle to evaluate cosine. $\cos\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$

Substitute the value for cosine. $\sec\left(\frac{11\pi}{6}\right) = \frac{1}{-\frac{1}{2}}$

Simplify. $\sec\left(\frac{11\pi}{6}\right) = -2$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 71–76, determine whether each statement is true or false.

71. $\sin(2n\pi + \theta) = \sin\theta$, n an integer.

72. $\cos(2n\pi + \theta) = \cos\theta$, n an integer.

73. $\sin\theta = 1$ when $\theta = \frac{(2n+1)\pi}{2}$, n an integer.

74. $\cos\theta = 1$ when $\theta = n\pi$, n an integer.

75. $\tan(\theta + 2n\pi) = \tan\theta$, n an integer.

76. $\tan\theta = 0$ if and only if $\theta = \frac{(2n+1)\pi}{2}$, n an integer.

77. Is cosecant an even or an odd function? Justify your answer.

78. Is tangent an even or an odd function? Justify your answer.

CHALLENGE

79. Find all the values of θ, $0 \le \theta \le 2\pi$, for which the equation $\sin\theta = \cos\theta$ is true.

80. Find all the values of θ (θ is any real number) for which the equation $\sin\theta = \cos\theta$ is true.

81. Find all the values of θ, $0 \le \theta \le 2\pi$, for which the equation $2\sin\theta = \csc\theta$ is true.

82. Find all the values of θ, $0 \le \theta \le 2\pi$, for which the equation $\cos\theta = \frac{1}{4}\sec\theta$ is true.

83. Find all the values of θ (θ is any real number) for which the equation $3\csc\theta = 4\sin\theta$ is true.

84. Find all the values of θ (θ is any real number) for which the equation $4\cos\theta = 3\sec\theta$ is true.

85. Does there exist an angle $0 \le \theta < 2\pi$ such that $\tan\theta = \cot\theta$?

86. Does there exist an angle $0 \le \theta < 2\pi$ such that $\sec\theta = \csc(-\theta)$?

TECHNOLOGY

87. Use a calculator to approximate $\sin 423°$. What do you expect $\sin(-423°)$ to be? Verify your answer with a calculator.

88. Use a calculator to approximate $\cos 227°$. What do you expect $\cos(-227°)$ to be? Verify your answer with a calculator.

89. Use a calculator to approximate $\tan 81°$. What do you expect $\tan(-81°)$ to be? Verify your answer with a calculator.

90. Use a calculator to approximate $\csc 211°$. What do you expect $\csc(-211°)$ to be? Verify your answer with a calculator.

For Exercises 91–94, refer to the following:

Set the calculator in parametric and radian modes and let

$$X_1 = \cos T$$
$$Y_1 = \sin T$$

Set the window so that $0 \le T \le 2\pi$, step $= \frac{\pi}{15}$, $-2 \le X \le 2$, and $-2 \le Y \le 2$. To approximate the sine or cosine of a T value, use the $\boxed{\text{TRACE}}$ key, type in the T value, and read the corresponding coordinates from the screen.

91. Approximate $\cos\left(\frac{\pi}{3}\right)$, take 5 steps of $\frac{\pi}{15}$ each, and read the x-coordinate.

92. Approximate $\sin\left(\frac{\pi}{3}\right)$, take 5 steps of $\frac{\pi}{15}$ each, and read the y-coordinate.

93. Approximate $\sin\left(\frac{2\pi}{3}\right)$ to four decimal places.

94. Approximate $\cos\left(\frac{5\pi}{4}\right)$ to four decimal places.

PREVIEW TO CALCULUS

The Fundamental Theorem of Calculus establishes that the definite integral $\int_a^b f(x)dx$ equals $F(b) - F(a)$, where F is any antiderivative of a continuous function f.

In Exercises 95–98, use the information below to find the exact value of each definite integral.

FUNCTION	$\sin x$	$\cos x$	$\sec^2 x$	$\csc x \cot x$
ANTIDERIVATIVE	$-\cos x$	$\sin x$	$\tan x$	$-\csc x$

95. $\int_0^{\pi} \sin x\, dx$

96. $\int_{\pi/4}^{5\pi/6} \cos x\, dx$

97. $\int_{7\pi/6}^{5\pi/4} \sec^2 x\, dx$

98. $\int_{5\pi/3}^{11\pi/6} \csc x \cot x\, dx$

SECTION 5.2 GRAPHS OF SINE AND COSINE FUNCTIONS

SKILLS OBJECTIVES

- Graph the sine and cosine functions.
- Determine the domain and range of the sine and cosine functions.
- Determine the amplitude and period of sinusoidal functions.
- Determine the phase shift of a sinusoidal function.
- Solve harmonic motion problems.
- Graph sums of functions.

CONCEPTUAL OBJECTIVES

- Understand why the graphs of the sine and cosine functions are called sinusoidal graphs.
- Understand the cyclic nature of periodic functions.
- Visualize harmonic motion as a sinusoidal function.

The Graphs of Sinusoidal Functions

The following are examples of things that repeat in a predictable way (are roughly periodic):

- heartbeat
- tide levels
- time of sunrise
- average outdoor temperature for the time of year

The trigonometric functions are *strictly* periodic. In the unit circle, the value of any of the trigonometric functions is the same for any coterminal angle (same initial and terminal sides no matter how many full rotations the angle makes). For example, if we add (or subtract) multiples of 2π to (from) the angle θ, the values for sine and cosine are unchanged.

$$\sin(\theta + 2n\pi) = \sin\theta \qquad \text{or} \qquad \cos(\theta + 2n\pi) = \cos\theta \quad (n \text{ is any integer})$$

DEFINITION **Periodic Function**

A function f is called a **periodic function** if there is a positive number p such that

$$f(x + p) = f(x) \qquad \text{for all } x \text{ in the domain of } f$$

If p is the smallest such number for which this equation holds, then p is called the **fundamental period**.

You will see in this chapter that sine, cosine, secant, and cosecant have fundamental period 2π, but that tangent and cotangent have fundamental period π.

The Graph of $f(x) = \sin x$

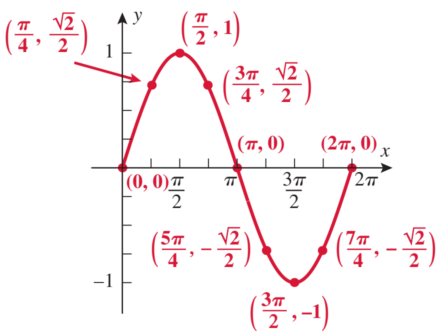
Let us start by point-plotting the sine function. We select special values for the sine function that we already know.

x	$f(x) = \sin x$	(x, y)
0	$\sin 0 = 0$	$(0, 0)$
$\frac{\pi}{4}$	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{4}$	$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}\right)$
π	$\sin \pi = 0$	$(\pi, 0)$
$\frac{5\pi}{4}$	$\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	$\sin\left(\frac{3\pi}{2}\right) = -1$	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{7\pi}{4}$	$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
2π	$\sin(2\pi) = 0$	$(2\pi, 0)$

Study Tip

Note that either notation, $y = \sin x$ or $f(x) = \sin x$, can be used.

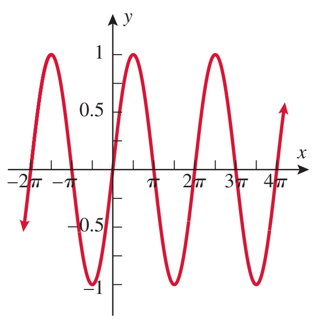
By plotting the above coordinates (x, y), we can obtain the graph of one **period**, or **cycle**, of the graph of $y = \sin x$. Note that $\frac{\sqrt{2}}{2} \approx 0.7$.



Study Tip

Looking at the graph of $f(x) = \sin x$, we are reminded that $\sin x > 0$ when $0 < x < \pi$ and $\sin x < 0$ when $\pi < x < 2\pi$. We also see that when $x = \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$, the reference angle is $\frac{\pi}{4}$.

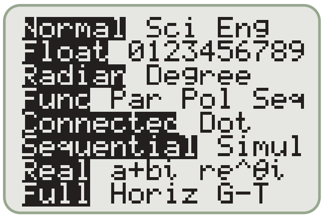
We can extend the graph horizontally in both directions (left and right) since the domain of the sine function is the set of all real numbers.



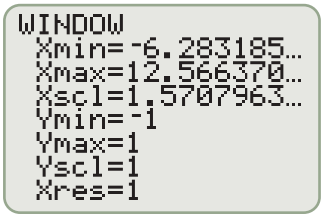
From here on, we are no longer showing angles on the unit circle but are now showing angles as *real numbers* in radians on the x-axis of the *Cartesian* graph. Therefore, we no longer illustrate a "terminal side" to an angle—the physical arcs and angles no longer exist; only their measures exist, as values of the x-coordinate.

Technology Tip

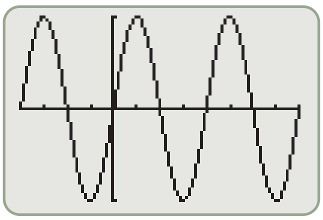
Set a TI/scientific calculator to radian mode by typing MODE.



Set the window at Xmin at -2π, Xmax at 4π, Xscl at $\frac{\pi}{2}$, Ymin at -1, Ymax at 1, and Yscl at 1. Setting Xscl at $\frac{\pi}{2}$ will mark the labels on the x-axis in terms of multiples of $\frac{\pi}{2}$.



Use Y= to enter the function sin(X).



If we graph the function $f(x) = \sin x$, the x-intercepts correspond to values of x at which the sine function is equal to zero.

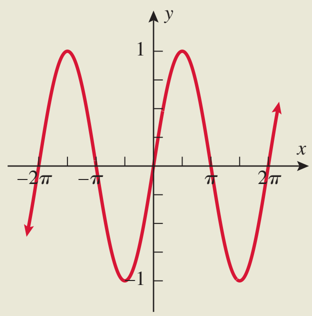
x	$f(x) = \sin x$	(x, y)
0	$\sin 0 = 0$	$(0, 0)$
π	$\sin \pi = 0$	$(\pi, 0)$
2π	$\sin(2\pi) = 0$	$(2\pi, 0)$
3π	$\sin(3\pi) = 0$	$(3\pi, 0)$
4π	$\sin(4\pi) = 0$	$(4\pi, 0)$
. . .		
$n\pi$	$\sin(n\pi) = 0$	$(n\pi, 0)$, n is an integer.

Notice that the point (0, 0) is both a y-intercept and an x-intercept but all x-intercepts have the form $(n\pi, 0)$. The maximum value of the sine function is 1, and the minimum value of the sine function is -1, which occurs at odd multiples of $\frac{\pi}{2}$.

x	$f(x) = \sin x$	(x, y)
$\frac{\pi}{2}$	$\sin\left(\frac{\pi}{2}\right) = 1$	$\left(\frac{\pi}{2}, 1\right)$
$\frac{3\pi}{2}$	$\sin\left(\frac{3\pi}{2}\right) = -1$	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{5\pi}{2}$	$\sin\left(\frac{5\pi}{2}\right) = 1$	$\left(\frac{5\pi}{2}, 1\right)$
$\frac{7\pi}{2}$	$\sin\left(\frac{7\pi}{2}\right) = -1$	$\left(\frac{7\pi}{2}, -1\right)$
. . .	. . .	. . .
$\frac{(2n+1)\pi}{2}$	$\sin\left(\frac{(2n+1)\pi}{2}\right) = \pm 1$	$\left(\frac{(2n+1)\pi}{2}, \pm 1\right)$, n is an integer.

The following box summarizes the sine function:

SINE FUNCTION $f(x) = \sin x$



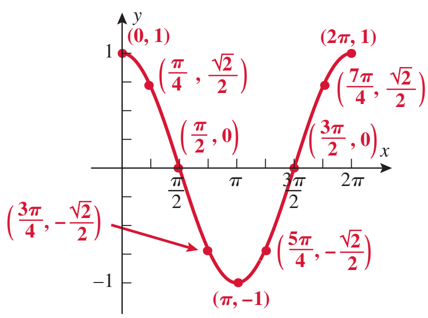
- Domain: $(-\infty, \infty)$ or $-\infty < x < \infty$
- Range: $[-1, 1]$ or $-1 \le y \le 1$
- The sine function is an odd function:
  - symmetric about the origin
  - $\sin(-x) = -\sin x$
- The sine function is a periodic function with fundamental period 2π.
- The x-intercepts, $0, \pm\pi, \pm 2\pi, \ldots$, are of the form $n\pi$, where n is an integer.
- The maximum (1) and minimum (-1) values of the sine function correspond to x-values of the form $\frac{(2n+1)\pi}{2}$, such as $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

The Graph of $f(x) = \cos x$

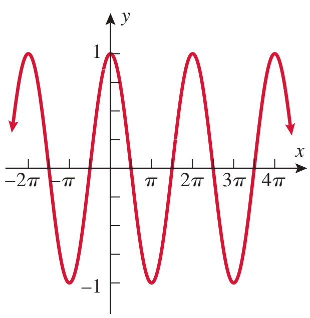
Let us start by point-plotting the cosine function.

x	$f(x) = \cos x$	(x, y)
0	$\cos 0 = 1$	$(0, 1)$
$\frac{\pi}{4}$	$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{2}$	$\cos\left(\frac{\pi}{2}\right) = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$	$\left(\frac{3\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
π	$\cos \pi = -1$	$(\pi, -1)$
$\frac{5\pi}{4}$	$\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$	$\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
$\frac{3\pi}{2}$	$\cos\left(\frac{3\pi}{2}\right) = 0$	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{7\pi}{4}$	$\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$	$\left(\frac{7\pi}{4}, \frac{\sqrt{2}}{2}\right)$
2π	$\cos(2\pi) = 1$	$(2\pi, 1)$

By plotting the above coordinates (x, y), we can obtain the graph of one period, or cycle, of the graph of $y = \cos x$. Note that $\frac{\sqrt{2}}{2} \approx 0.7$.



We can extend the graph horizontally in both directions (left and right) since the domain of the cosine function is all real numbers.



If we graph the function $f(x) = \cos x$, the x-intercepts correspond to values of x at which the cosine function is equal to zero.

Study Tip

Note that either notation, $y = \cos x$ or $f(x) = \cos x$, can be used.

Study Tip

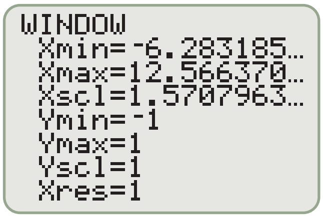
Looking at the graph of $f(x) = \cos x$, we are reminded that $\cos x > 0$ when

$0 < \pi < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$

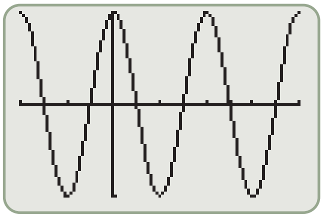
and $\cos x < 0$ when $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

Technology Tip

Set the window at Xmin at -2π, Xmax at 4π, Xscl at $\frac{\pi}{2}$, Ymin at -1, Ymax at 1, and Yscl at 1. Setting Xscl at $\frac{\pi}{2}$ will mark the labels on the x-axis in terms of multiples of $\frac{\pi}{2}$.



Use $\boxed{\text{Y=}}$ to enter the function cos(X).



x	$f(x) = \cos x$	(x, y)
$\frac{\pi}{2}$	$\cos\left(\frac{\pi}{2}\right) = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{2}$	$\cos\left(\frac{3\pi}{2}\right) = 0$	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{5\pi}{2}$	$\cos\left(\frac{5\pi}{2}\right) = 0$	$\left(\frac{5\pi}{2}, 0\right)$
$\frac{7\pi}{2}$	$\cos\left(\frac{7\pi}{2}\right) = 0$	$\left(\frac{7\pi}{2}, 0\right)$
. . .	. . .	. . .
$\frac{(2n+1)\pi}{2}$	$\cos\left(\frac{(2n+1)\pi}{2}\right) = 0$	$\left(\frac{(2n+1)\pi}{2}, 0\right)$, n is an integer.

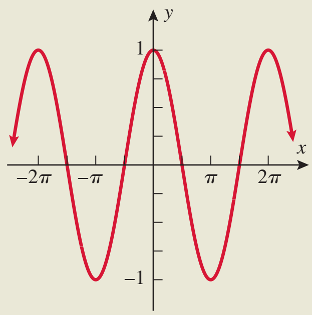
The point (0, 1) is the y-intercept, and there are several x-intercepts of the form $\left(\frac{(2n+1)\pi}{2}, 0\right)$. The maximum value of the cosine function is 1 and the minimum value of the cosine function is -1; these values occur at integer multiples of π, i.e., $n\pi$.

x	$f(x) = \cos x$	(x, y)
0	$\cos 0 = 1$	$(0, 1)$
π	$\cos \pi = -1$	$(\pi, -1)$
2π	$\cos(2\pi) = 1$	$(2\pi, 1)$
3π	$\cos(3\pi) = -1$	$(3\pi, -1)$
4π	$\cos(4\pi) = 1$	$(4\pi, 1)$
. . .	. . .	. . .
$n\pi$	$\cos(n\pi) = \pm 1$	$(n\pi, \pm 1)$, n is an integer.

The following box summarizes the cosine function:

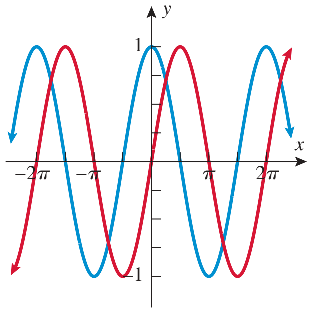
Cosine function $f(x) = \cos x$

- Domain: $(-\infty, \infty)$ or $-\infty < x < \infty$
- Range: $[-1, 1]$ or $-1 \le y \le 1$
- The cosine function is an even function:
  - symmetric about the y-axis
  - $\cos(-x) = \cos x$
- The cosine function is a periodic function with fundamental period 2π.
- The x-intercepts, $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$, are odd integer multiples of $\frac{\pi}{2}$ that have the form $\frac{(2n+1)\pi}{2}$, where n is an integer.
- The maximum (1) and minimum (-1) values of the cosine function correspond to x-values of the form $n\pi$, such as $0, \pm\pi, \pm 2\pi, \ldots$.



The Amplitude and Period of Sinusoidal Graphs

In mathematics, the word **sinusoidal** means "resembling the sine function." Let us start by graphing $f(x) = \sin x$ and $f(x) = \cos x$ on the same graph. Notice that they have similar characteristics (domain, range, period, and shape).



In fact, if we were to shift the cosine graph to the right $\frac{\pi}{2}$ units, the two graphs would be identical. For that reason we refer to any graphs of the form $y = \cos x$ or $y = \sin x$ as **sinusoidal functions**.

We now turn our attention to graphs of the form $y = A\sin(Bx)$ and $y = A\cos(Bx)$, which are graphs like $y = \sin x$ and $y = \cos x$ that have been stretched or compressed vertically and horizontally.

EXAMPLE 1 **Vertical Stretching and Compressing**

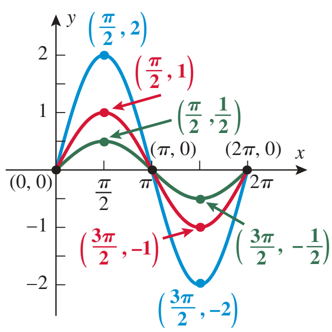
Plot the functions $y = 2\sin x$ and $y = \frac{1}{2}\sin x$ on the same graph with $y = \sin x$ on the interval $-4\pi \le x \le 4\pi$.

Solution:

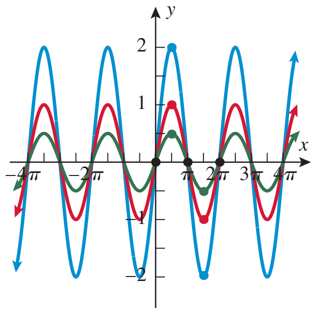
STEP 1 Make a table with the coordinate values of the graphs.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin x$	0	1	0	−1	0
$2\sin x$	0	2	0	−2	0
$\frac{1}{2}\sin x$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0

STEP 2 Label the points on the graph and connect with a smooth curve over one period, $0 \le x \le 2\pi$.

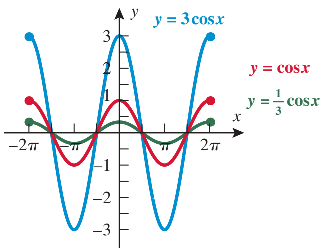


STEP 3 Extend the graph in both directions (repeat every 2π).



■ **YOUR TURN** Plot the functions $y = 3\cos x$ and $y = \frac{1}{3}\cos x$ on the same graph with $y = \cos x$ on the interval $-2\pi \le x \le 2\pi$.

■ **Answer:**



Notice in Example 1 and the corresponding Your Turn that:

- $y = 2\sin x$ has the shape and period of $y = \sin x$ but is stretched vertically.
- $y = \frac{1}{2}\sin x$ has the shape and period of $y = \sin x$ but is compressed vertically.
- $y = 3\cos x$ has the shape and period of $y = \cos x$ but is stretched vertically.
- $y = \frac{1}{3}\cos x$ has the shape and period of $y = \cos x$ but is compressed vertically.

In general, functions of the form $y = A\sin x$ and $y = A\cos x$ are stretched vertically when $|A| > 1$ and compressed vertically when $|A| < 1$.

The **amplitude** of a periodic function is half the difference between the maximum value of the function and the minimum value of the function. For the functions $y = \sin x$ and $y = \cos x$, the maximum value is 1 and the minimum value is -1. Therefore, the amplitude of each of these two functions is $|A| = \frac{1}{2}|1 - (-1)| = 1$.

AMPLITUDE OF SINUSOIDAL FUNCTIONS

For sinusoidal functions of the form $y = A\sin(Bx)$ and $y = A\cos(Bx)$, the **amplitude** is $|A|$. When $|A| < 1$, the graph is compressed vertically, and when $|A| > 1$, the graph is stretched vertically.

EXAMPLE 2 Finding the Amplitude of Sinusoidal Functions

State the amplitude of

a. $f(x) = -4\cos x$

b. $g(x) = \frac{1}{5}\sin x$

Solution (a): The amplitude is the magnitude of -4. $\quad A = |-4| = \boxed{4}$

Solution (b): The amplitude is the magnitude of $\frac{1}{5}$. $\quad A = \left|\frac{1}{5}\right| = \boxed{\frac{1}{5}}$

EXAMPLE 3 Horizontal Stretching and Compressing

Plot the functions $y = \cos(2x)$ and $y = \cos\left(\frac{1}{2}x\right)$ on the same graph with $y = \cos x$ on the interval $-2\pi \le x \le 2\pi$.

Solution:

STEP 1 Make a table with the coordinate values of the graphs. It is necessary only to select the points that correspond to x-intercepts, ($y = 0$), and maximum and minimum points, ($y = \pm 1$). Usually, the period is divided into four subintervals (which you will see in Examples 5 to 7).

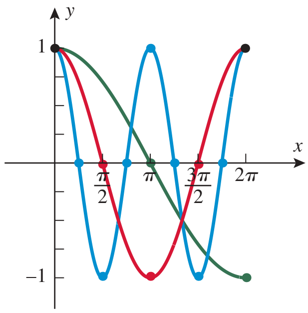
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos x$	1		0		-1		0		1
$\cos(2x)$	1	0	-1	0	1	0	-1	0	1
$\cos\left(\frac{1}{2}x\right)$	1				0				-1

STEP 2 Label the points on the graph and connect with a smooth curve.

$y = \cos x$

$y = \cos(2x)$

$y = \cos\left(\dfrac{1}{2}x\right)$

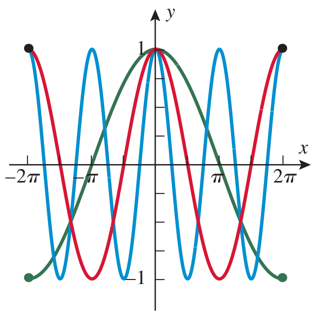


STEP 3 Extend the graph to cover the entire interval: $-2\pi \le x \le 2\pi$.

$y = \cos x$

$y = \cos(2x)$

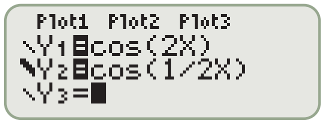
$y = \cos\left(\dfrac{1}{2}x\right)$

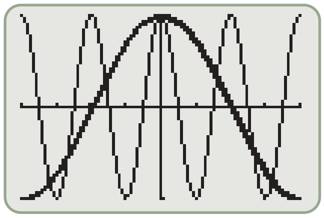


YOUR TURN Plot the functions $y = \sin(2x)$ and $y = \sin\left(\frac{1}{2}x\right)$ on the same graph with $y = \sin x$ on the interval $-2\pi \le x \le 2\pi$.

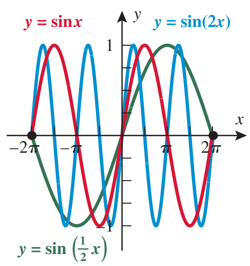
Technology Tip

Set the window at Xmin at -2π, Xmax at 2π, Xscl at $\frac{\pi}{2}$, Ymin at -1, Ymax at 1, and Yscl at 1. Setting Xscl at $\frac{\pi}{2}$ will mark the labels on the x-axis in terms of multiples of $\frac{\pi}{2}$.





Answer:



Notice in Example 3 and the corresponding Your Turn that:

- $y = \cos(2x)$ has the shape and amplitude of $y = \cos x$ but is compressed horizontally.
- $y = \cos\left(\frac{1}{2}x\right)$ has the shape and amplitude of $y = \cos x$ but is stretched horizontally.
- $y = \sin(2x)$ has the shape and amplitude of $y = \sin x$ but is compressed horizontally.
- $y = \sin\left(\frac{1}{2}x\right)$ has the shape and amplitude of $y = \sin x$ but is stretched horizontally.

In general, functions of the form $y = \sin(Bx)$ and $y = \cos(Bx)$, with $B > 0$, are compressed horizontally when $B > 1$ and stretched horizontally when $0 < B < 1$. Negative arguments ($B < 0$) are included in the context of *reflections*.

The period of the functions $y = \sin x$ and $y = \cos x$ is 2π. To find the period of a function of the form $y = A\sin(Bx)$ or $y = A\cos(Bx)$, set Bx equal to 2π and solve for x.

$$Bx = 2\pi$$
$$x = \frac{2\pi}{B}$$

PERIOD OF SINUSOIDAL FUNCTIONS

For sinusoidal functions of the form $y = A\sin(Bx)$ and $y = A\cos(Bx)$, with $B > 0$, the **period** is $\frac{2\pi}{B}$. When $0 < B < 1$, the graph is stretched horizontally, and when $B > 1$, the graph is compressed horizontally since the period is smaller than 2π.

Study Tip

When B is negative, the period is $\frac{2\pi}{|B|}$.

EXAMPLE 4 Finding the Period of a Sinusoidal Function

State the period of

a. $y = \cos(4x)$

b. $y = \sin\left(\frac{1}{3}x\right)$

Solution (a):

Compare $\cos(4x)$ with $\cos(Bx)$ to identify B. $\quad B = 4$

Calculate the period of $\cos(4x)$, using $p = \frac{2\pi}{B}$. $\quad p = \frac{2\pi}{4} = \frac{\pi}{2}$

The period of $\cos(4x)$ is $\boxed{p = \frac{\pi}{2}}$.

Solution (b):

Compare $\sin\left(\frac{1}{3}x\right)$ with $\sin(Bx)$ to identify B. $\quad B = \frac{1}{3}$

Calculate the period of $\sin\left(\frac{1}{3}x\right)$, using $p = \frac{2\pi}{B}$. $\quad p = \frac{2\pi}{\frac{1}{3}} = 6\pi$

The period of $\sin\left(\frac{1}{3}x\right)$ is $\boxed{p = 6\pi}$.

YOUR TURN State the period of

a. $y = \sin(3x)$ **b.** $y = \cos\left(\frac{1}{2}x\right)$

Answer:

a. $p = \frac{2\pi}{3}$ **b.** $p = 4\pi$

Now that you know the basic graphs of $y = \sin x$ and $y = \cos x$, you can sketch one cycle (period) of these graphs with the following x-values: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. For a period of 2π, we used steps of $\frac{\pi}{2}$. Therefore, for functions of the form $y = A\sin(Bx)$ or $y = A\cos(Bx)$, when we start at the origin and as long as we include these four basic values during one period, we are able to sketch the graphs.

Study Tip

Divide the period by 4 to get the key values along the x-axis for graphing.

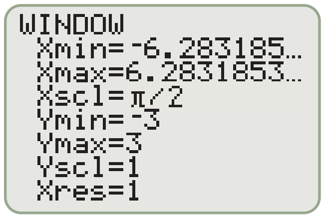
Strategy for Sketching Graphs of Sinusoidal Functions

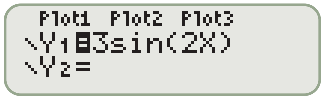
To graph $y = A\sin(Bx)$ or $y = A\cos(Bx)$ with $B > 0$:

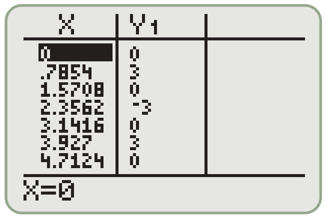
Step 1: Find the amplitude $|A|$ and period $\frac{2\pi}{B}$.
Step 2: Divide the period into four subintervals of equal lengths.
Step 3: Make a table and evaluate the function for x-values from Step 2 starting at $x = 0$.
Step 4: Draw the xy-plane (label the y-axis from $-|A|$ to $|A|$) and plot the points found in Step 3.
Step 5: Connect the points with a sinusoidal curve (with amplitude $|A|$).
Step 6: Extend the graph over one or two additional periods in both directions (left and right).

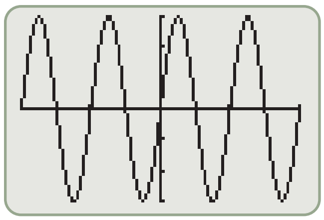
Technology Tip

Use a TI calculator to check the graph of $y = 3\sin(2x)$.









EXAMPLE 5 Graphing Sinusoidal Functions of the Form $y = A\sin(Bx)$

Use the strategy for graphing a sinusoidal function to graph $y = 3\sin(2x)$.

Solution:

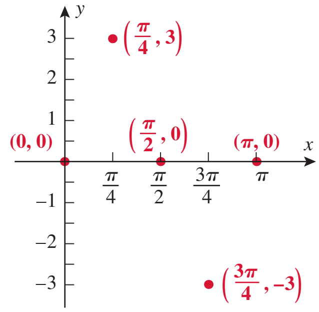
STEP 1 Find the amplitude and period for $A = 3$ and $B = 2$. $\quad |A| = |3| = 3 \quad$ and $\quad p = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

STEP 2 Divide the period π into four equal steps. $\quad \frac{\pi}{4}$

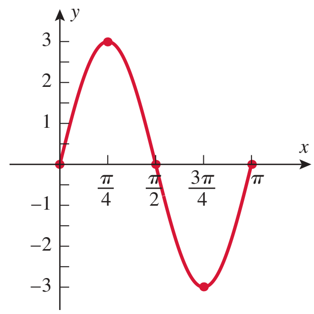
STEP 3 Make a table starting at $x = 0$ to the period $x = \pi$ in steps of $\frac{\pi}{4}$.

x	$y = 3\sin(2x)$	(x, y)
0	$3[\sin 0] = 3[0] = 0$	$(0, 0)$
$\frac{\pi}{4}$	$3\left[\sin\left(\frac{\pi}{2}\right)\right] = 3[1] = 3$	$\left(\frac{\pi}{4}, 3\right)$
$\frac{\pi}{2}$	$3[\sin \pi] = 3[0] = 0$	$\left(\frac{\pi}{2}, 0\right)$
$\frac{3\pi}{4}$	$3\left[\sin\left(\frac{3\pi}{2}\right)\right] = 3[-1] = -3$	$\left(\frac{3\pi}{4}, -3\right)$
π	$3[\sin(2\pi)] = 3[0] = 0$	$(\pi, 0)$

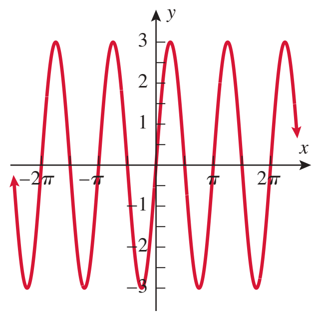
STEP 4 Draw the xy-plane and label the points in the table.



STEP 5 Connect the points with a sinusoidal curve.

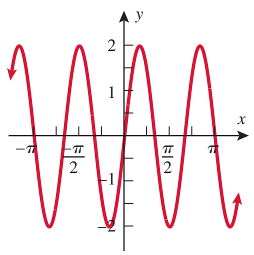


STEP 6 Repeat over several periods (to the left and right).



■ **YOUR TURN** Use the strategy for graphing sinusoidal functions to graph $y = 2\sin(3x)$.

■ **Answer:**



EXAMPLE 6 Graphing Sinusoidal Functions of the Form $y = A\cos(Bx)$

Use the strategy for graphing a sinusoidal function to graph $y = -2\cos\left(\frac{1}{3}x\right)$.

Solution:

STEP 1 Find the amplitude for $A = -2$. $\quad |A| = |-2| = 2$

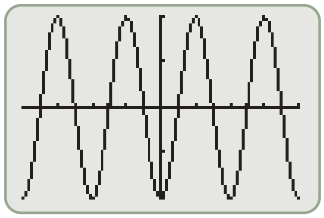
Find the period for $B = \frac{1}{3}$. $\quad p = \dfrac{2\pi}{B} = \dfrac{2\pi}{\frac{1}{3}} = 6\pi$

STEP 2 Divide the period 6π into four equal steps. $\quad \dfrac{6\pi}{4} = \dfrac{3\pi}{2}$

Technology Tip

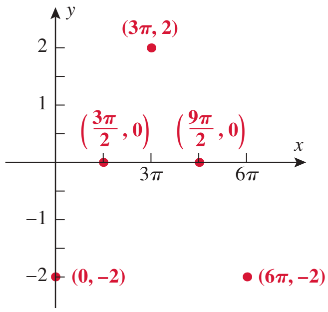
Use a TI calculator to check the graph of $y = -2\cos\left(\frac{1}{3}x\right)$. Set the window at Xmin at -12π, Xmax at 12π, Xscl at $\frac{3\pi}{2}$, Ymin at -2, Ymax at 2, and Yscl at 1. Setting Xscl at $\frac{3\pi}{2}$ will mark the labels on the x-axis in terms of multiples of $\frac{3\pi}{2}$.

STEP 3 Make a table starting at $x = 0$ and completing one period of 6π in steps of $\frac{3\pi}{2}$.

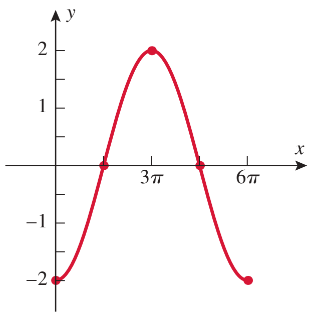


x	$y = -2\cos\left(\frac{1}{3}x\right)$	(x, y)
0	$-2[\cos 0] = -2[1] = -2$	$(0, -2)$
$\frac{3\pi}{2}$	$-2\left[\cos\left(\frac{\pi}{2}\right)\right] = -2[0] = 0$	$\left(\frac{3\pi}{2}, 0\right)$
3π	$-2[\cos\pi] = -2[-1] = 2$	$(3\pi, 2)$
$\frac{9\pi}{2}$	$-2\left[\cos\left(\frac{3\pi}{2}\right)\right] = -2[0] = 0$	$\left(\frac{9\pi}{2}, 0\right)$
6π	$-2[\cos(2\pi)] = -2[1] = -2$	$(6\pi, -2)$

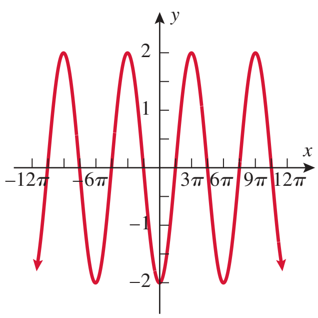
STEP 4 Draw the xy-plane and label the points in the table.



STEP 5 Connect the points with a sinusoidal curve.

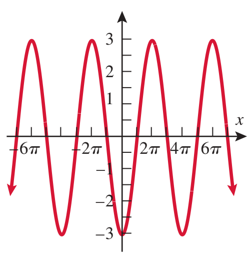


STEP 6 Repeat over several periods (to the left and right).



■ **YOUR TURN** Use the strategy for graphing a sinusoidal function to graph $y = -3\cos\left(\frac{1}{2}x\right)$.

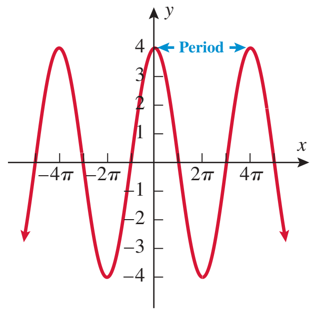
■ **Answer:**



Notice in Example 6 and the corresponding Your Turn that when A is negative, the result is a reflection of the original function (sine or cosine) about the x-axis.

EXAMPLE 7 **Finding an Equation for a Sinusoidal Graph**

Find an equation for the graph.



Solution:

This graph represents a cosine function. $y = A\cos(Bx)$

The amplitude is 4 (half the maximum spread). $|A| = 4$

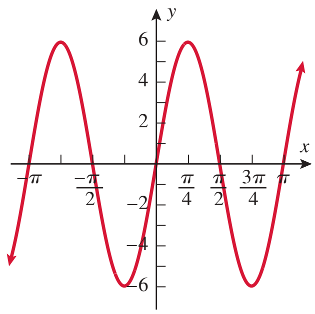
The period $\frac{2\pi}{B}$ is equal to 4π. $\frac{2\pi}{B} = 4\pi$

Solve for B. $B = \frac{1}{2}$

Substitute $A = 4$ and $B = \frac{1}{2}$ into $y = A\cos(Bx)$. $\boxed{y = 4\cos\left(\frac{1}{2}x\right)}$

Answer: $y = 6\sin(2x)$

YOUR TURN Find an equation for the graph.



Graphing a Shifted Sinusoidal Function: $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$

Recall from Section 1.3 that we graph functions using horizontal and vertical translations (shifts) in the following way ($c > 0$):

- To graph $f(x + c)$, shift $f(x)$ to the **left** c units.
- To graph $f(x - c)$, shift $f(x)$ to the **right** c units.
- To graph $f(x) + c$, shift $f(x)$ **up** c units.
- To graph $f(x) - c$, shift $f(x)$ **down** c units.

To graph functions of the form $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$, utilize the strategy below.

Strategy for graphing $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$

A strategy for graphing $y = A\sin(Bx + C) + D$ is outlined below. The same strategy can be used to graph $y = A\cos(Bx + C) + D$.

Step 1: Find the amplitude $|A|$.

Step 2: Find the period $\frac{2\pi}{B}$ and **phase shift** $-\frac{C}{B}$.

Step 3: Graph $y = A\sin(Bx + C)$ over one period $\left(\text{from } -\frac{C}{B} \text{ to } -\frac{C}{B} + \frac{2\pi}{B}\right)$.

Step 4: Extend the graph over several periods.

Step 5: Shift the graph of $y = A\sin(Bx + C)$ vertically D units.

Note: If we rewrite the function in **standard form**, we get

$$y = A\sin\left[B\left(x + \frac{C}{B}\right)\right] + D$$

which makes it easier to identify the **phase shift**.

If $B < 0$, we can use properties of even and odd functions

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

to rewrite the function with $B > 0$.

Study Tip

Rewriting in standard form

$$y = A\sin\left[B\left(x + \frac{C}{B}\right)\right]$$

makes identifying the phase shift easier.

Example 8 Graphing Functions of the Form $y = A\cos(Bx \pm C)$

Graph $y = 5\cos(4x + \pi)$ over one period.

Solution:

Step 1 Find the amplitude. $|A| = |5| = 5$

Step 2 Calculate the period and phase shift.

The interval for one period is from 0 to 2π. $\quad 4x + \pi = 0 \quad$ to $\quad 4x + \pi = 2\pi$

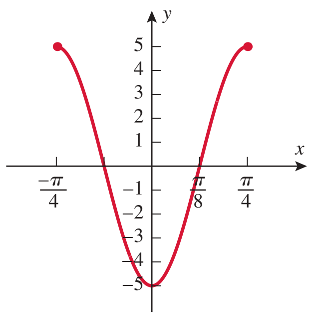
Solve for x. $\quad x = -\frac{\pi}{4} \quad$ to $\quad x = -\frac{\pi}{4} + \frac{\pi}{2}$

Identify the phase shift. $\quad -\frac{C}{B} = -\frac{\pi}{4}$

Identify the period $\frac{2\pi}{B}$. $\quad \frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$

Step 3 Graph.

Draw a cosine function starting at $x = -\frac{\pi}{4}$ with period $\frac{\pi}{2}$ and amplitude 5.



■ **Your Turn** Graph $y = 3\cos(2x - \pi)$ over one period.

Study Tip

An alternative method for finding the period and phase shift is to first write the function in standard form.

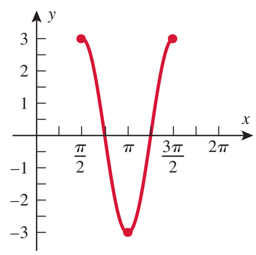
$$y = 5\cos\left[4\left(x + \frac{\pi}{4}\right)\right]$$

$B = 4$

Period $= \frac{2\pi}{B} = \frac{2\pi}{4} = \frac{\pi}{2}$.

Phase shift $= \frac{\pi}{4}$ units to the left.

■ **Answer:**

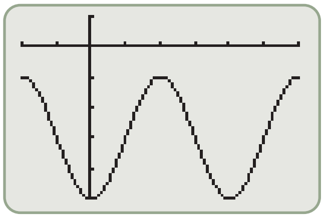


Technology Tip

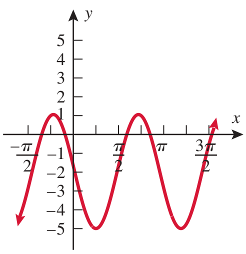
Use a TI calculator to check the graph of $y = -3 + 2\cos(2x - \pi)$.

Set the window at Xmin at $-\frac{\pi}{2}$, Xmax at $\frac{3\pi}{2}$, Xscl at $\frac{\pi}{4}$, Ymin at -5, Ymax at 1, and Yscl at 1.

Setting Xscl at $\frac{\pi}{4}$ will mark the labels on the x-axis in terms of multiples of $\frac{\pi}{4}$.



Answer:



EXAMPLE 9 Graphing Sinusoidal Functions

Graph $y = -3 + 2\cos(2x - \pi)$.

Solution:

STEP 1 Find the amplitude. $|A| = |2| = 2$

STEP 2 Find the phase shift and period.

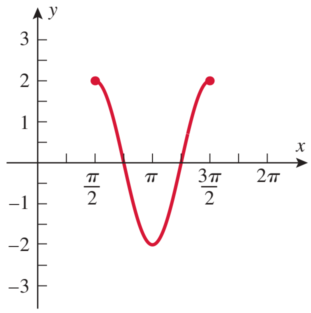
Set $2x - \pi$ equal to 0 and 2π. $2x - \pi = 0$ to $2x - \pi = 2\pi$

Solve for x. $x = \frac{\pi}{2}$ to $x = \frac{\pi}{2} + \pi$

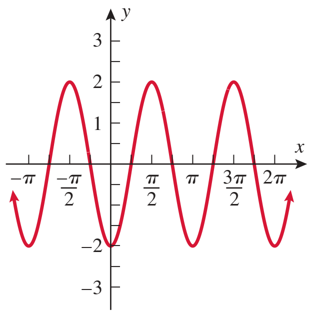
Identify phase shift and period. $-\frac{C}{B} = \frac{\pi}{2}$

$\frac{2\pi}{B} = \pi$

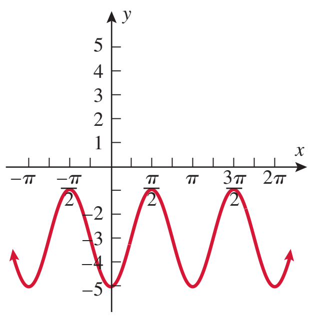
STEP 3 Graph $y = 2\cos(2x - \pi)$ starting at $x = \frac{\pi}{2}$ over one period, π.



STEP 4 Extend the graph of $y = 2\cos(2x - \pi)$ over several periods.



STEP 5 Shift the graph of $y = 2\cos(2x - \pi)$ down three units to arrive at the graph of $y = -3 + 2\cos(2x - \pi)$.



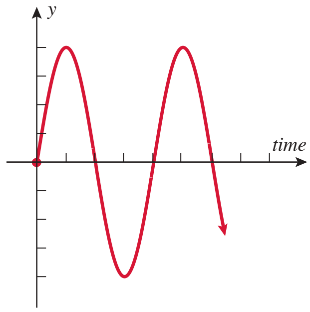
■ **YOUR TURN** Graph $y = -2 + 3\sin(2x + \pi)$.

Harmonic Motion

One of the most important applications of sinusoidal functions is in describing *harmonic motion*, which we define as the symmetric periodic movement of an object or quantity about a center (equilibrium) position or value. The oscillation of a pendulum is a form of harmonic motion. Other examples are the recoil of a spring balance scale when a weight is placed on the tray and the variation of current or voltage within an AC circuit.

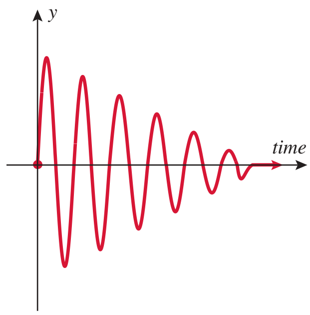
There are three types of harmonic motion: **simple harmonic motion**, **damped harmonic motion**, and **resonance**.

Simple Harmonic Motion



Simple harmonic motion is the kind of *unvarying* periodic motion that would occur in an ideal situation in which no resistive forces, such as friction, cause the amplitude of oscillation to decrease over time: the amplitude stays in exactly the same range in each period as time—the variable on the horizontal axis—increases. It will also occur if energy is being supplied at the correct rate to overcome resistive forces. Simple harmonic motion occurs, for example, in an AC electric circuit when a power source is consistently supplying energy. When you are swinging on a swing and "pumping" energy into the swing to keep it in motion at a constant period and amplitude, you are sustaining simple harmonic motion.

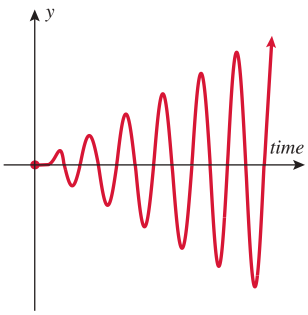
Damped Harmonic Motion



In damped harmonic motion, the amplitude of the periodic motion decreases as time increases. If you are on a moving swing and stop "pumping" new energy into the swing, the swing will continue moving with a constant period, but the amplitude—the height to which the swing will rise—will diminish with each cycle as the swing is slowed down by friction with the air or between its own moving parts.

Keystone/Stringer/Getty Images

Topham/The Image Works



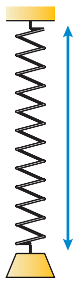
Resonance

Resonance occurs when the amplitude of periodic motion increases as time increases. It is caused when the energy applied to an oscillating object or system is more than what is needed to oppose friction or other forces and sustain simple harmonic motion. Instead, the applied energy *increases* the amplitude of harmonic motion with each cycle. With resonance, eventually, the amplitude becomes unbounded and the result is disastrous. Bridges have collapsed because of resonance. On the previous page are pictures of the Tacoma Narrows Bridge (near Seattle, Washington) that collapsed due to high winds resulting in resonance. Military soldiers know that when they march across a bridge, they must break cadence to prevent resonance.

Examples of Harmonic Motion

If we hang a weight from a spring, then while the resulting "system" is at rest we say it is in the equilibrium position.

If we then pull down on the weight and release it, the elasticity in the spring pulls the weight up and causes it to start oscillating up and down.



If we neglect friction and air resistance, we can imagine that the combination of the weight and the spring will oscillate indefinitely; the height of the weight with respect to the equilibrium position can be modeled by a simple sinusoidal function. This is an example of **simple harmonic motion**.

SIMPLE HARMONIC MOTION

The position of a point oscillating around an equilibrium position at time t is modeled by the sinusoidal function

$$y = A\sin(\omega t) \qquad \text{or} \qquad y = A\cos(\omega t)$$

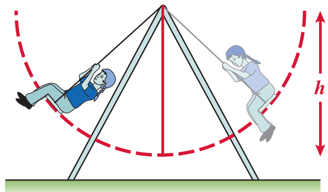
Here $|A|$ is the amplitude and the period is $\dfrac{2\pi}{\omega}$, where $\omega > 0$.

Note: The symbol ω (Greek lowercase omega) represents the angular frequency.

EXAMPLE 10 Simple Harmonic Motion

Let the height of the seat of a swing be equal to zero when the swing is at rest. Assume that a child starts swinging until she reaches the highest she can swing and keeps her effort constant. Suppose the height $h(t)$ of the seat can be given by

$$h(t) = 8\sin\left(\frac{\pi}{2}t\right)$$



where t is time in seconds and h is the height in feet. Note that positive h indicates height reached swinging forward and negative h indicates height reached swinging backward. Assume that $t = 0$ is when the child passes through the equilibrium position swinging forward.

a. Graph the height function $h(t)$ for $0 \le t \le 4$.

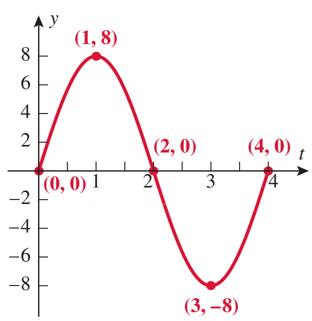
b. What is the maximum height above the resting level reached by the seat of the swing?

c. What is the period of the swinging child?

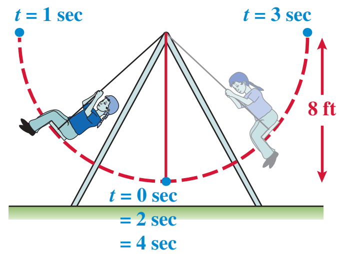
Solution (a):

Make a table with integer values of t. $\quad 0 \le t \le 4$

t (SECONDS)	$y = h(t) = 8\sin\left(\frac{\pi}{2}t\right)$ (FEET)	(t, y)
0	$8\sin 0 = 0$	(0, 0)
1	$8\sin\left(\frac{\pi}{2}\right) = 8$	(1, 8)
2	$8\sin\pi = 0$	(2, 0)
3	$8\sin\left(\frac{3\pi}{2}\right) = -8$	(3, −8)
4	$8\sin(2\pi) = 0$	(4, 0)



Labeling the time and height on the original diagram, we see that the maximum height is 8 feet and the period is 4 seconds.



Solutions (b) and (c): $h(t) = 8\sin\left(\frac{\pi}{2}t\right)$

The maximum height above the equilibrium is the amplitude. $\boxed{A = 8 \text{ ft}}$

The period is $\frac{2\pi}{\omega}$. $\boxed{p = \frac{2\pi}{\frac{\pi}{2}} = 4 \text{ sec}}$

Damped harmonic motion can be modeled by a sinusoidal function whose amplitude decreases as time increases. If we again hang a weight from a spring so that it is suspended at rest and then pull down on the weight and release, the weight will oscillate about the equilibrium point. This time we will not neglect friction and air resistance: The weight will oscillate closer and closer to the equilibrium point over time until the weight eventually comes to rest at the equilibrium point. This is an example of damped harmonic motion.

The product of any decreasing function and the original periodic function will describe damped oscillatory motion. Here are two examples of functions that describe damped harmonic motion:

$$y = \frac{1}{t}\sin(\omega t) \qquad y = e^{-t}\cos(\omega t)$$

where e^{-t} is a decreasing exponential function (exponential decay).

EXAMPLE 11 Damped Harmonic Motion

Assume that the child in Example 10 decides to stop pumping and allows the swing to continue until she eventually comes to rest. Assume that

$$h(t) = \frac{8}{t}\cos\left(\frac{\pi}{2}t\right)$$

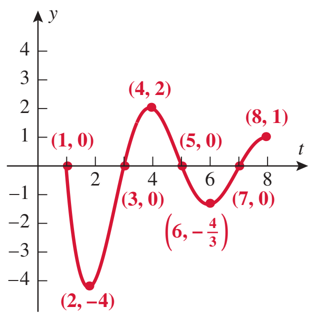
where t is time in seconds and h is the height in feet above the resting position. Note that positive h indicates height reached swinging forward and negative h indicates height reached swinging backward, assuming that $t = 1$ is when the child passes through the equilibrium position swinging backward and stops "pumping."

a. Graph the height function $h(t)$ for $1 \le t \le 8$.
b. What is the height above the resting level at 4 seconds? At 8 seconds? After 1 minute?

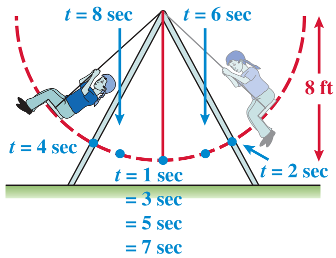
Solution (a):

Make a table with integer values of t. $1 \le t \le 8$

t (SECONDS)	$y = h(t) = \frac{8}{t}\cos\left(\frac{\pi}{2}t\right)$ (FEET)	(t, y)
1	$\frac{8}{1}\cos\left(\frac{\pi}{2}\right) = 0$	$(1, 0)$
2	$\frac{8}{2}\cos\pi = -4$	$(2, -4)$
3	$\frac{8}{3}\cos\left(\frac{3\pi}{2}\right) = 0$	$(3, 0)$
4	$\frac{8}{4}\cos(2\pi) = 2$	$(4, 2)$
5	$\frac{8}{5}\cos\left(\frac{5\pi}{2}\right) = 0$	$(5, 0)$
6	$\frac{8}{6}\cos(3\pi) = -\frac{4}{3}$	$\left(6, -\frac{4}{3}\right)$
7	$\frac{8}{7}\cos\left(\frac{7\pi}{2}\right) = 0$	$(7, 0)$
8	$\frac{8}{8}\cos(4\pi) = 1$	$(8, 1)$



Solution (b):



The height is 2 feet when t is 4 seconds. $\frac{8}{4}\cos(2\pi) = 2$

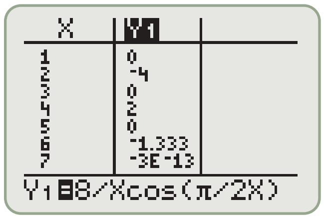
The height is 1 foot when t is 8 seconds. $\frac{8}{8}\cos(4\pi) = 1$

The height is 0.13 feet when t is 1 minute (60 seconds). $\frac{8}{60}\cos(30\pi) = 0.1333$

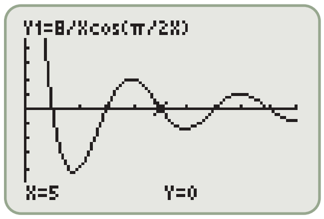
Technology Tip

To set up a table for $y = \frac{8}{t}\cos\left(\frac{\pi}{2}t\right)$, enter $Y_1 = \frac{8}{x}\cos\left(\frac{\pi}{2}x\right)$ and select TBLSET.

Press 2nd TABLE.



Now graph the function in the [0, 10] by [−6, 6] viewing rectangle.



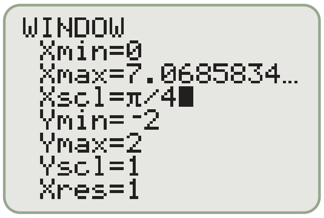
Resonance can be represented by the product of any increasing function and the original sinusoidal function. Here are two examples of functions that result in resonance as time increases:

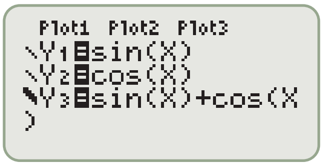
$$y = t\cos(\omega t) \qquad y = e^{t}\sin(\omega t)$$

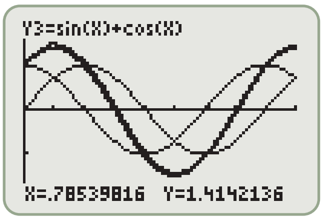
Technology Tip

To display the graphs of $\sin x$, $\cos x$, and $\sin x + \cos x$ in the same $\left[0, \frac{9\pi}{4}\right]$ by $[-2, 2]$ viewing window, enter $Y_1 = \sin x$, $Y_2 = \cos x$, and $Y_3 = \sin x + \cos x$.

To graph Y_3 using a thicker line, use the ◀ key to go to the left of Y_3, press ENTER, and select the thicker line.







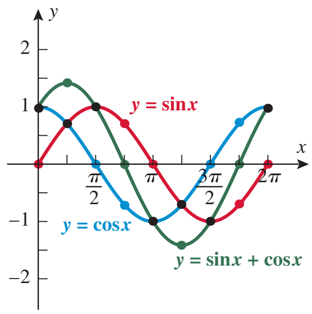
Graphing Sums of Functions: Addition of Ordinates

Since you have the ability to graph sinusoidal functions, let us now consider graphing sums of functions such as

$$y = x - \sin\left(\frac{\pi x}{2}\right) \qquad y = \sin x + \cos x \qquad y = 3\sin x + \cos(2x)$$

The method for graphing these sums is called the **addition of ordinates**, because we add the corresponding y-values (ordinates). The following table illustrates the ordinates (y-values) of the two sinusoidal functions $\sin x$ and $\cos x$; adding the corresponding ordinates leads to the y-values of $y = \sin x + \cos x$.

x	$\sin x$	$\cos x$	$y = \sin x + \cos x$
0	0	1	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}$
$\frac{\pi}{2}$	1	0	1
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	0
π	0	-1	-1
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2}$
$\frac{3\pi}{2}$	-1	0	-1
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	0
2π	0	1	1



Using a graphing utility, we can graph $Y_1 = \sin X$, $Y_2 = \cos X$, and $Y_3 = Y_1 + Y_2$.

EXAMPLE 12 Graphing Sums of Functions

Graph $y = x - \sin\left(\frac{\pi x}{2}\right)$ on the interval $0 \le x \le 4$.

Solution:

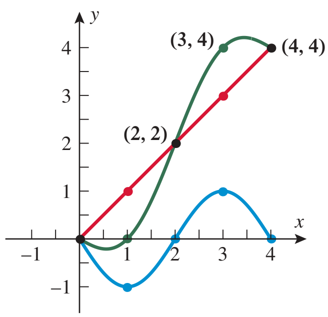
Let $y_1 = x$ and $y_2 = -\sin\left(\frac{\pi x}{2}\right)$.

State the amplitude and period of the graph of y_2. $|A| = |-1| = 1, p = 4$

Make a table of x- and y-values of y_1, y_2, and $y = y_1 + y_2$.

x	$y_1 = x$	$y_2 = -\sin\left(\frac{\pi x}{2}\right)$	$y = x + \left[-\sin\left(\frac{\pi x}{2}\right)\right]$
0	**0**	**0**	**0**
1	**1**	**−1**	**0**
2	**2**	**0**	**2**
3	**3**	**1**	**4**
4	**4**	**0**	**4**

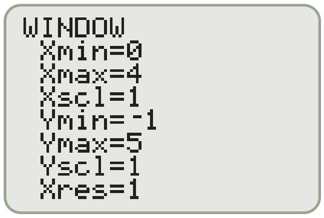
Graph $y_1 = x$, $y_2 = -\sin\left(\frac{\pi x}{2}\right)$, and $y = x - \sin\left(\frac{\pi x}{2}\right)$.

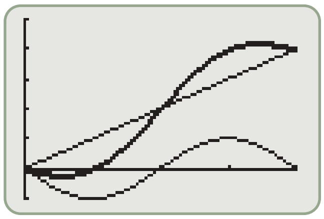


Technology Tip

To display the graphs of x, $-\sin\left(\frac{\pi x}{2}\right)$, and $x - \sin\left(\frac{\pi x}{2}\right)$ in the same [0, 4] by [−1, 5] viewing window, enter $Y_1 = x$, $Y_2 = -\sin\left(\frac{\pi x}{2}\right)$, and $Y_3 = x - \sin\left(\frac{\pi x}{2}\right)$.

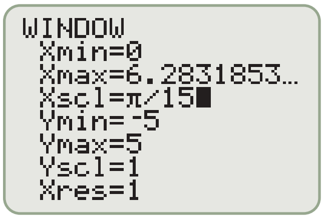
To graph Y_3 using a thicker line, use the ◀ key to go to the left of Y_3, press ENTER, and select the thicker line.

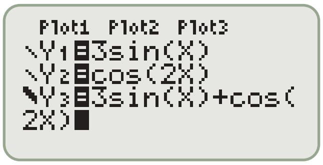


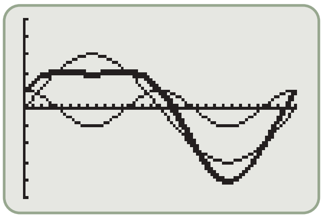


Technology Tip

To display the graphs of $3\sin x$, $\cos(2x)$, and $3\sin x + \cos(2x)$ in the same $[0, 2\pi]$ by $[-5, 5]$ viewing window, enter $Y_1 = 3\sin x$, $Y_2 = \cos(2x)$, and $Y_3 = 3\sin x + \cos(2x)$.







EXAMPLE 13 Graphing Sums of Sine and Cosine Functions

Graph $y = 3\sin x + \cos(2x)$ on the interval $0 \le x \le 2\pi$.

Solution:

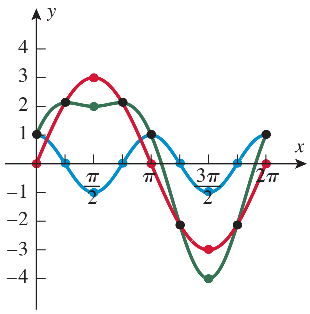
Let $y_1 = 3\sin x$, and state the amplitude and period of its graph. $|A| = 3, p = 2\pi$

Let $y_2 = \cos(2x)$, and state the amplitude and period of its graph. $|A| = 1, p = \pi$

Make a table of x- and y-values of y_1, y_2, and $y = y_1 + y_2$.

x	$y_1 = 3\sin x$	$y_2 = \cos(2x)$	$y = 3\sin x + \cos(2x)$
0	0	1	1
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}$	0	$\frac{3\sqrt{2}}{2}$
$\frac{\pi}{2}$	3	-1	2
$\frac{3\pi}{4}$	$\frac{3\sqrt{2}}{2}$	0	$\frac{3\sqrt{2}}{2}$
π	0	1	1
$\frac{5\pi}{4}$	$-\frac{3\sqrt{2}}{2}$	0	$-\frac{3\sqrt{2}}{2}$
$\frac{3\pi}{2}$	-3	-1	-4
$\frac{7\pi}{4}$	$-\frac{3\sqrt{2}}{2}$	0	$-\frac{3\sqrt{2}}{2}$
2π	0	1	1

Graph $y_1 = 3\sin x$, $y_2 = \cos(2x)$, and $y = 3\sin x + \cos(2x)$.



EXAMPLE 14 Graphing Sums of Cosine Functions

Graph $y = \cos\left(\frac{x}{2}\right) - \cos x$ on the interval $0 \le x \le 4\pi$.

Solution:

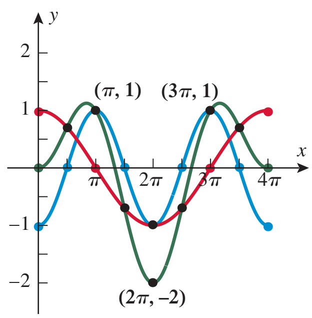
Let $y_1 = \cos\left(\frac{x}{2}\right)$ and state the amplitude and period of its graph. $|A| = 1, p = 4\pi$

Let $y_2 = -\cos x$ and state the amplitude and period of its graph. $|A| = |-1| = 1, p = 2\pi$

Make a table of x- and y-values of y_1, y_2, and $y = y_1 + y_2$.

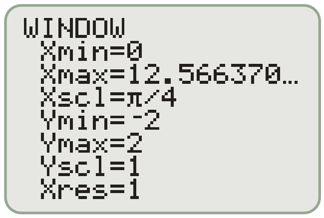
x	$y_1 = \cos\left(\frac{x}{2}\right)$	$y_2 = -\cos x$	$y = \cos\left(\frac{x}{2}\right) + (-\cos x)$
0	1	−1	0
$\frac{\pi}{2}$	$\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$
π	0	1	1
$\frac{3\pi}{2}$	$-\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$
2π	−1	−1	−2
$\frac{5\pi}{2}$	$-\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$
3π	0	1	1
$\frac{7\pi}{2}$	$\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$
4π	1	−1	0

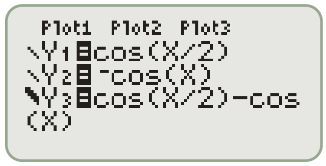
Graph $y_1 = \cos\left(\frac{x}{2}\right)$, $y_2 = -\cos x$, and $y = \cos\left(\frac{x}{2}\right) - \cos x$.

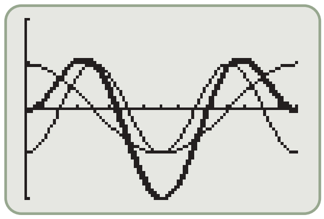


Technology Tip

To display the graphs of $\cos\left(\frac{x}{2}\right)$, $-\cos x$, and $\cos\left(\frac{x}{2}\right) - \cos x$ in the same $[0, 4\pi]$ by $[-2, 2]$ viewing window, enter $Y_1 = \cos\left(\frac{x}{2}\right)$, $Y_2 = -\cos x$, and $Y_3 = \cos\left(\frac{x}{2}\right) - \cos x$.







SECTION 5.2 SUMMARY

The sine function is an odd function, and its graph is symmetric about the origin. The cosine function is an even function, and its graph is symmetric about the y-axis. Graphs of the form $y = A\sin(Bx)$ and $y = A\cos(Bx)$ have amplitude $|A|$ and period $\frac{2\pi}{B}$.

To graph sinusoidal functions, point-plotting can be used. A more efficient way is to first determine the amplitude and period. Divide the period into four equal parts and choose the values of the division points starting at 0 for x. Make a table of those four points and graph them (this is the graph of one period) by labeling the four coordinates and drawing a smooth sinusoidal curve. Extend the graph to the left and right.

To find an equation of a sinusoidal function given its graph, start by first finding the amplitude (half the distance between the maximum and minimum values) so you can find A. Then determine the period so you can find B. Graphs of the form $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$ can be graphed using graph-shifting techniques.

Harmonic motion is one of the primary applications of sinusoidal functions. To graph combinations of trigonometric functions, add the corresponding y-values of the individual functions.

SECTION 5.2 EXERCISES

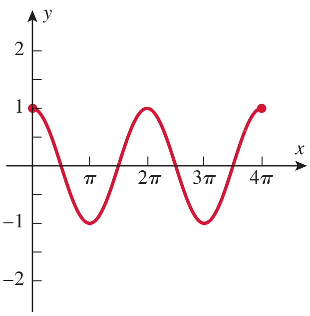
SKILLS

In Exercises 1–10, match the function with its graph (a–j).

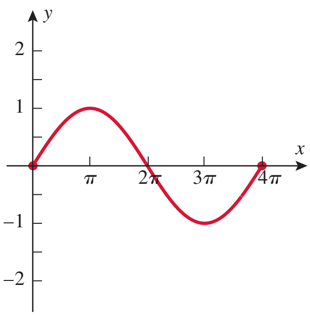
1. $y = -\sin x$ **2.** $y = \sin x$ **3.** $y = \cos x$ **4.** $y = -\cos x$ **5.** $y = 2\sin x$

6. $y = 2\cos x$ **7.** $y = \sin\left(\frac{1}{2}x\right)$ **8.** $y = \cos\left(\frac{1}{2}x\right)$ **9.** $y = -2\cos\left(\frac{1}{2}x\right)$ **10.** $y = -2\sin\left(\frac{1}{2}x\right)$

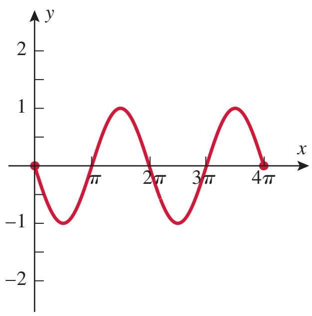
a.



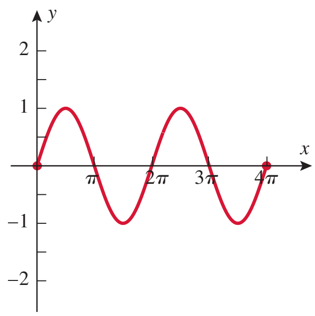
b.



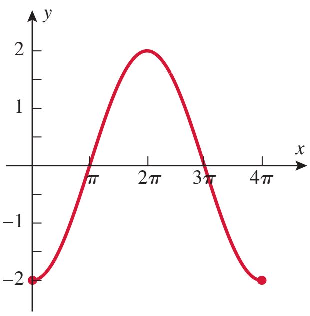
c.



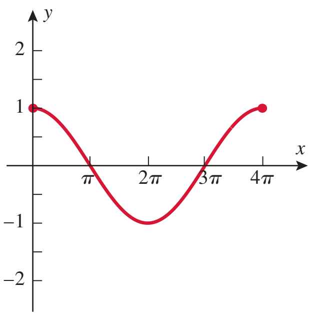
d.



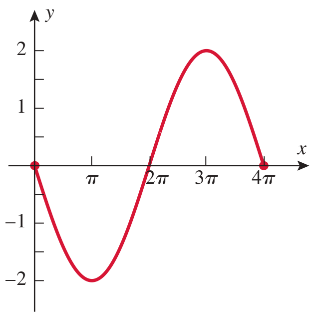
e.



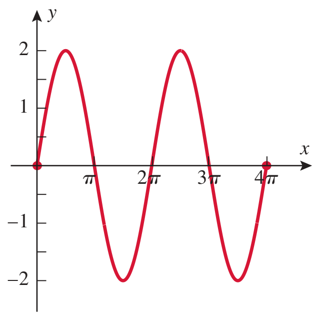
f.



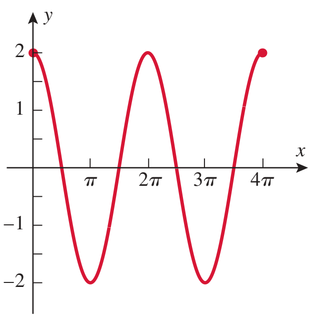
g.



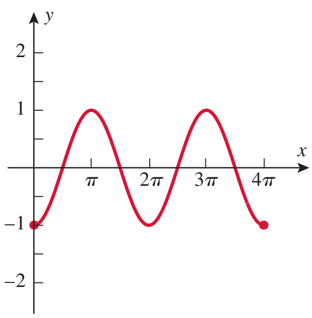
h.



i.



j.



In Exercises 11–20, state the amplitude and period of each function.

11. $y = \frac{3}{2}\cos(3x)$ **12.** $y = \frac{2}{3}\sin(4x)$ **13.** $y = -\sin(5x)$ **14.** $y = -\cos(7x)$ **15.** $y = \frac{2}{3}\cos\left(\frac{3}{2}x\right)$

16. $y = \frac{3}{2}\sin\left(\frac{2}{3}x\right)$ **17.** $y = -3\cos(\pi x)$ **18.** $y = -2\sin(\pi x)$ **19.** $y = 5\sin\left(\frac{\pi}{3}x\right)$ **20.** $y = 4\cos\left(\frac{\pi}{4}x\right)$

In Exercises 21–32, graph the given function over one period.

21. $y = 8\cos x$ **22.** $y = 7\sin x$ **23.** $y = \sin(4x)$ **24.** $y = \cos(3x)$

25. $y = -3\cos\left(\frac{1}{2}x\right)$ **26.** $y = -2\sin\left(\frac{1}{4}x\right)$ **27.** $y = -3\sin(\pi x)$ **28.** $y = -2\cos(\pi x)$

29. $y = 5\cos(2\pi x)$ **30.** $y = 4\sin(2\pi x)$ **31.** $y = -3\sin\left(\frac{\pi}{4}x\right)$ **32.** $y = -4\sin\left(\frac{\pi}{2}x\right)$

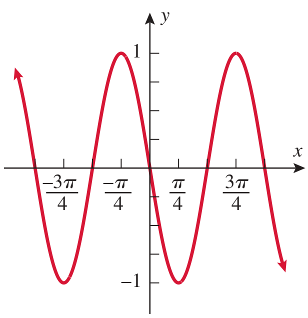
In Exercises 33–40, graph the given function over the interval [−2*p*, 2*p*], where *p* is the period of the function.

33. $y = -4\cos\left(\frac{1}{2}x\right)$ **34.** $y = -5\sin\left(\frac{1}{2}x\right)$ **35.** $y = -\sin(6x)$ **36.** $y = -\cos(4x)$

37. $y = 3\cos\left(\frac{\pi}{4}x\right)$ **38.** $y = 4\sin\left(\frac{\pi}{4}x\right)$ **39.** $y = \sin(4\pi x)$ **40.** $y = \cos(6\pi x)$

In Exercises 41–48, find the equation for each graph.

41.



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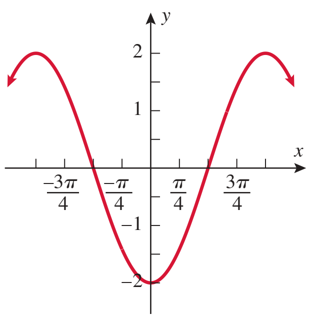

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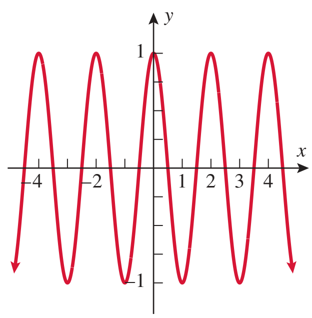

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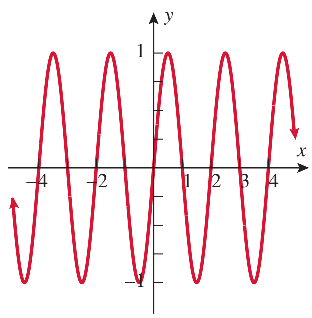

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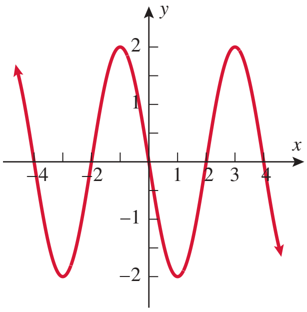

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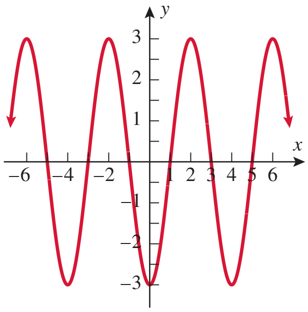

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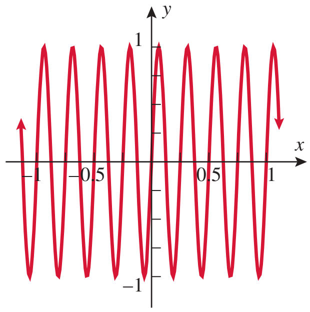

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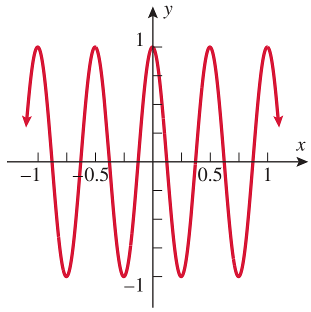


In Exercises 49–60, state the amplitude, period, and phase shift (including direction) of the given function and graph.

49. $y = 2\sin(\pi x - 1)$

50. $y = 4\cos(x + \pi)$

51. $y = -5\cos(3x + 2)$

52. $y = -7\sin(4x - 3)$

53. $y = 6\sin[-\pi(x + 2)]$

54. $y = 3\sin\left[-\dfrac{\pi}{2}(x - 1)\right]$

55. $y = 3\sin(2x + \pi)$

56. $y = -4\cos(2x - \pi)$

57. $y = -\dfrac{1}{4}\cos\left(\dfrac{1}{4}x - \dfrac{\pi}{2}\right)$

58. $y = \dfrac{1}{2}\sin\left(\dfrac{1}{3}x + \pi\right)$

59. $y = 2\cos\left[\dfrac{\pi}{2}(x - 4)\right]$

60. $y = -5\sin[-\pi(x + 1)]$

In Exercises 61–66, sketch the graph of the function over the indicated interval.

61. $y = \dfrac{1}{2} + \dfrac{3}{2}\cos(2x + \pi),\ \left[-\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right]$

62. $y = \dfrac{1}{3} + \dfrac{2}{3}\sin(2x - \pi),\ \left[-\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right]$

63. $y = \dfrac{1}{2} - \dfrac{1}{2}\sin\left(\dfrac{1}{2}x - \dfrac{\pi}{4}\right),\ \left[-\dfrac{7\pi}{2}, \dfrac{9\pi}{2}\right]$

64. $y = -\dfrac{1}{2} + \dfrac{1}{2}\cos\left(\dfrac{1}{2}x + \dfrac{\pi}{4}\right),\ \left[-\dfrac{9\pi}{2}, \dfrac{7\pi}{2}\right]$

65. $y = -3 + 4\sin[\pi(x - 2)],\ [0, 4]$

66. $y = 4 - 3\cos[\pi(x + 1)],\ [-1, 3]$

In Exercises 67–94, add the ordinates of the individual functions to graph each summed function on the indicated interval.

67. $y = 2x - \cos(\pi x),\ 0 \le x \le 4$

68. $y = 3x - 2\cos(\pi x),\ 0 \le x \le 4$

69. $y = \dfrac{1}{3}x + 2\cos(2x),\ 0 \le x \le 2\pi$

70. $y = \dfrac{1}{4}x + 3\cos\left(\dfrac{x}{2}\right),\ 0 \le x \le 4\pi$

71. $y = x - \cos\left(\dfrac{3\pi}{2}x\right),\ 0 \le x \le 6$

72. $y = -2x + 2\sin\left(\dfrac{\pi}{2}x\right),\ -2 \le x \le 2$

73. $y = \dfrac{1}{4}x - \dfrac{1}{2}\cos[\pi(x - 1)],\ 2 \le x \le 6$

74. $y = -\dfrac{1}{3}x + \dfrac{1}{3}\sin\left[\dfrac{\pi}{6}(x + 2)\right],\ -2 \le x \le 10$

75. $y = \sin x - \cos x,\ 0 \le x \le 2\pi$

76. $y = \cos x - \sin x,\ 0 \le x \le 2\pi$

77. $y = 3\cos x + \sin x,\ 0 \le x \le 2\pi$

78. $y = 3\sin x - \cos x,\ 0 \le x \le 2\pi$

79. $y = 4\cos x - \sin(2x),\ 0 \le x \le 2\pi$

80. $y = \dfrac{1}{2}\sin x + 2\cos(4x),\ -\pi \le x \le \pi$

81. $y = 2\sin[\pi(x - 1)] - 2\cos[\pi(x + 1)],\ -1 \le x \le 2$

82. $y = \sin\left[\dfrac{\pi}{4}(x + 2)\right] + 3\cos\left[\dfrac{3\pi}{3}(x - 1)\right],\ 1 \le x \le 5$

83. $y = \cos\left(\dfrac{x}{2}\right) + \cos(2x),\ 0 \le x \le 4\pi$

84. $y = \sin(2x) + \sin(3x),\ -\pi \le x \le \pi$

85. $y = \sin\left(\frac{x}{2}\right) + \sin(2x),\ 0 \le x \le 4\pi$

86. $y = -\sin\left(\frac{\pi}{4}x\right) - 3\sin\left(\frac{5\pi}{4}x\right),\ 0 \le x \le 4$

87. $y = -\frac{1}{3}\sin\left(\frac{\pi}{6}x\right) + \frac{2}{3}\sin\left(\frac{5\pi}{6}x\right),\ 0 \le x \le 3$

88. $y = 8\cos x - 6\cos\left(\frac{1}{2}x\right),\ -2\pi \le x \le 2\pi$

89. $y = -\frac{1}{4}\cos\left(\frac{\pi}{6}x\right) - \frac{1}{2}\cos\left(\frac{\pi}{3}x\right),\ 0 \le x \le 12$

90. $y = 2\cos\left(\frac{3}{2}x\right) - \cos\left(\frac{1}{2}x\right),\ -2\pi \le x \le 2\pi$

91. $y = 2\sin\left(\frac{x}{2}\right) - \cos(2x),\ 0 \le x \le 4\pi$

92. $y = 2\cos\left(\frac{x}{2}\right) + \sin(2x),\ 0 \le x \le 4\pi$

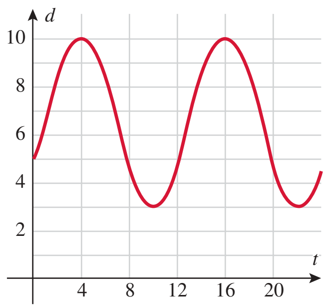
93. $y = 2\sin[\pi(x-1)] + 3\sin\left[2\pi\left(x + \frac{1}{2}\right)\right],\ -2 \le x \le 2$

94. $y = -\frac{1}{2}\cos\left(x + \frac{\pi}{3}\right) - 2\cos\left(x - \frac{\pi}{6}\right),\ -\pi \le x \le \pi$

APPLICATIONS

For Exercises 95 and 96, refer to the following:

An analysis of demand d for widgets manufactured by WidgetsRUs (measured in thousands of units per week) indicates that demand can be modeled by the graph below, where t is time in months since January 2010 (note that $t = 0$ corresponds to January 2010).



95. Business. Find the amplitude of the graph.

96. Business. Find the period of the graph.

For Exercises 97 and 98, refer to the following:

Researchers have been monitoring oxygen levels (milligrams per liter) in the water of a lake and have found that the oxygen levels fluctuate with an eight-week period. The following tables illustrate data from eight weeks.

97. Environment. Find the amplitude of the oxygen level fluctuations.

Week: t	0 (initial measurement)	1	2	3	4	5	6	7	8
Oxygen levels: mg/L	7	7.7	8	7.7	7	6.3	6	6.3	7

98. Environment. Find the amplitude of the oxygen level fluctuations.

Week: t	0 (initial measurement)	1	2	3	4	5	6	7	8
Oxygen levels: mg/L	7	8.4	9	8.4	7	5.6	5	5.6	7

For Exercises 99–102, refer to the following:

A weight hanging on a spring will oscillate up and down about its equilibrium position after it is pulled down and released.

This is an example of simple harmonic motion. This motion would continue forever if there were not any friction or air resistance. Simple harmonic motion can be described with the function $y = A\cos\left(t\sqrt{\frac{k}{m}}\right)$, where $|A|$ is the amplitude, t is the time in seconds, m is the mass of the weight, and k is a constant particular to the spring.

99. Simple Harmonic Motion. If the height of the spring is measured in centimeters and the mass in grams, then what are the amplitude and mass if $y = 4\cos\left(\frac{t\sqrt{k}}{2}\right)$?

100. Simple Harmonic Motion. If a spring is measured in centimeters and the mass in grams, then what are the amplitude and mass if $y = 3\cos\left(3t\sqrt{k}\right)$?

101. Frequency of Oscillations. The frequency of the oscillations in cycles per second is determined by $f = \frac{1}{p}$, where p is the period. What is the frequency for the oscillation modeled by $y = 3\cos\left(\frac{t}{2}\right)$?

102. Frequency of Oscillations. The frequency of the oscillations f is given by $f = \frac{1}{p}$, where p is the period. What is the frequency of oscillation modeled by $y = 3.5\cos(3t)$?

103. Sound Waves. A pure tone created by a vibrating tuning fork shows up as a sine wave on an oscilloscope's screen. A tuning fork vibrating at 256 hertz (Hz) gives the tone middle C and can have the equation $y = 0.005\sin[(2\pi)(256t)]$, where the amplitude is in centimeters (cm) and the time t in seconds. What are the amplitude and frequency of the wave where the frequency is $\frac{1}{p}$ in cycles per second?

Note: 1 hertz = 1 cycle per second.

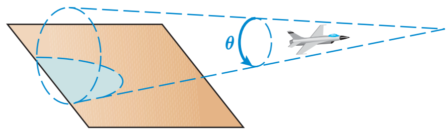
104. Sound Waves. A pure tone created by a vibrating tuning fork shows up as a sine wave on an oscilloscope's screen. A tuning fork vibrating at 288 hertz gives the tone D and can have the equation $y = 0.005\sin[(2\pi)(288t)]$, where the amplitude is in centimeters (cm) and the time t in seconds. What are the amplitude and frequency of the wave where the frequency is $\frac{1}{p}$ in cycles per second?

105. Sound Waves. If a sound wave is represented by $y = 0.008\sin(750\pi t)$ cm, what are its amplitude and frequency? See Exercise 103.

106. Sound Waves. If a sound wave is represented by $y = 0.006\cos(1000\pi t)$ cm, what are its amplitude and frequency? See Exercise 103.

For Exercises 107–110, refer to the following:

When an airplane flies faster than the speed of sound, the sound waves that are formed take on a cone shape, and where the cone hits the ground, a sonic boom is heard. If θ is the angle of the vertex of the cone, then $\sin\left(\frac{\theta}{2}\right) = \frac{330 \text{ m/sec}}{V} = \frac{1}{M}$, where V is the speed of the plane and M is the Mach number.



107. Sonic Booms. What is the speed of the plane if the plane is flying at Mach 2?

108. Sonic Booms. What is the Mach number if the plane is flying at 990 meters per second?

109. Sonic Booms. What is the speed of the plane if the cone angle is 60°?

110. Sonic Booms. What is the speed of the plane if the cone angle is 30°?

For Exercises 111 and 112, refer to the following:

With the advent of summer come fireflies. They are intriguing because they emit a flashing luminescence that beckons their mate to them. It is known that the speed and intensity of the flashing are related to the temperature—the higher the temperature, the quicker and more intense the flashing becomes. If you ever watch a single firefly, you will see that the intensity of the flashing is periodic with time. The intensity of light emitted is measured in *candelas per square meter* (of firefly). To give an idea of this unit of measure, the intensity of a picture on a typical TV screen is about 450 candelas per square meter. The measurement for the intensity of the light emitted by a typical firefly at its brightest moment is about 50 candelas per square meter. Assume that a typical cycle of this flashing is 4 seconds and that the intensity is essentially zero candelas at the beginning and ending of a cycle.

111. Bioluminescence in Fireflies. Find an equation that describes this flashing. What is the intensity of the flashing at 4 minutes?

112. Bioluminescence in Fireflies. Graph the equation from Exercise 111 for a period of 30 seconds.

CATCH THE MISTAKE

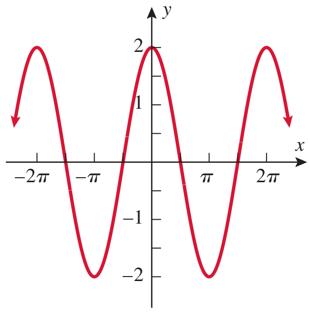
In Exercises 113 and 114, explain the mistake that is made.

113. Graph the function $y = -2\cos x$.

Solution:

Find the amplitude. $|A| = |-2| = 2$

The graph of $y = -2\cos x$ is similar to the graph of $y = \cos x$ with amplitude 2.



This is incorrect. What mistake was made?

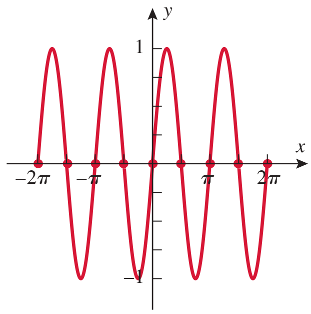
114. Graph the function $y = -\sin(2x)$.

Solution:

Make a table with values.

x	$y = -\sin(2x)$	(x, y)
0	$y = -\sin 0 = 0$	$(0, 0)$
$\frac{\pi}{2}$	$y = -\sin \pi = 0$	$(0, 0)$
π	$y = -\sin(2\pi) = 0$	$(0, 0)$
$\frac{3\pi}{2}$	$y = -\sin(3\pi) = 0$	$(0, 0)$
2π	$y = -\sin(4\pi) = 0$	$(0, 0)$

Graph the function by plotting these points and connecting them with a sinusoidal curve.



This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 115–118, determine whether each statement is true or false. (A and B are positive real numbers.)

115. The graph of $y = -A\cos(Bx)$ is the graph of $y = A\cos(Bx)$ reflected about the x-axis.

116. The graph of $y = A\sin(-Bx)$ is the graph of $y = A\sin(Bx)$ reflected about the x-axis.

117. The graph of $y = -A\cos(-Bx)$ is the graph of $y = A\cos(Bx)$.

118. The graph of $y = -A\sin(-Bx)$ is the graph of $y = A\sin(Bx)$.

In Exercises 119–122, A and B are positive real numbers.

119. Find the y-intercept of the function $y = A\cos(Bx)$.

120. Find the y-intercept of the function $y = A\sin(Bx)$.

121. Find the x-intercepts of the function $y = A\sin(Bx)$.

122. Find the x-intercepts of the function $y = A\cos(Bx)$.

CHALLENGE

123. Find the y-intercept of $y = -A\sin\left(Bx + \frac{\pi}{6}\right)$.

124. Find the y-intercept of $y = A\cos(Bx - \pi) + C$.

125. Find the x-intercept(s) of $y = A\sin(Bx) + A$.

126. Find an expression involving C and A that describes the values of C for which the graph of $y = A\cos(Bx) + C$ does not cross the x-axis. (Assume that $A > 0$.)

127. What is the range of $y = 2A\sin(Bx + C) - \frac{A}{2}$?

128. Can the y-coordinate of a point on the graph of $y = A\sin(Bx) + 3A\cos\left(\frac{B}{2}x\right)$ exceed $4A$? Explain. (Assume that $A > 0$.)

TECHNOLOGY

129. Use a graphing calculator to graph $Y_1 = 5\sin x$ and $Y_2 = \sin(5x)$. Is the following statement true based on what you see? $y = \sin(cx)$ has the same graph as $y = c\sin x$.

130. Use a graphing calculator to graph $Y_1 = 3\cos x$ and $Y_2 = \cos(3x)$. Is the following statement true based on what you see? $y = \cos(cx)$ has the same graph as $y = c\cos x$.

131. Use a graphing calculator to graph $Y_1 = \sin x$ and $Y_2 = \cos\left(x - \frac{\pi}{2}\right)$. What do you notice?

132. Use a graphing calculator to graph $Y_1 = \cos x$ and $Y_2 = \sin\left(x + \frac{\pi}{2}\right)$. What do you notice?

133. Use a graphing calculator to graph $Y_1 = \cos x$ and $Y_2 = \cos(x + c)$, where

a. $c = \frac{\pi}{3}$, and explain the relationship between Y_2 and Y_1.

b. $c = -\frac{\pi}{3}$, and explain the relationship between Y_2 and Y_1.

134. Use a graphing calculator to graph $Y_1 = \sin x$ and $Y_2 = \sin(x + c)$, where

a. $c = \frac{\pi}{3}$, and explain the relationship between Y_2 and Y_1.

b. $c = -\frac{\pi}{3}$, and explain the relationship between Y_2 and Y_1.

For Exercises 135 and 136, refer to the following:

Damped oscillatory motion, or *damped oscillation*, occurs when things in oscillatory motion experience friction or resistance. The friction causes the amplitude to decrease as a function of time. Mathematically, we can use a negative exponential function to damp the oscillations in the form of

$$f(t) = e^{-t}\sin t$$

135. Damped Oscillation. Graph the functions $Y_1 = e^{-t}$, $Y_2 = \sin t$, and $Y_3 = e^{-t}\sin t$ in the same viewing window (let t range from 0 to 2π). What happens as t increases?

136. Damped Oscillation. Graph $Y_1 = e^{-t}\sin t$, $Y_2 = e^{-2t}\sin t$, and $Y_3 = e^{-4t}\sin t$ in the same viewing window. What happens to $Y = e^{-kt}\sin t$ as k increases?

137. Use a graphing calculator to graph $Y_1 = \sin x$ and $Y_2 = \sin x + c$, where

a. $c = 1$, and explain the relationship between Y_2 and Y_1.

b. $c = -1$, and explain the relationship between Y_2 and Y_1.

138. Use a graphing calculator to graph $Y_1 = \cos x$ and $Y_2 = \cos x + c$, where

a. $c = \frac{1}{2}$, and explain the relationship between Y_2 and Y_1.

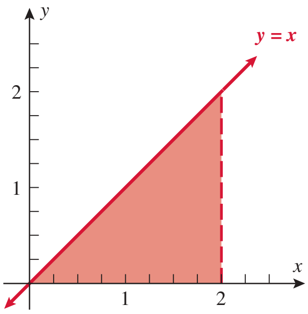
b. $c = -\frac{1}{2}$, and explain the relationship between Y_2 and Y_1.

139. What is the amplitude of the function $y = 3\cos x + 4\sin x$? Use a graphing calculator to graph $Y_1 = 3\cos x$, $Y_2 = 4\sin x$, and $Y_3 = 3\cos x + 4\sin x$ in the same viewing window.

140. What is the amplitude of the function $y = \sqrt{3}\cos x - \sin x$? Use a graphing calculator to graph $Y_1 = \sqrt{3}\cos x$, $Y_2 = \sin x$, and $Y_3 = \sqrt{3}\cos x - \sin x$ in the same viewing window.

PREVIEW TO CALCULUS

In calculus, the definite integral $\int_a^b f(x)dx$ is used to find the area below the graph of f, above the x-axis, between $x = a$ and $x = b$. For example, $\int_0^2 x\,dx = 2$, as you can see in the following figure:



The Fundamental Theorem of Calculus establishes that the definite integral $\int_a^b f(x)dx$ equals $F(b) - F(a)$, where F is any antiderivative of a continuous function f.

In Exercises 141–144, first shade the area corresponding to the definite integral and then use the information below to find the exact value of the area.

FUNCTION	$\sin x$	$\cos x$
ANTIDERIVATIVE	$-\cos x$	$\sin x$

141. $\displaystyle\int_0^{\pi} \sin x\,dx$

142. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\,dx$

143. $\displaystyle\int_0^{\pi/2} \cos x\,dx$

144. $\displaystyle\int_0^{\pi/2} \sin x\,dx$

SECTION 5.3 GRAPHS OF OTHER TRIGONOMETRIC FUNCTIONS

SKILLS OBJECTIVES

- Determine the domain and range of the tangent, cotangent, secant, and cosecant functions.
- Graph basic tangent, cotangent, secant, and cosecant functions.
- Determine the period of tangent, cotangent, secant, and cosecant functions.
- Graph translated tangent, cotangent, secant, and cosecant functions.

CONCEPTUAL OBJECTIVES

- Relate domain restrictions to vertical asymptotes.
- Understand the pattern that vertical asymptotes follow.
- Understand the relationships between the graphs of the cosine and secant functions and the sine and cosecant functions.

Graphing the Tangent, Cotangent, Secant, and Cosecant Functions

Section 5.2 focused on graphing sinusoidal functions (sine and cosine). We now turn our attention to graphing the other trigonometric functions: tangent, cotangent, secant, and cosecant. We know the graphs of the sine and cosine functions, and we can get the graphs of the other trigonometric functions from the sinusoidal functions. Recall the reciprocal and quotient identities:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

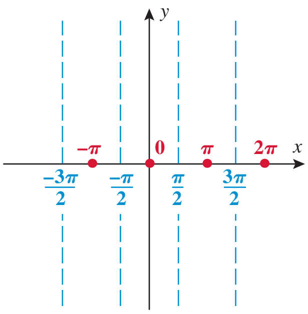
Recall that in graphing rational functions, a *vertical asymptote* is found by setting the denominator of the rational function equal to zero (as long as the numerator and denominator have no common factors). As you will see in this section, tangent and secant functions have graphs with vertical asymptotes at the x-values where cosine is equal to zero, and cotangent and cosecant functions have graphs with vertical asymptotes at the x-values where sine is equal to zero.

One important difference between the sinusoidal functions, $y = \sin x$ and $y = \cos x$, and the other four trigonometric functions ($y = \tan x$, $y = \sec x$, $y = \csc x$, and $y = \cot x$) is that the sinusoidal functions have defined amplitudes, whereas the other four trigonometric functions do not (since they are unbounded vertically).

The Tangent Function

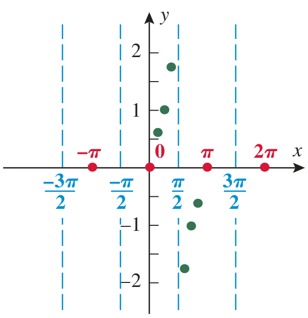
Since the tangent function is a quotient that relies on the sine and cosine functions, let us start with a table of values for the quadrantal angles.

x	$\sin x$	$\cos x$	$\tan x = \frac{\sin x}{\cos x}$	(x, y) OR ASYMPTOTE
0	0	1	0	$(0, 0)$
$\frac{\pi}{2}$	1	0	undefined	vertical asymptote: $x = \frac{\pi}{2}$
π	0	-1	0	$(\pi, 0)$
$\frac{3\pi}{2}$	-1	0	undefined	vertical asymptote: $x = \frac{3\pi}{2}$
2π	0	1	0	$(2\pi, 0)$



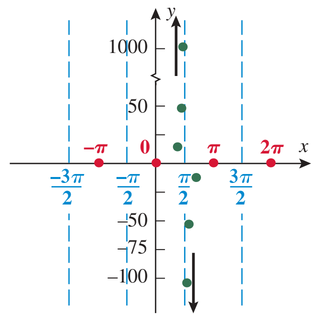
Notice that the x-intercepts correspond to integer multiples of π and vertical asymptotes correspond to odd integer multiples of $\frac{\pi}{2}$.

We know that the graph of the tangent function is undefined at the odd integer multiples of $\frac{\pi}{2}$, so its graph cannot cross the vertical asymptotes. The question is, what happens between the asymptotes? We know the x-intercepts, so let us now make a table for special values of x.



x	$\sin x$	$\cos x$	$\tan x = \frac{\sin x}{\cos x}$	(x, y)
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.577$	$\left(\frac{\pi}{6}, 0.577\right)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3} \approx 1.732$	$\left(\frac{\pi}{3}, 1.732\right)$
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3} \approx -1.732$	$\left(\frac{2\pi}{3}, -1.732\right)$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \approx -0.577$	$\left(\frac{5\pi}{6}, -0.577\right)$

What happens to $\tan x$ as x approaches $\frac{\pi}{2}$? We know $\tan x$ is undefined at $x = \frac{\pi}{2}$ but we must consider x-values both larger and smaller than $\frac{\pi}{2} \approx 1.571$.



approaching from the left → $\frac{\pi}{2}$ ← approaching from the right

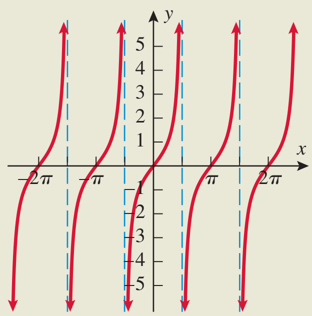
x	1.5	1.55	1.57	1.571	1.58	1.59	1.65
$\tan x$	14.1	48.1	1255.8	undefined	−108.6	−52.1	−12.6

$\tan x$ gets larger (1.5 to 1.57); $\tan x$ gets more negative (1.58 to 1.65)

The arrows on the graph on the right indicate increasing without bound (in the positive and negative directions).

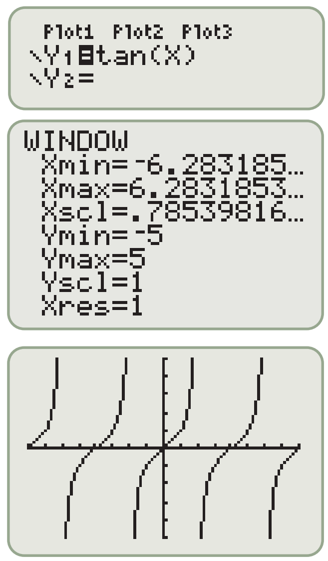
Graph of $y = \tan x$

1. The x-intercepts occur at multiples of π. $\quad (n\pi, 0)$
2. Vertical asymptotes occur at odd integer multiples of $\frac{\pi}{2}$. $\quad x = \frac{(2n+1)\pi}{2}$
3. The domain is the set of all real numbers except odd integer multiples of $\frac{\pi}{2}$. $\quad x \neq \frac{(2n+1)\pi}{2}$
4. The range is the set of all real numbers. $\quad (-\infty, \infty)$
5. $y = \tan x$ has period π.
6. $y = \tan x$ is an odd function (symmetric about the origin). $\quad \tan(-x) = -\tan x$
7. The graph has no defined amplitude, since the funtion is unbounded.



Note: n is an integer.

Technology Tip



The Cotangent Function

The cotangent function is similar to the tangent function in that it is a quotient involving the sine and cosine functions. The difference is that cotangent has cosine in the numerator and sine in the denominator: $\cot x = \frac{\cos x}{\sin x}$. The graph of $y = \tan x$ has x-intercepts corresponding to integer multiples of π and vertical asymptotes corresponding to odd integer multiples of $\frac{\pi}{2}$.

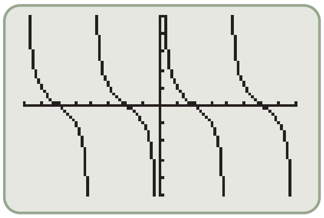
Study Tip

The graphs of $y = \tan x$ and $y = \cot x$ both have period π, and neither of them has defined amplitude.

The graph of the cotangent function is the reverse in that it has x-intercepts corresponding to odd integer multiples of $\frac{\pi}{2}$ and vertical asymptotes corresponding to integer multiples of π. This is because the x-intercepts occur when the numerator, $\cos x$, is equal to 0 and the vertical asymptotes occur when the denominator, $\sin x$, is equal to 0.

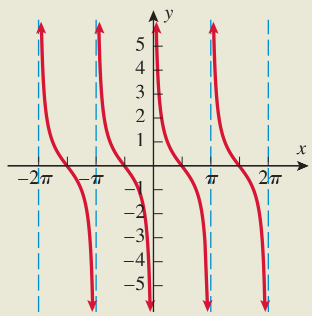
Technology Tip 

To graph $y = \cot x$, use the reciprocal property, $1/\tan x$.



Graph of $y = \cot x$

1. The x-intercepts occur at odd integer multiples of $\frac{\pi}{2}$. $\left(\frac{(2n+1)\pi}{2}, 0\right)$
2. Vertical asymptotes occur at integer multiples of π. $x = n\pi$
3. The domain is the set of all real numbers except integer multiples of π. $x \neq n\pi$
4. The range is the set of all real numbers. $(-\infty, \infty)$
5. $y = \cot x$ has period π.
6. $y = \cot x$ is an odd function (symmetric about the origin). $\cot(-x) = -\cot x$
7. The graph has no defined amplitude, since the funtion is unbounded.



Note: n is an integer.

The Secant Function

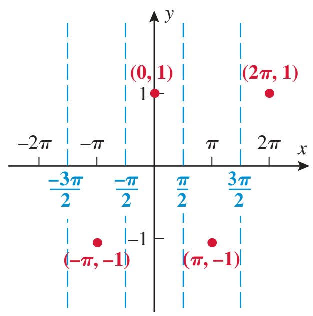
Since $y = \cos x$ has period 2π, the secant function, which is the reciprocal of the cosine function, $\sec x = \frac{1}{\cos x}$, also has period 2π. We now illustrate values of the secant function with a table.

x	$\cos x$	$\sec x = \frac{1}{\cos x}$	(x, y) OR ASYMPTOTE
0	1	1	**(0, 1)**
$\frac{\pi}{2}$	0	undefined	vertical asymptote: $x = \frac{\pi}{2}$
π	-1	-1	$(\pi, -1)$
$\frac{3\pi}{2}$	0	undefined	vertical asymptote: $x = \frac{3\pi}{2}$
2π	1	1	$(2\pi, 1)$

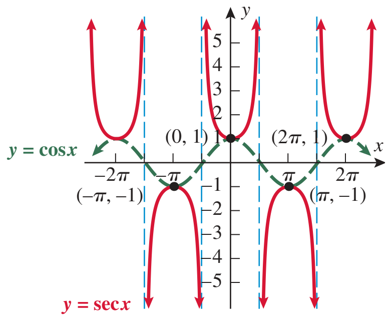
Again, we ask the same question: What happens as x approaches the vertical asymptotes? The secant function grows without bound in either the positive or negative direction.

If we graph $y = \cos x$ (the "guide" function) and $y = \sec x$ on the same graph, we notice the following:

- The x-intercepts of $y = \cos x$ correspond to the vertical asymptotes of $y = \sec x$.
- The range of cosine is $[-1, 1]$ and the range of secant is $(-\infty, -1] \cup [1, \infty)$.
- When cosine is positive, secant is positive, and when one is negative, the other is negative.

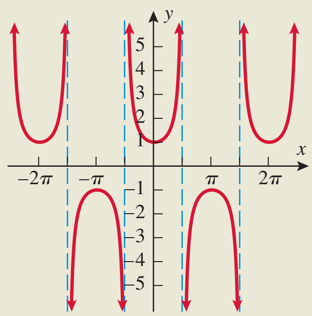


The cosine function is used as the guide function to graph the secant function.



Graph of $y = \sec x$

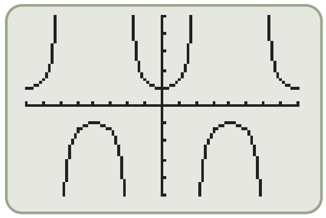
1. There are no x-intercepts.	$\dfrac{1}{\cos x} \neq 0$
2. Vertical asymptotes occur at odd integer multiples of $\dfrac{\pi}{2}$.	$x = \dfrac{(2n + 1)\pi}{2}$
3. The domain is the set of all real numbers except odd integer multiples of $\dfrac{\pi}{2}$.	$x \neq \dfrac{(2n + 1)\pi}{2}$
4. The range is $(-\infty, -1] \cup [1, \infty)$.	
5. $y = \sec x$ has period 2π.	
6. $y = \sec x$ is an even function (symmetric about the y-axis).	$\sec(-x) = \sec x$
7. The graph has no defined amplitude, since the function is unbounded.	



Note: n is an integer.

Technology Tip

To graph $y = \sec x$, enter as $1/\cos x$.



The Cosecant Function

Since $y = \sin x$ has period 2π, the cosecant function, which is the reciprocal of the sine function, $\csc x = \dfrac{1}{\sin x}$, also has period 2π. We now illustrate values of cosecant with a table.

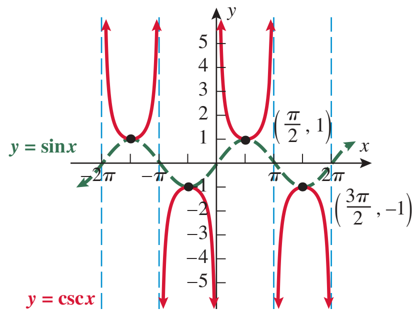
x	$\sin x$	$\csc x = \dfrac{1}{\sin x}$	(x, y) OR ASYMPTOTE
0	0	undefined	vertical asymptote: $x = 0$
$\dfrac{\pi}{2}$	1	1	$\left(\dfrac{\pi}{2}, 1\right)$
π	0	undefined	vertical asymptote: $x = \pi$
$\dfrac{3\pi}{2}$	-1	-1	$\left(\dfrac{3\pi}{2}, -1\right)$
2π	0	undefined	vertical asymptote: $x = 2\pi$

Again, we ask the same question: What happens as x approaches the vertical asymptotes? The cosecant function grows without bound in either the positive or negative direction.

If we graph $y = \sin x$ (the "guide" function) and $y = \csc x$ on the same graph, we notice the following:

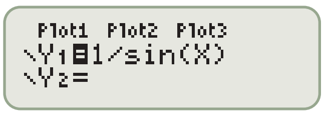
- The x-intercepts of $y = \sin x$ correspond to the vertical asymptotes of $y = \csc x$.
- The range of sine is $[-1, 1]$ and the range of cosecant is $(-\infty, -1] \cup [1, \infty)$.
- When sine is positive, cosecant is positive, and when one is negative, the other is negative.

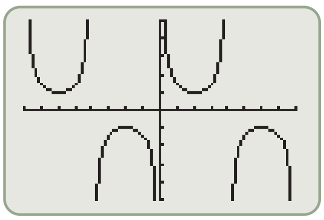
The sine function is used as the guide function to graph the cosecant function.



Technology Tip

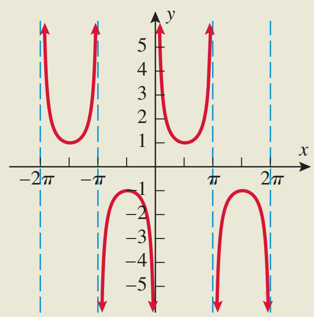
To graph $y = \csc x$, use $1/\sin x$.





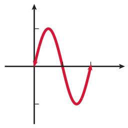
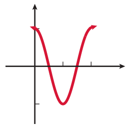
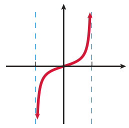
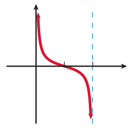
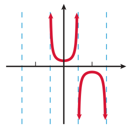
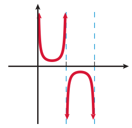
GRAPH OF $y = \csc x$

1. There are no x-intercepts. $\dfrac{1}{\sin x} \neq 0$
2. Vertical asymptotes occur at integer multiples of π. $x = n\pi$
3. The domain is the set of all real numbers except integer multiples of π. $x \neq n\pi$
4. The range is $(-\infty, -1] \cup [1, \infty)$.
5. $y = \csc x$ has period 2π.
6. $y = \csc x$ is an odd function (symmetric about the origin). $\csc(-x) = -\csc x$
7. The graph has no defined amplitude, since the function is unbounded.



Note: n is an integer.

Graphing More General Tangent, Cotangent, Secant, and Cosecant Functions

FUNCTION	$y = \sin x$	$y = \cos x$	$y = \tan x$	$y = \cot x$	$y = \sec x$	$y = \csc x$
Graph of One Period						
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	$x \neq \frac{(2n+1)\pi}{2}$	$x \neq n\pi$	$x \neq \frac{(2n+1)\pi}{2}$	$x \neq n\pi$
Range	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
Amplitude	1	1	none	none	none	none
Period	2π	2π	π	π	2π	2π
x-intercepts	$(n\pi, 0)$	$\left(\frac{(2n+1)\pi}{2}, 0\right)$	$(n\pi, 0)$	$\left(\frac{(2n+1)\pi}{2}, 0\right)$	none	none
Vertical Asymptotes	none	none	$x = \frac{(2n+1)\pi}{2}$	$x = n\pi$	$x = \frac{(2n+1)\pi}{2}$	$x = n\pi$

Note: n is an integer.

We use these basic functions as the starting point for graphing general tangent, cotangent, secant, and cosecant functions.

GRAPHING TANGENT AND COTANGENT FUNCTIONS

Graphs of $y = A\tan(Bx)$ and $y = A\cot(Bx)$ can be obtained using the following steps (assume $B > 0$):

Step 1: Calculate the period $\frac{\pi}{B}$.

Step 2: Find two neighboring vertical asymptotes.

For $y = A\tan(Bx)$: $Bx = -\frac{\pi}{2}$ and $Bx = \frac{\pi}{2}$

For $y = A\cot(Bx)$: $Bx = 0$ and $Bx = \pi$

Step 3: Find the x-intercept between the two asymptotes.

For $y = A\tan(Bx)$: Solve for x: $Bx = 0 \Rightarrow x = 0$

For $y = A\cot(Bx)$: Solve for x: $Bx = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2B}$

Step 4: Draw the vertical asymptotes and label the x-intercept.

Step 5: Divide the interval between the asymptotes into four equal parts. Set up a table with coordinates corresponding to the points in the interval.

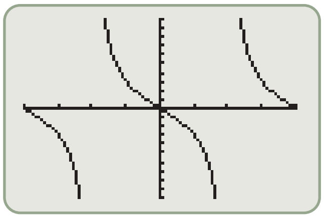
Step 6: Connect the points with a smooth curve. Use arrows to indicate the behavior toward the asymptotes.

- If $A > 0$
  - $y = A\tan(Bx)$ increases from left to right.
  - $y = A\cot(Bx)$ decreases from left to right.
- If $A < 0$
  - $y = A\tan(Bx)$ decreases from left to right.
  - $y = A\cot(Bx)$ increases from left to right.

Technology Tip

Graph $y = -3\tan(2x)$ on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

```
Plot1 Plot2 Plot3
\Y1=-3tan(2X)
\Y2=
```



EXAMPLE 1 Graphing $y = A\tan(Bx)$

Graph $y = -3\tan(2x)$ on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Solution: $A = -3$, $B = 2$

STEP 1 Calculate the period. $\quad \frac{\pi}{B} = \frac{\pi}{2}$

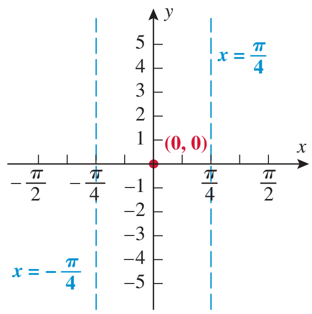
STEP 2 Find two vertical asymptotes. $\quad Bx = -\frac{\pi}{2}$ and $Bx = \frac{\pi}{2}$

Substitute $B = 2$ and solve for x. $\quad x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$

STEP 3 Find the x-intercept between the asymptotes. $\quad Bx = 0$

$x = 0$

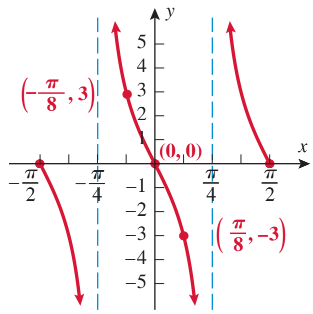
STEP 4 Draw the vertical asymptotes $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ and label the x-intercept **(0, 0)**.



STEP 5 Divide the period $\frac{\pi}{2}$ into four equal parts, in steps of $\frac{\pi}{8}$. Set up a table with coordinates corresponding to values of $y = -3\tan(2x)$.

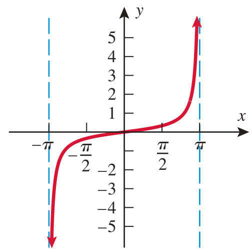
x	$y = -3\tan(2x)$	(x, y)
$-\frac{\pi}{4}$	undefined	vertical asymptote, $x = -\frac{\pi}{4}$
$-\frac{\pi}{8}$	3	$\left(-\frac{\pi}{8}, 3\right)$
0	**0**	**(0, 0)**
$\frac{\pi}{8}$	-3	$\left(\frac{\pi}{8}, -3\right)$
$\frac{\pi}{4}$	undefined	vertical asymptote, $x = \frac{\pi}{4}$

STEP 6 Graph the points from the table and connect with a smooth curve. Repeat to the right and left until you reach the interval endpoints. Notice the distance between the vertical asymptotes is the period length $\frac{\pi}{2}$.



■ **YOUR TURN** Graph $y = \frac{1}{3}\tan\left(\frac{1}{2}x\right)$ on the interval $-\pi \le x \le \pi$.

■ **Answer:**



EXAMPLE 2 Graphing $y = A\cot(Bx)$

Graph $y = 4\cot\left(\frac{1}{2}x\right)$ on the interval $-2\pi \le x \le 2\pi$.

Solution: $A = 4,\ B = \frac{1}{2}$

STEP 1 Calculate the period. $\frac{\pi}{B} = 2\pi$

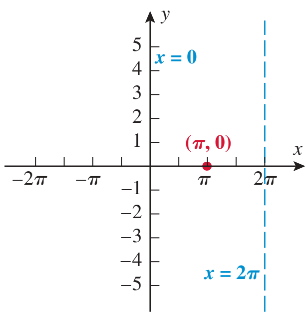
STEP 2 Find two vertical asymptotes. $Bx = 0$ and $Bx = \pi$

Substitute $B = \frac{1}{2}$ and solve for x. $x = 0$ and $x = 2\pi$

STEP 3 Find the x-intercept between the asymptotes. $Bx = \frac{\pi}{2}$

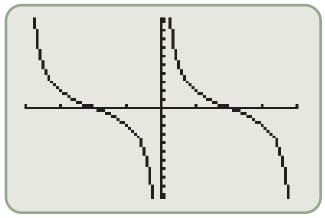
$x = \pi$

STEP 4 Draw the vertical asymptotes $x = 0$ and $x = 2\pi$ and label the x-intercept $(\pi, 0)$.



Technology Tip

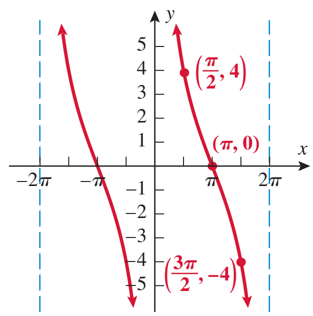
To graph $y = 4\cot\left(\frac{1}{2}x\right)$ on the interval $-2\pi \le x \le 2\pi$, enter $y = 4\left[\tan\left(\frac{1}{2}x\right)\right]^{-1}$.



STEP 5 Divide the period 2π into four equal parts, in steps of $\frac{\pi}{2}$. Set up a table with coordinates corresponding to values of $y = 4\cot\left(\frac{1}{2}x\right)$.

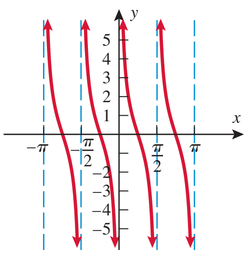
x	$y = 4\cot\left(\frac{1}{2}x\right)$	(x, y)
0	undefined	vertical asymptote, $x = 0$
$\frac{\pi}{2}$	4	$\left(\frac{\pi}{2}, 4\right)$
π	0	$(\pi, 0)$
$\frac{3\pi}{2}$	-4	$\left(\frac{3\pi}{2}, -4\right)$
2π	undefined	vertical asymptote, $x = 2\pi$

STEP 6 Graph the points from the table and connect with a smooth curve. Repeat to the right and left until you reach the interval endpoints.



■ **YOUR TURN** Graph $y = 2\cot(2x)$ on the interval $-\pi \le x \le \pi$.

■ **Answer:**



GRAPHING SECANT AND COSECANT FUNCTIONS

Graphs of $y = A\sec(Bx)$ and $y = A\csc(Bx)$ can be obtained using the following steps:

Step 1: Graph the corresponding guide function with a dashed curve.

For $y = A\sec(Bx)$, use $y = A\cos(Bx)$ as a guide.
For $y = A\csc(Bx)$, use $y = A\sin(Bx)$ as a guide.

Step 2: Draw the asymptotes, which correspond to the x-intercepts of the guide function.

Step 3: Draw the U shape between the asymptotes. If the guide function has a positive value between the asymptotes, the U opens upward; and if the guide function has a negative value, the U opens downward.

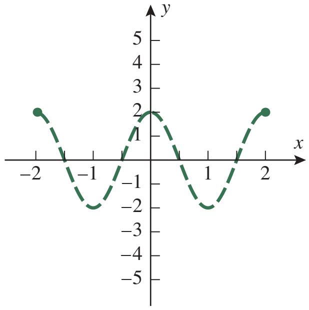
EXAMPLE 3 Graphing $y = A\sec(Bx)$

Graph $y = 2\sec(\pi x)$ on the interval $-2 \le x \le 2$.

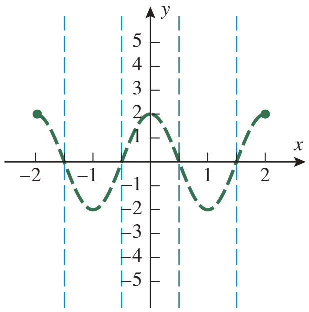
Solution:

STEP 1 Graph the corresponding guide function with a dashed curve.

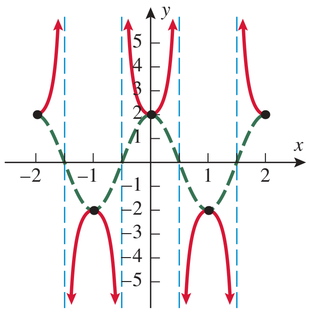
For $y = 2\sec(\pi x)$, use $y = 2\cos(\pi x)$ as a guide.



STEP 2 Draw the **asymptotes**, which correspond to the x-intercepts of the guide function.



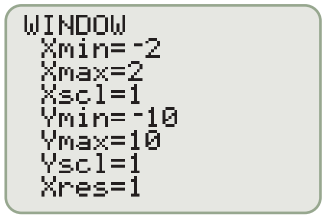
STEP 3 Use the U shape between the asymptotes. If the guide function is positive, the U opens upward, and if the guide function is negative, the U opens downward.

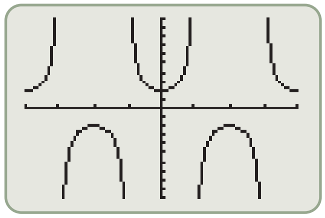


YOUR TURN Graph $y = -\sec(2\pi x)$ on the interval $-1 \le x \le 1$.

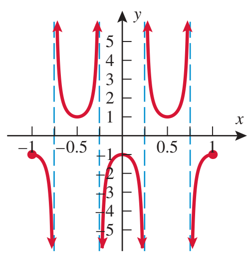
Technology Tip

To graph $y = 2\sec(\pi x)$ on the interval $-2 \le x \le 2$, enter $y = [2\cos(\pi x)]^{-1}$.





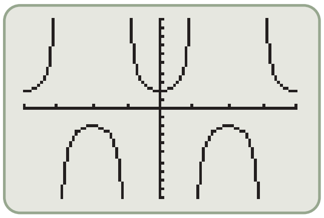
Answer:



Technology Tip

To graph $y = -3\csc(2\pi x)$ on the interval $-1 \le x \le 1$, enter $y = -3[\sin(2\pi x)]^{-1}$.

```
Plot1 Plot2 Plot3
\Y1 = -3(sin(2πX))
-1
```



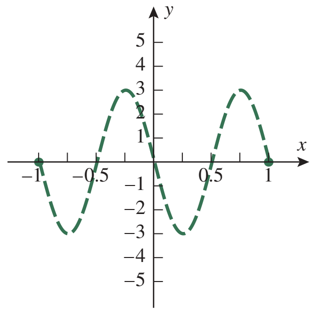
EXAMPLE 4 Graphing $y = A\csc(Bx)$

Graph $y = -3\csc(2\pi x)$ on the interval $-1 \le x \le 1$.

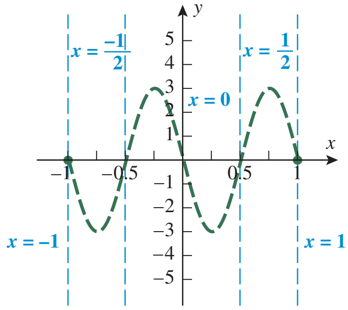
Solution:

STEP 1 Graph the corresponding guide function with a dashed curve.

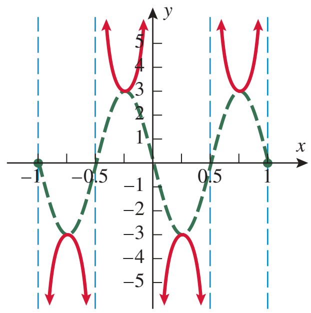
For $y = -3\csc(2\pi x)$, use $y = -3\sin(2\pi x)$ as a guide.



STEP 2 Draw the **asymptotes**, which correspond to the x-intercepts of the guide function.

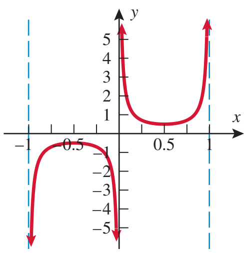


STEP 3 Use the U shape between the asymptotes. If the guide function is positive, the U opens upward, and if the guide function is negative, the U opens downward.



■ **YOUR TURN** Graph $y = \frac{1}{2}\csc(\pi x)$ on the interval $-1 \le x \le 1$.

■ **Answer:**



Translations of Trigonometric Functions

Vertical translations and horizontal translations (phase shifts) of the tangent, cotangent, secant, and cosecant functions are graphed the same way as vertical and horizontal translations of sinusoidal graphs. For tangent and cotangent functions, we follow the same procedure as we did with sinusoidal functions. For secant and cosecant functions, we graph the guide function first and then translate up or down depending on the sign of the vertical shift.

EXAMPLE 5 Graphing $y = A\tan(Bx + C) + D$

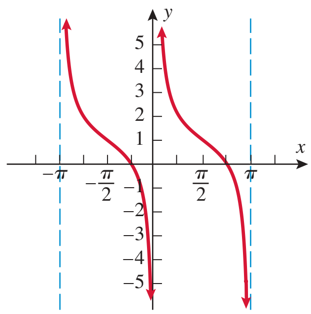
Graph $y = 1 - \tan\left(x - \frac{\pi}{2}\right)$ on $-\pi \le x \le \pi$. State the domain and range on the interval.

There are two ways to approach graphing this function. Both will be illustrated.

Solution (1):

Plot $y = \tan x$, and then do the following:

- Shift the curve to the right $\frac{\pi}{2}$ units. $\quad y = \tan\left(x - \frac{\pi}{2}\right)$
- Reflect the curve about the x-axis (because of the negative sign). $\quad y = -\tan\left(x - \frac{\pi}{2}\right)$
- Shift the entire graph up one unit. $\quad y = 1 - \tan\left(x - \frac{\pi}{2}\right)$



Solution (2): Graph $y = -\tan\left(x - \frac{\pi}{2}\right)$, and then shift the entire graph up one unit, because $D = 1$.

STEP 1 Calculate the period. $\quad \frac{\pi}{B} = \pi$

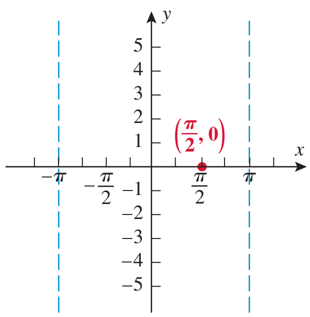
STEP 2 Find two vertical asymptotes. $\quad x - \frac{\pi}{2} = -\frac{\pi}{2}$ and $x - \frac{\pi}{2} = \frac{\pi}{2}$

Solve for x. $\quad \mathbf{x = 0}$ and $\mathbf{x = \pi}$

STEP 3 Find the x-intercept between the asymptotes. $\quad x - \frac{\pi}{2} = 0$

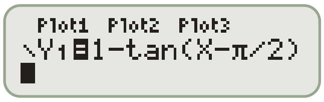
$\mathbf{x = \frac{\pi}{2}}$

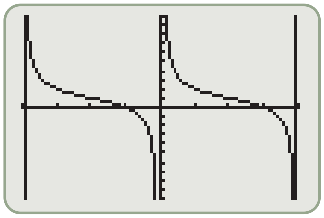
STEP 4 Draw the vertical asymptotes $\mathbf{x = 0}$ and $\mathbf{x = \pi}$ and label the x-intercept $\left(\frac{\pi}{2}, 0\right)$.



Technology Tip 

To obtain the graph of $y = 1 - \tan\left(x - \frac{\pi}{2}\right)$ on the interval $-\pi \le x \le \pi$, enter $y = 1 - \tan\left(x - \frac{\pi}{2}\right)$.

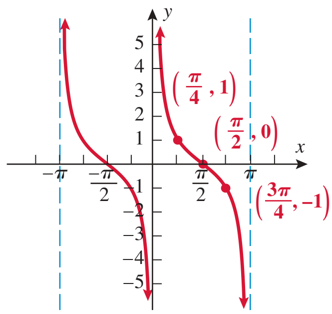




Step 5 Divide the period π into four equal parts, in steps of $\frac{\pi}{4}$. Set up a table with coordinates corresponding to values of $y = -\tan\left(x - \frac{\pi}{2}\right)$ between the two asymptotes.

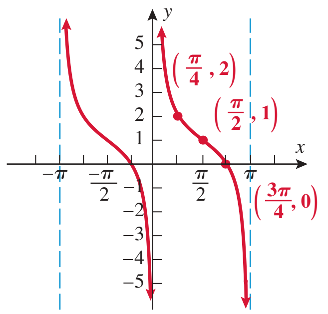
x	$y = -\tan\left(x - \frac{\pi}{2}\right)$	(x, y)
$x = 0$	undefined	vertical asymptote, $x = 0$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$ (x-intercept)
$\frac{3\pi}{4}$	-1	$\left(\frac{3\pi}{4}, -1\right)$
$x = \pi$	undefined	vertical asymptote, $x = \pi$

Step 6 Graph the points from the table and connect with a smooth curve. Repeat to the right and left until reaching the interval endpoints.



Step 7 Shift the entire graph up one unit to arrive at the graph of

$y = 1 - \tan\left(x - \frac{\pi}{2}\right)$.



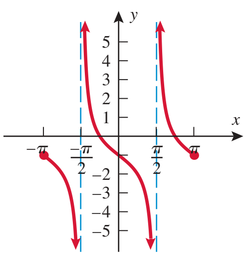
Step 8 State the domain and range on the interval.

Domain: $(-\pi, 0) \cup (0, \pi)$
Range: $(-\infty, \infty)$

■ **YOUR TURN** Graph $y = -1 + \cot\left(x + \frac{\pi}{2}\right)$ on $-\pi \le x \le \pi$. State the domain and range on the interval.

■ **Answer:**
Domain: $\left[-\pi, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Range: $(-\infty, \infty)$



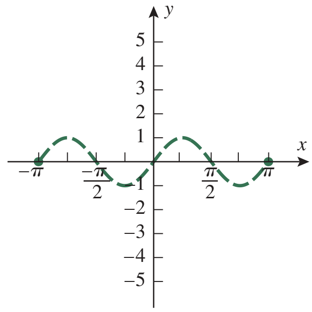
EXAMPLE 6 Graphing $y = A\csc(Bx + C) + D$

Graph $y = 1 - \csc(2x - \pi)$ on $-\pi \le x \le \pi$. State the domain and range on the interval.

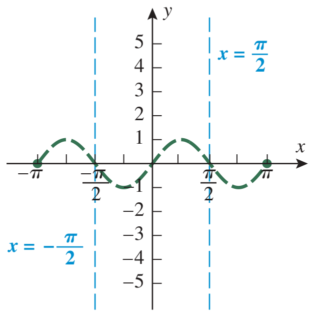
Solution:

Graph $y = -\csc(2x - \pi)$, and shift the entire graph up one unit to arrive at the graph of $y = 1 - \csc(2x - \pi)$.

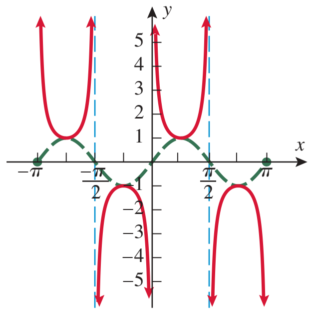
STEP 1 Draw the guide function, $y = -\sin(2x - \pi)$.



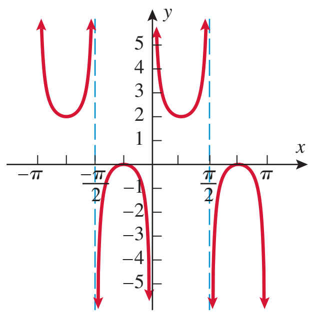
STEP 2 Draw the **vertical asymptotes** of $y = -\csc(2x - \pi)$ that correspond to the x-intercepts of $y = -\sin(2x - \pi)$.



STEP 3 Use the U shape between the asymptotes. If the guide function is positive, the U opens upward, and if the guide function is negative, the U opens downward.



STEP 4 Shift the entire graph up one unit to arrive at the graph of $y = 1 - \csc(2x - \pi)$.



STEP 5 State the domain and range on the interval.

Domain: $\left(-\pi, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

Range: $(-\infty, 0] \cup [2, \infty)$

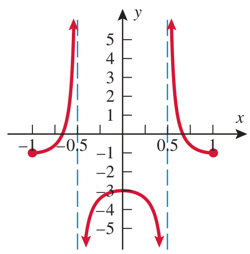
■ **YOUR TURN** Graph $y = -2 + \sec(\pi x - \pi)$ on $-1 \le x \le 1$. State the domain and range on the interval.

■ **Answer:**

Domain:

$\left[-1, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$

Range: $(-\infty, -3] \cup [-1, \infty)$



SECTION 5.3 SUMMARY

The tangent and cotangent functions have period π, whereas the secant and cosecant functions have period 2π. To graph the tangent and cotangent functions, first identify the vertical asymptotes and x-intercepts, and then find values of the function within a period (i.e., between the asymptotes). To find graphs of secant and cosecant functions, first graph their guide functions (cosine and sine, respectively), and then label vertical asymptotes that correspond to x-intercepts of the guide function. The graphs of the secant and cosecant functions resemble the letter U opening up or down. The secant and cosecant functions are positive when their guide function is positive and negative when their guide function is negative.

SECTION 5.3 EXERCISES

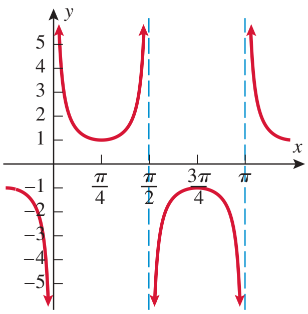
SKILLS

In Exercises 1–8, match the graphs to the functions (a–h).

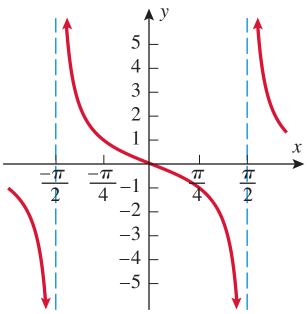
1. $y = -\tan x$ **2.** $y = -\csc x$ **3.** $y = \sec(2x)$ **4.** $y = \csc(2x)$

5. $y = \cot(\pi x)$ **6.** $y = -\cot(\pi x)$ **7.** $y = 3 \sec x$ **8.** $y = 3 \csc x$

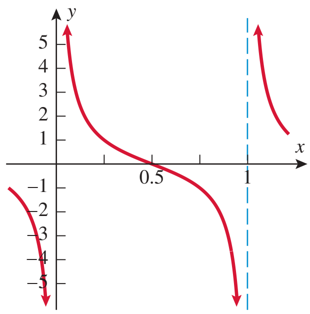
a.



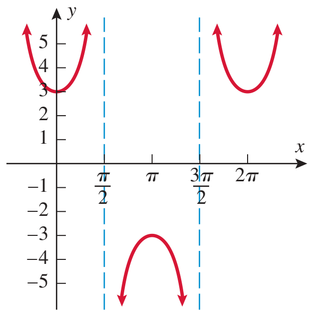
b.



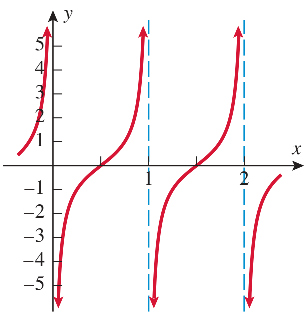
c.



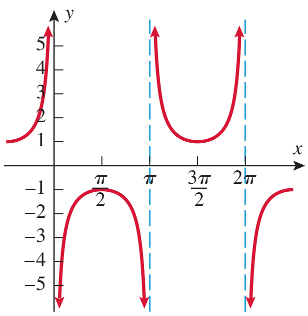
d.



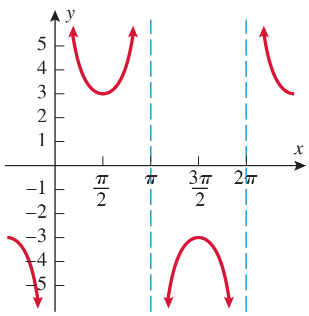
e.



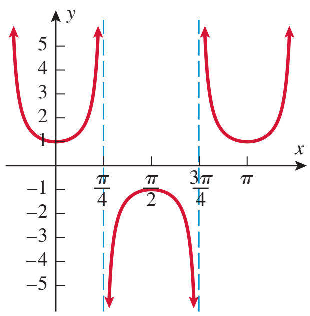
f.



g.



h.



In Exercises 9–28, graph the functions over the indicated intervals.

9. $y = \tan\left(\frac{1}{2}x\right),\ -2\pi \le x \le 2\pi$

10. $y = \cot\left(\frac{1}{2}x\right),\ -2\pi \le x \le 2\pi$

11. $y = -\cot(2\pi x),\ -1 \le x \le 1$

12. $y = -\tan(2\pi x),\ -1 \le x \le 1$

13. $y = 2\tan(3x),\ -\pi \le x \le \pi$

14. $y = 2\tan\left(\frac{1}{3}x\right),\ -3\pi \le x \le 3\pi$

15. $y = -\frac{1}{4}\cot\left(\frac{x}{2}\right)$, $-2\pi \le x \le 2\pi$

16. $y = -\frac{1}{2}\tan\left(\frac{x}{4}\right)$, $-4\pi \le x \le 4\pi$

17. $y = -\tan\left(x - \frac{\pi}{2}\right)$, $-\pi \le x \le \pi$

18. $y = \tan\left(x + \frac{\pi}{4}\right)$, $-\pi \le x \le \pi$

19. $y = 2\tan\left(x + \frac{\pi}{6}\right)$, $-\pi \le x \le \pi$

20. $y = -\frac{1}{2}\tan(x + \pi)$, $-\pi \le x \le \pi$

21. $y = \cot\left(x - \frac{\pi}{4}\right)$, $-\pi \le x \le \pi$

22. $y = -\cot\left(x + \frac{\pi}{2}\right)$, $-\pi \le x \le \pi$

23. $y = -\frac{1}{2}\cot\left(x + \frac{\pi}{3}\right)$, $-\pi \le x \le \pi$

24. $y = 3\cot\left(x - \frac{\pi}{6}\right)$, $-\pi \le x \le \pi$

25. $y = \tan(2x - \pi)$, $-2\pi \le x \le 2\pi$

26. $y = \cot(2x - \pi)$, $-2\pi \le x \le 2\pi$

27. $y = \cot\left(\frac{x}{2} + \frac{\pi}{4}\right)$, $-\pi \le x \le \pi$

28. $y = \tan\left(\frac{x}{3} - \frac{\pi}{3}\right)$, $-\pi \le x \le \pi$

In Exercises 29–46, graph the functions over the indicated intervals.

29. $y = \sec\left(\frac{1}{2}x\right)$, $-2\pi \le x \le 2\pi$

30. $y = \csc\left(\frac{1}{2}x\right)$, $-2\pi \le x \le 2\pi$

31. $y = -\csc(2\pi x)$, $-1 \le x \le 1$

32. $y = -\sec(2\pi x)$, $-1 \le x \le 1$

33. $y = \frac{1}{3}\sec\left(\frac{\pi}{2}x\right)$, $-4 \le x \le 4$

34. $y = \frac{1}{2}\csc\left(\frac{\pi}{3}x\right)$, $-6 \le x \le 6$

35. $y = -3\csc\left(\frac{x}{3}\right)$, $-6\pi \le x \le 0$

36. $y = -4\sec\left(\frac{x}{2}\right)$, $-4\pi \le x \le 4\pi$

37. $y = 2\sec(3x)$, $0 \le x \le 2\pi$

38. $y = 2\csc\left(\frac{1}{3}x\right)$, $-3\pi \le x \le 3\pi$

39. $y = -3\csc\left(x - \frac{\pi}{2}\right)$, over at least one period

40. $y = 5\sec\left(x + \frac{\pi}{4}\right)$, over at least one period

41. $y = \frac{1}{2}\sec(x - \pi)$, over at least one period

42. $y = -4\csc(x + \pi)$, over at least one period

43. $y = 2\sec(2x - \pi)$, $-2\pi \le x \le 2\pi$

44. $y = 2\csc(2x + \pi)$, $-2\pi \le x \le 2\pi$

45. $y = -\frac{1}{4}\sec(3x + \pi)$

46. $y = -\frac{2}{3}\csc\left(4x - \frac{\pi}{2}\right)$, $-\pi \le x \le \pi$

In Exercises 47–56, graph the functions over at least one period.

47. $y = 3 - 2\sec\left(x - \frac{\pi}{2}\right)$

48. $y = -3 + 2\csc\left(x + \frac{\pi}{2}\right)$

49. $y = \frac{1}{2} + \frac{1}{2}\tan\left(x - \frac{\pi}{2}\right)$

50. $y = \frac{3}{4} - \frac{1}{4}\cot\left(x + \frac{\pi}{2}\right)$

51. $y = -2 + 3\csc(2x - \pi)$

52. $y = -1 + 4\sec(2x + \pi)$

53. $y = -1 - \sec\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

54. $y = -2 + \csc\left(\frac{1}{2}x + \frac{\pi}{4}\right)$

55. $y = -2 - 3\cot\left(2x - \frac{\pi}{4}\right)$, $-\pi \le x \le \pi$

56. $y = -\frac{1}{4} + \frac{1}{2}\sec\left(\pi x + \frac{\pi}{4}\right)$, $-2 \le x \le 2$

In Exercises 57–66, state the domain and range of the functions.

57. $y = \tan\left(\pi x - \frac{\pi}{2}\right)$

58. $y = \cot\left(x - \frac{\pi}{2}\right)$

59. $y = 2\sec(5x)$

60. $y = -4\sec(3x)$

61. $y = 2 - \csc\left(\frac{1}{2}x - \pi\right)$

62. $y = 1 - 2\sec\left(\frac{1}{2}x + \pi\right)$

63. $y = -3\tan\left(\frac{\pi}{4}x - \pi\right) + 1$

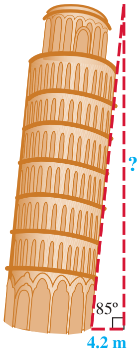
64. $y = \frac{1}{4}\cot\left(2\pi x + \frac{\pi}{3}\right) - 3$

65. $y = -2 + \frac{1}{2}\sec\left(\pi x + \frac{\pi}{2}\right)$

66. $y = \frac{1}{2} - \frac{1}{3}\csc\left(3x - \frac{\pi}{2}\right)$

APPLICATIONS

67. Tower of Pisa. The angle between the ground and the Tower of Pisa is about 85°. Its inclination measured at the base is 4.2 meters. What is the vertical distance from the top of the tower to the ground?



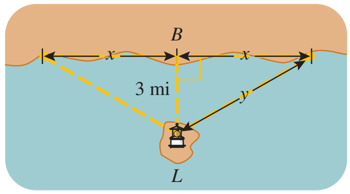
68. Architecture. The angle of elevation from the top of a building 40 feet tall to the top of another building 75 feet tall is $\frac{\pi}{6}$. What is the distance between the buildings?

69. Lighthouse. A lighthouse is located on a small island 3 miles offshore. The distance x is given by $x = 3\tan(\pi t)$, where t is the time measured in seconds. Suppose that at midnight the light beam forms a straight angle with the shoreline. Find x at

a. $t = \frac{2}{3}$ s **b.** $t = \frac{3}{4}$ s **c.** 1 s

d. $t = \frac{5}{4}$ s **e.** $t = \frac{4}{3}$ s

Round to the nearest length.



70. Lighthouse. If the length of the light beam is determined by $y = 3|\sec(\pi t)|$, find y at

a. $t = \frac{2}{3}$ s **b.** $t = \frac{3}{4}$ s **c.** 1 s

d. $t = \frac{5}{4}$ s **e.** $t = \frac{4}{3}$ s

Round to the nearest length.

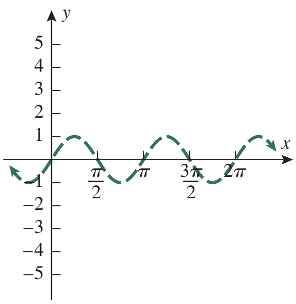
CATCH THE MISTAKE

In Exercises 71 and 72, explain the mistake that is made.

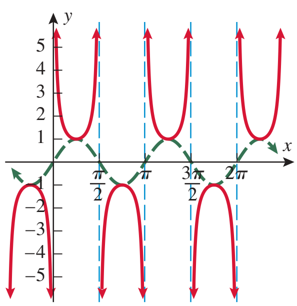
71. Graph $y = 3\csc(2x)$.

Solution:

Graph the guide function, $y = \sin(2x)$.



Draw vertical asymptotes at x-values that correspond to x-intercepts of the guide function.



Draw the cosecant function.

This is incorrect. What mistake was made?

72. Graph $y = \tan(4x)$.

Solution:

Step 1: Calculate the period. $\frac{\pi}{B} = \frac{\pi}{4}$

Step 2: Find two vertical asymptotes. $4x = 0$ and $4x = \pi$

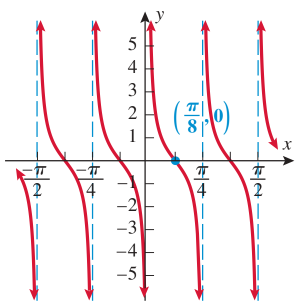
Solve for x. $x = 0$ and $x = \frac{\pi}{4}$

Step 3: Find the x-intercept between the asymptotes. $4x = \frac{\pi}{2}$

$x = \frac{\pi}{8}$

Step 4: Draw the vertical asymptotes $x = 0$ and $x = \frac{\pi}{4}$ and label the x-intercept $\left(\frac{\pi}{8}, 0\right)$.

Step 5: Graph.



This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 73 and 74, determine whether each statement is true or false.

73. $\sec\left(x - \dfrac{\pi}{2}\right) = \csc x$

74. $\csc\left(x - \dfrac{\pi}{2}\right) = \sec x$

75. For what values of n do $y = \tan x$ and $y = \tan(x - n\pi)$ have the same graph?

76. For what values of n do $y = \csc x$ and $y = \csc(x - n\pi)$ have the same graph?

77. Solve the equation $\tan(2x - \pi) = 0$ for x in the interval $[-\pi, \pi]$ by graphing.

78. Solve the equation $\csc(2x + \pi) = 0$ for x in the interval $[-\pi, \pi]$ by graphing.

79. Find the x-intercepts of $y = A\tan(Bx + C)$.

80. For what x-values does the graph of $y = -A\sec\left(\dfrac{\pi}{2}x\right)$ lie above the x-axis? (Assume $A > 0$.)

81. How many solutions are there to the equation $\tan x = x$? Explain.

82. For what values of A do the graphs of $y = A\sin(Bx + C)$ and $y = -2\csc\left(\dfrac{\pi}{6}x - \pi\right)$ never intersect?

TECHNOLOGY

83. What is the amplitude of the function $y = \cos x + \sin x$? Use a graphing calculator to graph $Y_1 = \cos x$, $Y_2 = \sin x$, and $Y_3 = \cos x + \sin x$ in the same viewing window.

84. Graph $Y_1 = \cos x + \sin x$ and $Y_2 = \sec x + \csc x$ in the same viewing window. Based on what you see, is $Y_1 = \cos x + \sin x$ the guide function for $Y_2 = \sec x + \csc x$?

85. What is the period of the function $y = \tan x + \cot x$? Use a graphing calculator to graph $Y_1 = \tan x + \cot x$ and $Y_3 = 2\csc(2x)$ in the same viewing window.

86. What is the period of the function $y = \tan\left(2x + \dfrac{\pi}{2}\right)$? Use a graphing calculator to graph $Y_1 = \tan\left(2x + \dfrac{\pi}{2}\right)$, $Y_2 = \tan\left(2x - \dfrac{\pi}{2}\right)$, and $Y_3 = \tan\left(-2x + \dfrac{\pi}{2}\right)$ in the same viewing window. Describe the relationships of Y_1 and Y_2 and Y_2 and Y_3.

PREVIEW TO CALCULUS

In calculus, the definite integral $\int_a^b f(x)\,dx$ is used to find the area below the graph of a continuous function f, above the x-axis, and between $x = a$ and $x = b$. The Fundamental Theorem of Calculus establishes that the definite integral $\int_a^b f(x)\,dx$ equals $F(b) - F(a)$, where F is any antiderivative of a continuous function f.

In Exercises 87–90, first shade the area corresponding to the definite integral and then use the information below to find the exact value of the area.

Function	$\tan x$	$\cot x$	$\sec x$	$\csc x$
Antiderivative	$-\ln\lvert\cos x\rvert$	$\ln\lvert\sin x\rvert$	$\ln\lvert\sec x + \tan x\rvert$	$-\ln\lvert\csc x + \cot x\rvert$

87. $\displaystyle\int_0^{\pi/4} \tan x\,dx$

88. $\displaystyle\int_{\pi/4}^{\pi/2} \cot x\,dx$

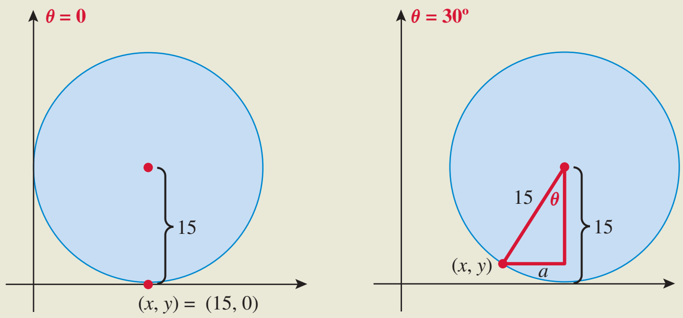
89. $\displaystyle\int_0^{\pi/4} \sec x\,dx$

90. $\displaystyle\int_{\pi/4}^{\pi/2} \csc x\,dx$

CHAPTER 5 INQUIRY-BASED LEARNING PROJECT

Perhaps in previous chapters you have solved problems involving how fast a car travels if the tires rotate at 300 rpm's or how to calculate the angular velocity if you know the linear velocity. The following problem uses the rolling wheel idea but in a different way. You will be asked to trace the path of a speck on the outer edge of a wheel as it rolls down the road. The connection to this chapter is that you will need to find the (x, y) coordinates of the speck using functions involving $\sin(\theta)$ and $\cos(\theta)$. Historically, this problem has two significant results. It is connected to the solution of how to build a perfect clock (a major breakthrough) and the fastest path from a point A to a point B, where B is below A but not directly below A.

In order to trace the path of a point (x, y) on the outer edge of a tire, we will assume the tire has a radius of 15 inches and rolls toward the right. To get started, (x, y) will be the point touching the x-axis and θ will be 0 (see the diagram below). θ stands for how many degrees the tire has rotated. You will do this for one full revolution in 30° increments.



1. Fill in the chart. See the diagrams above for help. (Use one decimal.)

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
x													
y													

Getting started:

- The height y can be found using right triangle trigonometry with the side b and the fact that the radius of the tires is 15 inches.
- The x-coordinate is a units behind (or ahead of) the center of the circle (tire). Identify where the center is located and then subtract (or add) a from this result. Notice that a is also calculated using right triangle trigonometry.

2. On a sheet of graph paper, plot the points (x, y) in your chart. You should see a smooth curve; otherwise, something is off.

3. Find the lateral position of the speck x in terms of θ: $x(\theta) =$

4. Find the vertical position of the speck y in terms of θ: $y(\theta) =$

MODELING OUR WORLD

Some would argue that temperatures are oscillatory in nature: that we are not experiencing global warming at all, but instead a natural cycle in Earth's temperature. In the Modeling Our World features in Chapters 1—3, you modeled mean temperatures with linear, polynomial, and logarithmic models—all of which modeled increasing temperatures. Looking at the data another way, we can demonstrate—over a short period of time—how it might appear to be oscillatory, which we can model with a sinusoidal curve. Be warned, however, that even the stock market, which increases gradually over time, looks sinusoidal over a short period.

The following table summarizes average yearly temperature in degree Fahrenheit (°F) and carbon dioxide emissions in parts per million (ppm) for the Mauna Loa Observatory in Hawaii.

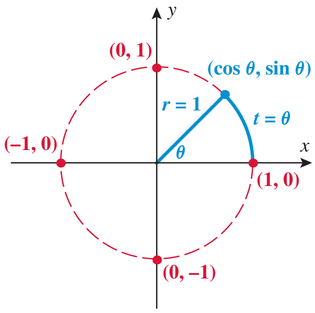
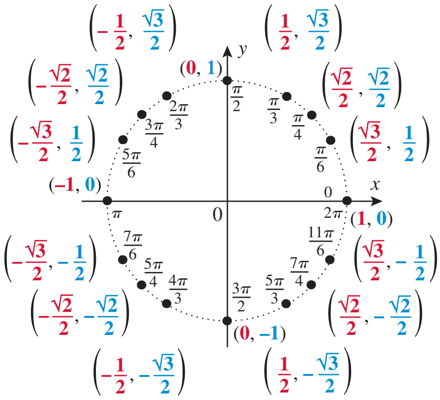
Year	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005
Temperature	44.45	43.29	43.61	43.35	46.66	45.71	45.53	47.53	45.86	46.23
CO_2 Emissions (ppm)	316.9	320.0	325.7	331.1	338.7	345.9	354.2	360.6	369.4	379.7

1. Find a *sinusoidal function* of the form $f(t) = k + A\sin(Bt)$ that models the temperature in Mauna Loa. Assume that the peak amplitude occurs in 1980 and again in 2010 (at 46.66°). Let $t = 0$ correspond to 1960.

2. Do your models support the claim of global warming? Explain.

3. You have modeled these same data with different types of models—-oscillatory and nonoscillatory. Over the short 30-year period from 1980 to 2010, you have a sinusoidal model that can fit the data, but does it prove anything in the long term?

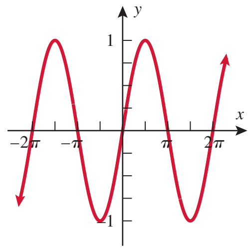
4. Would a resonance-type sinusoidal function of the form $f(t) = k + A\sin(Bt)$ perhaps be a better fit for 1960 to 2005? Develop a function of this form that models the data.

SECTION	CONCEPT	KEY IDEAS/FORMULAS
5.1	**Trigonometric functions: The unit circle approach**	
	Trigonometric functions and the unit circle	 
	Properties of trigonometric (circular) functions	Cosine is an even function. $\cos(-\theta) = \cos\theta$ Sine is an odd function. $\sin(-\theta) = -\sin\theta$

SECTION	CONCEPT	KEY IDEAS/FORMULAS
5.2	**Graphs of sine and cosine functions**	

The graphs of sinusoidal functions

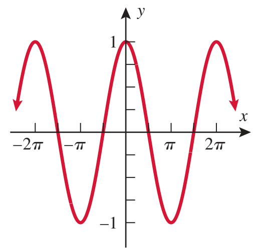
$f(x) = \sin x$



Odd Function

$\sin(-x) = -\sin x$

$f(x) = \cos x$



Even Function

$\cos(-x) = \cos(x)$

The amplitude and period of sinusoidal graphs

$y = A\sin(Bx)$ or $y = A\cos(Bx)$, $B > 0$

Amplitude $= |A|$

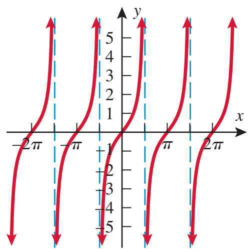
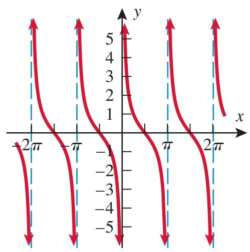
- $|A| > 1$ stretch vertically.
- $|A| < 1$ compress vertically.

Period $= \dfrac{2\pi}{B}$

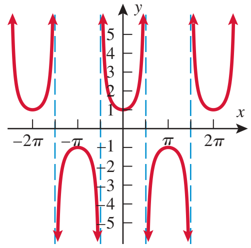
- $B > 1$ compress horizontally.
- $B < 1$ stretch horizontally.

Graphing a shifted sinusoidal function:
$y = A\sin(Bx + C) + D$
and
$y = A\cos(Bx + C) + D$

- $y = A\sin(Bx \pm C) = A\sin\left[B\left(x \pm \dfrac{C}{B}\right)\right]$ has period $\dfrac{2\pi}{B}$ and a phase shift of $\dfrac{C}{B}$ units to the left $(+)$ or the right $(-)$.
- $y = A\cos(Bx \pm C) = A\cos\left[B\left(x \pm \dfrac{C}{B}\right)\right]$ has period $\dfrac{2\pi}{B}$ and a phase shift of $\dfrac{C}{B}$ units to the left $(+)$ or the right $(-)$.
- To graph $y = A\sin(Bx + C) + D$ or $y = A\cos(Bx + C) + D$, start with the graph of $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ and shift up or down D units.

SECTION	CONCEPT	KEY IDEAS/FORMULAS
	Harmonic motion	■ Simple ■ Damped ■ Resonance
	Graphing sums of functions: Addition of ordinates	
5.3	**Graphs of other trigonometric functions**	
	Graphing the tangent, cotangent, secant, and cosecant functions	**The tangent function**  x-intercepts: $(n\pi, 0)$ Asymptotes: $x = \dfrac{(2n + 1)\pi}{2}$ Period: π Amplitude: none **The cotangent function**  Asymptotes: $x = n\pi$ x-intercepts: $\left(\dfrac{(2n + 1)\pi}{2}, 0\right)$ Period: π Amplitude: none

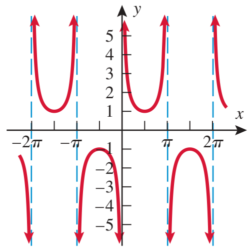
The secant function



Asymptotes: $x = \dfrac{(2n + 1)\pi}{2}$

Period: 2π
Amplitude: none
x-intercepts: none

The cosecant function



Asymptotes: $x = n\pi$
Period: 2π
Amplitude: none
x-intercepts: none

Graphing more general tangent, cotangent, secant, and cosecant functions

$y = A\tan(Bx)$, $y = A\cot(Bx)$, $y = A\sec(Bx)$, $y = A\csc(Bx)$

Section	Concept	Key Ideas/Formulas
	Translations of trigonometric functions	$y = A\tan(Bx + C)$ or $y = A\cot(Bx + C)$

To find asymptotes, set $Bx + C$ equal to

- $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ for tangent.
- 0 and π for cotangent.

To find x-intercepts, set $Bx + C$ equal to

- 0 for tangent.
- $\dfrac{\pi}{2}$ for cotangent.

$y = A\sec(Bx + C)$ or $y = A\csc(Bx + C)$

To graph $y = A\sec(Bx + C)$, use $y = A\cos(Bx + C)$ as the guide.

To graph $y = A\csc(Bx + C)$, use $y = A\sin(Bx + C)$ as the guide.

Intercepts on the guide function correspond to vertical asymptotes of secant or cosecant functions.

5.1 Trigonometric Functions: The Unit Circle Approach

Find each trigonometric function value in *exact* form.

1. $\tan\left(\frac{5\pi}{6}\right)$ **2.** $\cos\left(\frac{5\pi}{6}\right)$ **3.** $\sin\left(\frac{11\pi}{6}\right)$

4. $\sec\left(\frac{11\pi}{6}\right)$ **5.** $\cot\left(\frac{5\pi}{4}\right)$ **6.** $\csc\left(\frac{5\pi}{4}\right)$

7. $\sin\left(\frac{3\pi}{2}\right)$ **8.** $\cos\left(\frac{3\pi}{2}\right)$ **9.** $\cos\pi$

10. $\tan\left(\frac{7\pi}{4}\right)$ **11.** $\cos\left(\frac{\pi}{3}\right)$ **12.** $\sin\left(\frac{11\pi}{6}\right)$

13. $\sin\left(-\frac{7\pi}{4}\right)$ **14.** $\tan\left(-\frac{2\pi}{3}\right)$ **15.** $\csc\left(-\frac{3\pi}{2}\right)$

16. $\cot\left(-\frac{5\pi}{6}\right)$ **17.** $\cos\left(-\frac{7\pi}{6}\right)$ **18.** $\sec\left(-\frac{3\pi}{4}\right)$

19. $\tan\left(-\frac{13\pi}{6}\right)$ **20.** $\cos\left(-\frac{14\pi}{3}\right)$

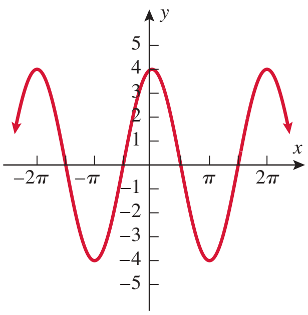
5.2 Graphs of Sine and Cosine Functions

Refer to the graph of the sinusoidal function to answer the questions.

21. Determine the period of the function.

22. Determine the amplitude of the function.

23. Write an equation for the sinusoidal function.

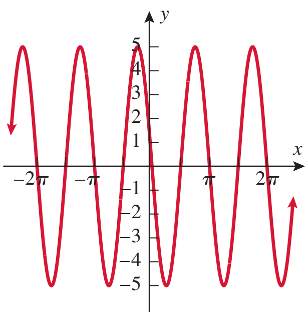


Refer to the graph of the sinusoidal function to answer the questions.

24. Determine the period of the function.

25. Determine the amplitude of the function.

26. Write an equation for the sinusoidal function.



Determine the amplitude and period of each function.

27. $y = -2\cos(2\pi x)$ **28.** $y = \frac{1}{3}\sin\left(\frac{\pi}{2}x\right)$

29. $y = \frac{1}{5}\sin(3x)$ **30.** $y = -\frac{7}{6}\cos(6x)$

Graph each function from -2π to 2π.

31. $y = -2\sin\left(\frac{x}{2}\right)$ **32.** $y = 3\sin(3x)$

33. $y = \frac{1}{2}\cos(2x)$ **34.** $y = -\frac{1}{4}\cos\left(\frac{x}{2}\right)$

State the amplitude, period, phase shift, and vertical shift of each function.

35. $y = 2 + 3\sin\left(x - \frac{\pi}{2}\right)$

36. $y = 3 - \frac{1}{2}\sin\left(x + \frac{\pi}{4}\right)$

37. $y = -2 - 4\cos\left[3\left(x + \frac{\pi}{4}\right)\right]$

38. $y = -1 + 2\cos\left[2\left(x - \frac{\pi}{3}\right)\right]$

39. $y = -\frac{1}{2} + \frac{1}{3}\cos\left(\pi x - \frac{1}{2}\right)$

40. $y = \frac{3}{4} - \frac{1}{6}\sin\left(\frac{\pi}{6}x + \frac{\pi}{3}\right)$

Graph each function from $-\pi$ to π.

41. $y = 3x - \cos(2x)$

42. $y = -\frac{1}{2}\cos(4x) + \frac{1}{2}\cos(2x)$

43. $y = 2\sin\left(\frac{1}{3}x\right) - 3\sin(3x)$

44. $y = 5\cos x + 3\sin\left(\frac{x}{2}\right)$

5.3 Graphs of Other Trigonometric Functions

State the domain and range of each function.

45. $y = 4\tan\left(x + \frac{\pi}{2}\right)$

46. $y = \cot 2\left(x - \frac{\pi}{2}\right)$

47. $y = 3\sec(2x)$

48. $y = 1 + 2\csc x$

49. $y = -\frac{1}{2} + \frac{1}{4}\sec\left(\pi x - \frac{2\pi}{3}\right)$

50. $y = 3 - \frac{1}{2}\csc(2x - \pi)$

Graph each function on the interval $[-2\pi, 2\pi]$.

51. $y = -\tan\left(x - \frac{\pi}{4}\right)$

52. $y = 1 + \cot(2x)$

53. $y = 2 + \sec(x - \pi)$

54. $y = -\csc\left(x + \frac{\pi}{4}\right)$

55. $y = \frac{1}{2} + 2\csc\left(2x - \frac{\pi}{2}\right)$

56. $y = -1 - \frac{1}{2}\sec\left(\pi x - \frac{3\pi}{4}\right)$

Technology Exercises

Section 5.1

In Exercises 57 and 58, refer to the following:

A graphing calculator can be used to graph the unit circle with parametric equations (these will be covered in more detail in Section 9.9). For now, set the calculator in parametric and radian modes and let

$$X_1 = \cos T$$
$$Y_1 = \sin T$$

Set the window so that $0 \le T \le 2\pi$, step $= \frac{\pi}{15}$, $-2 \le X \le 2$, and $-2 \le Y \le 2$. To approximate the sine or cosines of a T value, use the $\boxed{\text{TRACE}}$ key, enter the T value, and read the corresponding coordinates from the screen.

57. Use the above steps to approximate $\cos\left(\frac{13\pi}{12}\right)$ to four decimal places.

58. Use the above steps to approximate $\sin\left(\frac{5\pi}{6}\right)$ to four decimal places.

Section 5.2

59. Use a graphing calculator to graph $Y_1 = \cos x$ and $Y_2 = \cos(x + c)$, where

a. $c = \frac{\pi}{6}$, and explain the relationship between Y_2 and Y_1.

b. $c = -\frac{\pi}{6}$, and explain the relationship between Y_2 and Y_1.

60. Use a graphing calculator to graph $Y_1 = \sin x$ and $Y_2 = \sin x + c$, where

a. $c = \frac{1}{2}$, and explain the relationship between Y_2 and Y_1.

b. $c = -\frac{1}{2}$, and explain the relationship between Y_2 and Y_1.

Section 5.3

61. What is the amplitude of the function $y = 4\cos x - 3\sin x$? Use a graphing calculator to graph $Y_1 = 4\cos x$, $Y_2 = 3\sin x$, and $Y_3 = 4\cos x - 3\sin x$ in the same viewing window.

62. What is the amplitude of the function $y = \sqrt{3}\sin x + \cos x$? Use a graphing calculator to graph $Y_1 = \sqrt{3}\sin x$, $Y_2 = \cos x$, and $Y_3 = \sqrt{3}\sin x + \cos x$ in the same viewing window.

1. State the amplitude and period of $y = -5\sin(3x)$.
2. Graph $y = -2\cos\left(\frac{1}{2}x\right)$ on the interval $-4\pi \leq x \leq 4\pi$.
3. Graph $y = 1 + 3\sin(x + \pi)$ on the interval $-3\pi \leq x \leq 3\pi$.
4. Graph $y = 4 - \sin\left(x - \dfrac{\pi}{2}\right)$ on the interval $-6\pi \leq x \leq 6\pi$.
5. Graph $y = -2 - \cos\left(x + \dfrac{\pi}{2}\right)$ on the interval $-4\pi \leq x \leq 4\pi$.
6. Graph $y = 3 + 2\cos\left(x + \dfrac{3\pi}{2}\right)$ on the interval $-5\pi \leq x \leq 5\pi$.
7. Graph $y = \tan\left(\pi x - \dfrac{\pi}{2}\right)$ over two periods.
8. The vertical asymptotes of $y = 2\csc(3x - \pi)$ correspond to the __________ of $y = 2\sin(3x - \pi)$.
9. State the x-intercepts of $y = \tan(2x)$ for all x.
10. State the phase shift and vertical shift for $y = -\cot\left(\dfrac{\pi}{3}x - \pi\right)$.
11. State the range of $y = -3\sec\left(2x + \dfrac{\pi}{3}\right) - 1$.
12. State the domain of $y = \tan\left(2x - \dfrac{\pi}{6}\right) + 3$.
13. Graph $y = -2\csc\left(x + \dfrac{\pi}{2}\right)$ over two periods.
14. Find the x-intercept(s) of $y = \dfrac{6}{\sqrt{3}} - 3\sec\left(6x - \dfrac{5\pi}{6}\right)$.
15. True or false: The equation $2\sin\theta = 2.0001$ has no solution.
16. On what x-intervals does the graph of $y = \cos(2x)$ lie below the x-axis?
17. Write the equation of a sine function that has amplitude 4, vertical shift $\frac{1}{2}$ down, phase shift $\frac{3}{2}$ to the left, and period π.
18. Write the equation of a cotangent function that has period π, vertical shift 0.01 up, and no phase shift.
19. Graph $y = \cos(3x) - \frac{1}{2}\sin(3x)$ for $0 \leq x \leq \pi$.
20. $y = -\dfrac{1}{5}\cos\left(\dfrac{x}{3}\right)$
    a. Graph the function over one period.
    b. Determine the amplitude, period, and phase shift.
21. $y = 4\sin(2\pi x)$
    a. Graph the function over one period.
    b. Determine the amplitude, period, and phase shift.
22. $y = -2\sin(3x + 4\pi) + 1$
    a. Write the sinusoidal function in standard form.
    b. Determine its amplitude, period, and phase shift.
    c. Graph the function over one period.
23. $y = 6 + 5\cos(2x - \pi)$
    a. Write the sinusoidal function in standard form.
    b. Determine its amplitude, period, and phase shift.
    c. Graph the function over one period.
24. Graph the function $y = 2\cos x - \sin x$ on the interval $[-\pi, \pi]$ by adding the ordinates of each individual function.

1. Find the domain of $f(x) = \dfrac{4}{\sqrt{15 + 3x}}$. Express the domain in interval notation.

2. On February 24, the gasoline price (per gallon) was \$1.94; on May 24, its price per gallon was \$2.39. Find the average rate of change per month in the gasoline price from February 24 to May 24.

3. Write the function whose graph is the graph of $y = |x|$, but stretched by a factor of 2, shifted up four units, and shifted to the left six units. Graph the function on the interval $[-10, 10]$.

4. Given $f(x) = 2x - 5$ and $g(x) = x^2 + 7$, find

   a. $f + g$

   b. $f - g$

   c. $f \cdot g$

   d. f/g

   e. $f \circ g$

5. Find the inverse function of the one-to-one function $f(x) = \dfrac{x - 2}{3x + 5}$. Find the domain and range of both f and f^{-1}.

6. For $f(x) = -x^2(2x - 6)^3(x + 5)^4$

   a. list each zero and its multiplicity.

   b. sketch the graph.

7. Divide $(6x^4 - 5x^3 + 6x^2 + 7x - 4)$ by $(2x^2 - 1)$ using long division. Express the answer in the form $Q(x) = ?$ and $r(x) = ?$.

8. For the polynomial function $f(x) = x^4 + x^3 - 7x^2 - x + 6$

   a. factor as a product of linear and/or irreducible quadratic factors.

   b. graph the function.

9. Factor the polynomial $P(x) = x^4 - 4x^2 - 5$ as a product of linear factors.

10. Given $f(x) = \dfrac{x^2 - 5x - 14}{2x^2 + 14x + 20}$,

    a. determine the vertical, horizontal, or slant asymptotes (if they exist).

    b. graph the function.

11. If \$3000 is deposited into an account paying 1.2% compounded quarterly, how much will you have in the account in 10 years?

12. Approximate $\log_2 19$ utilizing a calculator. Round to two decimal places.

13. Write $\ln\left(\dfrac{a^3}{b^2c^5}\right)$ as a sum or difference of constant multiples of logarithms.

14. Solve the exponential equation $4^{3x-2} + 5 = 23$. Round your answer to three decimal places.

15. Solve the exponential equation $\log(x + 2) + \log(x + 3) = \log(2x + 10)$. Round your answer to three decimal places.

16. Find the area of a circular sector with radius $r = 6.5$ cm and central angle $\theta = \dfrac{4\pi}{5}$. Round your answer to the nearest integer.

17. Solve the right triangle $\beta = 27°$, $c = 14$ in. Round your answer to the nearest hundredth.

18. The terminal side of an angle θ in standard position passes through the point $(3, -2)$. Calculate the exact value of the six trigonometric functions for angle θ.

19. Solve the triangle $\alpha = 68°$, $a = 24$ m, and $b = 24.5$ m.

20. Solve the triangle $a = 5$, $b = 6$, and $c = 7$.

21. Given that $\sin\theta = \dfrac{1}{2}$ and $\dfrac{\pi}{2} < \theta < \pi$, find the exact value of all the other trigonometric functions.

22. Graph the function $y = 4\cos(2x + \pi)$ over one period.

23. The frequency of the oscillations f is given by $f = \dfrac{1}{p}$, where p is the period. What is the frequency of the oscillations modeled by $y = 1.14\sin(4t)$?

24. Graph the function $y = -2 + 5\csc\left(4x - \dfrac{\pi}{2}\right)$ over two periods. State the range of the function.

CUMULATIVE TEST

6

Analytic Trigonometry

When you press a touch-tone button to dial a phone number, how does the phone system know which key you have pressed? Dual Tone Multi-Frequency (DTMF), also known as Touch-Tone dialing, was developed by Bell Labs in the 1960s. The Touch-Tone system also introduced a standardized *keypad* layout.

The keypad is laid out in a 4×3 matrix, with each row representing a low frequency and each column representing a high frequency.

Frequency	1209 Hz	1336 Hz	1477 Hz
697 Hz	1	2	3
770 Hz	4	5	6
852 Hz	7	8	9
941 Hz	*	0	#

When you press the number 8, the phone sends a sinusoidal tone that combines a low-frequency tone of 852 hertz and a high-frequency tone of 1336 hertz. The result can be found using sum-to-product *trigonometric identities*.

IN THIS CHAPTER we will verify trigonometric identities. Specific identities that we will discuss are sum and difference, double-angle and half-angle, and product-to-sum and sum-to-product. Inverse trigonometric functions will be defined. Trigonometric identities and inverse trigonometric functions will be used to solve trigonometric equations.

ANALYTIC TRIGONOMETRY

6.1 Verifying Trigonometric Identities

- Fundamental Identities
- Simplifying Trigonometric Expressions Using Identities
- Verifying Identities

6.2 Sum and Difference Identities

- Sum and Difference Identities for the Cosine Function
- Sum and Difference Identities for the Sine Function
- Sum and Difference Identities for the Tangent Function

6.3 Double-Angle and Half-Angle Identities

- Double-Angle Identities
- Half-Angle Identities

6.4 Product-to-Sum and Sum-to-Product Identities

- Product-to-Sum Identities
- Sum-to-Product Identities

6.5 Inverse Trigonometric Functions

- Inverse Sine Function
- Inverse Cosine Function
- Inverse Tangent Function
- Remaining Inverse Trigonometric Functions
- Finding Exact Values for Expressions Involving Inverse Trigonometric Functions

6.6 Trigonometric Equations

- Solving Trigonometric Equations by Inspection
- Solving Trigonometric Equations Using Algebraic Techniques
- Solving Trigonometric Equations That Require the Use of Inverse Functions
- Using Trigonometric Identities to Solve Trigonometric Equations

LEARNING OBJECTIVES

- Verify trigonometric identities.
- Use the sum and difference identities to simplify trigonometric expressions.
- Use the double-angle and half-angle identities to simplify trigonometric expressions.
- Use the product-to-sum and sum-to-product identities to simplify trigonometric expressions.
- Evaluate the inverse trigonometric functions for specific values.
- Solve trigonometric equations.

SECTION 6.1 VERIFYING TRIGONOMETRIC IDENTITIES

SKILLS OBJECTIVES

- Apply fundamental identities.
- Simplify trigonometric expressions using identities.
- Verify trigonometric identities.

CONCEPTUAL OBJECTIVES

- Understand that there is more than one way to verify an identity.
- Understand that identities must hold for all values in the domain of the functions that are related by the identities.

Fundamental Identities

In mathematics, an **identity** is an equation that is true for *all* values of the variable for which the expressions in the equation are defined. If an equation is true for only *some* values of the variable, it is a **conditional equation**.

The following boxes summarize trigonometric identities that have been discussed in Chapters 4 and 5.

Study Tip

Just because an equation is true for *some* values of x does not mean it is an identity. An equation has to be true for **all** values of the variable for it to be an **identity**.

RECIPROCAL IDENTITIES

RECIPROCAL IDENTITIES	EQUIVALENT FORMS	DOMAIN RESTRICTIONS
$\csc x = \dfrac{1}{\sin x}$	$\sin x = \dfrac{1}{\csc x}$	$x \neq n\pi \quad n =$ integer
$\sec x = \dfrac{1}{\cos x}$	$\cos x = \dfrac{1}{\sec x}$	$x \neq \dfrac{n\pi}{2} \quad n =$ odd integer
$\cot x = \dfrac{1}{\tan x}$	$\tan x = \dfrac{1}{\cot x}$	$x \neq \dfrac{n\pi}{2} \quad n =$ integer

QUOTIENT IDENTITIES

QUOTIENT IDENTITIES	DOMAIN RESTRICTIONS
$\tan x = \dfrac{\sin x}{\cos x}$	$\cos x \neq 0 \quad x \neq \dfrac{n\pi}{2} \quad n =$ odd integer
$\cot x = \dfrac{\cos x}{\sin x}$	$\sin x \neq 0 \quad x \neq n\pi \quad n =$ integer

PYTHAGOREAN IDENTITIES

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

In Chapter 5, we discussed even and odd trigonometric functions, which like even and odd functions in general have these respective properties:

TYPE OF FUNCTION	ALGEBRAIC IDENTITY	GRAPH
Even	$f(-x) = f(x)$	Symmetry about the y-axis
Odd	$f(-x) = -f(x)$	Symmetry about the origin

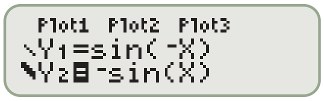
We already learned in Chapter 5 that the sine function is an odd function and the cosine function is an even function. Combining this knowledge with the reciprocal and quotient identities, we arrive at the *even-odd identities*, which we can add to our list of basic identities.

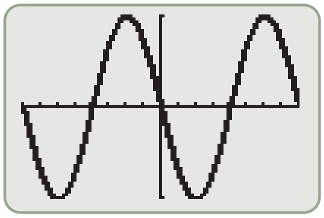
EVEN-ODD IDENTITIES

$$\text{Odd}\begin{cases}\sin(-x) = -\sin x\\ \csc(-x) = -\csc x\\ \tan(-x) = -\tan x\\ \cot(-x) = -\cot x\end{cases}\qquad \text{Even}\begin{cases}\cos(-x) = \cos x\\ \sec(-x) = \sec x\end{cases}$$

Technology Tip

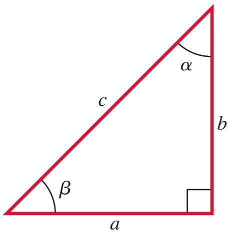
Graphs of $y_1 = \sin(-x)$ and $y_2 = -\sin x$.





Cofunctions

Recall complementary angles (Section 4.1). Notice the *co* in *co*sine, *co*secant, and *co*tangent functions. These *cofunctions* are based on the relationship of *co*mplementary angles. Let us look at a right triangle with labled sides and angles.



$$\left.\begin{aligned}\sin\beta &= \frac{\text{opposite of }\beta}{\text{hypotenuse}} = \frac{b}{c}\\ \cos\alpha &= \frac{\text{adjacent to }\alpha}{\text{hypotenuse}} = \frac{b}{c}\end{aligned}\right\}\sin\beta = \cos\alpha$$

Recall that the sum of the measures of the three angles in a triangle is 180°. In a right triangle, one angle is 90°; therefore, the two acute angles are complementary angles (the measures sum to 90°). You can see in the triangle above that β and α are *co*mplementary angles. In other words, the sine of an angle is the same as the *co*sine of the *co*mplement of that angle. This is true for all *trigonometric cofunction* pairs.

COFUNCTION THEOREM

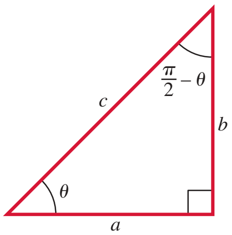
A trigonometric function of an angle is always equal to the cofunction of the complement of the angle. If $\alpha + \beta = 90°$ $\left(\text{or } \alpha + \beta = \frac{\pi}{2}\right)$, then

$$\sin\beta = \cos\alpha$$
$$\sec\beta = \csc\alpha$$
$$\tan\beta = \cot\alpha$$

Study Tip

Alternate form of cofunction identities:

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$
$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$
$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$
$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$
$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$$
$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$



COFUNCTION IDENTITIES

$$\sin\theta = \cos(90° - \theta) \qquad \cos\theta = \sin(90° - \theta)$$
$$\tan\theta = \cot(90° - \theta) \qquad \cot\theta = \tan(90° - \theta)$$
$$\sec\theta = \csc(90° - \theta) \qquad \csc\theta = \sec(90° - \theta)$$

EXAMPLE 1 Writing Trigonometric Function Values in Terms of Their Cofunctions

Write each function or function value in terms of its cofunction.

a. $\sin 30°$ **b.** $\tan x$ **c.** $\csc 40°$

Solution (a):

Cosine is the cofunction of sine. $\sin\theta = \cos(90° - \theta)$

Substitute $\theta = 30°$. $\sin 30° = \cos(90° - 30°)$

Simplify. $\boxed{\sin 30° = \cos 60°}$

Solution (b):

Cotangent is the cofunction of tangent. $\tan\theta = \cot(90° - \theta)$

Substitute $\theta = x$. $\boxed{\tan x = \cot(90° - x)}$

Solution (c):

Cosecant is the cofunction of secant. $\csc\theta = \sec(90° - \theta)$

Substitute $\theta = 40°$. $\csc 40° = \sec(90° - 40°)$

Simplify. $\boxed{\csc 40° = \sec 50°}$

■ **YOUR TURN** Write each function or function value in terms of its cofunction.

a. $\cos 45°$ **b.** $\csc y$

■ **Answer:** **a.** $\sin 45°$ **b.** $\sec(90° - y)$

Simplifying Trigonometric Expressions Using Identities

We can use the fundamental identities and algebraic manipulation to simplify more complicated trigonometric expressions. In simplifying trigonometric expressions, one approach is to first convert all expressions into sines and cosines and then simplify.

EXAMPLE 2 Simplifying Trigonometric Expressions

Simplify $\tan x \sin x + \cos x$.

Solution:

Write the tangent function in terms of the sine and cosine functions: $\tan x = \dfrac{\sin x}{\cos x}$.

$$\tan x \cdot \sin x + \cos x$$
$$= \frac{\sin x}{\cos x}\sin x + \cos x$$

Simplify.

$$= \frac{\sin^2 x}{\cos x} + \cos x$$

Write as a fraction with a single quotient by finding a common denominator, $\cos x$.

$$= \frac{\sin^2 x + \cos^2 x}{\cos x}$$

Use the Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

$$= \frac{1}{\cos x}$$

Use the reciprocal identity $\sec x = \dfrac{1}{\cos x}$.

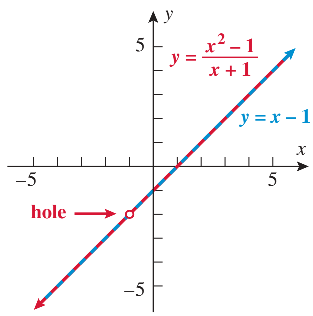
$$= \boxed{\sec x}$$

■ **YOUR TURN** Simplify $\cot x \cos x + \sin x$.

■ **Answer:** $\csc x$

In Example 2, $\tan x$ and $\sec x$ are not defined for odd integer multiples of $\frac{\pi}{2}$. In the Your Turn, $\cot x$ and $\csc x$ are not defined for integer multiples of π. Both the original expression and the simplified form are governed by the same restrictions. There are times when the original expression is subject to more domain restrictions than the simplified form and thus special attention must be given to domain restrictions.

For example, the algebraic expression $\dfrac{x^2 - 1}{x + 1}$ is under the domain restriction $x \neq -1$ because that value for x makes the value of the denominator equal to zero. If we forget to state the domain restrictions, we might simplify the algebraic expression as $\dfrac{x^2 - 1}{x + 1} = \dfrac{(x - 1)(x + 1)}{(x + 1)} = x - 1$ and assume this is true for all values of x. The correct simplification is $\dfrac{x^2 - 1}{x + 1} = x - 1$ for $x \neq -1$. In fact, if we were to graph both the original expression $y = \dfrac{x^2 - 1}{x + 1}$ and the line $y = x - 1$, they would coincide, except that the graph of the original expression would have a "hole" or discontinuity at $x = -1$. In this chapter, it is assumed that the domain of the simplified expression is the same as the domain of the original expression.



EXAMPLE 3 Simplifying Trigonometric Expressions

Simplify $\dfrac{1}{\csc^2 x} + \dfrac{1}{\sec^2 x}$.

Solution:

Rewrite the expression in terms of quotients squared.

$$\frac{1}{\csc^2 x} + \frac{1}{\sec^2 x} = \left(\frac{1}{\csc x}\right)^2 + \left(\frac{1}{\sec x}\right)^2$$

Use the reciprocal identities to write the cosecant and secant functions in terms of sines and cosines: $\sin x = \dfrac{1}{\csc x}$ and $\cos x = \dfrac{1}{\sec x}$.

$$= \sin^2 x + \cos^2 x$$

Use the Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

$$= \boxed{1}$$

■ **YOUR TURN** Simplify $\dfrac{1}{\cos^2 x} - 1$.

■ **Answer:** $\tan^2 x$

Verifying Identities

We will now use the trigonometric identities to verify, or establish, other trigonometric identities. For example, verify that

$$(\sin x - \cos x)^2 - 1 = -2\sin x \cos x$$

The good news is that we will know we are done when we get there, since we know the desired identity. But how do we get there? How do we verify that the identity is true? Remember that it must be true for *all* x, not just some x. Therefore, it is not enough to simply select values for x and show it is true for those specific values.

Words	**Math**
Start with one side of the equation (the more complicated side).	$(\sin x - \cos x)^2 - 1$
Remember that $(a - b)^2 = a^2 - 2ab + b^2$ and expand $(\sin x - \cos x)^2$.	$= \sin^2 x - 2\sin x \cos x + \cos^2 x - 1$
Group the $\sin^2 x$ and $\cos^2 x$ terms and use the Pythagorean identity.	$= -2\sin x \cos x + \underbrace{(\sin^2 x + \cos^2 x)}_{1} - 1$
Simplify.	$= -2\sin x \cos x$

When we arrive at the right side of the equation, then we have succeeded in verifying the identity. In verifying trigonometric identities, there is no one procedure that works for all identities. You must manipulate one side of the equation until it looks like the other side.

The following suggestions help guide the way in verifying trigonometric identities.

Guidelines for Verifying Trigonometric Identities

- Start with the more complicated side of the equation.
- Combine all sums and differences of fractions (quotients) into a single fraction (quotient).
- Use fundamental trigonometric identities.
- Use algebraic techniques to manipulate one side of the equation until the other side of the equation is achieved.
- Sometimes it is helpful to convert all trigonometric functions into sines and cosines.

It is important to note that trigonometric identities must be valid for all values of the independent variable (usually, x or θ) for which the expressions in the equation are defined (domain of the equation).

EXAMPLE 4 Verifying Trigonometric Identities

Verify the identity $\dfrac{\tan x - \cot x}{\tan x + \cot x} = \sin^2 x - \cos^2 x$.

Solution:

Start with the more complicated side of the equation.

$$\frac{\tan x - \cot x}{\tan x + \cot x}$$

Use the quotient identity to write the tangent and cotangent functions in terms of the sine and cosine functions.

$$= \frac{\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}$$

Multiply by $\dfrac{\sin x \cos x}{\sin x \cos x}$.

$$= \left(\frac{\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}\right)\left(\frac{\sin x \cos x}{\sin x \cos x}\right)$$

Simplify.

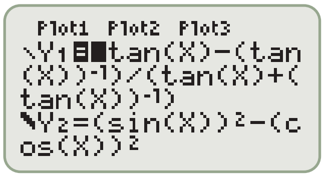
$$= \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x}$$

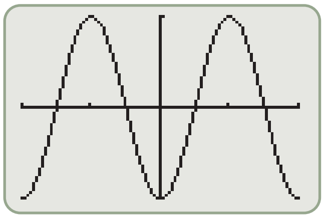
Use the Pythagorean identity: $\sin^2 x + \cos^2 x = 1$.

$$= \boxed{\sin^2 x - \cos^2 x}$$

Technology Tip

Graphs of $y_1 = \dfrac{\tan x - \cot x}{\tan x + \cot x}$ and $y_2 = \sin^2 x - \cos^2 x$.





EXAMPLE 5 Determining Whether a Trigonometric Equation Is an Identity

Determine whether $\left(1 - \cos^2 x\right)\left(1 + \cot^2 x\right) = 0$, is an identity, a conditional equation, or a contradiction.

Solution:

Use the quotient identity to write the cotangent function in terms of the sine and cosine functions.

$$(1 - \cos^2 x)(1 + \cot^2 x)$$

$$= \left(1 - \cos^2 x\right)\left(1 + \frac{\cos^2 x}{\sin^2 x}\right)$$

Combine the expression in the second parentheses so that it is a single quotient.

$$= \left(1 - \cos^2 x\right)\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right)$$

Use the Pythagorean identity.

$$= \underbrace{\left(1 - \cos^2 x\right)}_{\sin^2 x}\left(\frac{\overbrace{\sin^2 x + \cos^2 x}^{1}}{\sin^2 x}\right)$$

Eliminate the parentheses.

$$= \frac{\sin^2 x}{\sin^2 x}$$

Simplify.

$$= 1$$

Since $1 \neq 0$, this is not an identity, but rather a $\boxed{\text{contradiction}}$.

EXAMPLE 6 Determine Whether a Trigonometric Equation Is an Identity

Determine whether $\sin^4 x - \cos^4 x = 0$ is an identity, a conditional equation, or a contradiction.

Solution:

Factor the left side of the equation. $\sin^4 x - \cos^4 x$

$$= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x) = 0$$

Apply the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$. $= (\sin^2 x - \cos^2 x) = 0$

Factor the left side of the equation again. $= (\sin x - \cos x)(\sin x + \cos x) = 0$

Solve. $\sin x = \cos x \quad \text{or} \quad \sin x = -\cos x$

Notice that when $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ the equation is satisfied, but for $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ the equation is not satisfied. Since this equation is true for some (but not all values) of x, it is a $\boxed{\text{conditional equation}}$.

EXAMPLE 7 Verifying Trigonometric Identities

Verify that $\dfrac{\sin(-x)}{\cos(-x)\tan(-x)} = 1$.

Solution:

Start with the left side of the equation. $\dfrac{\sin(-x)}{\cos(-x)\tan(-x)}$

Use the even-odd identities. $= \dfrac{-\sin x}{-\cos x \tan x}$

Simplify. $= \dfrac{\sin x}{\cos x \tan x}$

Use the quotient identity to write the tangent function in terms of the sine and cosine functions. $= \dfrac{\sin x}{\cos x\left(\dfrac{\sin x}{\cos x}\right)}$

Divide out the cosine term in the denominator. $= \dfrac{\sin x}{\sin x}$

Simplify. $= 1$

We have verified that $\dfrac{\sin(-x)}{\cos(-x)\tan(-x)} = 1$.

Study Tip

Start with the more complicated expression (side) and manipulate until reaching the simpler expression (on the other side).

So far we have discussed working with only one side of the identity until arriving at the other side. Another method for verifying identities is to work with (simplify) each side separately and use identities and algebraic techniques to arrive at the same result on both sides.

EXAMPLE 8 Verifying an Identity by Simplifying Both Sides Separately

Verify that $\dfrac{\sin x + 1}{\sin x} = -\dfrac{\cot^2 x}{1 - \csc x}$.

Solution:

Left-hand side: $\dfrac{\sin x + 1}{\sin x} = \dfrac{\sin x}{\sin x} + \dfrac{1}{\sin x} = 1 + \csc x$

Right-hand side: $\dfrac{-\cot^2 x}{1 - \csc x} = \dfrac{1 - \csc^2 x}{1 - \csc x} = \dfrac{(1 - \csc x)(1 + \csc x)}{(1 - \csc x)} = 1 + \csc x$

Since the left-hand side equals the right-hand side, the equation is an identity.

SECTION 6.1 SUMMARY

We combined the fundamental trigonometric identities—reciprocal, quotient, Pythagorean, even-odd, and cofunction—with algebraic techniques to simplify trigonometric expressions and verify more complex trigonometric identities. Two steps that we often use in both simplifying trigonometric expressions and verifying trigonometric identities are: (1) writing all trigonometric functions in terms of the sine and cosine functions, and (2) combining sums or differences of quotients into a single quotient.

When verifying trigonometric identities, we typically work with the more complicated side (keeping the other side in mind as our goal). Another approach to verifying trigonometric identities is to work on each side separately and arrive at the same result.

SECTION 6.1 EXERCISES

SKILLS

In Exercises 1–6, use the cofunction identities to fill in the blanks.

1. $\sin 60° = \cos$________
2. $\sin 45° = \cos$________
3. $\cos x = \sin$________
4. $\cot A = \tan$________
5. $\csc 30° = \sec$________
6. $\sec B = \csc$________

In Exercises 7–14, write the trigonometric function values in terms of its cofunction.

7. $\sin(x + y)$
8. $\sin(60° - x)$
9. $\cos(20° + A)$
10. $\cos(A + B)$
11. $\cot(45° - x)$
12. $\sec(30° - \theta)$
13. $\csc(60° - \theta)$
14. $\tan(40° + \theta)$

In Exercises 15–38, simplify each of the trigonometric expressions.

15. $\sin x \csc x$
16. $\tan x \cot x$
17. $\sec(-x)\cot x$
18. $\tan(-x)\cos(-x)$
19. $\csc(-x)\sin x$
20. $\cot(-x)\tan x$
21. $\sec x \cos(-x) + \tan^2 x$
22. $\sec(-x)\tan(-x)\cos(-x)$
23. $(\sin^2 x)(\cot^2 x + 1)$
24. $(\cos^2 x)(\tan^2 x + 1)$
25. $(\sin x - \cos x)(\sin x + \cos x)$
26. $(\sin x + \cos x)^2$

27. $\dfrac{\csc x}{\cot x}$

28. $\dfrac{\sec x}{\tan x}$

29. $\dfrac{1 - \cot(-x)}{1 + \cot x}$

30. $\sec^2 x - \tan^2(-x)$

31. $\dfrac{1 - \cos^4 x}{1 + \cos^2 x}$

32. $\dfrac{1 - \sin^4 x}{1 + \sin^2 x}$

33. $\dfrac{1 - \cot^4 x}{1 - \cot^2 x}$

34. $\dfrac{1 - \tan^4(-x)}{1 - \tan^2 x}$

35. $1 - \dfrac{\sin^2 x}{1 - \cos x}$

36. $1 - \dfrac{\cos^2 x}{1 + \sin x}$

37. $\dfrac{\tan x - \cot x}{\tan x + \cot x} + 2\cos^2 x$

38. $\dfrac{\tan x - \cot x}{\tan x + \cot x} + \cos^2 x$

In Exercises 39–64, verify each of the trigonometric identities.

39. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

40. $(1 - \sin x)(1 + \sin x) = \cos^2 x$

41. $(\csc x + 1)(\csc x - 1) = \cot^2 x$

42. $(\sec x + 1)(\sec x - 1) = \tan^2 x$

43. $\tan x + \cot x = \csc x \sec x$

44. $\csc x - \sin x = \cot x \cos x$

45. $\dfrac{2 - \sin^2 x}{\cos x} = \sec x + \cos x$

46. $\dfrac{2 - \cos^2 x}{\sin x} = \csc x + \sin x$

47. $[\cos(-x) - 1][1 + \cos x] = -\sin^2 x$

48. $\tan(-x)\cot x = -1$

49. $\dfrac{\sec(-x)\cot x}{\csc(-x)} = -1$

50. $\csc(-x) - 1 = \dfrac{\cot^2 x}{\csc(-x) + 1}$

51. $\dfrac{1}{\csc^2 x} + \dfrac{1}{\sec^2 x} = 1$

52. $\dfrac{1}{\cot^2 x} - \dfrac{1}{\tan^2 x} = \sec^2 x - \csc^2 x$

53. $\dfrac{1}{1 - \sin x} + \dfrac{1}{1 + \sin x} = 2\sec^2 x$

54. $\dfrac{1}{1 - \cos x} + \dfrac{1}{1 + \cos x} = 2\csc^2 x$

55. $\dfrac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

56. $\dfrac{\cos^2 x}{1 - \sin x} = 1 + \sin x$

57. $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}$

58. $\csc x + \cot x = \dfrac{1}{\csc x - \cot x}$

59. $\dfrac{\csc x - \tan x}{\sec x + \cot x} = \dfrac{\cos x - \sin^2 x}{\sin x + \cos^2 x}$

60. $\dfrac{\sec x + \tan x}{\csc x + 1} = \tan x$

61. $\dfrac{\cos^2 x + 1 + \sin x}{\cos^2 x + 3} = \dfrac{1 + \sin x}{2 + \sin x}$

62. $\dfrac{\sin x + 1 - \cos^2 x}{\cos^2 x} = \dfrac{\sin x}{1 - \sin x}$

63. $\sec x(\tan x + \cot x) = \dfrac{\csc x}{\cos^2 x}$

64. $\tan x(\csc x - \sin x) = \cos x$

In Exercises 65–78, determine whether each equation is a conditional equation or an identity.

65. $\cos^2 x(\tan x - \sec x)(\tan x + \sec x) = 1$

66. $\cos^2 x(\tan x - \sec x)(\tan x + \sec x) = \sin^2 x - 1$

67. $\dfrac{\csc x \cot x}{\sec x \tan x} = \cot^3 x$

68. $\sin x \cos x = 0$

69. $\sin x + \cos x = \sqrt{2}$

70. $\sin^2 x + \cos^2 x = 1$

71. $\tan^2 x - \sec^2 x = 1$

72. $\sec^2 x - \tan^2 x = 1$

73. $\sin x = \sqrt{1 - \cos^2 x}$

74. $\csc x = \sqrt{1 + \cot^2 x}$

75. $\sqrt{\sin^2 x + \cos^2 x} = 1$

76. $\sqrt{\sin^2 x + \cos^2 x} = \sin x + \cos x$

77. $(\sin x - \cos x)^2 = \sin^2 x - \cos^2 x$

78. $[\sin(-x) - 1][\sin(-x) + 1] = \cos^2 x$

APPLICATIONS

79. Area of a Circle. Show that the area of a circle with radius $r = \sec x$ is equal to $\pi + \pi(\tan x)^2$.

80. Area of a Triangle. Show that the area of a triangle with base $b = \cos x$ and height $h = \sec x$ is equal to $\frac{1}{2}$.

81. Pythagorean Theorem. Find the length of the hypotenuse of a right triangle whose legs have lengths 1 and $\tan\theta$.

82. Pythagorean Theorem. Find the length of the hypotenuse of a right triangle whose legs have lengths 1 and $\cot\theta$.

CATCH THE MISTAKE

In Exercises 83–86, explain the mistake that is made.

83. Verify the identity $\dfrac{\cos x}{1-\tan x}+\dfrac{\sin x}{1-\cot x}=\sin x+\cos x$.

Solution:

Start with the left side of the equation. $\dfrac{\cos x}{1-\tan x}+\dfrac{\sin x}{1-\cot x}$

Write the tangent and cotangent functions in terms of sines and cosines. $=\dfrac{\cos x}{1-\dfrac{\sin x}{\cos x}}+\dfrac{\sin x}{1-\dfrac{\cos x}{\sin x}}$

Cancel the common cosine in the first term and sine in the second term. $=\dfrac{1}{1-\sin x}+\dfrac{1}{1-\cos x}$

This is incorrect. What mistake was made?

84. Verify the identity $\dfrac{\cos^3 x\sec x}{1-\sin x}=1+\sin x$.

Solution:

Start with the equation on the left. $\dfrac{\cos^3 x\sec x}{1-\sin x}$

Rewrite secant in terms of sine. $=\dfrac{\cos^3 x\dfrac{1}{\sin x}}{1-\sin x}$

Simplify. $=\dfrac{\cos^3 x}{1-\sin^2 x}$

Use the Pythagorean identity. $=\dfrac{\cos^3 x}{\cos^2 x}$

Simplify. $=\cos x$

This is incorrect. What mistakes were made?

85. Determine whether the equation is a conditional equation or an identity: $\dfrac{\tan x}{\cot x}=1$.

Solution:

Start with the left side. $\dfrac{\tan x}{\cot x}$

Rewrite the tangent and cotangent functions in terms of sines and cosines. $=\dfrac{\dfrac{\sin x}{\cos x}}{\dfrac{\cos x}{\sin x}}$

Simplify. $=\dfrac{\sin^2 x}{\cos^2 x}=\tan^2 x$

Let $x=\dfrac{\pi}{4}$. *Note:* $\tan\left(\dfrac{\pi}{4}\right)=1$. $=1$

Since $\dfrac{\tan x}{\cot x}=1$, this equation is an identity.

This is incorrect. What mistake was made?

86. Determine whether the equation is a conditional equation or an identity: $|\sin x|-\cos x=1$.

Solution:

Start with the left side of the equation. $|\sin x|-\cos x$

Let $x=\dfrac{n\pi}{2}$, where n is an odd integer. $\left|\sin\left(\dfrac{n\pi}{2}\right)\right|-\cos\left(\dfrac{n\pi}{2}\right)$

Simplify. $|\pm 1|-0=1$

Since $|\sin x|-\cos x=1$, this is an identity.

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 87 and 88, determine whether each statement is true or false.

87. If an equation is true for some values (but not all values), then it is still an identity.

88. If an equation has an infinite number of solutions, then it is an identity.

89. In which quadrants is the equation $\cos\theta=\sqrt{1-\sin^2\theta}$ true?

90. In which quadrants is the equation $-\cos\theta=\sqrt{1-\sin^2\theta}$ true?

91. In which quadrants is the equation $\csc\theta=-\sqrt{1+\cot^2\theta}$ true?

92. In which quadrants is the equation $\sec\theta=\sqrt{1+\tan^2\theta}$ true?

93. Do you think that $\sin(A+B)=\sin A+\sin B$? Why?

94. Do you think that $\cos\left(\frac{1}{2}A\right)=\frac{1}{2}\cos A$? Why?

95. Do you think $\tan(2A)=2\tan A$? Why?

96. Do you think $\cot(A^2)=(\cot A)^2$? Why?

CHALLENGE

97. Simplify $(a\sin x+b\cos x)^2+(b\sin x-a\cos x)^2$.

98. Simplify $\dfrac{1+\cot^3 x}{1+\cot x}+\cot x$.

99. Show that $\csc\left(\dfrac{\pi}{2}+\theta+2n\pi\right)=\sec\theta$, n an integer.

100. Show that $\sec\left(\dfrac{\pi}{2}-\theta-2n\pi\right)=\csc\theta$, n an integer.

101. Simplify $\csc\left(2\pi-\dfrac{\pi}{2}-\theta\right)\cdot\sec\left(\theta-\dfrac{\pi}{2}\right)\cdot\sin(-\theta)$.

102. Simplify $\tan\theta\cdot\cot(2\pi-\theta)$.

TECHNOLOGY

In the next section, you will learn the sum and difference identities. In Exercises 103–106, we illustrate these identities with graphing calculators.

103. Determine the correct sign (+ or −) for $\cos(A + B) = \cos A \cos B \underbrace{\pm}_{?} \sin A \sin B$ by graphing $Y_1 = \cos(A + B)$, $Y_2 = \cos A \cos B + \sin A \sin B$, and $Y_2 = \cos A \cos B - \sin A \sin B$ in the same viewing rectangle for several values of A and B.

104. Determine the correct sign (+ or −) for $\cos(A - B) = \cos A \cos B \underbrace{\pm}_{?} \sin A \sin B$ by graphing $Y_1 = \cos(A - B)$, $Y_2 = \cos A \cos B + \sin A \sin B$, and $Y_2 = \cos A \cos B - \sin A \sin B$ in the same viewing rectangle for several values of A and B.

105. Determine the correct sign (+ or −) for $\sin(A + B) = \sin A \cos B \underbrace{\pm}_{?} \cos A \sin B$ by graphing $Y_1 = \sin(A + B)$, $Y_2 = \sin A \cos B + \cos A \sin B$, and $Y_2 = \sin A \cos B - \cos A \sin B$ in the same viewing rectangle for several values of A and B.

106. Determine the correct sign (+ or −) for $\sin(A - B) = \sin A \cos B \underbrace{\pm}_{?} \cos A \sin B$ by graphing $Y_1 = \sin(A - B)$, $Y_2 = \sin A \cos B + \cos A \sin B$, and $Y_2 = \sin A \cos B - \cos A \sin B$ in the same viewing rectangle for several values of A and B.

PREVIEW TO CALCULUS

For Exercises 107–110, refer to the following:

In calculus, when integrating expressions such as $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, trigonometric functions are used as "dummy" functions to eliminate the radical. Once the integration is performed, the trigonometric function is "unsubstituted." These trigonometric substitutions (and corresponding trigonometric identities) are used to simplify these types of expressions.

When simplifying, it is important to remember that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Expressions	Substitution		Trigonometric Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

107. Start with the expression $\sqrt{a^2 - x^2}$ and let $x = a\sin\theta$, assuming $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Simplify the original expression so that it contains no radicals.

108. Start with the expression $\sqrt{a^2 + x^2}$ and let $x = a\tan\theta$, assuming $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Simplify the original expression so that it contains no radicals.

109. Start with the expression $\sqrt{x^2 - a^2}$ and let $x = a\sec\theta$, assuming $0 \leq \theta < \frac{\pi}{2}$. Simplify the original expression so that it contains no radicals.

110. Use a trigonometric substitution to simplify the expression $\sqrt{9 - x^2}$ so that it contains no radicals.

SECTION 6.2 SUM AND DIFFERENCE IDENTITIES

SKILLS OBJECTIVES

- Find exact values of trigonometric functions of certain rational multiples of π by using the sum and difference identities.
- Develop new identities from the sum and difference identities.

CONCEPTUAL OBJECTIVE

- Understand that a trigonometric function of a sum is not the sum of the trigonometric functions.

In this section, we will consider trigonometric functions with arguments that are sums and differences. In general, $f(A + B) \neq f(A) + f(B)$. First, it is important to note that function notation is not distributive:

$$\cos(A + B) \neq \cos A + \cos B$$

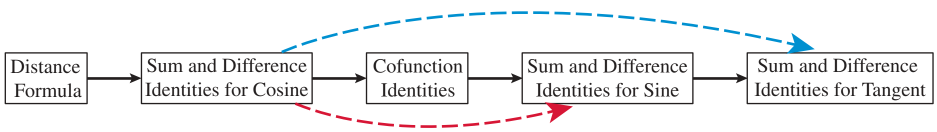
This principle is easy to prove. Let $A = \pi$ and $B = 0$; then

$$\cos(A + B) = \cos(\pi + 0) = \cos(\pi) = -1$$
$$\cos A + \cos B = \cos \pi + \cos 0 = -1 + 1 = 0$$

In this section, we will derive some new and important identities.

- Sum and difference identities for the cosine, sine, and tangent functions
- Cofunction identities

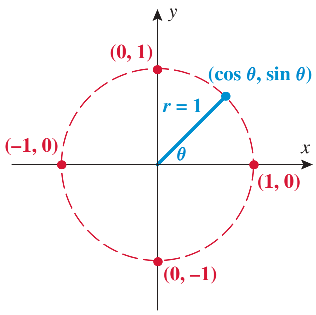
We begin with the familiar distance formula, from which we can derive the sum and difference identities for the cosine function. From there we can derive the sum and difference formulas for the sine and tangent functions.



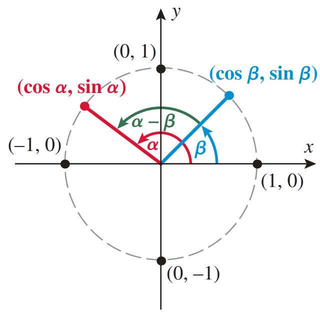
Before we start deriving and working with trigonometric sum and difference identities, let us first discuss why these are important. Sum and difference (and later product-to-sum and sum-to-product) identities are important because they allow calculation in functional (analytic) form and often lead to evaluating expressions *exactly* (as opposed to approximating them with calculators). The identities developed in this chapter are useful in such applications as musical sound, where they allow the determination of the "beat" frequency. In calculus, these identities will simplify the integration and differentiation processes.

Sum and Difference Identities for the Cosine Function

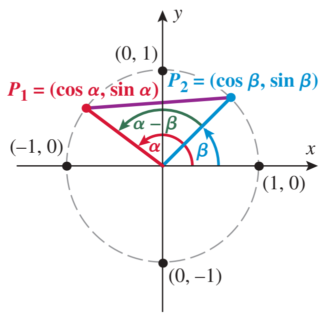
Recall from Section 5.1 that the unit circle approach gave the relationship between the coordinates along the unit circle and the sine and cosine functions. Specifically, the x-coordinate corresponded to the value of the cosine function and the y-coordinate corresponded to the value of the sine function.



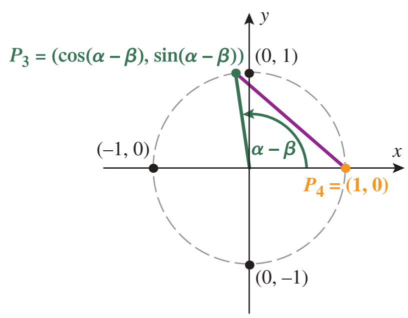
Let us now draw the unit circle with two angles α and β, realizing that the two terminal sides of these angles form a third angle, $\alpha - \beta$.



If we label the points $P_1 = (\cos\alpha, \sin\alpha)$ and $P_2 = (\cos\beta, \sin\beta)$, we can then draw a **segment** connecting points P_1 and P_2.



If we rotate the angle clockwise so the central angle $\alpha - \beta$ is in standard position, then the two points where the initial and terminal sides intersect the unit circle are $P_4 = (1, 0)$ and $P_3 = (\cos(\alpha - \beta), \sin(\alpha - \beta))$, respectively.



The distance from P_1 to P_2 is equal to the length of the **segment** joining the points. Similarly, the distance from P_3 to P_4 is equal to the length of the **segment** joining the points. Since the lengths of the **segments** are equal, we say that the distances are equal: $d(P_1, P_2) = d(P_3, P_4)$.

Study Tip

The distance from point $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$ is given by the distance formula $d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

WORDS	MATH
Set the distances (segment lengths) equal.	$d(P_1, P_2) = d(P_3, P_4)$
Apply the distance formula	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2}$

Substitute $P_1 = (x_1, y_1) = (\cos\alpha, \sin\alpha)$ and $P_2 = (x_2, y_2) = (\cos\beta, \sin\beta)$ into the left side of the equation and $P_3 = (x_3, y_3) = (\cos(\alpha - \beta), \sin(\alpha - \beta))$ and $P_4 = (x_4, y_4) = (1, 0)$ into the right side of the equation.

$$\sqrt{[\cos\beta - \cos\alpha]^2 + [\sin\beta - \sin\alpha]^2} = \sqrt{[1 - \cos(\alpha - \beta)]^2 + [0 - \sin(\alpha - \beta)]^2}$$

Square both sides of the equation.	$[\cos\beta - \cos\alpha]^2 + [\sin\beta - \sin\alpha]^2 = [1 - \cos(\alpha - \beta)]^2 + [0 - \sin(\alpha - \beta)]^2$
Eliminate the brackets.	$\cos^2\beta - 2\cos\beta\cos\alpha + \cos^2\alpha + \sin^2\beta - 2\sin\beta\sin\alpha + \sin^2\alpha$ $= 1 - 2\cos(\alpha - \beta) + \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)$
Regroup terms on each side and use the Pythagorean identity.	$\underbrace{\cos^2\alpha + \sin^2\alpha}_{1} - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta + \underbrace{\cos^2\beta + \sin^2\beta}_{1}$ $= 1 - 2\cos(\alpha - \beta) + \underbrace{\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)}_{1}$
Simplify.	$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$
Subtract 2 from both sides.	$-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = -2\cos(\alpha - \beta)$
Divide by -2.	$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$
Write the **difference identity for the cosine function**.	$\boxed{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$

We can now derive the sum identity for the cosine function from the difference identity for the cosine function and the properties of even and odd functions.

WORDS	MATH
Apply the difference identity.	$\cos(\alpha + \beta) = \cos[\alpha - (-\beta)]$ $\cos(\alpha + \beta) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$
Simplify the left side and use properties of even and odd functions on the right side.	$\cos(\alpha + \beta) = \cos\alpha(\cos\beta) + \sin\alpha(-\sin\beta)$
Write the **sum identity for the cosine function**.	$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$

SUM AND DIFFERENCE IDENTITIES FOR THE COSINE FUNCTION

Sum	$\cos(A + B) = \cos A\cos B - \sin A\sin B$
Difference	$\cos(A - B) = \cos A\cos B + \sin A\sin B$

Technology Tip

a. Use a calculator to check the values for $\cos\left(\frac{7\pi}{12}\right)$ and $\frac{\sqrt{2}-\sqrt{6}}{4}$.

```
cos(7π/12)
        -.2588190451
(√(2)-√(6))/4
        -.2588190451
```

b. Use a calculator to check the values of $\cos 15°$ and $\frac{\sqrt{2}+\sqrt{6}}{4}$. Be sure the calculator is set in degree mode.

```
Normal Sci Eng
Float 0123456789
Radian Degree
Func Par Pol Seq
Connected Dot
Sequential Simul
Real a+bi re^θi
Full Horiz G-T
```

```
cos(15)
        .9659258263
(√(2)+√(6))/4
        .9659258263
```

■ **Answer:** **a.** $\frac{\sqrt{6}-\sqrt{2}}{4}$

b. $\frac{\sqrt{6}-\sqrt{2}}{4}$

EXAMPLE 1 Finding Exact Values for the Cosine Function

Evaluate each of the following cosine expressions exactly:

a. $\cos\left(\frac{7\pi}{12}\right)$

b. $\cos 15°$

Solution (a):

Write $\frac{7\pi}{12}$ as a sum of known "special" angles. $\quad \cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{4\pi}{12} + \frac{3\pi}{12}\right)$

Simplify. $\quad \cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$

Write the sum identity for the cosine function. $\quad \cos(A + B) = \cos A \cos B - \sin A \sin B$

Substitute $A = \frac{\pi}{3}$ and $B = \frac{\pi}{4}$. $\quad \cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

Evaluate the expressions on the right exactly. $\quad \cos\left(\frac{7\pi}{12}\right) = \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}$

Simplify. $\quad \boxed{\cos\left(\frac{7\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}}$

Solution (b):

Write 15° as a difference of known "special" angles. $\quad \cos 15° = \cos(45° - 30°)$

Write the difference identity for the cosine function. $\quad \cos(A - B) = \cos A \cos B + \sin A \sin B$

Substitute $A = 45°$ and $B = 30°$. $\quad \cos 15° = \cos 45° \cos 30° + \sin 45° \sin 30°$

Evaluate the expressions on the right exactly. $\quad \cos 15° = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}$

Simplify. $\quad \boxed{\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}}$

■ **YOUR TURN** Use the sum or difference identities for the cosine function to evaluate each cosine expression exactly.

a. $\cos\left(\frac{5\pi}{12}\right)$ **b.** $\cos 75°$

Example 1 illustrates an important characteristic of the sum and difference identities: that we can now find the exact trigonometric function value of angles that are multiples of $15°\left(\text{or, equivalently, } \frac{\pi}{12}\right)$, since each of these can be written as a sum or difference of angles for which we know the trigonometric function values exactly.

EXAMPLE 2 **Writing a Sum or Difference as a Single Cosine Expression**

Use the sum or the difference identity for the cosine function to write each of the following expressions as a single cosine expression:

a. $\sin(5x)\sin(2x) + \cos(5x)\cos(2x)$

b. $\cos x \cos(3x) - \sin x \sin(3x)$

Solution (a):

Because of the positive sign, this will be a cosine of a difference.

Reverse the expression and write the formula. $\cos A \cos B + \sin A \sin B = \cos(A - B)$

Identify A and B. $A = 5x$ and $B = 2x$

Substitute $A = 5x$ and $B = 2x$ into the difference identity. $\cos(5x)\cos(2x) + \sin(5x)\sin(2x) = \cos(5x - 2x)$

Simplify. $\cos(5x)\cos(2x) + \sin(5x)\sin(2x) = \boxed{\cos(3x)}$

Notice that if we had selected $A = 2x$ and $B = 5x$ instead, the result would have been $\cos(-3x)$, but since the cosine function is an even function, this would have simplified to $\cos(3x)$.

Solution (b):

Because of the negative sign, this will be a cosine of a sum.

Reverse the expression and write the formula. $\cos A \cos B - \sin A \sin B = \cos(A + B)$

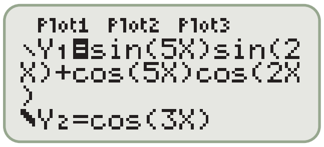
Identify A and B. $A = x$ and $B = 3x$

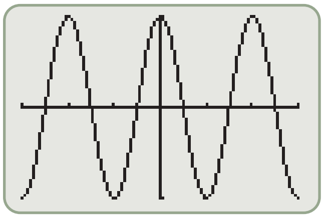
Substitute $A = x$ and $B = 3x$ into the sum identity. $\cos x \cos(3x) - \sin x \sin(3x) = \cos(x + 3x)$

Simplify. $\cos x \cos(3x) - \sin x \sin(3x) = \boxed{\cos(4x)}$

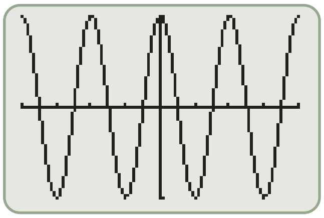
Technology Tip

a.





b.



YOUR TURN Write as a single cosine expression.

$$\cos(4x)\cos(7x) + \sin(4x)\sin(7x)$$

Answer: $\cos(3x)$

Sum and Difference Identities for the Sine Function

We can now use the cofunction identities (Section 6.1) together with the sum and difference identities for the cosine function to develop the sum and difference identities for the sine function.

WORDS	**MATH**
Start with the cofunction identity.	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
Let $\theta = A + B$.	$\sin(A + B) = \cos\left[\frac{\pi}{2} - (A + B)\right]$
Regroup the terms in the cosine expression.	$\sin(A + B) = \cos\left[\left(\frac{\pi}{2} - A\right) - B\right]$
Use the difference identity for the cosine function.	$\sin(A + B) = \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B$
Use the cofunction identities.	$\sin(A + B) = \underbrace{\cos\left(\frac{\pi}{2} - A\right)}_{\sin A}\cos B + \underbrace{\sin\left(\frac{\pi}{2} - A\right)}_{\cos A}\sin B$
Simplify.	$\boxed{\sin(A + B) = \sin A\cos B + \cos A\sin B}$

Now we can derive the difference identity for the sine function using the sum identity for the sine function and the properties of even and odd functions.

WORDS	**MATH**
Replace B with $-B$ in the sum identity.	$\sin(A + (-B)) = \sin A\cos(-B) + \cos A\sin(-B)$
Simplify using even and odd identities.	$\boxed{\sin(A - B) = \sin A\cos B - \cos A\sin B}$

SUM AND DIFFERENCE IDENTITIES FOR THE SINE FUNCTION

Sum	$\sin(A + B) = \sin A\cos B + \cos A\sin B$
Difference	$\sin(A - B) = \sin A\cos B - \cos A\sin B$

EXAMPLE 3 Finding Exact Values for the Sine Function

Use the sum or the difference identity for the sine function to evaluate each sine expression exactly.

a. $\sin\left(\frac{5\pi}{12}\right)$ **b.** $\sin 75°$

Solution (a):

Write $\frac{5\pi}{12}$ as a sum of known "special" angles.	$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right)$
Simplify.	$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$
Write the sum identity for the sine function.	$\sin(A + B) = \sin A\cos B + \cos A\sin B$
Substitute $A = \frac{\pi}{6}$ and $B = \frac{\pi}{4}$.	$\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)$

Evaluate the expressions on the right exactly. $\quad \sin\left(\dfrac{5\pi}{12}\right) = \left(\dfrac{1}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right) + \left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{2}}{2}\right)$

Simplify. $\quad \boxed{\sin\left(\dfrac{5\pi}{12}\right) = \dfrac{\sqrt{2}+\sqrt{6}}{4}}$

Solution (b):

Write 75° as a sum of known "special" angles. $\quad \sin 75° = \sin(45° + 30°)$

Write the sum identity for the sine function. $\quad \sin(A + B) = \sin A \cos B + \cos A \sin B$

Substitute $A = 45°$ and $B = 30°$. $\quad \sin 75° = \sin 45° \cos 30° + \cos 45° \sin 30°$

Evaluate the expressions on the right exactly. $\quad \sin 75° = \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{\sqrt{2}}{2}\right)\left(\dfrac{1}{2}\right)$

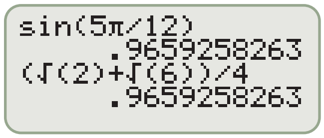
Simplify. $\quad \boxed{\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}}$

■ **YOUR TURN** Use the sum or the difference identity for the sine function to evaluate each sine expression exactly.

a. $\sin\left(\dfrac{7\pi}{12}\right)$ **b.** $\sin 15°$

Technology Tip

Use a calculator to check the values of $\sin\left(\dfrac{5\pi}{12}\right)$ and $\dfrac{\sqrt{2}+\sqrt{6}}{4}$. Be sure the calculator is in radian mode.



■ **Answer: a.** $\dfrac{\sqrt{6}+\sqrt{2}}{4}$ **b.** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

We see in Example 3 that the sum and difference identities allow us to calculate exact values for trigonometric functions of angles that are multiples of 15° $\left(\text{or, equivalently, } \dfrac{\pi}{12}\right)$, as we saw with the cosine function.

EXAMPLE 4 Writing a Sum or Difference as a Single Sine Expression

Graph $y = 3\sin x\cos(3x) + 3\cos x\sin(3x)$.

Solution:

Use the sum identity for the sine function to write the expression as a single sine expression.

Factor out the common 3. $\quad y = 3[\sin x\cos(3x) + \cos x\sin(3x)]$

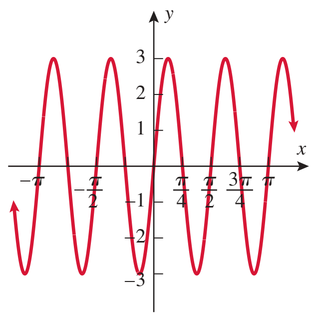
Write the sum identity for the sine function. $\quad \sin A\cos B + \cos A\sin B = \sin(A + B)$

Identify A and B. $\quad A = x \quad \text{and} \quad B = 3x$

Substitute $A = x$ and $B = 3x$ into the sum identity. $\quad \sin x\cos(3x) + \cos x\sin(3x) = \sin(x + 3x) = \sin(4x)$

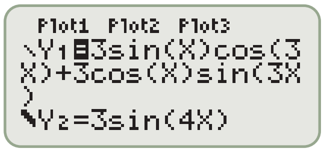
Simplify. $\quad y = 3[\underbrace{\sin x\cos(3x) + \cos x\sin(3x)}_{\sin(4x)}]$

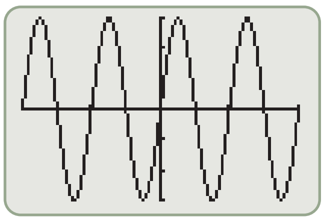
Graph $y = 3\sin(4x)$.



Technology Tip

Graphs of $y_1 = 3\sin x\cos(3x) + 3\cos x\sin(3x)$ and $y_2 = 3\sin(4x)$.





Sum and Difference Identities for the Tangent Function

We now develop the sum and difference identities for the tangent function.

Words	Math
Start with the quotient identity.	$\tan x = \dfrac{\sin x}{\cos x}$
Let $x = A + B$.	$\tan(A + B) = \dfrac{\sin(A + B)}{\cos(A + B)}$
Use the sum identities for the sine and cosine functions.	$\tan(A + B) = \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$
Multiply the numerator and denominator by $\dfrac{1}{\cos A \cos B}$.	$\tan(A + B) = \dfrac{\dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}} = \dfrac{\dfrac{\sin A \cancel{\cos B}}{\cos A \cancel{\cos B}} + \dfrac{\cancel{\cos A} \sin B}{\cancel{\cos A} \cos B}}{\dfrac{\cancel{\cos A}\cancel{\cos B}}{\cancel{\cos A}\cancel{\cos B}} - \dfrac{\sin A \sin B}{\cos A \cos B}}$
Simplify.	$\tan(A + B) = \dfrac{\left(\dfrac{\sin A}{\cos A}\right) + \left(\dfrac{\sin B}{\cos B}\right)}{1 - \left(\dfrac{\sin A}{\cos A}\right)\left(\dfrac{\sin B}{\cos B}\right)}$
Write the expressions inside the parentheses in terms of the tangent function.	$\boxed{\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}}$
Replace B with $-B$.	$\tan(A - B) = \dfrac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$
Since the tangent function is an odd function, $\tan(-B) = -\tan B$.	$\boxed{\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}}$

Sum and Difference Identities for the Tangent Function

Sum	$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
Difference	$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

EXAMPLE 5 Finding Exact Values for the Tangent Function

Find the exact value of $\tan(\alpha + \beta)$ if $\sin\alpha = -\frac{1}{3}$ and $\cos\beta = -\frac{1}{4}$ and the terminal side of α lies in quadrant III and the terminal side of β lies in quadrant II.

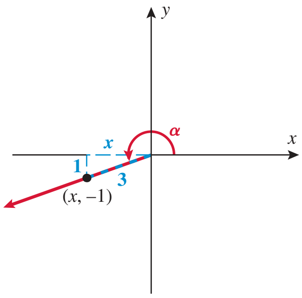
Solution:

STEP 1 Write the sum identity for the tangent function.
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

STEP 2 Find $\tan\alpha$.

The terminal side of α lies in quadrant III.

$$\sin\alpha = \frac{y}{r} = -\frac{1}{3}.$$



Solve for x. (Recall $x^2 + y^2 = r^2$.)
$$x^2 + (-1)^2 = 3^2$$
$$x = \pm\sqrt{8}$$

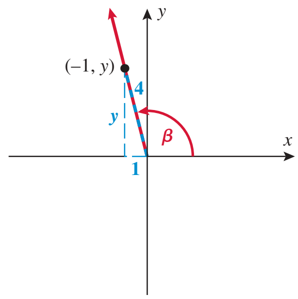
Take the negative sign since x is negative in quadrant III.
$$x = -2\sqrt{2}$$

Find $\tan\alpha$.
$$\tan\alpha = \frac{y}{x} = \frac{-1}{-2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

STEP 3 Find $\tan\beta$.

The terminal side of β lies in quadrant II.

$$\cos\beta = -\frac{1}{4} = \frac{x}{r}.$$



Solve for y. (Recall $x^2 + y^2 = r^2$.)
$$(-1)^2 + y^2 = 4^2$$
$$y = \pm\sqrt{15}$$

Take the positive sign since y is positive in quadrant II.
$$y = \sqrt{15}$$

Find $\tan\beta$.
$$\tan\beta = \frac{y}{x} = \frac{\sqrt{15}}{-1} = -\sqrt{15}$$

STEP 4 Substitute $\tan\alpha = \frac{\sqrt{2}}{4}$ and $\tan\beta = -\sqrt{15}$ into the sum identity for the tangent function.
$$\tan(\alpha + \beta) = \frac{\frac{\sqrt{2}}{4} - \sqrt{15}}{1 - \left(\frac{\sqrt{2}}{4}\right)\left(-\sqrt{15}\right)}$$

Multiply the numerator and the denominator by 4.
$$\tan(\alpha + \beta) = \frac{4}{4} \frac{\left(\frac{\sqrt{2}}{4} - \sqrt{15}\right)}{\left(1 + \frac{\sqrt{30}}{4}\right)}$$
$$= \frac{\sqrt{2} - 4\sqrt{15}}{4 + \sqrt{30}}$$

The expression $\boxed{\tan(\alpha + \beta) = \frac{\sqrt{2} - 4\sqrt{15}}{4 + \sqrt{30}}}$ can be simplified further if we rationalize the denominator.

Technology Tip

If $\sin\alpha = -\frac{1}{3}$ and α is in QIII, then $\alpha = \pi + \sin^{-1}\left(\frac{1}{3}\right)$. If $\cos\beta = -\frac{1}{4}$ and β is in QII, then $\beta = \pi - \cos^{-1}\left(\frac{1}{4}\right)$. Now use the graphing calculator to find $\tan(\alpha + \beta)$ by entering

```
tan(π+sin⁻¹(1/3)+
π-cos⁻¹(1/4))
          -1.485426269
```

In Step 4, use the graphing calculator to evaluate both expressions

$$\frac{\frac{\sqrt{2}}{4} - \sqrt{15}}{1 - \left(\frac{\sqrt{2}}{4}\right)\left(-\sqrt{15}\right)} \text{ and } \frac{\sqrt{2} - 4\sqrt{15}}{4 + \sqrt{30}} \text{ by entering}$$

```
(√(2)/4-√(15))/(
1-√(2)/4*-√(15))
          -1.485426269
(√(2)-4√(15))/(4
+√(30))
          -1.485426269
█
```

It is important to note in Example 5 that right triangles have been superimposed in the Cartesian plane. The coordinate pair (x, y) can have positive or negative values, but the radius r is always positive. When right triangles are superimposed, with one vertex at the point (x, y) and another vertex at the origin, it is important to understand that triangles have positive side lengths.

SECTION 6.2 SUMMARY

In this section, we derived the sum and difference identities for the cosine function using the distance formula. The cofunction identities and sum and difference identities for the cosine function were used to derive the sum and difference identities for the sine function. We combined the sine and cosine sum and difference identities to determine the tangent sum and difference identities. The sum and difference identities enabled us to evaluate a trigonometric expression exactly for any multiple of $15° \left(\text{i.e., } \frac{\pi}{12}\right)$.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

SECTION 6.2 EXERCISES

SKILLS

In Exercises 1–16, find the exact value for each trigonometric expression.

1. $\sin\left(\frac{\pi}{12}\right)$ **2.** $\cos\left(\frac{\pi}{12}\right)$ **3.** $\cos\left(-\frac{5\pi}{12}\right)$ **4.** $\sin\left(-\frac{5\pi}{12}\right)$ **5.** $\tan\left(-\frac{\pi}{12}\right)$ **6.** $\tan\left(\frac{13\pi}{12}\right)$

7. $\sin 105°$ **8.** $\cos 195°$ **9.** $\tan(-105°)$ **10.** $\tan 165°$ **11.** $\cot\left(\frac{\pi}{12}\right)$ **12.** $\cot\left(-\frac{5\pi}{12}\right)$

13. $\sec\left(-\frac{11\pi}{12}\right)$ **14.** $\sec\left(-\frac{13\pi}{12}\right)$ **15.** $\csc(-255°)$ **16.** $\csc(-15°)$

In Exercises 17–28, write each expression as a single trigonometric function.

17. $\sin(2x)\sin(3x) + \cos(2x)\cos(3x)$

18. $\sin x \sin(2x) - \cos x \cos(2x)$

19. $\sin x \cos(2x) - \cos x \sin(2x)$

20. $\sin(2x)\cos(3x) + \cos(2x)\sin(3x)$

21. $\cos(\pi - x)\sin x + \sin(\pi - x)\cos x$

22. $\sin\left(\frac{\pi}{3}x\right)\cos\left(-\frac{\pi}{2}x\right) - \cos\left(\frac{\pi}{3}x\right)\sin\left(-\frac{\pi}{2}x\right)$

23. $(\sin A - \sin B)^2 + (\cos A - \cos B)^2 - 2$

24. $(\sin A + \sin B)^2 + (\cos A + \cos B)^2 - 2$

25. $2 - (\sin A + \cos B)^2 - (\cos A + \sin B)^2$

26. $2 - (\sin A - \cos B)^2 - (\cos A + \sin B)^2$

27. $\dfrac{\tan 49° - \tan 23°}{1 + \tan 49° \tan 23°}$

28. $\dfrac{\tan 49° + \tan 23°}{1 - \tan 49° \tan 23°}$

In Exercises 29–34, find the exact value of the indicated expression using the given information and identities.

29. Find the exact value of $\cos(\alpha + \beta)$ if $\cos\alpha = -\frac{1}{3}$ and $\cos\beta = -\frac{1}{4}$ and the terminal side of α lies in quadrant III and the terminal side of β lies in quadrant II.

30. Find the exact value of $\cos(\alpha - \beta)$ if $\cos\alpha = \frac{1}{3}$ and $\cos\beta = -\frac{1}{4}$ and the terminal side of α lies in quadrant IV and the terminal side of β lies in quadrant II.

31. Find the exact value of $\sin(\alpha - \beta)$ if $\sin\alpha = -\frac{3}{5}$ and $\sin\beta = \frac{1}{5}$ and the terminal side of α lies in quadrant III and the terminal side of β lies in quadrant I.

32. Find the exact value of $\sin(\alpha + \beta)$ if $\sin\alpha = -\frac{3}{5}$ and $\sin\beta = \frac{1}{5}$ and the terminal side of α lies in quadrant III and the terminal side of β lies in quadrant II.

33. Find the exact value of $\tan(\alpha + \beta)$ if $\sin\alpha = -\frac{3}{5}$ and $\cos\beta = -\frac{1}{4}$ and the terminal side of α lies in quadrant III and the terminal side of β lies in quadrant II.

34. Find the exact value of $\tan(\alpha - \beta)$ if $\sin\alpha = -\frac{3}{5}$ and $\cos\beta = -\frac{1}{4}$ and the terminal side of α lies in quadrant III and the terminal side of β lies in quadrant II.

In Exercises 35–52, determine whether each equation is a conditional equation or an identity.

35. $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$

36. $\cos(A + B) + \cos(A - B) = 2\cos A\cos B$

37. $\sin\left(x - \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$

38. $\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$

39. $\frac{\sqrt{2}}{2}(\sin x + \cos x) = \sin\left(x + \frac{\pi}{4}\right)$

40. $\sqrt{3}\cos x + \sin x = 2\cos\left(x + \frac{\pi}{3}\right)$

41. $\sin^2 x = \frac{1 - \cos(2x)}{2}$

42. $\cos^2 x = \frac{1 + \cos(2x)}{2}$

43. $\sin(2x) = 2\sin x\cos x$

44. $\cos(2x) = \cos^2 x - \sin^2 x$

45. $\sin(A + B) = \sin A + \sin B$

46. $\cos(A + B) = \cos A + \cos B$

47. $\tan(\pi + B) = \tan B$

48. $\tan(A - \pi) = \tan A$

49. $\cot(3\pi + x) = \frac{1}{\tan x}$

50. $\csc(2x) = 2\sec x\csc x$

51. $\frac{1 + \tan x}{1 - \tan x} = \tan\left(x - \frac{\pi}{4}\right)$

52. $\cot\left(x + \frac{\pi}{4}\right) = \frac{1 - \tan x}{1 + \tan x}$

In Exercises 53–62, graph each of the functions by first rewriting it as a sine, cosine, or tangent of a difference or sum.

53. $y = \cos\left(\frac{\pi}{3}\right)\sin x + \cos x\sin\left(\frac{\pi}{3}\right)$

54. $y = \cos\left(\frac{\pi}{3}\right)\sin x - \cos x\sin\left(\frac{\pi}{3}\right)$

55. $y = \sin x\sin\left(\frac{\pi}{4}\right) + \cos x\cos\left(\frac{\pi}{4}\right)$

56. $y = \sin x\sin\left(\frac{\pi}{4}\right) - \cos x\cos\left(\frac{\pi}{4}\right)$

57. $y = -\sin x\cos(3x) - \cos x\sin(3x)$

58. $y = \sin x\sin(3x) + \cos x\cos(3x)$

59. $y = \frac{1 + \tan x}{1 - \tan x}$

60. $y = \frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}$

61. $y = \frac{1 + \sqrt{3}\tan x}{\sqrt{3} - \tan x}$

62. $y = \frac{1 - \tan x}{1 + \tan x}$

APPLICATIONS

In Exercises 63 and 64, refer to the following:

Sum and difference identities can be used to simplify more complicated expressions. For instance, the sine and cosine function can be represented by infinite polynomials called power series.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

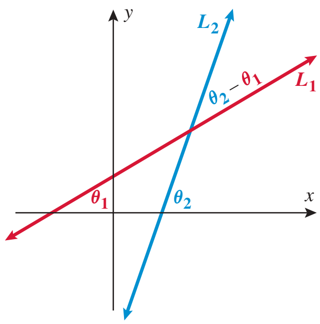
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

63. Power Series. Find the power series that represents $\cos\left(x - \frac{\pi}{4}\right)$.

64. Power Series. Find the power series that represents $\sin\left(x + \frac{3\pi}{2}\right)$.

For Exercises 65 and 66, use the following:

A nonvertical line makes an angle with the x-axis. In the figure, we see that the line L_1 makes an acute angle θ_1 with the x-axis. Similarly, the line L_2 makes an acute angle θ_2 with the x-axis.



$\tan\theta_1 = \text{slope of } L_1 = m_1$

$\tan\theta_2 = \text{slope of } L_2 = m_2$

65. Angle Between Two Lines. Show that

$$\tan(\theta_2 - \theta_1) = \frac{m_2 - m_1}{1 + m_1 m_2}$$

66. Relating Tangent and Slope. Show that

$$\tan(\theta_1 - \theta_2) = \frac{m_1 - m_2}{1 + m_1 m_2}$$

For Exercises 67 and 68, refer to the following:

An electric field E of a wave with constant amplitude A, propagating a distance z, is given by

$$E = A\cos(kz - ct)$$

where k is the propagation wave number, which is related to the wavelength λ by $k = \frac{2\pi}{\lambda}$, and where $c = 3.0 \times 10^8$ m/s is the speed of light in a vacuum, and t is time in seconds.

67. Electromagnetic Wave Propagation. Use the cosine difference identity to express the electric field in terms of both sine and cosine functions. When the quotient of the propagation distance z and the wavelength λ are equal to an integer, what do you notice?

68. Electromagnetic Wave Propagation. Use the cosine difference identity to express the electric field in terms of both sine and cosine functions. When $t = 0$, what do you notice?

69. Biology. By analyzing available empirical data, it has been determined that the body temperature of a species fluctuates according to the model

$$T(t) = 38 - 2.5\cos\left[\frac{\pi}{6}(t - 3)\right], \quad 0 \le t \le 24$$

where T represents temperature in degrees Celsius and t represents time (in hours) measured from 12:00 P.M. (noon). Use an identity to express $T(t)$ in terms of the sine function.

70. Health/Medicine. During the course of treatment of an illness, the concentration of a drug (in micrograms per milliliter) in the bloodstream fluctuates during the dosing period of 8 hours according to the model

$$C(t) = 15.4 - 4.7\sin\left(\frac{\pi}{4}t + \frac{\pi}{2}\right), \quad 0 \le t \le 8$$

Use an identity to express the concentration $C(t)$ in terms of the cosine function.

Note: This model does not apply to the first dose of the medication as there will be no medication in the bloodstream.

CATCH THE MISTAKE

In Exercises 71 and 72, explain the mistake that is made.

71. Find the exact value of $\tan\left(\frac{5\pi}{12}\right)$.

Solution:

Write $\frac{5\pi}{12}$ as a sum.	$\tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$
Distribute.	$\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{6}\right)$
Evaluate the tangent function for $\frac{\pi}{4}$ and $\frac{\pi}{6}$.	$1 + \frac{\sqrt{3}}{3}$

This is incorrect. What mistake was made?

72. Find the exact value of $\tan\left(-\frac{7\pi}{6}\right)$.

Solution:

The tangent function is an even function.	$\tan\left(\frac{7\pi}{6}\right)$
Write $\frac{7\pi}{6}$ as a sum.	$\tan\left(\pi + \frac{\pi}{6}\right)$
Use the tangent sum identity, $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.	$\frac{\tan\pi + \tan\left(\frac{\pi}{6}\right)}{1 - \tan\pi\tan\left(\frac{\pi}{6}\right)}$
Evaluate the tangent functions on the right.	$\frac{0 + \frac{1}{\sqrt{3}}}{1 - 0}$
Simplify.	$\frac{\sqrt{3}}{3}$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 73–76, determine whether each statement is true or false.

73. $\cos 15° = \cos 45° - \cos 30°$

74. $\sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{6}\right)$

75. $\tan\left(x + \frac{\pi}{4}\right) = 1 + \tan x$

76. $\cot\left(\frac{\pi}{4} - x\right) = \dfrac{1 + \tan x}{1 - \tan x}$

CHALLENGE

77. Verify that $\sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$.

78. Verify that $\cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$.

79. Although in general the statement $\sin(A - B) = \sin A - \sin B$ is not true, it is true for some values. Determine some values of A and B that make this statement true.

80. Although in general the statement $\sin(A + B) = \sin A + \sin B$ is not true, it is true for some values. Determine some values of A and B that make this statement true.

TECHNOLOGY

81. In Exercise 63, you showed that the difference quotient for $f(x) = \sin x$ is $\cos x\left(\frac{\sin h}{h}\right) - \sin x\left(\frac{1 - \cos h}{h}\right)$.

Plot $Y_1 = \cos x\left(\frac{\sin h}{h}\right) - \sin x\left(\frac{1 - \cos h}{h}\right)$ for

a. $h = 1$ **b.** $h = 0.1$ **c.** $h = 0.01$

What function does the difference quotient for $f(x) = \sin x$ resemble when h approaches zero?

82. Show that the difference quotient for $f(x) = \cos x$ is $-\sin x\left(\frac{\sin h}{h}\right) - \cos x\left(\frac{1 - \cos h}{h}\right)$.

Plot $Y_1 = -\sin x\left(\frac{\sin h}{h}\right) - \cos x\left(\frac{1 - \cos h}{h}\right)$ for

a. $h = 1$ **b.** $h = 0.1$ **c.** $h = 0.01$

What function does the difference quotient for $f(x) = \cos x$ resemble when h approaches zero?

83. Show that the difference quotient for $f(x) = \sin(2x)$ is $\cos(2x)\left[\frac{\sin(2h)}{h}\right] - \sin(2x)\left[\frac{1 - \cos(2h)}{h}\right]$.

Plot $Y_1 = \cos(2x)\left[\frac{\sin(2h)}{h}\right] - \sin(2x)\left[\frac{1 - \cos(2h)}{h}\right]$ for

a. $h = 1$ **b.** $h = 0.1$ **c.** $h = 0.01$

What function does the difference quotient for $f(x) = \sin(2x)$ resemble when h approaches zero?

84. Show that the difference quotient for $f(x) = \cos(2x)$ is $-\sin(2x)\left[\frac{\sin(2h)}{h}\right] - \cos(2x)\left[\frac{1 - \cos(2h)}{h}\right]$.

Plot $Y_1 = -\sin(2x)\left[\frac{\sin(2h)}{h}\right] - \cos(2x)\left[\frac{1 - \cos(2h)}{h}\right]$ for

a. $h = 1$ **b.** $h = 0.1$ **c.** $h = 0.01$

What function does the difference quotient for $f(x) = \cos(2x)$ resemble when h approaches zero?

PREVIEW TO CALCULUS

In calculus, one technique used to solve differential equations consists of the separation of variables. For example, consider the equation $x^2 + 3y\frac{f(y)}{g(x)} = 0$, which is equivalent to $3yf(y) = -x^2g(x)$. Here each side of the equation contains only one type of variable, either x or y.

In Exercises 85–88, use the sum and difference identities to separate the variables in each equation.

85. $\sin(x + y) = 0$

86. $\cos(x - y) = 0$

87. $\tan(x + y) = 2$

88. $\cos(x + y) = \sin y$

SECTION 6.3 DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

SKILLS OBJECTIVES

- Use the double-angle identities to find exact values of certain trigonometric functions.
- Use the double-angle identities to help in verifying identities.
- Use the half-angle identities to find exact values of certain trigonometric functions.
- Use half-angle identities to help in verifying identities.

CONCEPTUAL OBJECTIVES

- Understand that the double-angle identities are derived from the sum identities.
- Understand that the half-angle identities are derived from the double-angle identities.

Double-Angle Identities

In previous chapters, we could only evaluate trigonometric functions exactly for reference angles of 30°, 45°, and 60° or $\frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$; note that as of the previous section, we now can include multiples of $\frac{\pi}{12}$ among these "special" angles. Now, we can use *double-angle identities* to also evaluate the trigonometric function values for other angles that are even integer multiples of the special angles or to verify other trigonometric identities. One important distinction now is that we will be able to find exact values of many functions using the double-angle identities without needing to know the value of the angle.

Derivation of Double-Angle Identities

To derive the double-angle identities, we let $A = B$ in the sum identities.

WORDS	MATH
Write the identity for the sine of a sum.	$\sin(A + B) = \sin A \cos B + \cos A \sin B$
Let $B = A$.	$\sin(A + A) = \sin A \cos A + \cos A \sin A$
Simplify.	$\boxed{\sin(2A) = 2 \sin A \cos A}$
Write the identity for the cosine of a sum.	$\cos(A + B) = \cos A \cos B - \sin A \sin B$
Let $B = A$.	$\cos(A + A) = \cos A \cos A - \sin A \sin A$
Simplify.	$\boxed{\cos(2A) = \cos^2 A - \sin^2 A}$

We can write the double-angle identity for the cosine function two other ways if we use the Pythagorean identity:

1. Write the identity for the cosine function of a double angle. $\cos(2A) = \cos^2 A - \sin^2 A$

   Use the Pythagorean identity for the cosine function. $\cos(2A) = \underbrace{\cos^2 A}_{1-\sin^2 A} - \sin^2 A$

   Simplify. $\boxed{\cos(2A) = 1 - 2\sin^2 A}$

2. Write the identity for the cosine function of a double angle. $\cos(2A) = \cos^2 A - \sin^2 A$

Use the Pythagorean identity for the sine function. $\cos(2A) = \cos^2 A - \underbrace{\sin^2 A}_{1-\cos^2 A}$

Simplify. $\boxed{\cos(2A) = 2\cos^2 A - 1}$

The tangent function can always be written as a quotient, $\tan(2A) = \dfrac{\sin(2A)}{\cos(2A)}$, if $\sin(2A)$ and $\cos(2A)$ are known. Here we write the double-angle identity for the tangent function in terms of only the tangent function.

Write the tangent of a sum identity. $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Let $B = A$. $\tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A}$

Simplify. $\boxed{\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}}$

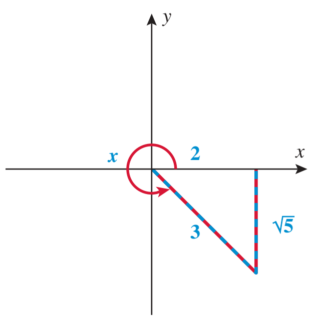
DOUBLE-ANGLE IDENTITIES

SINE	COSINE	TANGENT
$\sin(2A) = 2\sin A\cos A$	$\cos(2A) = \cos^2 A - \sin^2 A$	$\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}$
	$\cos(2A) = 1 - 2\sin^2 A$	
	$\cos(2A) = 2\cos^2 A - 1$	

Applying Double-Angle Identities

EXAMPLE 1 Finding Exact Values Using Double-Angle Identities

If $\cos x = \frac{2}{3}$, find $\sin(2x)$ given $\sin x < 0$.



Solution:

Find $\sin x$.

Use the Pythagorean identity. $\sin^2 x + \cos^2 x = 1$

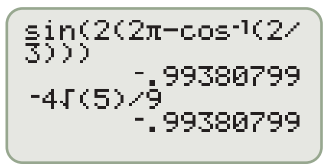
Substitute $\cos x = \frac{2}{3}$. $\sin^2 x + \left(\dfrac{2}{3}\right)^2 = 1$

Solve for $\sin x$, which is negative. $\sin x = -\sqrt{1 - \dfrac{4}{9}}$

Simplify. $\sin x = -\dfrac{\sqrt{5}}{3}$

Technology Tip

If $\cos x = \frac{2}{3}$ and $\sin x < 0$, then x is in QIV and one value for x is $x = 2\pi - \cos^{-1}\left(\frac{2}{3}\right)$. Now use the graphing calculator to find $\sin(2x)$, entering $\sin\left\{2\left[2\pi - \cos^{-1}\left(\frac{2}{3}\right)\right]\right\}$, and compare that value to $-\dfrac{4\sqrt{5}}{9}$.



Find $\sin(2x)$.

Use the double-angle formula for the sine function. $\quad \sin(2x) = 2\sin x\cos x$

Substitute $\sin x = -\frac{\sqrt{5}}{3}$ and $\cos x = \frac{2}{3}$. $\quad \sin(2x) = 2\left(-\frac{\sqrt{5}}{3}\right)\left(\frac{2}{3}\right)$

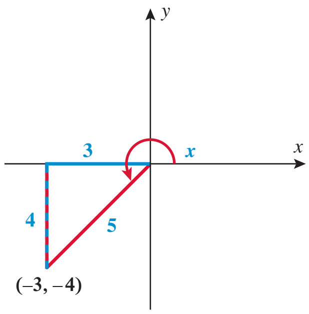
Simplify. $\quad \boxed{\sin(2x) = -\frac{4\sqrt{5}}{9}}$

■ **Answer:** $\sin(2x) = \frac{4\sqrt{2}}{9}$

■ **YOUR TURN** If $\cos x = -\frac{1}{3}$, find $\sin(2x)$ given $\sin x < 0$.

EXAMPLE 2 Finding Exact Values Using Double-Angle Identities

If $\sin x = -\frac{4}{5}$ and $\cos x < 0$, find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$.



Solution:

Solve for $\cos x$.

Use the Pythagorean identity. $\quad \sin^2 x + \cos^2 x = 1$

Substitute $\sin x = -\frac{4}{5}$. $\quad \left(-\frac{4}{5}\right)^2 + \cos^2 x = 1$

Simplify. $\quad \cos^2 x = \frac{9}{25}$

Solve for $\cos x$, which is negative. $\quad \cos x = -\sqrt{\frac{9}{25}} = -\frac{3}{5}$

Find $\sin(2x)$.

Use the double-angle identity for the sine function. $\quad \sin(2x) = 2\sin x\cos x$

Substitute $\sin x = -\frac{4}{5}$ and $\cos x = -\frac{3}{5}$. $\quad \sin(2x) = 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right)$

Simplify. $\quad \boxed{\sin(2x) = \frac{24}{25}}$

Find $\cos(2x)$.

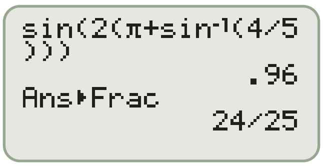
Use the double-angle identity for the cosine function. $\quad \cos(2x) = \cos^2 x - \sin^2 x$

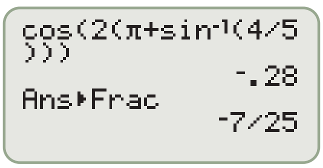
Substitute $\sin x = -\frac{4}{5}$ and $\cos x = -\frac{3}{5}$. $\quad \cos(2x) = \left(-\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2$

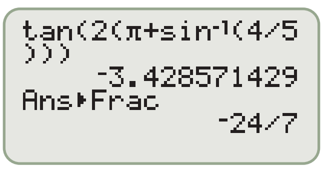
Simplify. $\quad \boxed{\cos(2x) = -\frac{7}{25}}$

Technology Tip

If $\sin x = -\frac{4}{5}$ and $\cos x < 0$, then x is in QIII and a value for x is $x = \pi + \sin^{-1}\left(\frac{4}{5}\right)$. Now use the graphing calculator to find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$.







Find $\tan(2x)$.

Use the quotient identity. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Let $\theta = 2x$. $\tan(2x) = \dfrac{\sin(2x)}{\cos(2x)}$

Substitute $\sin(2x) = \frac{24}{25}$ and $\cos(2x) = -\frac{7}{25}$. $\tan(2x) = \dfrac{\frac{24}{25}}{-\frac{7}{25}}$

Simplify. $\boxed{\tan(2x) = -\dfrac{24}{7}}$

Note: We could also have found $\tan(2x)$ first by finding $\tan x = \dfrac{\sin x}{\cos x}$ and then using the value for $\tan x$ in the double-angle identity, $\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}$.

YOUR TURN If $\cos x = \frac{3}{5}$ and $\sin x < 0$, find $\sin(2x)$, $\cos(2x)$, and $\tan(2x)$.

Answer: $\sin(2x) = -\frac{24}{25}$, $\cos(2x) = -\frac{7}{25}$, $\tan(2x) = \frac{24}{7}$

EXAMPLE 3 Verifying Trigonometric Identities Using Double-Angle Identities

Verify the identity $(\sin x - \cos x)^2 = 1 - \sin(2x)$.

Solution:

Start with the left side of the equation. $\boxed{(\sin x - \cos x)^2}$

Expand by squaring. $= \sin^2 x - 2\sin x\cos x + \cos^2 x$

Group the $\sin^2 x$ and $\cos^2 x$ terms. $= \sin^2 x + \cos^2 x - 2\sin x\cos x$

Apply the Pythagorean identity. $= \underbrace{\sin^2 x + \cos^2 x}_{1} - 2\sin x\cos x$

Apply the sine double-angle identity. $= 1 - \underbrace{2\sin x\cos x}_{\sin(2x)}$

Simplify. $= \boxed{1 - \sin(2x)}$

EXAMPLE 4 Verifying Multiple-Angle Identities

Verify the identity $\cos(3x) = (1 - 4\sin^2 x)\cos x$.

Solution:

Write the cosine of a sum identity. $\cos(A + B) = \cos A\cos B - \sin A\sin B$

Let $A = 2x$ and $B = x$. $\cos(2x + x) = \cos(2x)\cos x - \sin(2x)\sin x$

Apply the double-angle identities. $\cos(3x) = \underbrace{\cos(2x)}_{1-2\sin^2 x}\cos x - \underbrace{\sin(2x)}_{2\sin x\cos x}\sin x$

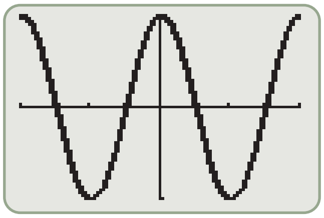
Simplify. $\cos(3x) = \cos x - 2\sin^2 x\cos x - 2\sin^2 x\cos x$

$\cos(3x) = \cos x - 4\sin^2 x\cos x$

Factor out the common cosine term. $\boxed{\cos(3x) = (1 - 4\sin^2 x)\cos x}$

Technology Tip

Graphs of $y = \dfrac{\cot x - \tan x}{\cot x + \tan x}$ and $y = \cos(2x)$.



EXAMPLE 5 Simplifying Trigonometric Expressions Using Double-Angle Identities

Graph $y = \dfrac{\cot x - \tan x}{\cot x + \tan x}$.

Solution:

Simplify $y = \dfrac{\cot x - \tan x}{\cot x + \tan x}$ first.

Write the cotangent and tangent functions in terms of the sine and cosine functions.

$$y = \frac{\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}}{\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}}$$

Multiply the numerator and the denominator by $\sin x \cos x$.

$$y = \left(\frac{\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}}{\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x}}\right)\left(\frac{\sin x \cos x}{\sin x \cos x}\right)$$

Simplify.

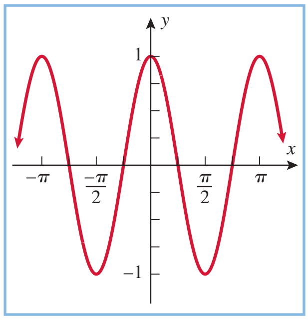
$$y = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x}$$

Use the double-angle and Pythagorean identities.

$$y = \frac{\overbrace{\cos^2 x - \sin^2 x}^{\cos(2x)}}{\underbrace{\cos^2 x + \sin^2 x}_{1}}$$

$$\boxed{y = \cos(2x)}$$

Graph $y = \cos(2x)$.



Half-Angle Identities

We now use the *double-angle identities* to develop the *half-angle identities*. Like the double-angle identities, the half-angle identities will allow us to find certain exact values of trigonometric functions and to verify other trigonometric identities. The *half-angle identities* come directly from the double-angle identities. We start by rewriting the second and third forms of the cosine double-angle identity to obtain identities for the square of the sine and cosine functions, $\sin^2$ and $\cos^2$.

WORDS	MATH
Write the second form of the cosine double-angle identity.	$\cos(2A) = 1 - 2\sin^2 A$
Find $\sin^2 A$.	
Isolate the $2\sin^2 A$ term on one side of the equation.	$2\sin^2 A = 1 - \cos(2A)$
Divide both sides by 2.	$\boxed{\sin^2 A = \dfrac{1 - \cos(2A)}{2}}$

Find $\cos^2 A$.

Write the third form of the cosine double-angle identity. $\qquad \cos(2A) = 2\cos^2 A - 1$

Isolate the $2\cos^2 A$ term on one side of the equation. $\qquad 2\cos^2 A = 1 + \cos^2 A$

Divide both sides by 2. $\qquad \boxed{\cos^2 A = \dfrac{1 + \cos(2A)}{2}}$

Find $\tan^2 A$.

Taking the quotient of these leads us to another identity.

$$\tan^2 A = \frac{\sin^2 A}{\cos^2 A} = \frac{\dfrac{1 - \cos(2A)}{2}}{\dfrac{1 + \cos(2A)}{2}}$$

Simplify. $\qquad \boxed{\tan^2 A = \dfrac{1 - \cos(2A)}{1 + \cos(2A)}}$

These three identities for the squared functions—really, alternative forms of the double-angle identities—are used in calculus as power reduction formulas (identities that allow us to reduce the power of the trigonometric function from 2 to 1):

$$\boxed{\sin^2 A = \frac{1 - \cos(2A)}{2}} \qquad \boxed{\cos^2 A = \frac{1 + \cos(2A)}{2}} \qquad \boxed{\tan^2 A = \frac{1 - \cos(2A)}{1 + \cos(2A)}}$$

We can now use these forms of the double-angle identities to derive the *half-angle identities*.

Words	Math
For the *sine half-angle identity*, start with the double-angle formula involving both the sine and cosine functions, $\cos(2x) = 1 - 2\sin^2 x$, and solve for $\sin^2 x$.	$\sin^2 x = \dfrac{1 - \cos(2x)}{2}$
Solve for $\sin x$.	$\sin x = \pm\sqrt{\dfrac{1 - \cos(2x)}{2}}$
Let $x = \dfrac{A}{2}$.	$\sin\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1 - \cos\left(2 \cdot \dfrac{A}{2}\right)}{2}}$
Simplify.	$\boxed{\sin\left(\dfrac{A}{2}\right) = \pm\sqrt{\dfrac{1 - \cos A}{2}}}$

For the *cosine half-angle identity*, start with the double-angle formula involving only the cosine function, $\cos(2x) = 2\cos^2 x - 1$, and solve for $\cos^2 x$.
$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Solve for $\cos x$.
$$\cos x = \pm\sqrt{\frac{1 + \cos(2x)}{2}}$$

Let $x = \frac{A}{2}$.
$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos\left(2 \cdot \frac{A}{2}\right)}{2}}$$

Simplify.
$$\boxed{\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos A}{2}}}$$

For the *tangent half-angle identity*, start with the quotient identity.
$$\tan\left(\frac{A}{2}\right) = \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

Substitute half-angle identities for the sine and cosine functions.
$$\tan\left(\frac{A}{2}\right) = \frac{\pm\sqrt{\frac{1 - \cos A}{2}}}{\pm\sqrt{\frac{1 + \cos A}{2}}}$$

Simplify.
$$\boxed{\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}}$$

Note: We can also find $\tan\left(\frac{A}{2}\right)$ by starting with the identity $\tan^2 x = \frac{1 - \cos(2x)}{1 + \cos(2x)}$, solving for $\tan x$, and letting $x = \frac{A}{2}$. The tangent function also has two other similar forms for $\tan\left(\frac{A}{2}\right)$ (see Exercises 131 and 132).

HALF-ANGLE IDENTITIES

SINE	COSINE	TANGENT
$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos A}{2}}$	$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos A}{2}}$	$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$
		$\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1 + \cos A}$
		$\tan\left(\frac{A}{2}\right) = \frac{1 - \cos A}{\sin A}$

Study Tip

The sign $+$ or $-$ is determined by what quadrant contains $\frac{A}{2}$ and what the sign of the particular trigonometric function is in that quadrant.

It is important to note that these identities hold for any real number A or any angle with either degree measure or radian measure A as long as both sides of the equation are defined. The sign ($+$ or $-$) is determined by the sign of the trigonometric function in the quadrant that contains $\frac{A}{2}$.

EXAMPLE 6 Finding Exact Values Using Half-Angle Identities

Use a half-angle identity to find $\cos 15°$.

Solution:

Write $\cos 15°$ in terms of a half angle.

$$\cos 15° = \cos\left(\frac{30°}{2}\right)$$

Write the half-angle identity for the cosine function.

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos A}{2}}$$

Substitute $A = 30°$.

$$\cos\left(\frac{30°}{2}\right) = \pm\sqrt{\frac{1 + \cos 30°}{2}}$$

Simplify.

$$\cos 15° = \pm\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

$15°$ is in quadrant I, where the cosine function is positive.

$$\cos 15° = \sqrt{\frac{2 + \sqrt{3}}{4}} = \boxed{\frac{\sqrt{2 + \sqrt{3}}}{2}}$$

■ **YOUR TURN** Use a half-angle identity to find $\sin 22.5°$.

Technology Tip

Use a TI calculator to check the values of $\cos 15°$ and $\sqrt{\frac{2 + \sqrt{3}}{4}}$. Be sure the calculator is in degree mode.

```
cos(15)
         .9659258263
√((2+√(3))/4)
         .9659258263
```

■ **Answer:** $\sin 22.5° = \frac{\sqrt{2 - \sqrt{2}}}{2}$

EXAMPLE 7 Finding Exact Values Using Half-Angle Identities

Use a half-angle identity to find $\tan\left(\frac{11\pi}{12}\right)$.

Solution:

Write $\tan\left(\frac{11\pi}{12}\right)$ in terms of a half angle.

$$\tan\left(\frac{11\pi}{12}\right) = \tan\left(\frac{\frac{11\pi}{6}}{2}\right)$$

Write the half-angle identity for the tangent function.*

$$\tan\left(\frac{A}{2}\right) = \frac{1 - \cos A}{\sin A}$$

Substitute $A = \frac{11\pi}{6}$.

$$\tan\left(\frac{\frac{11\pi}{6}}{2}\right) = \frac{1 - \cos\left(\frac{11\pi}{6}\right)}{\sin\left(\frac{11\pi}{6}\right)}$$

Simplify.

$$\tan\left(\frac{11\pi}{12}\right) = \frac{1 - \frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$

$$\tan\left(\frac{11\pi}{12}\right) = \boxed{\sqrt{3} - 2}$$

$\frac{11\pi}{12}$ is in quadrant II, where tangent is negative. Notice that if we approximate $\tan\left(\frac{11\pi}{12}\right)$ with a calculator, we find that $\tan\left(\frac{11\pi}{12}\right) \approx -0.2679$ and $\sqrt{3} - 2 \approx -0.2679$.

*This form of the tangent half-angle identity was selected because of mathematical simplicity. If we had selected either of the other forms, we would have obtained an expression that had a square root within a square root or a radical in the denominator (requiring rationalization).

■ **YOUR TURN** Use a half-angle identity to find $\tan\left(\frac{\pi}{8}\right)$.

Technology Tip

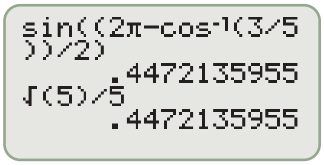
Use a TI calculator to check the values of $\tan\left(\frac{11\pi}{12}\right)$ and $\sqrt{3} - 2$. Be sure the calculator is in radian mode.

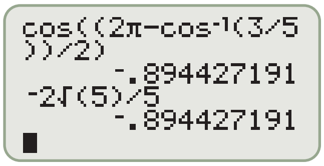
```
tan(11π/12)
        -.2679491924
√(3)-2
        -.2679491924
```

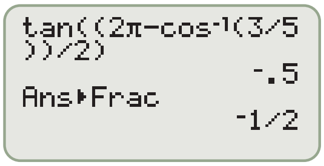
■ **Answer:** $\frac{\sqrt{2}}{2 + \sqrt{2}}$ or $\sqrt{2} - 1$

Technology Tip

If $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, then x is in QIV, $\frac{x}{2}$ is in QII, and $x = 2\pi - \cos^{-1}\left(\frac{3}{5}\right)$. Now use the graphing calculator to find $\sin\left(\frac{x}{2}\right)$, $\cos\left(\frac{x}{2}\right)$, and $\tan\left(\frac{x}{2}\right)$.







EXAMPLE 8 Finding Exact Values Using Half-Angle Identities

If $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin\left(\frac{x}{2}\right)$, $\cos\left(\frac{x}{2}\right)$, and $\tan\left(\frac{x}{2}\right)$.

Solution:

Determine in which quadrant $\frac{x}{2}$ lies.

Since $\frac{3\pi}{2} < x < 2\pi$, we divide by 2. $\qquad \frac{3\pi}{4} < \frac{x}{2} < \pi$

$\frac{x}{2}$ lies in quadrant II; therefore, the sine function is positive and both the cosine and tangent functions are negative.

Write the half-angle identity for the sine function. $\qquad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

Substitute $\cos x = \frac{3}{5}$. $\qquad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\frac{3}{5}}{2}}$

Simplify. $\qquad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1}{5}} = \pm\frac{\sqrt{5}}{5}$

Since $\frac{x}{2}$ lies in quadrant II, choose the positive value for the sine function. $\qquad \boxed{\sin\left(\frac{x}{2}\right) = \frac{\sqrt{5}}{5}}$

Write the half-angle identity for the cosine function. $\qquad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$

Substitute $\cos x = \frac{3}{5}$. $\qquad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\frac{3}{5}}{2}}$

Simplify. $\qquad \cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{4}{5}} = \pm\frac{2\sqrt{5}}{5}$

Since $\frac{x}{2}$ lies in quadrant II, choose the negative value for the cosine function. $\qquad \boxed{\cos\left(\frac{x}{2}\right) = -\frac{2\sqrt{5}}{5}}$

Use the quotient identity for tangent. $\qquad \tan\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$

Substitute $\sin\left(\frac{x}{2}\right) = \frac{\sqrt{5}}{5}$ and $\cos\left(\frac{x}{2}\right) = -\frac{2\sqrt{5}}{5}$. $\qquad \tan\left(\frac{x}{2}\right) = \frac{\frac{\sqrt{5}}{5}}{-\frac{2\sqrt{5}}{5}}$

Simplify. $\qquad \boxed{\tan\left(\frac{x}{2}\right) = -\frac{1}{2}}$

■ **YOUR TURN** If $\cos x = -\frac{3}{5}$ and $\pi < x < \frac{3\pi}{2}$, find $\sin\left(\frac{x}{2}\right)$, $\cos\left(\frac{x}{2}\right)$, and $\tan\left(\frac{x}{2}\right)$.

■ **Answer:** $\sin\left(\frac{x}{2}\right) = \frac{2\sqrt{5}}{5}$, $\cos\left(\frac{x}{2}\right) = -\frac{\sqrt{5}}{5}$, $\tan\left(\frac{x}{2}\right) = -2$

EXAMPLE 9 Using Half-Angle Identities to Verify Other Identities

Verify the identity $\cos^2\left(\dfrac{x}{2}\right) = \dfrac{\tan x + \sin x}{2\tan x}$.

Solution:

Write the cosine half-angle identity. $\quad \cos\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{1 + \cos x}{2}}$

Square both sides of the equation. $\quad \cos^2\left(\dfrac{x}{2}\right) = \dfrac{1 + \cos x}{2}$

Multiply the numerator and denominator on the right side by $\tan x$. $\quad \cos^2\left(\dfrac{x}{2}\right) = \left(\dfrac{1 + \cos x}{2}\right)\left(\dfrac{\tan x}{\tan x}\right)$

Simplify. $\quad \cos^2\left(\dfrac{x}{2}\right) = \dfrac{\tan x + \cos x \tan x}{2\tan x}$

Note that $\cos x \tan x = \sin x$. $\quad \boxed{\cos^2\left(\dfrac{x}{2}\right) = \dfrac{\tan x + \sin x}{2\tan x}}$

An alternative solution is to start with the right-hand side.

Solution (alternative):

Start with the right-hand side. $\quad \dfrac{\tan x + \sin x}{2\tan x}$

Write this expression as the sum of two expressions. $\quad = \dfrac{\tan x}{2\tan x} + \dfrac{\sin x}{2\tan x}$

Simplify. $\quad = \dfrac{1}{2} + \dfrac{1}{2}\dfrac{\sin x}{\tan x}$

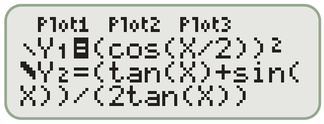
Write $\tan x = \dfrac{\sin x}{\cos x}$. $\quad = \dfrac{1}{2} + \dfrac{1}{2}\dfrac{\sin x}{\dfrac{\sin x}{\cos x}}$

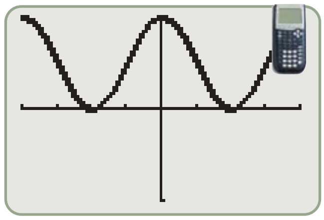
$$= \frac{1}{2}(1 + \cos x)$$

$$= \cos^2\left(\frac{x}{2}\right)$$

Technology Tip

Graphs of $y_1 = \cos^2\left(\dfrac{x}{2}\right)$ and $y_2 = \dfrac{\tan x + \sin x}{2\tan x}$.





EXAMPLE 10 Using Half-Angle Identities to Verify Other Trigonometric Identities

Verify the identity $\tan x = \csc(2x) - \cot(2x)$.

Solution:

Write the third half-angle formula for the tangent function. $\quad \tan\left(\dfrac{A}{2}\right) = \dfrac{1 - \cos A}{\sin A}$

Write the right side as a difference of two expressions having the same denominator. $\quad \tan\left(\dfrac{A}{2}\right) = \dfrac{1}{\sin A} - \dfrac{\cos A}{\sin A}$

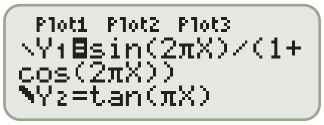
Substitute the reciprocal and quotient identities, respectively, on the right. $\quad \tan\left(\dfrac{A}{2}\right) = \csc A - \cot A$

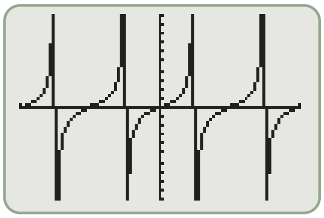
Let $A = 2x$. $\quad \boxed{\tan x = \csc(2x) - \cot(2x).}$

Notice in Example 10 that we started with the third half-angle identity for the tangent function. In Example 11 we will start with the second half-angle identity for the tangent function. In general, you select the form that appears to lead to the desired expression.

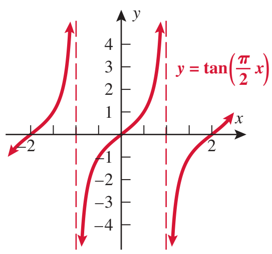
Technology Tip

Graphs of $y_1 = \dfrac{\sin(2\pi x)}{1 + \cos(2\pi x)}$ and $y_2 = \tan(\pi x)$.





■ **Answer:**



EXAMPLE 11 Using Half-Angle Identities to Simplify Trigonometric Expressions

Graph $y = \dfrac{\sin(2\pi x)}{1 + \cos(2\pi x)}$.

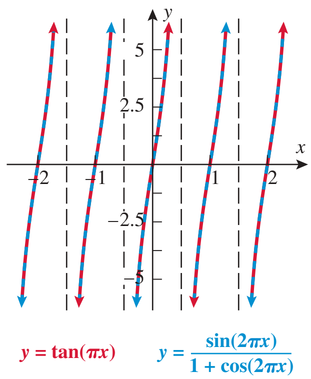
Solution:

Simplify the trigonometric expression using a half-angle identity for the tangent function.

Write the second half-angle identity for the tangent function. $\tan\left(\dfrac{A}{2}\right) = \dfrac{\sin A}{1 + \cos A}$

Let $A = 2\pi x$. $\tan(\pi x) = \dfrac{\sin(2\pi x)}{1 + \cos(2\pi x)}$

Graph $y = \tan(\pi x)$.



■ **YOUR TURN** Graph $y = \dfrac{1 - \cos(\pi x)}{\sin(\pi x)}$.

SECTION 6.3 SUMMARY

In this section, we derived the double-angle identities from the sum identities. We then used the double-angle identities to find exact values of trigonometric functions, to verify other trigonometric identities, and to simplify trigonometric expressions.

$$\begin{aligned}\sin(2A) &= 2\sin A\cos A\\ \cos(2A) &= \cos^2 A - \sin^2 A\\ &= 1 - 2\sin^2 A\\ &= 2\cos^2 A - 1\\ \tan(2A) &= \frac{2\tan A}{1 - \tan^2 A}\end{aligned}$$

The double-angle identities were used to derive the half-angle identities. We then used the half-angle identities to find certain exact values of trigonometric functions, verify other trigonometric identities, and simplify trigonometric expressions.

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos A}{2}} \qquad \cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 + \cos A}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$$

We determine the sign, $+$ or $-$, by first deciding which quadrant contains $\dfrac{A}{2}$ and then finding the sign of the indicated trigonometric function in that quadrant.

Recall that there are three forms of the tangent half-angle identity. There is no need to memorize the other forms of the tangent half-angle identity, since they can be derived by first using the Pythagorean identity and algebraic manipulation.

SECTION 6.3 EXERCISES

SKILLS

In Exercises 1–12, use the double-angle identities to answer the following questions:

1. If $\sin x = \frac{1}{\sqrt{5}}$ and $\cos x < 0$, find $\sin(2x)$.
2. If $\sin x = \frac{1}{\sqrt{5}}$ and $\cos x < 0$, find $\cos(2x)$.
3. If $\cos x = \frac{5}{13}$ and $\sin x < 0$, find $\tan(2x)$.
4. If $\cos x = -\frac{5}{13}$ and $\sin x < 0$, find $\tan(2x)$.
5. If $\tan x = \frac{12}{5}$ and $\pi < x < \frac{3\pi}{2}$, find $\sin(2x)$.
6. If $\tan x = \frac{12}{5}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos(2x)$.
7. If $\sec x = \sqrt{5}$ and $\sin x > 0$, find $\tan(2x)$.
8. If $\sec x = \sqrt{3}$ and $\sin x < 0$, find $\tan(2x)$.
9. If $\csc x = -2\sqrt{5}$ and $\cos x < 0$, find $\sin(2x)$.
10. If $\csc x = -\sqrt{13}$ and $\cos x > 0$, find $\sin(2x)$.
11. If $\cos x = -\frac{12}{13}$ and $\csc x < 0$, find $\cot(2x)$.
12. If $\sin x = \frac{12}{13}$ and $\cot x < 0$, find $\csc(2x)$.

In Exercises 13–24, simplify each expression. Evaluate the resulting expression exactly, if possible.

13. $\dfrac{2\tan 15°}{1 - \tan^2 15°}$
14. $\dfrac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$
15. $\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$
16. $\sin 15° \cos 15°$
17. $\cos^2(2x) - \sin^2(2x)$
18. $\cos^2(x + 2) - \sin^2(x + 2)$
19. $\dfrac{2\tan\left(\frac{5\pi}{12}\right)}{1 - \tan^2\left(\frac{5\pi}{12}\right)}$
20. $\dfrac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}$
21. $1 - 2\sin^2\left(\frac{7\pi}{12}\right)$
22. $2\sin^2\left(-\frac{5\pi}{8}\right) - 1$
23. $2\cos^2\left(-\frac{7\pi}{12}\right) - 1$
24. $1 - 2\cos^2\left(-\frac{\pi}{8}\right)$

In Exercises 25–40, use the double-angle identities to verify each identity.

25. $\csc(2A) = \frac{1}{2}\csc A \sec A$
26. $\cot(2A) = \frac{1}{2}(\cot A - \tan A)$
27. $(\sin x - \cos x)(\cos x + \sin x) = -\cos(2x)$
28. $(\sin x + \cos x)^2 = 1 + \sin(2x)$
29. $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$
30. $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$
31. $\cos^4 x - \sin^4 x = \cos(2x)$
32. $\cos^4 x + \sin^4 x = 1 - \frac{1}{2}\sin^2(2x)$
33. $8\sin^2 x \cos^2 x = 1 - \cos(4x)$
34. $[\cos(2x) - \sin(2x)][\sin(2x) + \cos(2x)] = \cos(4x)$
35. $-\frac{1}{2}\sec^2 x = -2\sin^2 x \csc^2(2x)$
36. $4\csc(4x) = \dfrac{\sec x \csc x}{\cos(2x)}$
37. $\sin(3x) = \sin x(4\cos^2 x - 1)$
38. $\tan(3x) = \dfrac{\tan x(3 - \tan^2 x)}{(1 - 3\tan^2 x)}$
39. $\frac{1}{2}\sin(4x) = 2\sin x \cos x - 4\sin^3 x \cos x$
40. $\cos(4x) = [\cos(2x) - \sin(2x)][(\cos(2x) + \sin(2x)]$

In Exercises 41–50, graph the functions.

41. $y = \dfrac{\sin(2x)}{1 - \cos(2x)}$

42. $y = \dfrac{2\tan x}{2 - \sec^2 x}$

43. $y = \dfrac{\cot x + \tan x}{\cot x - \tan x}$

44. $y = \frac{1}{2}\tan x \cot x \sec x \csc x$

45. $y = \sin(2x)\cos(2x)$

46. $y = 3\sin(3x)\cos(-3x)$

47. $y = 1 - \dfrac{\tan x \cot x}{\sec x \csc x}$

48. $y = 3 - 2\dfrac{\sec(2x)}{\csc(2x)}$

49. $y = \dfrac{\sin(2x)}{\cos x} - 3\cos(2x)$

50. $y = 2 + \dfrac{\sin(2x)}{\cos x} - 3\cos(2x)$

In Exercises 51–66, use the half-angle identities to find the exact values of the trigonometric expressions.

51. $\sin 15°$

52. $\cos 22.5°$

53. $\cos\left(\dfrac{11\pi}{12}\right)$

54. $\sin\left(\dfrac{\pi}{8}\right)$

55. $\cos 75°$

56. $\sin 75°$

57. $\tan 67.5°$

58. $\tan 202.5°$

59. $\sec\left(-\dfrac{9\pi}{8}\right)$

60. $\csc\left(\dfrac{9\pi}{8}\right)$

61. $\cot\left(\dfrac{13\pi}{8}\right)$

62. $\cot\left(\dfrac{7\pi}{8}\right)$

63. $\sec\left(\dfrac{5\pi}{8}\right)$

64. $\csc\left(-\dfrac{5\pi}{8}\right)$

65. $\cot(-135°)$

66. $\cot 105°$

In Exercises 67–82, use the half-angle identities to find the desired function values. Assume that $x \geq 0$.

67. If $\cos x = \dfrac{5}{13}$ and $\sin x < 0$, find $\sin\left(\dfrac{x}{2}\right)$.

68. If $\cos x = -\dfrac{5}{13}$ and $\sin x < 0$, find $\cos\left(\dfrac{x}{2}\right)$.

69. If $\tan x = \dfrac{12}{5}$ and $\pi < x < \dfrac{3\pi}{2}$, find $\sin\left(\dfrac{x}{2}\right)$.

70. If $\tan x = \dfrac{12}{5}$ and $\pi < x < \dfrac{3\pi}{2}$, find $\cos\left(\dfrac{x}{2}\right)$.

71. If $\sec x = \sqrt{5}$ and $\sin x > 0$, find $\tan\left(\dfrac{x}{2}\right)$.

72. If $\sec x = \sqrt{3}$ and $\sin x < 0$, find $\tan\left(\dfrac{x}{2}\right)$.

73. If $\csc x = 3$ and $\cos x < 0$, find $\sin\left(\dfrac{x}{2}\right)$.

74. If $\csc x = -3$ and $\cos x > 0$, find $\cos\left(\dfrac{x}{2}\right)$.

75. If $\cos x = -\dfrac{1}{4}$ and $\csc x < 0$, find $\cot\left(\dfrac{x}{2}\right)$.

76. If $\cos x = \dfrac{1}{4}$ and $\cot x < 0$, find $\csc\left(\dfrac{x}{2}\right)$.

77. If $\cot x = -\dfrac{24}{5}$ and $\dfrac{\pi}{2} < x < \pi$, find $\cos\left(\dfrac{x}{2}\right)$.

78. If $\cot x = -\dfrac{24}{5}$ and $\dfrac{\pi}{2} < x < \pi$, find $\sin\left(\dfrac{x}{2}\right)$.

79. If $\sin x = -0.3$ and $\sec x > 0$, find $\tan\left(\dfrac{x}{2}\right)$.

80. If $\sin x = -0.3$ and $\sec x < 0$, find $\cot\left(\dfrac{x}{2}\right)$.

81. If $\sec x = 2.5$ and $\tan x > 0$, find $\cot\left(\dfrac{x}{2}\right)$.

82. If $\sec x = -3$ and $\cot x < 0$, find $\tan\left(\dfrac{x}{2}\right)$.

In Exercises 83–88, simplify each expression using half-angle identities. Do not evaluate.

83. $\sqrt{\dfrac{1 + \cos\left(\dfrac{5\pi}{6}\right)}{2}}$

84. $\sqrt{\dfrac{1 - \cos\left(\dfrac{\pi}{4}\right)}{2}}$

85. $\dfrac{\sin 150°}{1 + \cos 150°}$

86. $\dfrac{1 - \cos 150°}{\sin 150°}$

87. $\sqrt{\dfrac{1 - \cos\left(\dfrac{5\pi}{4}\right)}{1 + \cos\left(\dfrac{5\pi}{4}\right)}}$

88. $\sqrt{\dfrac{1 - \cos 15°}{1 + \cos 15°}}$

In Exercises 89–100, use the half-angle identities to verify the identities.

89. $\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) = 1$

90. $\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \cos x$

91. $\sin(-x) = -2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

92. $2\cos^2\left(\frac{x}{4}\right) = 1 + \cos\left(\frac{x}{2}\right)$

93. $\tan^2\left(\frac{x}{2}\right) = \frac{1 - \cos x}{1 + \cos x}$

94. $\tan^2\left(\frac{x}{2}\right) = (\csc x - \cot x)^2$

95. $\tan\left(\frac{A}{2}\right) + \cot\left(\frac{A}{2}\right) = 2\csc A$

96. $\cot\left(\frac{A}{2}\right) - \tan\left(\frac{A}{2}\right) = 2\cot A$

97. $\csc^2\left(\frac{A}{2}\right) = \frac{2(1 + \cos A)}{\sin^2 A}$

98. $\sec^2\left(\frac{A}{2}\right) = \frac{2(1 - \cos A)}{\sin^2 A}$

99. $\csc\left(\frac{A}{2}\right) = \pm|\csc A|\sqrt{2 + 2\cos A}$

100. $\sec\left(\frac{A}{2}\right) = \pm|\csc A|\sqrt{2 - 2\cos A}$

In Exercises 101–108, graph the functions.

101. $y = 4\cos^2\left(\frac{x}{2}\right)$

102. $y = -6\sin^2\left(\frac{x}{2}\right)$

103. $y = \dfrac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$

104. $y = 1 - \left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right]^2$

105. $y = 4\sin^2\left(\frac{x}{2}\right) - 1$

106. $y = -\frac{1}{6}\cos^2\left(\frac{x}{2}\right) + 2$

107. $y = \sqrt{\dfrac{1 - \cos(2x)}{1 + \cos(2x)}} \quad 0 \le x < \pi$

108. $y = \sqrt{\dfrac{1 + \cos(3x)}{2}} + 3 \quad 0 \le x \le \frac{\pi}{3}$

APPLICATIONS

109. Business/Economics. Annual cash flow of a stock fund (measured as a percentage of total assets) has fluctuated in cycles. The highs were roughly +12% of total assets and lows were roughly −8% of total assets. This cash flow can be modeled by the function

$$C(t) = 12 - 20\sin^2 t$$

Use a double-angle identity to express $C(t)$ in terms of the cosine function.

110. Business. Computer sales are generally subject to seasonal fluctuations. An analysis of the sales of a computer manufacturer during 2008–2010 is approximated by the function

$$s(t) = 0.098\cos^2 t + 0.387 \qquad 1 \le t \le 12$$

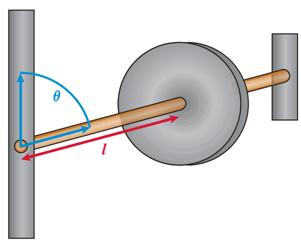
where t represents time in quarters ($t = 1$ represents the end of the first quarter of 2008), and $s(t)$ represents computer sales (quarterly revenue) in millions of dollars. Use a double-angle identity to express $s(t)$ in terms of the cosine function.

For Exercises 111 and 112, refer to the following:

An ore-crusher wheel consists of a heavy disk spinning on its axle. The normal (crushing) force F, in pounds, between the wheel and the inclined track is determined by

$$F = W\sin\theta + \frac{1}{2}\psi^2\left[\frac{C}{R}(1 - \cos 2\theta) + \frac{A}{l}\sin 2\theta\right]$$

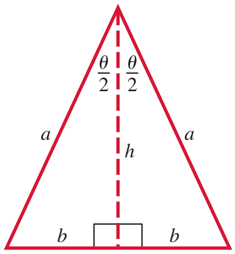
where W is the weight of the wheel in pounds, θ is the angle of the axis, C and A are moments of inertia, R is the radius of the wheel, l is the distance from the wheel to the pin where the axle is attached, and ψ is the speed in rpm that the wheel is spinning. The optimum crushing force occurs when the angle θ is between 45° and 90°.



111. Ore-Crusher Wheel. Find F if the angle is 60°, W is 500 lb, ψ is 200 rpm, $\frac{C}{R} = 750$, and $\frac{A}{l} = 3.75$.

112. Ore-Crusher Wheel. Find F if the angle is 75°, W is 500 lb, ψ is 200 rpm, $\frac{C}{R} = 750$, and $\frac{A}{l} = 3.75$.

113. Area of an Isosceles Triangle. Consider the triangle below, where the vertex angle measures θ, the equal sides measure a, the height is h, and half the base is b. (In an isosceles triangle, the perpendicular dropped from the vertex angle divides the triangle into two congruent triangles.) The two triangles formed are right triangles.

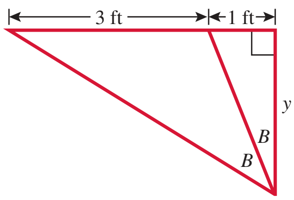


In the right triangles, $\sin\left(\frac{\theta}{2}\right) = \frac{b}{a}$ and $\cos\left(\frac{\theta}{2}\right) = \frac{h}{a}$. Multiply each side of each equation by a to get $b = a\sin\left(\frac{\theta}{2}\right)$, $h = a\cos\left(\frac{\theta}{2}\right)$.

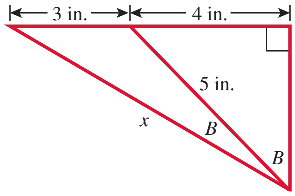
The area of the entire isosceles triangle is $A = \frac{1}{2}(2b)h = bh$. Substitute the values for b and h into the area formula. Show that the area is equivalent to $\left(\frac{a^2}{2}\right)\sin\theta$.

114. Area of an Isosceles Triangle. Use the results from Exercise 113 to find the area of an isosceles triangle whose equal sides measure 7 inches and whose base angles each measure 75°.

115. With the information given in the diagram below, compute y.



116. With the information given in the diagram below, compute x.

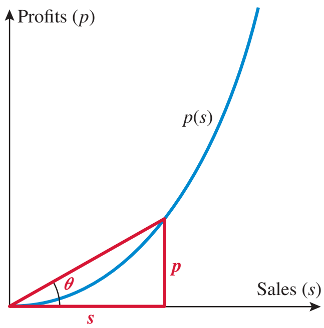


For Exercises 117 and 118, refer to the following:

Monthly profits can be expressed as a function of sales, that is, $p(s)$. A financial analysis of a company has determined that the sales s in thousands of dollars are also related to monthly profits p in thousands of dollars by the relationship:

$$\tan\theta = \frac{p}{s} \quad \text{for} \quad 0 \le s \le 50,\ 0 \le p < 40$$

Based on sales and profits, it can be determined that the domain for angle θ is $0 \le \theta \le 38°$.



117. Business. If monthly profits are \$3000 and monthly sales are \$4000, find $\tan\left(\frac{\theta}{2}\right)$.

118. Business. If monthly profits are p and monthly sales are s (where $p < s$), find $\tan\left(\frac{\theta}{2}\right)$.

CATCH THE MISTAKE

In Exercises 119–122, explain the mistake that is made.

119. If $\cos x = \frac{1}{3}$, find $\sin(2x)$ given $\sin x < 0$.

Solution:

Write the double-angle identity for the sine function. $\quad \sin(2x) = 2\sin x \cos x$

Solve for $\sin x$ using the Pythagorean identity. $\quad \sin^2 x + \left(\frac{1}{3}\right)^2 = 1$

$$\sin x = \frac{2\sqrt{2}}{3}$$

Substitute $\cos x = \frac{1}{3}$ and $\sin x = \frac{2\sqrt{2}}{3}$. $\quad \sin(2x) = 2\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{1}{3}\right)$

Simplify. $\quad \sin(2x) = \frac{4\sqrt{2}}{9}$

This is incorrect. What mistake was made?

120. If $\sin x = \frac{1}{3}$, find $\tan(2x)$ given $\cos x < 0$.

Solution:

Use the quotient identity. $\quad \tan(2x) = \frac{\sin(2x)}{\cos x}$

Use the double-angle formula for the sine function. $\quad \tan(2x) = \frac{2\sin x \cos x}{\cos x}$

Cancel the common cosine factors. $\quad \tan(2x) = 2\sin x$

Substitute $\sin x = \frac{1}{3}$. $\quad \tan(2x) = \frac{2}{3}$

This is incorrect. What mistake was made?

121. If $\cos x = -\frac{1}{3}$, find $\sin\left(\frac{x}{2}\right)$ given $\pi < x < \frac{3\pi}{2}$.

Solution:

Write the half-angle identity for the sine function. $\quad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$

Substitute $\cos x = -\frac{1}{3}$. $\quad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \frac{1}{3}}{2}}$

Simplify. $\quad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{2}}{\sqrt{3}}$

The sine function is negative. $\quad \sin\left(\frac{x}{2}\right) = -\frac{\sqrt{2}}{\sqrt{3}}$

This is incorrect. What mistake was made?

122. If $\cos x = \frac{1}{3}$, find $\tan^2\left(\frac{x}{2}\right)$.

Solution:

Use the quotient identity. $\quad \tan^2\left(\frac{x}{2}\right) = \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2 x}$

Use the half-angle identity for the sine function. $\quad \tan^2\left(\frac{x}{2}\right) = \frac{\frac{1 - \cos x}{2}}{\cos^2 x}$

Simplify. $\quad \tan^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(\frac{1}{\cos^2 x} - \frac{\cos x}{\cos^2 x}\right)$

$$\tan^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(\frac{1}{\cos^2 x} - \frac{1}{\cos x}\right)$$

Substitute $\cos x = \frac{1}{3}$. $\quad \tan^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(\frac{1}{\frac{1}{9}} - \frac{1}{\frac{1}{3}}\right)$

$$\tan^2\left(\frac{x}{2}\right) = 3$$

This is incorrect. What mistake was made?

CONCEPTUAL

For Exercises 123–130, determine whether each statement is true or false.

123. $\sin(2A) + \sin(2A) = \sin(4A)$

124. $\cos(4A) - \cos(2A) = \cos(2A)$

125. If $\tan x > 0$, then $\tan(2x) > 0$.

126. If $\sin x > 0$, then $\sin(2x) > 0$.

127. $\sin\left(\frac{A}{2}\right) + \sin\left(\frac{A}{2}\right) = \sin A$

128. $\cos\left(\frac{A}{2}\right) + \cos\left(\frac{A}{2}\right) = \cos A$

129. If $\tan x > 0$, then $\tan\left(\frac{x}{2}\right) > 0$.

130. If $\sin x > 0$, then $\sin\left(\frac{x}{2}\right) > 0$.

131. Given $\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$, verify $\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1 + \cos A}$. Substitute $A = \pi$ into the identity and explain your results.

132. Given $\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1 - \cos A}{1 + \cos A}}$, verify $\tan\left(\frac{A}{2}\right) = \frac{1 - \cos A}{\sin A}$. Substitute $A = \pi$ into the identity and explain your results.

CHALLENGE

133. Is the identity $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$ true for $x = \dfrac{\pi}{4}$? Explain.

134. Is the identity $2\csc(2x) = \dfrac{1 + \tan^2 x}{\tan x}$ true for $x = \dfrac{\pi}{2}$? Explain.

135. Prove that $\cot\left(\dfrac{A}{4}\right) = \pm\sqrt{\dfrac{1 + \cos\left(\frac{A}{2}\right)}{1 - \cos\left(\frac{A}{2}\right)}}$.

136. Prove that $\cot\left(-\dfrac{A}{2}\right)\sec\left(\dfrac{A}{2}\right)\csc\left(-\dfrac{A}{2}\right)\tan\left(\dfrac{A}{2}\right) = 2\csc A$.

137. Find the values of x in the interval $[0, 2\pi]$ for which $\tan\left(\dfrac{x}{2}\right) > 0$.

138. Find the values of x in the interval $[0, 2\pi]$ for which $\cot\left(\dfrac{x}{2}\right) \le 0$.

TECHNOLOGY

One cannot *prove* that an equation is an identity using technology, but rather one uses it as a first step to see whether or not the equation *seems* to be an identity.

139. With a graphing calculator, plot $Y_1 = 1 - \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!}$ and $Y_2 = \cos(2x)$ for x range $[-1, 1]$. Is Y_1 a good approximation to Y_2?

140. With a graphing calculator, plot $Y_1 = (2x) - \dfrac{(2x)^3}{3!} + \dfrac{(2x)^5}{5!}$ and $Y_2 = \sin(2x)$ for x range $[-1, 1]$. Is Y_1 a good approximation to Y_2?

141. With a graphing calculator, plot $Y_1 = \left(\dfrac{x}{2}\right) - \dfrac{\left(\frac{x}{2}\right)^3}{3!} + \dfrac{\left(\frac{x}{2}\right)^5}{5!}$ and $Y_2 = \sin\left(\dfrac{x}{2}\right)$ for x range $[-1, 1]$. Is Y_1 a good approximation to Y_2?

142. Using a graphing calculator, plot $Y_1 = 1 - \dfrac{\left(\frac{x}{2}\right)^2}{2!} + \dfrac{\left(\frac{x}{2}\right)^4}{4!}$ and $Y_2 = \cos\left(\dfrac{x}{2}\right)$ for x range $[-1, 1]$. Is Y_1 a good approximation to Y_2?

PREVIEW TO CALCULUS

In calculus, we work with the derivative of expressions containing trigonometric functions. Usually, it is better to work with a simplified version of these expressions.

In Exercises 143–146, simplify each expression using the double-angle and half-angle identities.

143. $\dfrac{\dfrac{2\sin x}{\cos x}}{\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x}}$

144. $\cos^4 x - 6\sin^2 x\cos^2 x + \sin^4 x$

145. $3\sin x\cos^2 x - \sin^3 x$

146. $\sqrt{\dfrac{1 - \sqrt{\dfrac{1 + \cos x}{2}}}{1 + \sqrt{\dfrac{1 + \cos x}{2}}}}$

SECTION 6.4 PRODUCT-TO-SUM AND SUM-TO-PRODUCT IDENTITIES

SKILLS OBJECTIVES

- Express products of trigonometric functions as sums of trigonometric functions.
- Express sums of trigonometric functions as products of trigonometric functions.

CONCEPTUAL OBJECTIVES

- Understand that the sum and difference identities are used to derive product-to-sum identities.
- Understand that the product-to-sum identities are used to derive the sum-to-product identities.

In calculus, often it is helpful to write products of trigonometric functions as sums of other trigonometric functions, and vice versa. In this section, we discuss the *product-to-sum identities*, which convert products to sums, and *sum-to-product identities*, which convert sums to products.

Product-to-Sum Identities

The *product-to-sum identities* are derived from the sum and difference identities.

WORDS	MATH
Write the identity for the cosine of a sum.	$\cos A\cos B - \sin A\sin B = \cos(A+B)$
Write the identity for the cosine of a difference.	$\cos A\cos B + \sin A\sin B = \cos(A-B)$
Add the two identities.	$2\cos A\cos B = \cos(A+B) + \cos(A-B)$
Divide both sides by 2.	$\boxed{\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]}$
	$\cos A\cos B + \sin A\sin B = \cos(A-B)$
Subtract the sum identity from the difference identity.	$-\cos A\cos B + \sin A\sin B = -\cos(A+B)$
	$2\sin A\sin B = \cos(A-B) - \cos(A+B)$
Divide both sides by 2.	$\boxed{\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]}$
Write the identity for the sine of a sum.	$\sin A\cos B + \cos A\sin B = \sin(A+B)$
Write the identity for the sine of a difference.	$\sin A\cos B - \cos A\sin B = \sin(A-B)$
Add the two identities.	$2\sin A\cos B = \sin(A+B) + \sin(A-B)$
Divide both sides by 2.	$\boxed{\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]}$

PRODUCT-TO-SUM IDENTITIES

1. $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
3. $\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

EXAMPLE 1 Illustrating a Product-to-Sum Identity for Specific Values

Show that product-to-sum identity (3) is true when $A = 30°$ and $B = 90°$.

Solution:

Write product-to-sum identity (3). $\sin A \cos B = \dfrac{1}{2}[\sin(A + B) + \sin(A - B)]$

Let $A = 30°$ and $B = 90°$. $\sin 30° \cos 90° = \dfrac{1}{2}[\sin(30° + 90°) + \sin(30° - 90°)]$

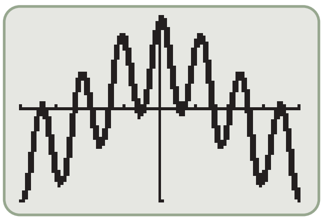
Simplify. $\sin 30° \cos 90° = \dfrac{1}{2}[\sin 120° + \sin(-60°)]$

Evaluate the trigonometric functions. $\dfrac{1}{2} \cdot 0 = \dfrac{1}{2}\left[\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2}\right]$

Simplify. $0 = 0$

Technology Tip

Graphs of $y_1 = \cos(4x)\cos(3x)$ and $y_2 = \frac{1}{2}[\cos(7x) + \cos x]$.



■ **Answer:** $\frac{1}{2}[\cos(7x) + \cos(3x)]$

EXAMPLE 2 Convert a Product to a Sum

Convert the product $\cos(4x)\cos(3x)$ to a sum.

Solution:

Write product-to-sum identity (1). $\cos A \cos B = \dfrac{1}{2}[\cos(A + B) + \cos(A - B)]$

Let $A = 4x$ and $B = 3x$. $\cos(4x)\cos(3x) = \dfrac{1}{2}[\cos(4x + 3x) + \cos(4x - 3x)]$

Simplify. $\boxed{\cos(4x)\cos(3x) = \dfrac{1}{2}[\cos(7x) + \cos x]}$

■ **YOUR TURN** Convert the product $\cos(2x)\cos(5x)$ to a sum.

EXAMPLE 3 **Converting Products to Sums**

Express $\sin(2x)\sin(3x)$ in terms of cosines.

COMMON MISTAKE

A common mistake that is often made is calling the product of two sines the square of a sine.

CORRECT

Write product-to-sum identity (2).

$$\sin A \sin B$$
$$= \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

Let $A = 2x$ and $B = 3x$.

$$\sin(2x)\sin(3x)$$
$$= \frac{1}{2}[\cos(2x - 3x) - \cos(2x + 3x)]$$

Simplify.

$$\sin(2x)\sin(3x)$$
$$= \frac{1}{2}[\cos(-x) - \cos(5x)]$$

The cosine function is an even function; thus,

$$\sin(2x)\sin(3x) = \frac{1}{2}[\cos x - \cos(5x)].$$

INCORRECT

Multiply the two sine functions.

$\sin(2x)\sin(3x) = \sin^2(6x^2)$ **ERROR**

CAUTION

1. $\sin A \sin B \neq \sin^2(AB)$.
2. The argument must be the same in order to use the identity: $\sin A \sin A = (\sin A)^2 = \sin^2 A$.

YOUR TURN Express $\sin x \sin(2x)$ in terms of cosines.

Answer: $\frac{1}{2}[\cos x - \cos(3x)]$

Sum-to-Product Identities

The *sum-to-product identities* can be obtained from the product-to-sum identities.

Words	Math
Write the identity for the product of the sine and cosine functions.	$\frac{1}{2}[\sin(x + y) + \sin(x - y)] = \sin x \cos y$
Let $x + y = A$ and $x - y = B$, then $x = \frac{A + B}{2}$ and $y = \frac{A - B}{2}$.	
Substitute these values into the identity.	$\frac{1}{2}[\sin A + \sin B] = \sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$
Multiply by 2.	$\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$

The other three *sum-to-product* identities can be found similarly. All are summarized in the box below.

SUM-TO-PRODUCT IDENTITIES

4. $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

5. $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

6. $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

7. $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

EXAMPLE 4 Illustrating a Sum-to-Product Identity for Specific Values

Show that sum-to-product identity (7) is true when $A = 30°$ and $B = 90°$.

Solution:

Write the sum-to-product identity (7).
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Let $A = 30°$ and $B = 90°$.
$$\cos 30° - \cos 90° = -2\sin\left(\frac{30° + 90°}{2}\right)\sin\left(\frac{30° - 90°}{2}\right)$$

Simplify.
$$\cos 30° - \cos 90° = -2\sin 60° \sin(-30°)$$

The sine function is an odd function.
$$\cos 30° - \cos 90° = 2\sin 60° \sin 30°$$

Evaluate the trigonometric functions.
$$\frac{\sqrt{3}}{2} - 0 = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

Simplify.
$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

EXAMPLE 5 Convert a Sum to a Product

Convert $-9[\sin(2x) - \sin(10x)]$, a trigonometric expression containing a sum, to a product.

Solution:

The expression inside the brackets is in the form of identity (5).
$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Let $A = 2x$ and $B = 10x$.
$$\sin(2x) - \sin(10x) = 2\sin\left(\frac{2x - 10x}{2}\right)\cos\left(\frac{2x + 10x}{2}\right)$$

Simplify.
$$\sin(2x) - \sin(10x) = 2\sin(-4x)\cos(6x)$$

The sine function is an odd function.
$$\sin(2x) - \sin(10x) = -2\sin(4x)\cos(6x)$$

Multiply both sides by -9.
$$\boxed{-9[\sin(2x) - \sin(10x)] = 18\sin(4x)\cos(6x)}$$

EXAMPLE 6 Simplifying a Trigonometric Expression

Simplify the expression $\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + \sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$.

Solution:

Use identities (4) and (5).

$$\underbrace{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}_{\frac{1}{2}[\sin x + \sin y]} + \underbrace{\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)}_{\frac{1}{2}[\sin x - \sin y]}$$

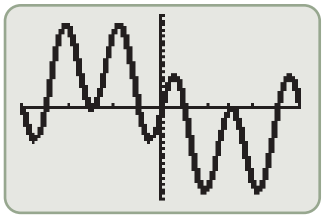
$$= \frac{1}{2}\sin x + \frac{1}{2}\sin y + \frac{1}{2}\sin x - \frac{1}{2}\sin y$$

Simplify. $= \boxed{\sin x}$

Technology Tip

Graphs of $y_1 = -9[\sin(2x) - \sin(10x)]$ and $y_2 = 18\sin(4x)\cos(6x)$.

```
Plot1 Plot2 Plot3
\Y1=-9(sin(2X)-sin(10X))
\Y2=18sin(4X)cos(6X)
```



Applications

In music, a note is a fixed pitch (frequency) that is given a name. If two notes are sounded simultaneously, then they combine to produce another note often called a "beat." The beat frequency is the difference of the two frequencies. The more rapid the beat, the further apart the two frequencies of the notes are. When musicians "tune" their instruments, they use a tuning fork to sound a note and then tune the instrument until the beat is eliminated; hence, the fork and instrument are in tune with each other. Mathematically, a note or tone is represented as $A\cos(2\pi ft)$, where A is the amplitude (loudness), f is the frequency in hertz, and t is time in seconds. The following figure summarizes common notes and frequencies:

EXAMPLE 7 Music

Express the musical tone when a C and G are simultaneously struck (assume with the same loudness).

Find the beat frequency $f_2 - f_1$. Assume uniform loudness, $A = 1$.

Solution:

Write the mathematical description of a C note. $\cos(2\pi f_1 t)$, $f_1 = 262\text{ Hz}$

Write the mathematical description of a G note. $\cos(2\pi f_2 t)$, $f_2 = 392\text{ Hz}$

Add the two notes. $\cos(524\pi t) + \cos(784\pi t)$

Use a sum-to-product identity: $\cos(524\pi t) + \cos(784\pi t)$.

$$= 2\cos\left(\frac{524\pi t + 784\pi t}{2}\right)\cos\left(\frac{524\pi t - 784\pi t}{2}\right)$$

Simplify. $= 2\cos(654\pi t)\cos(-130\pi t)$

Cosine is an even function: $\cos(-x) = \cos x$. $= 2\cos(654\pi t)\cos(130\pi t)$

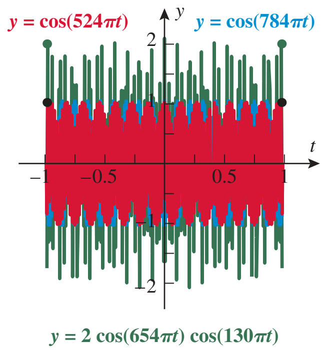
Identify average frequency and beat of the tone.

$$= 2\underbrace{\cos(327\pi t)}_{\text{average frequency}}\underbrace{\cos(130\pi t)}_{\text{beats per second}}$$

The beat frequency can also be found by subtracting f_1 from f_2.

$$f_2 - f_1 = 392 - 262 = 130 \text{ Hz}$$

Therefore, the tone of average frequency, 327 hertz, has a beat of 130 hertz (beats/per second).



SECTION 6.4 SUMMARY

In this section, we used the sum and difference identities to derive the product-to-sum identities. The product-to-sum identities allowed us to express products as sums.

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

We then used the product-to-sum identities to derive the sum-to-product identities. The sum-to-product identities allow us to express sums as products.

$$\sin A + \sin B = 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A - B}{2}\right)\cos\left(\frac{A + B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

SECTION 6.4 EXERCISES

SKILLS

In Exercises 1–14, write each product as a sum or difference of sines and/or cosines.

1. $\sin(2x)\cos x$

2. $\cos(10x)\sin(5x)$

3. $5\sin(4x)\sin(6x)$

4. $-3\sin(2x)\sin(4x)$

5. $4\cos(-x)\cos(2x)$

6. $-8\cos(3x)\cos(5x)$

7. $\sin\left(\frac{3x}{2}\right)\sin\left(\frac{5x}{2}\right)$

8. $\sin\left(\frac{\pi x}{2}\right)\sin\left(\frac{5\pi x}{2}\right)$

9. $\cos\left(\frac{2x}{3}\right)\cos\left(\frac{4x}{3}\right)$

10. $\sin\left(-\frac{\pi}{4}x\right)\cos\left(-\frac{\pi}{2}x\right)$

11. $-3\cos(0.4x)\cos(1.5x)$

12. $2\sin(2.1x)\sin(3.4x)$

13. $4\sin(-\sqrt{3}x)\cos(3\sqrt{3}x)$

14. $-5\cos\left(-\frac{\sqrt{2}}{3}x\right)\sin\left(\frac{5\sqrt{2}}{3}x\right)$

In Exercises 15–28, write each expression as a product of sines and/or cosines.

15. $\cos(5x) + \cos(3x)$

16. $\cos(2x) - \cos(4x)$

17. $\sin(3x) - \sin x$

18. $\sin(10x) + \sin(5x)$

19. $\sin\left(\frac{x}{2}\right) - \sin\left(\frac{5x}{2}\right)$

20. $\cos\left(\frac{x}{2}\right) - \cos\left(\frac{5x}{2}\right)$

21. $\cos\left(\frac{2}{3}x\right) + \cos\left(\frac{7}{3}x\right)$

22. $\sin\left(\frac{2}{3}x\right) + \sin\left(\frac{7}{3}x\right)$

23. $\sin(0.4x) + \sin(0.6x)$

24. $\cos(0.3x) - \cos(0.5x)$

25. $\sin(\sqrt{5}x) - \sin(3\sqrt{5}x)$

26. $\cos(-3\sqrt{7}x) - \cos(2\sqrt{7}x)$

27. $\cos\left(-\frac{\pi}{4}x\right) + \cos\left(\frac{\pi}{6}x\right)$

28. $\sin\left(\frac{3\pi}{4}x\right) + \sin\left(\frac{5\pi}{4}x\right)$

In Exercises 29–34, simplify the trigonometric expressions.

29. $\dfrac{\cos(3x) - \cos x}{\sin(3x) + \sin x}$

30. $\dfrac{\sin(4x) + \sin(2x)}{\cos(4x) - \cos(2x)}$

31. $\dfrac{\cos x - \cos(3x)}{\sin(3x) - \sin x}$

32. $\dfrac{\sin(4x) + \sin(2x)}{\cos(4x) + \cos(2x)}$

33. $\dfrac{\cos(5x) + \cos(2x)}{\sin(5x) - \sin(2x)}$

34. $\dfrac{\sin(7x) - \sin(2x)}{\cos(7x) - \cos(2x)}$

In Exercises 35–42, verify the identities.

35. $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A+B}{2}\right)$

36. $\dfrac{\sin A - \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A-B}{2}\right)$

37. $\dfrac{\cos A - \cos B}{\sin A + \sin B} = -\tan\left(\dfrac{A-B}{2}\right)$

38. $\dfrac{\cos A - \cos B}{\sin A - \sin B} = -\tan\left(\dfrac{A+B}{2}\right)$

39. $\dfrac{\sin A + \sin B}{\sin A - \sin B} = \tan\left(\dfrac{A+B}{2}\right)\cot\left(\dfrac{A-B}{2}\right)$

40. $\dfrac{\cos A - \cos B}{\cos A + \cos B} = -\tan\left(\dfrac{A+B}{2}\right)\tan\left(\dfrac{A-B}{2}\right)$

41. $\dfrac{\cos(A+B) + \cos(A-B)}{\sin(A+B) + \sin(A-B)} = \cot A$

42. $\dfrac{\cos(A-B) - \cos(A+B)}{\sin(A+B) + \sin(A-B)} = \tan B$

APPLICATIONS

43. Business. An analysis of the monthly costs and monthly revenues of a toy store indicates that monthly costs fluctuate (increase and decrease) according to the function

$$C(t) = \sin\left(\frac{\pi}{6}t + \pi\right)$$

and monthly revenues fluctuate (increase and decrease) according to the function

$$R(t) = \sin\left(\frac{\pi}{6}t + \frac{5\pi}{3}\right)$$

Find the function that describes how the monthly profits fluctuate: $P(t) = R(t) - C(t)$. Using identities in this section, express $P(t)$ in terms of a cosine function.

44. Business. An analysis of the monthly costs and monthly revenues of an electronics manufacturer indicates that monthly costs fluctuate (increase and decrease) according to the function

$$C(t) = \cos\left(\frac{\pi}{3}t + \frac{\pi}{3}\right)$$

and monthly revenues fluctuate (increase and decrease) according to the function

$$R(t) = \cos\left(\frac{\pi}{3}t\right)$$

Find the function that describes how the monthly profits fluctuate: $P(t) = R(t) - C(t)$. Using identities in this section, express $P(t)$ in terms of a sine function.

45. Music. Write a mathematical description of a tone that results from simultaneously playing a G and a B. What is the beat frequency? What is the average frequency?

46. Music. Write a mathematical description of a tone that results from simultaneously playing an F and an A. What is the beat frequency? What is the average frequency?

47. Optics. Two optical signals with uniform ($A = 1$) intensities and wavelengths of 1.55 μm and 0.63 μm are "beat" together. What is the resulting sum if their individual signals are given by $\sin\left(\frac{2\pi tc}{1.55\ \mu\text{m}}\right)$ and $\sin\left(\frac{2\pi tc}{0.63\ \mu\text{m}}\right)$, where $c = 3.0 \times 10^8$ m/s?

Note: $1\ \mu\text{m} = 10^{-6}$ m.

48. Optics. The two optical signals in Exercise 47 are beat together. What are the average frequency and the beat frequency?

For Exercises 49 and 50, refer to the following:

Touch-tone keypads have the following simultaneous low and high frequencies.

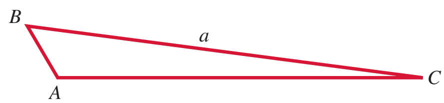
Frequency	1209 Hz	1336 Hz	1477 Hz
697 Hz	1	2	3
770 Hz	4	5	6
852 Hz	7	8	9
941 Hz	*	0	#

The signal given when a key is pressed is $\sin(2\pi f_1 t) + \sin(2\pi f_2 t)$, where f_1 is the low frequency and f_2 is the high frequency.

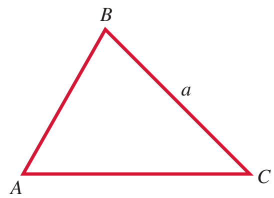
49. Touch-Tone Dialing. What is the mathematical function that models the sound of dialing 4?

50. Touch-Tone Dialing. What is the mathematical function that models the sound of dialing 3?

51. Area of a Triangle. A formula for finding the area of a triangle when given the measures of the angles and one side is Area $= \dfrac{a^2 \sin B \sin C}{2 \sin A}$, where a is the side opposite angle A. If the measures of angles B and C are 52.5° and 7.5°, respectively, and if $a = 10$ ft, use the appropriate product-to-sum identity to change the formula so that you can solve for the area of the triangle exactly.



52. Area of a Triangle. If the measures of angles B and C in Exercise 51 are 75° and 45°, respectively, and if $a = 12$ in., use the appropriate product-to-sum identity to change the formula so that you can solve for the area of the triangle exactly.



CATCH THE MISTAKE

In Exercises 53 and 54, explain the mistake that is made.

53. Simplify the expression $(\cos A - \cos B)^2 + (\sin A - \sin B)^2$.

Solution:

Expand by squaring.

$$\cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$

Group terms.

$$\cos^2 A + \sin^2 A - 2\cos A \cos B - 2\sin A \sin B + \cos^2 B + \sin^2 B$$

Simplify using the Pythagorean identity.

$$\underbrace{\cos^2 A + \sin^2 A}_{1} - 2\cos A \cos B - 2\sin A \sin B + \underbrace{\cos^2 B + \sin^2 B}_{1}$$

Factor the common 2. $\quad 2(1 - \cos A \cos B - \sin A \sin B)$

Simplify. $\quad 2(1 - \cos AB - \sin AB)$

This is incorrect. What mistakes were made?

54. Simplify the expression $(\sin A - \sin B)(\cos A + \cos B)$.

Solution:

Multiply the expressions using the distributive property.

$$\sin A \cos A + \sin A \cos B - \sin B \cos A - \sin B \cos B$$

Cancel the second and third terms.

$$\sin A \cos A - \sin B \cos B$$

Use the product-to-sum identity.

$$\underbrace{\sin A \cos A}_{\frac{1}{2}[\sin(A + A) + \sin(A - A)]} - \underbrace{\sin B \cos B}_{\frac{1}{2}[\sin(B + B) + \sin(B - B)]}$$

Simplify. $\quad = \dfrac{1}{2}\sin(2A) - \dfrac{1}{2}\sin(2B)$

This is incorrect. What mistake was made?

CONCEPTUAL

In Exercises 55–58, determine whether each statement is true or false.

55. $\cos A \cos B = \cos AB$

56. $\sin A \sin B = \sin AB$

57. The product of two cosine functions is a sum of two other cosine functions.

58. The product of two sine functions is a difference of two cosine functions.

59. Write $\sin A \sin B \sin C$ as a sum or difference of sines and cosines.

60. Write $\cos A \cos B \cos C$ as a sum or difference of sines and cosines.

CHALLENGE

61. Prove the addition formula $\cos(A + B) = \cos A \cos B - \sin A \sin B$ using the identities of this section.

62. Prove the difference formula $\sin(A - B) = \sin A \cos B - \sin B \cos A$ using the identities of this section.

63. Graph $y = 1 - 3\sin(\pi x)\sin\left(-\frac{\pi}{6}x\right)$.

64. Graph $y = 4\sin(2x - 1)\cos(2 - x)$.

65. Graph $y = -\cos\left(\frac{2\pi}{3}x\right)\cos\left(\frac{5\pi}{6}x\right)$.

66. Graph $y = x - \cos(2x)\sin(3x)$.

TECHNOLOGY

67. Suggest an identity $4\sin x\cos x\cos(2x) =$ _______ by graphing $Y_1 = 4\sin x\cos x\cos(2x)$ and determining the function based on the graph.

68. Suggest an identity $1 + \tan x\tan(2x) =$ _______ by graphing $Y_1 = 1 + \tan x\tan(2x)$ and determining the function based on the graph.

69. With a graphing calculator, plot $Y_1 = \sin(4x)\sin(2x)$, $Y_2 = \sin(6x)$, and $Y_3 = \frac{1}{2}[\cos(2x) - \cos(6x)]$ in the same viewing rectangle $[0, 2\pi]$ by $[-1, 1]$. Which graphs are the same?

70. With a graphing calculator, plot $Y_1 = \cos(4x)\cos(2x)$, $Y_2 = \cos(6x)$, and $Y_3 = \frac{1}{2}[\cos(6x) + \cos(2x)]$ in the same viewing rectangle $[0, 2\pi]$ by $[-1, 1]$. Which graphs are the same?

PREVIEW TO CALCULUS

In calculus, the method of separation of variables is used to solve certain differential equations. Given an equation with two variables, the method consists of writing the equation in such a way that each side of the equation contains only one type of variable.

In Exercises 71–74, use the product-to-sum and sum-to-product identities to separate the variables *x* and *y* in each equation.

71. $\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{1}{5}$

72. $\frac{1}{2} = \sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

73. $\sin(x + y) = 1 + \sin(x - y)$

74. $2 + \cos(x + y) = \cos(x - y)$

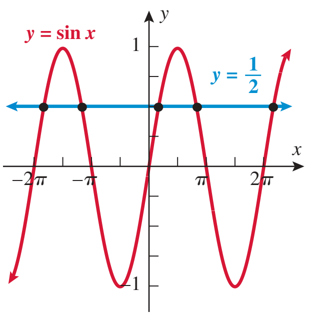
SECTION 6.5 INVERSE TRIGONOMETRIC FUNCTIONS

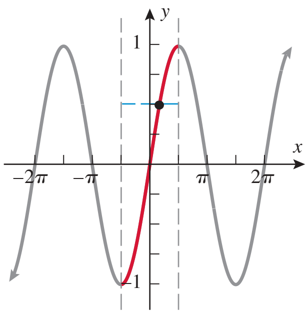
SKILLS OBJECTIVES

- Develop inverse trigonometric functions.
- Find values of inverse trigonometric functions.
- Graph inverse trigonometric functions.

CONCEPTUAL OBJECTIVES

- Understand the different notations for inverse trigonometric functions.
- Understand why domain restrictions on trigonometric functions are needed for inverse trigonometric functions to exist.
- Extend properties of inverse functions to develop inverse trigonometric identities.





In Section 1.5, we discussed one-to-one functions and inverse functions. Here we present a summary of that section. A function is one-to-one if it passes the horizontal line test: No two x-values map to the same y-value.

Notice that the sine function does not pass the horizontal line test. However, if we restrict the domain to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then the restricted function is one-to-one.

Recall that if $y = f(x)$, then $x = f^{-1}(y)$.

The following are the properties of inverse functions:

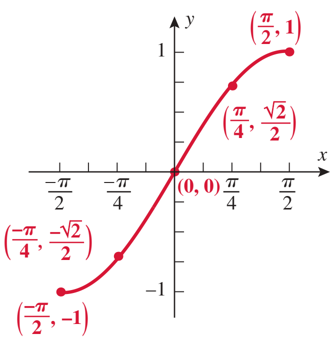
1. If f is a one-to-one function, then the inverse function f^{-1} exists.
2. The domain of f^{-1} = the range of f.
   The range of f^{-1} = the domain of f.
3. $f^{-1}(f(x)) = x$ for all x in the domain of f.
   $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.
4. The graph of f^{-1} is the reflection of the graph of f about the line $y = x$. If the point (a, b) lies on the graph of a function, then the point (b, a) lies on the graph of its inverse.

Inverse Sine Function

Let us start with the sine function with the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$y = \sin x$ Domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ Range: $[-1, 1]$

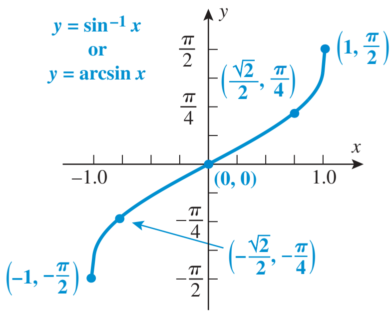
x	y
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$
0	0
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1



By the properties of inverse functions, the inverse sine function will have a domain of $[-1, 1]$ and a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. To find the inverse sine function, we interchange the x- and y-values of $y = \sin x$.

$$y = \sin^{-1}x \qquad \text{Domain: } [-1, 1] \qquad \text{Range: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

x	y
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
0	0
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	$\frac{\pi}{2}$



Notice that the inverse sine function, like the sine function, is an odd function (symmetric about the origin).

If the sine of an angle is known, and the angle is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, what is the measure of that angle? The inverse sine function determines that angle measure. Another notation for the inverse sine function is $\arcsin x$.

Inverse Sine Function

$$\underbrace{y = \sin^{-1}x \text{ or } y = \arcsin x}_{\text{“}y\text{ is the inverse sine of }x\text{”}} \qquad \text{means} \qquad \underbrace{x = \sin y}_{\text{“}y\text{ is the angle measure whose sine equals }x\text{”}}$$

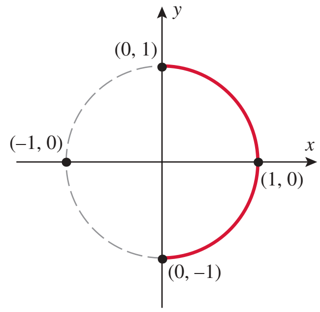
$$\text{where } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

It is important to note that the -1 as the superscript indicates an inverse function. Therefore, the inverse sine function should not be interpreted as a reciprocal:

$$\sin^{-1}x \neq \frac{1}{\sin x}$$

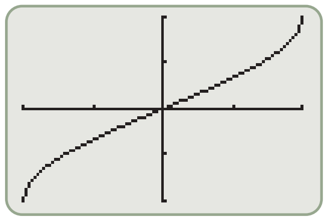
Study Tip

The inverse sine function gives an angle on the right half of the unit circle (QI and QIV).



Technology Tip

To graph $y = \sin^{-1}x$, use $[-1, 1]$ as the domain and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as the range.



Study Tip

Trigonometric functions take angle measures and return real numbers. Inverse trigonometric functions take real numbers and return angle measures.

EXAMPLE 1 Finding Exact Values of an Inverse Sine Function

Find the exact value of each of the following expressions:

a. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **b.** $\arcsin\left(-\frac{1}{2}\right)$

Solution (a):

Let $\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. $\qquad \sin\theta = \frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Which value of θ, in the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, corresponds to a sine value of $\frac{\sqrt{3}}{2}$?

- The range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ corresponds to quadrants I and IV.
- The sine function is positive in quadrant I.
- We look for a value of θ in quadrant I that has a sine value of $\frac{\sqrt{3}}{2}$. $\qquad \theta = \frac{\pi}{3}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\qquad \boxed{\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}}$

Calculator Confirmation: Since $\frac{\pi}{3} = 60°$, if our calculator is set in degree mode, we should find that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is equal to $60°$.

Solution (b):

Let $\theta = \arcsin\left(-\frac{1}{2}\right)$. $\qquad \sin\theta = -\frac{1}{2}$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Which value of θ, in the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, corresponds to a sine value of $-\frac{1}{2}$?

- The range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ corresponds to quadrants I and IV.
- The sine function is negative in quadrant IV.
- We look for a value of θ in quadrant IV that has a sine value of $-\frac{1}{2}$. $\qquad \theta = -\frac{\pi}{6}$

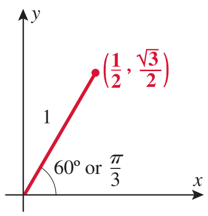
$\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. $\qquad \boxed{\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}}$

Calculator Confirmation: Since $-\frac{\pi}{6} = -30°$, if our calculator is set in degree mode, we should find that $\sin^{-1}\left(-\frac{1}{2}\right)$ is equal to $-30°$.

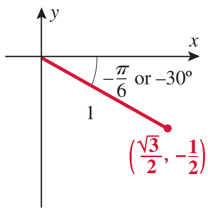
Study Tip

In Example 1, note that the graphs help identify the desired angles.

a.



b.



YOUR TURN Find the exact value of each of the following expressions:

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **b.** $\arcsin\left(\frac{1}{2}\right)$

Answer: a. $-\frac{\pi}{3}$ **b.** $\frac{\pi}{6}$

In Example 1, it is important to note that in part (a), both 60° and 120° correspond to the sine function equal to $\frac{\sqrt{3}}{2}$ and only one of them is valid, which is why the domain restrictions are necessary for inverse functions except for quadrantal angles. There are always two angles (values) from 0 to 360° or 0 to 2π (except that only 90°, or $\frac{\pi}{2}$, and 270°, or $\frac{3\pi}{2}$, correspond to 1 and -1, respectively) that correspond to the sine function equal to a particular value.

It is important to note that the inverse sine function has a domain $[-1, 1]$. For example, $\sin^{-1}3$ does not exist because 3 is not in the domain of the inverse sine function. Notice that calculator evaluation of $\sin^{-1}3$ says *error*. Calculators can be used to evaluate inverse sine functions when an exact evaluation is not feasible, just as they are for the basic trigonometric functions. For example, $\sin^{-1}0.3 \approx 17.46°$, or 0.305 radians.

We now state the properties relating the sine function and the inverse sine function that follow directly from properties of inverse functions.

Technology Tip

Use a TI calculator to find $\sin^{-1}3$ and $\sin^{-1}0.3$. Be sure to set the calculator in radian mode.

SINE-INVERSE SINE IDENTITIES

$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for} \quad -1 \le x \le 1$$

For example, $\sin^{-1}\left[\sin\left(\frac{\pi}{12}\right)\right] = \frac{\pi}{12}$, since $\frac{\pi}{12}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. However, you must be careful not to overlook the domain restriction for which these identities hold, as illustrated in the next example.

EXAMPLE 2 Using Inverse Identities to Evaluate Expressions Involving Inverse Sine Functions

Find the exact value of each of the following trigonometric expressions:

a. $\sin\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right]$ **b.** $\sin^{-1}\left[\sin\left(\frac{3\pi}{4}\right)\right]$

Solution (a):

Write the appropriate identity. $\sin(\sin^{-1}x) = x$ for $-1 \le x \le 1$

Let $x = \frac{\sqrt{2}}{2}$, which is in the interval $[-1, 1]$.

Since the domain restriction is met, the identity can be used. $\boxed{\sin\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right] = \frac{\sqrt{2}}{2}}$

Technology Tip

a. Check the answer of $\sin\left[\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right]$ with a calculator.

```
sin(sin⁻¹(√(2)/2)
)
          .7071067812
√(2)/2
          .7071067812
```

b. Check the answer of $\sin^{-1}\left[\sin\left(\frac{3\pi}{4}\right)\right]$ with a calculator in radian mode.

```
sin⁻¹(sin(3π/4))
          .7853981634
π/4
          .7853981634
```

Solution (b):

COMMON MISTAKE

Ignoring the domain restrictions on inverse identities.

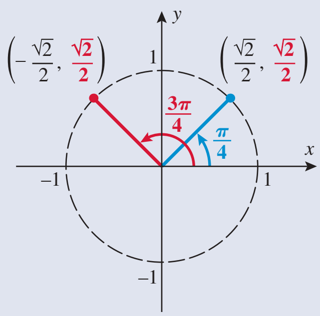
CORRECT

Write the appropriate identity.

$\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Let $x = \frac{3\pi}{4}$, which is *not* in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Since the domain restriction is not met, the identity cannot be used. Instead, we look for a value in the domain that corresponds to the same value of sine.



Substitute $\sin\frac{3\pi}{4} = \sin\frac{\pi}{4}$ into the expression.

$$\sin^{-1}\left[\sin\left(\frac{3\pi}{4}\right)\right] = \sin^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right]$$

Since $\frac{\pi}{4}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can use the identity.

$$\sin^{-1}\left[\sin\left(\frac{3\pi}{4}\right)\right] = \sin^{-1}\left[\sin\left(\frac{\pi}{4}\right)\right] = \boxed{\frac{\pi}{4}}$$

INCORRECT

$\sin^{-1}(\sin x) = x$ **ERROR**

Let $x = \frac{3\pi}{4}$.

(Forgot the domain restriction.)

$\sin^{-1}\left[\sin\left(\frac{3\pi}{4}\right)\right] = \frac{3\pi}{4}$

INCORRECT

YOUR TURN Find the exact value of each of the following trigonometric expressions:

a. $\sin\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$ **b.** $\sin^{-1}\left[\sin\left(\frac{5\pi}{6}\right)\right]$

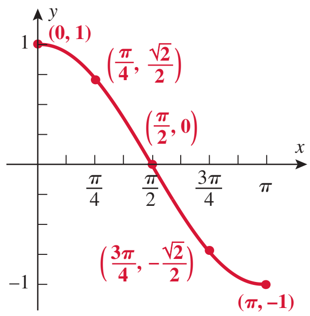
■ **Answer: a.** $-\frac{1}{2}$ **b.** $\frac{\pi}{6}$

Inverse Cosine Function

The cosine function is also not a one-to-one function, so we must restrict the domain in order to develop the inverse cosine function.

$$y = \cos x \qquad \text{Domain: } [0, \pi] \qquad \text{Range: } [-1, 1]$$

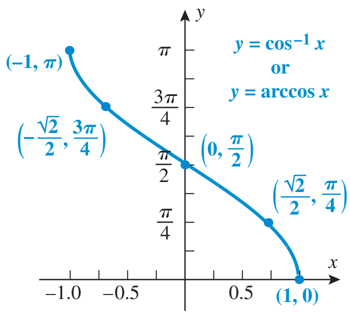
x	y
0	1
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$
π	-1



By the properties of inverses, the inverse cosine function will have a domain of $[-1,1]$ and a range of $[0, \pi]$. To find the inverse cosine function, we interchange the x- and y-values of $y = \cos x$.

$$y = \cos^{-1} x \qquad \text{Domain: } [-1, 1] \qquad \text{Range: } [0, \pi]$$

x	y
-1	π
$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	0



Notice that the inverse cosine function, unlike the cosine function, is not symmetric about the y-axis or the origin. Although the inverse sine and inverse cosine functions have the same domain, they behave differently. The inverse sine function increases on its domain (from left to right), whereas the inverse cosine function decreases on its domain (from left to right).

If the cosine of an angle is known and the angle is between 0 and π, what is the measure of that angle? The inverse cosine function determines that angle measure. Another notation for the inverse cosine function is $\arccos x$.

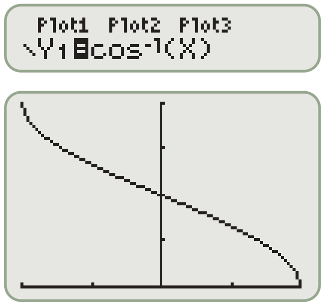
INVERSE COSINE FUNCTION

$$\underbrace{y = \cos^{-1} x \text{ or } y = \arccos x}_{\text{“}y\text{ is the inverse cosine of }x\text{”}} \qquad \text{means} \qquad \underbrace{x = \cos y}_{\text{“}y\text{ is the angle measure whose cosine equals }x\text{”}}$$

$$\text{where } -1 \le x \le 1 \text{ and } 0 \le y \le \pi$$

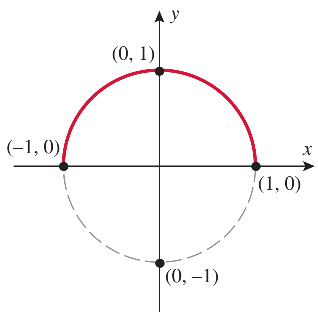
Technology Tip

To graph $y = \cos^{-1} x$, use $[-1, 1]$ as the domain and $[0, \pi]$ as the range.



Study Tip

The inverse cosine function gives an angle on the top half of the unit circle (QI and QII).



Study Tip

$$\cos^{-1} x \neq \frac{1}{\cos x}$$

Technology Tip

a. Check the answer of $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ with a calculator.

```
cos⁻¹(-√(2)/2)
             135
```

b. Check the answer of $\arccos 0$.

```
cos⁻¹(0)
              90
```

EXAMPLE 3 Finding Exact Values of an Inverse Cosine Function

Find the exact value of each of the following expressions:

a. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **b.** $\arccos 0$

Solution (a):

Let $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. $\cos\theta = -\frac{\sqrt{2}}{2}$ when $0 \le \theta \le \pi$

Which value of θ, in the range $0 \le \theta \le \pi$, corresponds to a cosine value of $-\frac{\sqrt{2}}{2}$?

- The range $0 \le \theta \le \pi$ corresponds to quadrants I and II.
- The cosine function is negative in quadrant II.
- We look for a value of θ in quadrant II that has a cosine value of $-\frac{\sqrt{2}}{2}$. $\theta = \frac{3\pi}{4}$

$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and $\frac{3\pi}{4}$ is in the interval $[0, \pi]$. $\boxed{\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}}$

Calculator Confirmation: Since $\frac{3\pi}{4} = 135°$, if our calculator is set in degree mode, we should find that $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ is equal to $135°$.

Solution (b):

Let $\theta = \arccos 0$. $\cos\theta = 0$ when $0 \le \theta \le \pi$

Which value of θ, in the range $0 \le \theta \le \pi$, corresponds to a cosine value of 0? $\theta = \frac{\pi}{2}$

$\cos\left(\frac{\pi}{2}\right) = 0$ and $\frac{\pi}{2}$ is in the interval $[0, \pi]$. $\boxed{\arccos 0 = \frac{\pi}{2}}$

Calculator Confirmation: Since $\frac{\pi}{2} = 90°$, if our calculator is set in degree mode, we should find that $\cos^{-1} 0$ is equal to $90°$.

YOUR TURN Find the exact value of each of the following expressions:

a. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ **b.** $\arccos 1$

Answer: a. $\frac{\pi}{4}$ **b.** 0

We now state the properties relating the cosine function and the inverse cosine function that follow directly from the properties of inverses.

COSINE-INVERSE COSINE IDENTITIES

$$\cos^{-1}(\cos x) = x \quad \text{for} \quad 0 \le x \le \pi$$

$$\cos(\cos^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$

As was the case with inverse identities for the sine function, you must be careful not to overlook the domain restrictions governing when each of these identities hold.

EXAMPLE 4 Using Inverse Identities to Evaluate Expressions Involving Inverse Cosine Functions

Find the exact value of each of the following trigonometric expressions:

a. $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ **b.** $\cos^{-1}\left[\cos\left(\frac{7\pi}{4}\right)\right]$

Solution (a):

Write the appropriate identity. $\cos(\cos^{-1}x) = x$ for $-1 \le x \le 1$

Let $x = -\frac{1}{2}$, which is in the interval $[-1, 1]$.

Since the domain restriction is met,

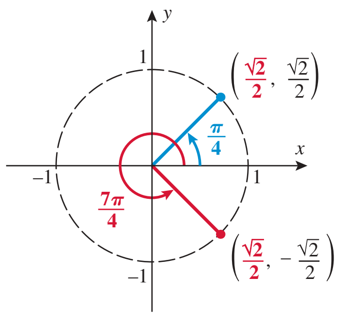
the identity can be used. $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right] = \boxed{-\frac{1}{2}}$

Solution (b):

Write the appropriate identity. $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

Let $x = \frac{7\pi}{4}$, which is *not* in the interval $[0, \pi]$.

Since the domain restriction is not met, the identity cannot be used.



Instead, we find another angle in the interval that has the same cosine value. $\cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$

Substitute $\cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$ into the expression. $\cos^{-1}\left[\cos\left(\frac{7\pi}{4}\right)\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{4}\right)\right]$

Since $\frac{\pi}{4}$ is in the interval $[0, \pi]$, we can use the identity. $= \boxed{\frac{\pi}{4}}$

YOUR TURN Find the exact value of each of the following trigonometric expressions:

a. $\cos\left[\cos^{-1}\left(\frac{1}{2}\right)\right]$ **b.** $\cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right]$

Technology Tip

a. Check the answer of $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ with a calculator.

```
cos(cos⁻¹(-1/2))
                -.5
```

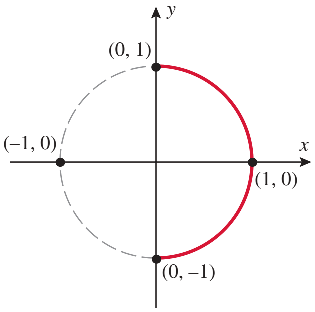
b. Check the answer of $\cos^{-1}\left[\cos\left(\frac{7\pi}{4}\right)\right]$. Be sure to set the calculator to radian mode.

```
cos⁻¹(cos(7π/4))
         .7853981634
π/4
         .7853981634
```

Answer: a. $\frac{1}{2}$ **b.** $\frac{\pi}{6}$

Study Tip

The inverse tangent function gives an angle on the right half of the unit circle (QI and QIV).

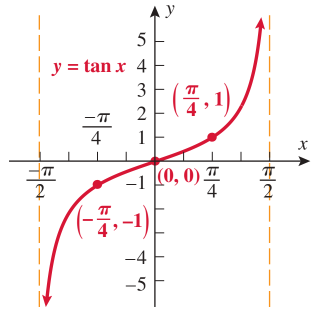


Inverse Tangent Function

The tangent function, too, is not a one-to-one function (it fails the horizontal line test). Let us start with the tangent function with a restricted domain:

$$y = \tan x \qquad \text{Domain: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad \text{Range: } (-\infty, \infty)$$

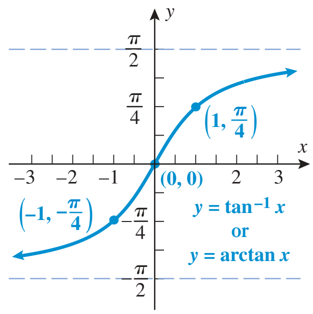
x	y
$-\frac{\pi}{2}$	$-\infty$
$-\frac{\pi}{4}$	-1
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	∞



By the properties of inverse functions, the inverse tangent function will have a domain of $(-\infty, \infty)$ and a range of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the inverse tangent function, interchange x and y values.

$$y = \tan^{-1}x \qquad \text{Domain: } (-\infty, \infty) \qquad \text{Range: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

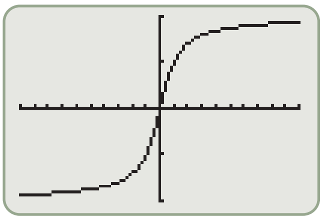
x	y
$-\infty$	$-\frac{\pi}{2}$
-1	$-\frac{\pi}{4}$
0	0
1	$\frac{\pi}{4}$
∞	$\frac{\pi}{2}$



Technology Tip

To graph $y = \tan^{-1}x$, use $(-\infty, \infty)$ as the domain and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as the range.

```
Plot1  Plot2  Plot3
\Y1■tan-1(X)
```



Notice that the inverse tangent function, like the tangent function, is an odd function (it is symmetric about the origin).

The inverse tangent function allows us to answer the question: If the tangent of an angle is known, what is the measure of that angle? Another notation for the inverse tangent function is $\arctan x$.

INVERSE TANGENT FUNCTION

$$\underbrace{y = \tan^{-1}x \text{ or } y = \arctan x}_{\text{“}y\text{ is the inverse tangent of }x\text{”}} \qquad \text{means} \qquad \underbrace{x = \tan y}_{\text{“}y\text{ is the angle measure whose tangent equals }x\text{”}}$$

$$\text{where } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

EXAMPLE 5 Finding Exact Values of an Inverse Tangent Function

Find the exact value of each of the following expressions:

a. $\tan^{-1}\left(\sqrt{3}\right)$ **b.** $\arctan 0$

Solution (a):

Let $\theta = \tan^{-1}\left(\sqrt{3}\right)$. $\tan\theta = \sqrt{3}$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Which value of θ, in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, corresponds to a tangent value of $\sqrt{3}$? $\theta = \frac{\pi}{3}$

$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$ and $\frac{\pi}{3}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\boxed{\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}}$

Calculator Confirmation: Since $\frac{\pi}{3} = 60°$, if our calculator is set in degree mode, we should find that $\tan^{-1}\left(\sqrt{3}\right)$ is equal to $60°$.

Solution (b):

Let $\theta = \arctan 0$. $\tan\theta = 0$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Which value of θ, in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, corresponds to a tangent value of 0? $\theta = 0$

$\tan 0 = 0$, and 0 is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\boxed{\arctan 0 = 0}$

Calculator Confirmation: $\tan^{-1} 0$ is equal to 0.

Technology Tip

a. Use a calculator to check the answer for $\tan^{-1}\left(\sqrt{3}\right)$.

```
tan⁻¹(√(3))
                60
```

Radian mode:

```
tan⁻¹(√(3))
       1.047197551
π/3
       1.047197551
```

b. Use a calculator to check the answer for $\arctan 0$.

```
tan⁻¹(0)
                 0
```

We now state the properties relating the tangent function and the inverse tangent function that follow directly from the properties of inverses.

TANGENT-INVERSE TANGENT IDENTITIES

$$\tan^{-1}(\tan x) = x \qquad \text{for} \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan(\tan^{-1} x) = x \qquad \text{for} \qquad -\infty < x < \infty$$

Technology Tip

a. Use a calculator to check the answer for $\tan(\tan^{-1}17)$.

```
tan(tan⁻¹(17))
              17
```

b. Use a calculator to check the answer for $\tan^{-1}\left[\tan\left(\frac{2\pi}{3}\right)\right]$.

```
tan⁻¹(tan(2π/3)
     -1.047197551
-π/3
     -1.047197551
```

EXAMPLE 6 Using Inverse Identities to Evaluate Expressions Involving Inverse Tangent Functions

Find the exact value of each of the following trigonometric expressions:

a. $\tan(\tan^{-1}17)$ **b.** $\tan^{-1}\left[\tan\left(\frac{2\pi}{3}\right)\right]$

Solution (a):

Write the appropriate identity. $\tan(\tan^{-1}x) = x$ for $-\infty < x < \infty$

Let $x = 17$, which is in the interval $(-\infty, \infty)$.

Since the domain restriction is met, the identity can be used. $\boxed{\tan(\tan^{-1}17) = 17}$

Solution (b):

Write the appropriate identity. $\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Let $x = \frac{2\pi}{3}$, which is *not* in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Since the domain restriction is not met, the identity cannot be used.

Instead, we find another angle in the interval that has the same tangent value. $\tan\left(\frac{2\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right)$

Substitute $\tan\left(\frac{2\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right)$ into the expression. $\tan^{-1}\left[\tan\left(\frac{2\pi}{3}\right)\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right]$

Since $-\frac{\pi}{3}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we can use the identity. $\tan^{-1}\left[\tan\left(\frac{2\pi}{3}\right)\right] = \boxed{-\frac{\pi}{3}}$

YOUR TURN Find the exact value of $\tan^{-1}\left[\tan\left(\frac{7\pi}{6}\right)\right]$.

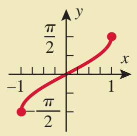
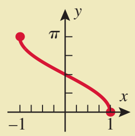
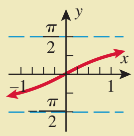
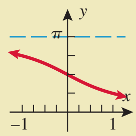
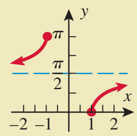
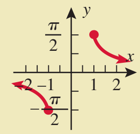
Answer: $\frac{\pi}{6}$

Remaining Inverse Trigonometric Functions

The remaining three inverse trigonometric functions are defined similarly to the previous ones.

- Inverse cotangent function: $\cot^{-1}x$ or $\operatorname{arccot} x$
- Inverse secant function: $\sec^{-1}x$ or $\operatorname{arcsec} x$
- Inverse cosecant function: $\csc^{-1}x$ or $\operatorname{arccsc} x$

A table summarizing all six of the inverse trigonometric functions is given below:

Inverse Function	$y = \sin^{-1}x$	$y = \cos^{-1}x$	$y = \tan^{-1}x$	$y = \cot^{-1}x$	$y = \sec^{-1}x$	$y = \csc^{-1}x$
Domain	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(0, \pi)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
Graph						

EXAMPLE 7 Finding the Exact Value of Inverse Trigonometric Functions

Find the exact value of the following expressions:

a. $\cot^{-1}(\sqrt{3})$ **b.** $\csc^{-1}(\sqrt{2})$ **c.** $\sec^{-1}(-\sqrt{2})$

Solution (a):

Let $\theta = \cot^{-1}(\sqrt{3})$. $\cot\theta = \sqrt{3}$ when $0 < \theta < \pi$

Which value of θ, in the range $0 < \theta < \pi$, corresponds to a cotangent value of $\sqrt{3}$? $\theta = \frac{\pi}{6}$

$\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$ and $\frac{\pi}{6}$ is in the interval $(0, \pi)$. $\boxed{\cot^{-1}(\sqrt{3}) = \frac{\pi}{6}}$

Solution (b):

Let $\theta = \csc^{-1}(\sqrt{2})$. $\csc\theta = \sqrt{2}$

Which value of θ, in the range $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, corresponds to a cosecant value of $\sqrt{2}$? $\theta = \frac{\pi}{4}$

$\csc\left(\frac{\pi}{4}\right) = \sqrt{2}$ and $\frac{\pi}{4}$ is in the interval $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. $\boxed{\csc^{-1}(\sqrt{2}) = \frac{\pi}{4}}$

Solution (c):

Let $\theta = \sec^{-1}(-\sqrt{2})$. $\sec\theta = -\sqrt{2}$

Which value of θ, in the range $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$, corresponds to a secant value of $-\sqrt{2}$? $\theta = \frac{3\pi}{4}$

$\sec\left(\frac{3\pi}{4}\right) = -\sqrt{2}$ and $\frac{3\pi}{4}$ is in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. $\boxed{\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}}$

Technology Tip

$\cot^{-1}(\sqrt{3}) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$,

$\csc^{-1}(\sqrt{2}) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$, and

$\sec^{-1}(-\sqrt{2}) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

```
tan⁻¹(1/√(3))
        .5235987756
π/6
        .5235987756
```

```
sin⁻¹(1/√(2))
        .7853981634
π/4
        .7853981634
```

```
cos⁻¹(-1/√(2))
          2.35619449
3π/4
          2.35619449
```

How do we approximate the inverse secant, inverse cosecant, and inverse cotangent functions with a calculator? Scientific calculators have keys ($\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$) for three of the inverse trigonometric functions but not for the other three. Recall that we find the cosecant, secant, and cotangent function values by taking sine, cosine, or tangent, and finding the reciprocal.

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

However, *the reciprocal approach cannot be used for inverse functions.* The three inverse trigonometric functions $\csc^{-1}x$, $\sec^{-1}x$, and $\cot^{-1}x$ cannot be found by finding the reciprocal of $\sin^{-1}x$, $\cos^{-1}x$, or $\tan^{-1}x$.

$$\csc^{-1}x \neq \frac{1}{\sin^{-1}x} \qquad \sec^{-1}x \neq \frac{1}{\cos^{-1}x} \qquad \cot^{-1}x \neq \frac{1}{\tan^{-1}x}$$

Instead, we seek the equivalent $\sin^{-1}x$, $\cos^{-1}x$, or $\tan^{-1}x$ values by algebraic means, always remembering to look within the correct domain and range.

Study Tip

$$\sec^{-1}x \neq \frac{1}{\cos^{-1}x}$$

$$\csc^{-1}x \neq \frac{1}{\sin^{-1}x}$$

$$\cot^{-1}x \neq \frac{1}{\tan^{-1}x}$$

WORDS	MATH
Start with the inverse secant function.	$y = \sec^{-1}x \quad$ for $\quad x \leq -1 \quad$ or $\quad x \geq 1$
Write the equivalent secant expression.	$\sec y = x \quad$ for $\quad 0 \leq y < \frac{\pi}{2} \quad$ or $\quad \frac{\pi}{2} < y \leq \pi$
Apply the reciprocal identity.	$\frac{1}{\cos y} = x$
Simplify using algebraic techniques.	$\cos y = \frac{1}{x}$
Write the result in terms of the inverse cosine function.	$y = \cos^{-1}\left(\frac{1}{x}\right)$
Therefore, we have the relationship:	$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \quad$ for $\quad x \leq -1 \quad$ or $\quad x \geq 1$

The other relationships will be found in the exercises and are summarized below:

INVERSE SECANT, INVERSE COSECANT, AND INVERSE COTANGENT IDENTITIES

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right) \qquad \text{for} \qquad x \leq -1 \text{ or } x \geq 1$$

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right) \qquad \text{for} \qquad x \leq -1 \text{ or } x \geq 1$$

$$\cot^{-1}x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) & \text{for} \quad x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right) & \text{for} \quad x < 0 \end{cases}$$

EXAMPLE 8 **Using Inverse Identities**

a. Find the exact value of $\sec^{-1} 2$.
b. Use a calculator to find the value of $\cot^{-1} 7$.

Solution (a):

Let $\theta = \sec^{-1} 2$. $\qquad \sec\theta = 2$ on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Substitute the reciprocal identity. $\qquad \dfrac{1}{\cos\theta} = 2$

Solve for $\cos\theta$. $\qquad \cos\theta = \dfrac{1}{2}$

The restricted interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ corresponds to quadrants I and II.

The cosine function is positive in quadrant I. $\qquad \theta = \dfrac{\pi}{3}$

$$\boxed{\sec^{-1} 2 = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}$$

Solution (b):

Since we do not know an exact value that would correspond to the cotangent function equal to 7, we proceed using identities and a calculator.

Select the correct identity, given that $x = 7 > 0$. $\qquad \cot^{-1} x = \tan^{-1}\left(\dfrac{1}{x}\right)$

Let $x = 7$. $\qquad \cot^{-1} 7 = \tan^{-1}\left(\dfrac{1}{7}\right)$

Evaluate the right side with a calculator. $\qquad \boxed{\cot^{-1} 7 \approx 8.13°}$

Technology Tip

Use the inverse trigonometry function identities to find

a. $\sec^{-1} 2 = \cos^{-1}\left(\frac{1}{2}\right)$

b. $\cot^{-1} 7 = \tan^{-1}\left(\frac{1}{7}\right)$

```
cos⁻¹(1/2)
        1.047197551
π/3
        1.047197551
```

```
tan⁻¹(1/7)
        .1418970546
Ans*180/π
        8.130102354
```

Finding Exact Values for Expressions Involving Inverse Trigonometric Functions

We will now find exact values of trigonometric expressions that involve inverse trigonometric functions.

EXAMPLE 9 Finding Exact Values of Trigonometric Expressions Involving Inverse Trigonometric Functions

Find the exact value of $\cos\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$.

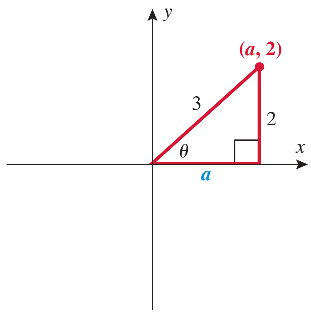
Solution:

STEP 1 Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$. $\sin\theta = \frac{2}{3}$ when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

The range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ corresponds to quadrants I and IV.

The sine function is positive in quadrant I.

STEP 2 Draw angle θ in quadrant I.



Label the sides known from the sine value. $\sin\theta = \frac{2}{3} = \frac{\text{opposite}}{\text{hypotenuse}}$

STEP 3 Find the unknown side length a. $a^2 + 2^2 = 3^2$

Solve for a. $a = \pm\sqrt{5}$

Since θ is in quadrant I, a is positive. $a = \sqrt{5}$

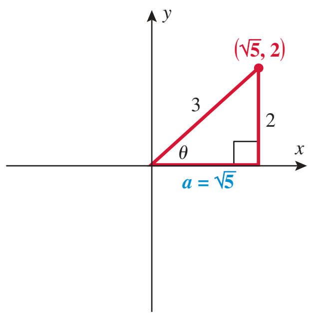
STEP 4 Find $\cos\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$.

Substitute $\theta = \sin^{-1}\left(\frac{2}{3}\right)$. $\cos\left[\sin^{-1}\left(\frac{2}{3}\right)\right] = \cos\theta$

Find $\cos\theta$.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3}$$

$$\boxed{\cos\left[\sin^{-1}\left(\frac{2}{3}\right)\right] = \frac{\sqrt{5}}{3}}$$



■ **YOUR TURN** Find the exact value of $\sin\left[\cos^{-1}\left(\frac{1}{3}\right)\right]$.

■ **Answer:** $\frac{2\sqrt{2}}{3}$

EXAMPLE 10 Finding Exact Values of Trigonometric Expressions Involving Inverse Trigonometric Functions

Find the exact value of $\tan\left[\cos^{-1}\left(-\frac{7}{12}\right)\right]$.

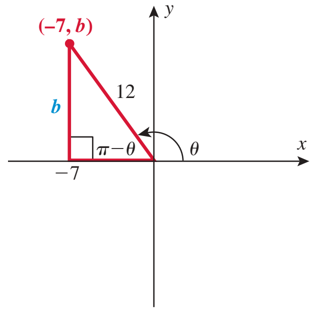
Solution:

STEP 1 Let $\theta = \cos^{-1}\left(-\frac{7}{12}\right)$. $\qquad \cos\theta = -\dfrac{7}{12}$ when $0 \le \theta \le \pi$

The range $0 \le \theta \le \pi$ corresponds to quadrants I and II.

The cosine function is negative in quadrant II.

STEP 2 Draw angle θ in quadrant II.



Label the sides known from the cosine value. $\qquad \cos\theta = -\dfrac{7}{12} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

STEP 3 Find the length of the unknown side b. $\qquad b^2 + (-7)^2 = 12^2$

Solve for b. $\qquad b = \pm\sqrt{95}$

Since θ is in quadrant II, b is positive. $\qquad b = \sqrt{95}$

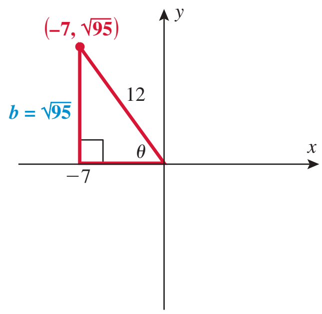
STEP 4 Find $\tan\left[\cos^{-1}\left(-\frac{7}{12}\right)\right]$.

Substitute $\theta = \cos^{-1}\left(-\frac{7}{12}\right)$. $\qquad \tan\left[\cos^{-1}\left(-\dfrac{7}{12}\right)\right] = \tan\theta$

Find $\tan\theta$.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{95}}{-7}$$

$$\boxed{\tan\left[\cos^{-1}\left(-\frac{7}{12}\right)\right] = -\frac{\sqrt{95}}{7}}$$



■ **YOUR TURN** Find the exact value of $\tan\left[\sin^{-1}\left(-\frac{3}{7}\right)\right]$.

■ **Answer:** $-\dfrac{3\sqrt{10}}{20}$

EXAMPLE 11 **Using Identities to Find Exact Values of Trigonometric Expressions Involving Inverse Trigonometric Functions**

Find the exact value of $\cos\left[\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}1\right]$.

Solution:

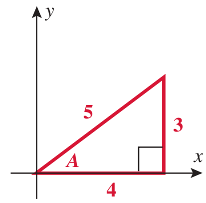
Recall the cosine sum identity: $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Let $A = \sin^{-1}\left(\frac{3}{5}\right)$ and $B = \tan^{-1}1$.

$$\cos\left[\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}1\right] = \cos\left[\sin^{-1}\left(\frac{3}{5}\right)\right]\cos(\tan^{-1}1) - \sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right]\sin(\tan^{-1}1)$$

From the figure,

$$A = \sin^{-1}\left(\tfrac{3}{5}\right) \Rightarrow \sin A = \tfrac{3}{5}$$



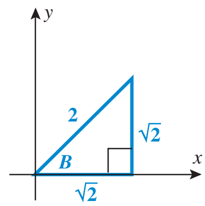
we see that

$$\cos\left[\sin^{-1}\left(\frac{3}{5}\right)\right] = \cos A = \frac{4}{5}$$

$$\sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right] = \sin A = \frac{3}{5}$$

From the figure,

$$B = \tan^{-1}(1) \Rightarrow \tan B = 1$$



we see that

$$\cos(\tan^{-1}1) = \cos B = \frac{\sqrt{2}}{2}$$

$$\sin(\tan^{-1}1) = \sin B = \frac{\sqrt{2}}{2}$$

Substitute these values into the cosine sum identity:

$$\cos\left[\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}1\right] = \left(\frac{4}{5}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{3}{5}\right)\left(\frac{\sqrt{2}}{2}\right)$$

Simplify.

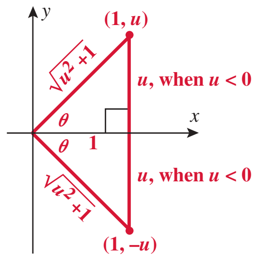
$$= \boxed{\frac{\sqrt{2}}{10}}$$

YOUR TURN Find the exact value of $\sin\left[\cos^{-1}\left(\frac{3}{5}\right) + \tan^{-1}1\right]$.

Answer: $\frac{7\sqrt{2}}{10}$

EXAMPLE 12 Writing Trigonometric Expressions Involving Inverse Trigonometric Functions in Terms of a Single Variable

Write the expression $\cos(\tan^{-1}u)$ as an equivalent expression in terms of only the variable u.



Solution: Let $\theta = \tan^{-1}u$; therefore, $\tan\theta = u = \frac{u}{1}$.

Realize that u can be positive or negative. Since the range of the inverse tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, sketch the angle θ in both quadrants I and IV and draw the corresponding two right triangles. Recalling that the tangent ratio is opposite over adjacent, we label those corresponding sides with u and 1, respectively. Then solving for the hypotenuse using the Pythagorean theorem gives $\sqrt{u^2+1}$.

Substitute $\theta = \tan^{-1} u$ into $\cos(\tan^{-1}u)$. $\qquad \cos(\tan^{-1}u) = \cos\theta$

Use the right triangle ratio for cosine: adjacent over hypotenuse. $\qquad = \dfrac{1}{\sqrt{u^2+1}}$

Rationalize the denominator. $\qquad \cos(\tan^{-1}u) = \dfrac{1}{\sqrt{u^2+1}} \cdot \dfrac{\sqrt{u^2+1}}{\sqrt{u^2+1}}$

$$= \boxed{\dfrac{\sqrt{u^2+1}}{u^2+1}}$$

YOUR TURN Write the expression $\sin(\tan^{-1}u)$ as an equivalent expression in terms of only the variable u.

Answer: $\dfrac{u\sqrt{u^2+1}}{u^2+1}$

SECTION 6.5 SUMMARY

If a trigonometric function value of an angle or of a real number is known, what is that number, as defined by the domain restriction? Inverse trigonometric functions determine the angle measure (or the value of the argument). To define the inverse trigonometric relations as functions, we first restrict the trigonometric functions to domains in which they are one-to-one functions. Exact values for inverse trigonometric functions can be found when the function values are those of the special angles. Inverse trigonometric functions also provide a means for evaluating one trigonometric function when we are given the value of another. It is important to note that the -1 as a superscript indicates an inverse function, not a reciprocal.

INVERSE FUNCTION	$y = \sin^{-1}x$	$y = \cos^{-1}x$	$y = \tan^{-1}x$	$y = \cot^{-1}x$	$y = \sec^{-1}x$	$y = \csc^{-1}x$
DOMAIN	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
RANGE	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(0, \pi)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
GRAPH